ASYMPTOTIC STUDY OF AN ANISOTROPIC FOKKER-PLANCK COLLISION OPERATOR IN A STRONG MAGNETIC FIELD

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Abstract. The present paper is concerned with the derivation, via asymptotic studies, of a reduced hybrid model describing the anisotropic fusion plasma dynamics in tokamaks. The parallel dynamics is governed by a kinetic equation, whereas the perpendicular dynamics is described by a Maxwellian distribution function, whose temperature $T_\perp$ satisfies an evolution equation, exchanging information with the parallel direction via some coupling terms. The reduced model is obtained from the underlying fully kinetic model, under the assumption of a strong magnetic field and strong collisionality in the perpendicular direction. From a numerical point of view, reduced models are very advantageous, permitting significant savings in computational times and memory. To improve the precision of the reduced description, we propose in this paper also first order correction terms with respect to the parameter describing the anisotropy, and discuss these terms from a physical point of view. This first order truncated model is new to our knowledge, meets the desired requirements of precision and efficiency, and its derivation is clearly exposed in this work, based on formal asymptotic studies.

Keywords: Anisotropic fusion plasma, strong magnetic fields, strong perpendicular collisionality, Hilbert expansion, asymptotic limits, truncation, hybrid kinetic/fluid model.

1. Introduction

Strong anisotropy is naturally present in a broad variety of plasma phenomena. For example, space and laboratory plasma in strong magnetic fields exhibit different properties parallel and perpendicular to the magnetic field lines. To accurately describe such situations, quantities are usually decomposed into distinct parallel and perpendicular components, such as $T_\parallel$ and $T_\perp$ for the temperature. These quantities can evolve according to rather different evolution equations, which are coupled in order to permit exchanges (of energy, momentum, *etc.*) between the parallel and perpendicular directions. For instance, in strongly-magnetized plasma, the diffusion of thermal energy along magnetic field lines can be orders of magnitude faster than across field lines [30], and it seems questionable to describe then such anisotropic dynamics with a set of equations based on isotropic assumptions. One of the first studies about the (anisotropic) form of the plasma distribution function in strong magnetic fields has been given in [18], leading to the famous Chew-Goldberger-Low (CGL) expression of the pressure tensor. Since then, many studies followed making use of specific anisotropic plasma distribution functions, of which we can mention here only a small sample: non-linear development of electromagnetic instabilities [21], Whistler instabilities [40], mirror and ion cyclotron instabilities [24], and transport equations for multi-component space plasmas [5], among many others. In many of these models, the anisotropy is manifest in the different temperatures of the velocity distribution function along and perpendicular to the magnetic field. Describing these anisotropic phenomena is crucial for the design of plasma fusion devices, as anisotropy strongly impacts heat diffusion and hence the heat deposition on the device wall [46]. Anisotropy also affects turbulence [45], instabilities, the propagation of energetic particles, *etc.* Briefly, anisotropy is one of the key aspects to be taken into account in electromagnetic plasma simulations.

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A precise study of such anisotropic plasma dynamics starts with the fully kinetic description, in which each plasma species of charge q_s and mass m_s is described by the Boltzmann-type equation,

$$\partial_t f_s + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_s = \sum_r Q_{sr}(f_s, f_r) \,, \tag{1}$$

for the particle distribution function $f_s(t, \mathbf{x}, \mathbf{v})$ defined on position-velocity phase-space. The long-range interactions between the species are mediated by the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, satisfying Maxwell's equations. The short-range interactions are modeled via collision operators Q_{sr}, accounting for the thermodynamic processes such as entropy decay and thermal equilibration. In the presence of a strong magnetic field $\mathbf{B}$, charged particles will execute helical movements around the field lines, their dynamics being thus constrained in the perpendicular direction with respect to the magnetic field, whereas in the parallel direction particles move freely. This is the simplest explanation of the creation of anisotropy.

The distribution functions f_s contain all the information about the plasma dynamics. However, solving numerically the whole kinetic system (1) coupled with Maxwell's equation is out of reach for today's computers. The multiscale nature of the problem requires indeed prohibitively high spatial and velocity resolutions when standard schemes are used. However, taking advantage of the high anisotropy of the problem could help to design multiscale schemes or to derive reduced models, yielding computational savings. One scenario where reduced models have been fruitfully employed is the study of the propagation of edge-localized modes (ELMs) in the scrape-off layer (SOL) of Tokamak fusion devices [33,35]. In the latest version of these works [20], the distribution functions of electrons and ions describing the ELMs were assumed of the following form:

$$f_s(t, x, v_\parallel, \mathbf{v}_\perp) = g_s(t, x, v_\parallel) \frac{m_s}{2\pi k_B T_{s,\perp}(t, x)} \exp\left(-\frac{m_s |\mathbf{v}_\perp|^2}{2 k_B T_{s,\perp}(t, x)} \right). \tag{2}$$

Here, f_s is the product of a reduced $1\mathrm{D}_x 1\mathrm{D}_v$ kinetic distribution function g_s in the parallel velocity direction and a $1\mathrm{D}_x 2\mathrm{D}_v$ Maxwellian distribution function in the perpendicular velocity direction. The constant $k_B > 0$ denotes the Boltzmann constant, and $m_s > 0$ denotes the mass of one element of the species s. In [20] the authors propose a coupled system of PDEs for the evolution of g_s and $T_{s,\perp}$, permitting the exchange of thermal energy via a simplified (BGK) collision operator. The above mentioned models were often introduced without a clear mathematical derivation. In this paper, we aim to derive the form of the distribution function f_s given in (2) as an asymptotic limit solution of equation (1) in a suitable scaling reflecting the strong anisotropy. Moreover, in this process we shall also derive first order corrections to the limit solution; these will include the well-known plasma drifts across the magnetic field, and thus enhance the physics content of the reduced model. The exact interplay between parallel kinetic and perpendicular fluid aspects of the model shall furthermore be underlined in this work.

What are the physical processes that could lead to a distribution function of the form (2)? First of all, Maxwellian velocity distributions arise from collisions. Therefore, we shall assume a high collisionality in the perpendicular velocity directions $\mathbf{v}_\perp$. By contrast, in the parallel velocity direction the distribution function $g_s(v_\parallel)$ is not necessarily in thermal equilibrium, thus subject to far less collisions. This points to an anisotropy in the collisional frequencies, which we shall take into account in our modelling. Moreover, we will investigate strongly magnetized plasma, where the Lorentz force is dominated by the magnetic field term.

A more precise description than (2) of a magnetized plasma is provided by gyrokinetic theory [16, 31, 42]. There the distribution function is assumed to be of the form $f(t, \mathbf{x}, v_\parallel, \mu)$, with $\mu := |\mathbf{v}_\perp|^2/(2|\mathbf{B}|)$ the magnetic moment. However in gyrokinetic theory it is not assumed that one has strong collisions in the perpendicular direction such that the whole distribution function remains kinetic. The effect of the strong magnetic field is merely the reduction to a $3\mathrm{D}_x\,2\mathrm{D}_v$ kinetic distribution function, where μ is however an adiabatic invariant. In situation (2) which is studied here, the distribution function in the direction perpendicular to $\mathbf{B}$ has a Maxwellian form, reducing thus further the complexity of the problem. Indeed, solving the gyrokinetic equation is more demanding (but also more precise) than solving the truncated hybrid kinetic/fluid model we shall present in this paper. Thus, the aim of our paper is to obtain via asymptotic arguments such a reduced hybrid model, which is often encountered in literature to further reduce the numerical complexity of the resolution of a full kinetic or gyrokinetic equation.

This is thus the physical context we are interested in. From now on, we shall consider only a single plasma species in a given electromagnetic field $\mathbf{E}$, $\mathbf{B} : \mathbb{R}_+ \times \mathbb{T}^3_x \to \mathbb{R}^3$ with $\mathbf{B} = B(t,\mathbf{x})\mathbf{e}_z$ pointing along the z-direction (periodic in $\mathbf{x}$). We shall assume that this magnetic field is non-vanishing, namely

$$B(t,\mathbf{x}) > 0 \qquad (t,\,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{T}^3_x\,. \tag{3}$$

The starting point of the present work is hence the following *Vlasov-Fokker-Planck equation* (VFP),

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \frac{q}{m}\left(\mathbf{E} + \mathbf{v}\times B\mathbf{e}_z\right)\cdot\nabla_{\mathbf{v}} f = \nu_\perp\, Q_\perp(f) + \nu_{\mathrm{r}}\, Q_{\mathrm{r}}(f)\,, \tag{4}$$

for the particle distribution function $f : \mathbb{R}_+ \times \mathbb{T}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$, with the collision frequencies $\nu_{\perp,\mathrm{r}} > 0$. As mentioned above, one key idea of this work is to single out the specific collisions in the plane perpendicular to the magnetic field lines, modelled here by a nonlinear *Fokker-Planck operator*,

$$Q_\perp(f) := \nabla_{\mathbf{v}_\perp}\cdot\left[(\mathbf{v}_\perp - \mathbf{u}_\perp)\,f + \frac{k_{\mathrm{B}}T_\perp}{m}\,\nabla_{\mathbf{v}_\perp} f\right], \tag{5}$$

where $\mathbf{v}_\perp := (v_x,\,v_y)^t \in \mathbb{R}^2_v$, while the mean perpendicular velocity $\mathbf{u}_\perp$ and the perpendicular temperature $T_\perp$ depend on f in the following way:

$$\begin{aligned} n(t,x) &:= \int_{\mathbb{R}^3_v} f\;\mathrm{d}\mathbf{v}\,,\\ n\,\mathbf{u}_\perp(t,\mathbf{x}) &:= \int_{\mathbb{R}^3_v} \mathbf{v}_\perp\, f\;\mathrm{d}\mathbf{v}\,,\\ n\,k_{\mathrm{B}}\,T_\perp(t,\mathbf{x}) &:= \frac{m}{2}\int_{\mathbb{R}^3_v} |\mathbf{v}_\perp - \mathbf{u}_\perp|^2\, f\,\mathrm{d}\mathbf{v}\,. \end{aligned} \tag{6}$$

The remaining collision operator Q_{r} will be chosen of the form

$$Q_{\mathrm{r}}(f) := \nabla_{\mathbf{v}}\cdot\left[(\mathbf{v} - \mathbf{u})\,f + \frac{k_{\mathrm{B}}T}{m}\,\nabla_{\mathbf{v}} f\right] - Q_\perp(f)\,, \tag{7}$$

where

$$\begin{aligned} n\,\mathbf{u}(t,\mathbf{x}) &:= \int_{\mathbb{R}^3_v} \mathbf{v}\, f\;\mathrm{d}\mathbf{v}\,,\\ \frac{3}{2} n\,k_{\mathrm{B}}\,T(t,\mathbf{x}) &:= \frac{m}{2}\int_{\mathbb{R}^3_v} |\mathbf{v} - \mathbf{u}|^2\, f\,\mathrm{d}\mathbf{v}\,. \end{aligned} \tag{8}$$

This operator is nothing else than a standard isotropic Fokker-Planck operator (leading in the long time limit to a full isotropisation) minus the just introduced perpendicular collision operator, such that if both $Q_\perp(f)$ and $Q_{\mathrm{r}}(f)$ scale equally, the right-hand side of (4) is a standard isotropic collision operator; however we shall focus in this work on an anisotropic regime, as mentioned above. One can choose for $Q_{\rm r}(f)$ more general collision operators than (7), which should however satisfy the following properties:

- Preservation of mass, momentum and energy:

$$\int_{\mathbb{R}^3_v} \begin{pmatrix} 1 \\ \mathbf{v} \\ \frac{|\mathbf{v}|^2}{2} \end{pmatrix} Q_{\rm r}(f)\,{\rm d}\mathbf{v} = 0\,; \tag{9}$$

- Thermalisation between parallel and perpendicular directions:

$$m \int_{\mathbb{R}^3_v} \frac{|\mathbf{v}_\perp|^2}{2} Q_{\rm r}(f)\,{\rm d}\mathbf{v} = \eta\, n\, k_B\,(T - T_\perp) = \frac{\eta}{3}\, n\, k_B\,(T_\parallel - T_\perp)\,, \tag{10}$$

where $T = (2\,T_\perp + T_\parallel)/3$, $\eta > 0$ is some given coefficient (equal to $\eta = 2$ for $Q_{\rm r}$ defined in (7)) and

$$n\, u_\parallel(t,\mathbf{x}) := \int_{\mathbb{R}^3_v} v_\parallel\, f\ {\rm d}\mathbf{v}\,, \qquad \frac{1}{2}\, n\, k_{\rm B}\, T_\parallel(t,\mathbf{x}) := \frac{m}{2} \int_{\mathbb{R}^3_v} |v_\parallel - u_\parallel|^2\, f\,{\rm d}\mathbf{v}\,. \tag{11}$$

Altogether, equation (4) models a magnetized plasma in a given electromagnetic field, undergoing anisotropic collisions, which lead on long time scales to complete thermalisation. However, on a short time scale the two parallel and perpendicular temperatures need not necessarily be equal, and this is reflected by the choice of our collision operator. On short times of order $\mathcal{O}(\nu_\perp^{-1})$, the perpendicular energy is conserved thanks to

$$\int_{\mathbb{R}^3_v} \frac{|\mathbf{v}_\perp|^2}{2}\, Q_\perp(f)\,{\rm d}\mathbf{v} = 0\,, \tag{12}$$

which is a natural property in a high magnetic field setting [41]. On long time scales, the perpendicular energy is however not anymore conserved because of the operator $Q_{\rm r}$, which thermalizes and ensures isotropisation of the temperatures as $t \to \infty$, c.f.(10). However, we recognize that our modeling is limited in the following ways:

- The electromagnetic fields are prescribed and not solved in a self-consistent manner;
- The magnetic field is pointing in a fixed direction, and has no curvature;
- $Q_\perp$ and $Q_{\rm r}$ are modeled by differential type Fokker-Planck operators (5)-(7), which is an approximation of the more physical Rosenbluth collision operators;
- The effects of multiple species are excluded;
- The space domain is periodic.

These simplifications are not so dramatic and can be easily removed. They have been made to simplify the analysis in order to focus on the main point, namely the effects of anisotropic collisions.

In this work, we set a physical scaling (given in Appendix A) reflecting the strongly magnetized nature of the plasma and the dominance of collisions perpendicular to $\mathbf{B}$. This scaling makes apparent a small parameter $\varepsilon \ll 1$, and the obtained adimensional model reads

$$\partial_t f^\varepsilon + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^\varepsilon + \mathbf{E}\cdot\nabla_{\mathbf{v}} f^\varepsilon + \frac{1}{\varepsilon}\,(\mathbf{v}\times B\mathbf{e}_z)\cdot\nabla_{\mathbf{v}} f^\varepsilon = \frac{\nu_\perp}{\varepsilon}\, Q_\perp(f^\varepsilon) + \nu_{\rm r}\, Q_{\rm r}(f^\varepsilon)\,. \tag{13}$$

Here, the rescaled operators $Q_\perp$, $Q_{\rm r}$ are of the following form

$$\begin{aligned} Q_\perp(f^\varepsilon) &= \nabla_{\mathbf{v}_\perp}\cdot\left[(\mathbf{v}_\perp - \mathbf{u}_\perp^\varepsilon) f^\varepsilon + T_\perp^\varepsilon\, \nabla_{\mathbf{v}_\perp} f^\varepsilon\right]\,, \\ Q_{\rm r}(f^\varepsilon) &:= \nabla_{\mathbf{v}}\cdot\left[(\mathbf{v} - \mathbf{u}^\varepsilon)\, f^\varepsilon + T^\varepsilon\,\nabla_{\mathbf{v}} f^\varepsilon\right] - Q_\perp(f^\varepsilon)\,, \end{aligned} \tag{14}$$

where $\mathbf{u}_\perp^\varepsilon$, $\mathbf{u}^\varepsilon$, $T_\perp^\varepsilon$ and $T^\varepsilon := (2T_\perp^\varepsilon + T_\parallel^\varepsilon)/3$ depend on f^ε in the following way:

$$\begin{aligned}
n^\varepsilon(t,x) &:= \int_{\mathbb{R}^3_v} f^\varepsilon \, \mathrm{d}\mathbf{v}\,, \\
n^\varepsilon\, \mathbf{u}_\perp^\varepsilon(t,\mathbf{x}) &:= \int_{\mathbb{R}^3_v} \mathbf{v}_\perp\, f^\varepsilon \, \mathrm{d}\mathbf{v}\,, \qquad & n^\varepsilon\, \mathbf{u}^\varepsilon(t,\mathbf{x}) &:= \int_{\mathbb{R}^3_v} \mathbf{v}\, f^\varepsilon \, \mathrm{d}\mathbf{v}\,, \\
n^\varepsilon\, T_\perp^\varepsilon(t,\mathbf{x}) &:= \frac{1}{2}\int_{\mathbb{R}^3_v} |\mathbf{v}_\perp - \mathbf{u}_\perp^\varepsilon|^2 \, f^\varepsilon \, \mathrm{d}\mathbf{v}\,, \qquad & \frac{1}{2} n^\varepsilon\, T_\parallel^\varepsilon(t,\mathbf{x}) &:= \frac{1}{2}\int_{\mathbb{R}^3_v} |v_\parallel - u_\parallel^\varepsilon|^2 \, f^\varepsilon \, \mathrm{d}\mathbf{v}\,.
\end{aligned} \tag{15}$$

This equation is supplemented with a suitable, well-prepared initial condition $f^\varepsilon(t=0) = f^\varepsilon_{\text{in}}$. We do not treat in this work with the possible occurrence of initial layers.

Our goal is to find approximate solutions to equation (13) in the regime $\varepsilon \ll 1$. The analysis performed in this work is mostly formal and based on a Hilbert expansion:

$$f^\varepsilon = f^0 + \varepsilon\, f^1 + \varepsilon^2\, f^2 + \dots$$

This ansatz leads to a hierarchy of coupled equations for the coefficients f^0, f^1, ... The analysis of this hierarchy requires a careful study of the dominant operator in equation (13), and its linearized version. The study of the latter is performed in a rigorous manner, using well known techniques coming from the isotropic functional analysis framework. A careful truncation permits to get a reduced model corresponding to (13) in the $\varepsilon \ll 1$ regime. The interested reader is referred to [22,34] for a comprehensive introduction on those methods and on Hilbert expansions.

The field of asymptotic analysis in strongly magnetized plasma is very active. One can for instance cite [27,44] for the study of the Vlasov-Poisson system in the case of strong magnetic field, [1,14] for Vlasov-Poisson-Boltzmann, or even [13] for the coupling with Maxwell equations. Multi-species plasma also feature a small electron-ion mass ratio and small Debye length; these singular parameters also can be taken into account in scaling assumptions. See [23,32,36,37] for the former, and [19,28,29] for instance for the latter. Various assumptions can be made about these parameters, leading to various types of scaling. Some of them are very well studied, such as the hydrodynamic scaling [7,10,22,26,39], the drift-diffusion scaling [2,3,6,8,9,25,43], or the high-field scaling [4,12,38].

The document is organized as follows: in Section 2 we state the main results, discuss them and give some notation. We then write in subsection 2.2 the Hilbert hierarchy. We give in Section 3 the proof of Theorem 1, which concerns the limit model. Section 4 deals with the proof of Theorem 2, concerning the first order correction. Finally in Section 5, we summarize and give an outlook on future applications and improvements of the current work. To simplify the presentation, the Appendix regroups the scaling procedure along with cumbersome proofs and computations.

2. Main results

This section contains the two main results of the present work:

(1) The first result concerns the derivation of the limit model which approximates the kinetic equation (13) in the asymptotics $\varepsilon \to 0$. The limit model is stated in Theorem 1, followed by a discussion of its key aspects.
(2) The second result addresses the derivation of first order correction terms to the limit model. The corresponding truncated system is stated in Theorem 2, followed by a discussion of its key aspects and novelties.

Finally, we decided to rewrite the first-order model (from Theorem 2), in physical units in section 2.3 (with ε-scaling removed).

In all of the following, we will denote by

$$\langle\theta\rangle_\perp := \int_{\mathbb{R}_v^2} \theta \, \mathrm{d}\mathbf{v}_\perp \,, \qquad \langle\theta\rangle_\parallel := \int_{\mathbb{R}_v} \theta \, \mathrm{d}v_\parallel \,, \qquad \langle\theta\rangle := \int_{\mathbb{R}_v^3} \theta \, \mathrm{d}\mathbf{v} \,,$$

the integration against the orthogonal, parallel and total velocity variable, respectively. One also introduces the following notation for any vector field X taking values in $\mathbb{R}^2$:

$$X^\top := X \times \mathbf{e}_z := \begin{pmatrix} v_y \\ -v_x \end{pmatrix} . \tag{16}$$

2.1. Main results.

Theorem 1 (Limit model). *In the limit $\varepsilon \to 0$, the solution f^ε of (13) converges (formally) to a function f^0 of the following factorized form*

$$f^0(t, \mathbf{x}, \mathbf{v}) = g^0(t, \mathbf{x}, v_\parallel) \, \mathcal{M}_\perp^{T_\perp^0(t,\mathbf{x})}(\mathbf{v}_\perp) \,, \tag{17a}$$

where the perpendicular Maxwellian $\mathcal{M}_\perp^{T_\perp^0(t,\mathbf{x})}$ is given by

$$\mathcal{M}_\perp^{T_\perp^0(t,\mathbf{x})}(\mathbf{v}_\perp) := \frac{1}{2\,\pi\, T_\perp^0(t,\mathbf{x})} \, e^{-\frac{|\mathbf{v}_\perp|^2}{2\, T_\perp^0(t,\mathbf{x})}} . \tag{17b}$$

The "reduced kinetic distribution g^0" and the perpendicular temperature $T_\perp^0$ satisfy the system

$$\begin{cases} \partial_t g^0 + v_\parallel \, \partial_z g^0 + E_\parallel \, \partial_{v_\parallel} g^0 = \nu_{\mathrm{r}} \, \partial_{v_\parallel} \left[(v_\parallel - u_\parallel^0) \, g^0 + T^0 \, \partial_{v_\parallel} g^0 \right] , \\[2ex] \partial_t (n^0 T_\perp^0) + \partial_z (n^0 \, T_\perp^0 \, u_\parallel^0) = \dfrac{2}{3} \, \nu_{\mathrm{r}} \, n^0 \, \left(T_\parallel^0 - T_\perp^0 \right) \,, \end{cases} \tag{17c}$$

where $T^0 = (T_\parallel^0 + 2\, T_\perp^0)/3$ and

$$n^0 = \langle g^0 \rangle_\parallel \,, \qquad n^0 \, u_\parallel^0 = \left\langle v_\parallel \, g^0 \right\rangle_\parallel \,, \qquad \frac{1}{2} n^0 \, T_\parallel^0 = \frac{1}{2} \left\langle (v_\parallel - u_\parallel^0)^2 \, g^0 \right\rangle_\parallel . \tag{18}$$

This system (17) is supplemented with the following well-prepared initial condition

$$f_{\mathrm{in}}^0(\mathbf{x}, \mathbf{v}) = g_{\mathrm{in}}^0(\mathbf{x}, v_\parallel) \, \mathcal{M}_\perp^{T_{\perp,\mathrm{in}}^0(\mathbf{x})}(\mathbf{v}_\perp) \,, \tag{19}$$

where $f_{\mathrm{in}}^0 := \lim_{\varepsilon\to 0} f_{\mathrm{in}}^\varepsilon$.

Let us now discuss some key aspects of Theorem 1:

- In the $\varepsilon \to 0$ limit, the particle distribution function f^ε decomposes exactly as a product of a reduced kinetic distribution function g^0 modelling the parallel transport, and a Maxwellian distribution in the perpendicular variable, depending only on the perpendicular temperature $T_\perp^0$ (17a).
- The two quantities g^0 and $T_\perp^0$ satisfy the coupled system of PDEs (17c). The reduced distribution function g^0 satisfies a $1\mathrm{D}_x 1\mathrm{D}_v$ kinetic equation along the magnetic field lines. The perpendicular temperature is advected along the field lines by the bulk velocity $u_\parallel^0$ associated with g^0.
- Moreover, there is a coupling responsible for the energy exchanges between the parallel and perpendicular directions, represented by the Fokker-Planck term and the relaxation term $n^0(T_\parallel^0 - T_\perp^0)$. On long time scales, these terms lead to isotropisation between $T_\parallel^0$ and $T_\perp^0$.

- Setting $E_\parallel \equiv 0$ in order to not inject energy into the system, the system (17c) exactly conserves the energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3_x}\left[\frac{3}{2}\,n^0\,T^0+\frac{1}{2}\,n^0\,(u^0_\parallel)^2\right]\,\mathrm{d}\mathbf{x}=0\,.$$

- The limit model (17c) is exactly the one described in [20], without source terms. The temperature equation can be rewritten in non-conservative form

$$\partial_t T^0_\perp + u^0_\parallel\,\partial_z T^0_\perp = \frac{2}{3}\,\nu_{\mathrm{r}}\,(T^0_\parallel - T^0_\perp)\,, \tag{20}$$

which is obtained from the kinetic equation in (17c) by using the conservation law

$$\partial_t n^0 + \partial_z(n^0\,u^0_\parallel) = 0\,. \tag{21}$$

This formulation corresponds to the one given in [20].

Theorem 2 (**First order corrections**). *Let f^ε be the solution of equation (13). Then, one can approximate the distribution function f^ε by $\widehat{f}^\varepsilon$, such that*

$$f^\varepsilon = \widehat{f}^\varepsilon + \mathcal{O}(\varepsilon^2)\,, \qquad as \qquad \varepsilon\to 0\,,$$

and $\widehat{f}^\varepsilon$ has the following form (for notational reasons we kept the index ε in the factorisation)

$$\widehat{f}^\varepsilon(t,\mathbf{x},\mathbf{v}) = \widehat{g}(t,\,\mathbf{x},\,v_\parallel)\,\mathcal{M}_\perp^{\widehat{T}_\perp(t,\mathbf{x})}(\mathbf{v}_\perp)\left(1+\varepsilon\,\Lambda_{\widehat{g},\widehat{T}_\perp}\right)\,. \tag{22a}$$

The "reduced kinetic distribution" $\widehat{g}$ and the perpendicular temperature $\widehat{T}_\perp$ satisfy the coupled system

$$\left\{\begin{aligned} &\partial_t\widehat{g} + v_\parallel\,\partial_z\widehat{g} + E_\parallel\,\partial_{v_\parallel}\widehat{g} + \varepsilon\,\nabla_{\mathbf{x}_\perp}\cdot(\widehat{\mathbf{u}}^{\mathrm{K}}_{\mathrm{drift}}\,\widehat{g}) = \nu_{\mathrm{r}}\,\partial_{v_\parallel}\left[(v_\parallel-\widehat{u}_\parallel)\,\widehat{g} + \widehat{T}\,\partial_{v_\parallel}\widehat{g}\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \varepsilon\,\nabla_{\mathbf{x}_\perp}\cdot\left(\widehat{n}\,\widehat{T}_\perp\,\mathbb{D}_2\nabla_{\mathbf{x}_\perp}\frac{\widehat{g}}{\widehat{n}}\right), \\ &\partial_t(\widehat{n}\,\widehat{T}_\perp) + \partial_z(\widehat{n}\,\widehat{T}_\perp\,\widehat{u}_\parallel) + \varepsilon\,\nabla_{\mathbf{x}_\perp}\cdot(2\,\widehat{n}\,\widehat{T}_\perp\,\widehat{\mathbf{u}}_{\mathrm{drift}}) + \varepsilon\,\nabla_{\mathbf{x}}\cdot\mathbf{q} = \varepsilon\,\widehat{n}\,\widehat{\mathbf{u}}_{\mathrm{drift}}\cdot\mathbf{E}_\perp \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \frac{2}{3}\,\nu_{\mathrm{r}}\,\widehat{n}\,(\widehat{T}_\parallel - \widehat{T}_\perp)\,. \end{aligned}\right. \tag{22b}$$

Here, the drift velocities are given by

$$\widehat{\mathbf{u}}^{\mathrm{K}}_{\mathrm{drift}} = \frac{\mathbf{E}_\perp\times\mathbf{B}}{|B|^2} - \frac{\nabla_{\mathbf{x}_\perp}(\widehat{g}\,\widehat{T}_\perp)\times\mathbf{B}}{\widehat{g}\,|B|^2}, \qquad \widehat{\mathbf{u}}_{\mathrm{drift}} = \frac{\mathbf{E}_\perp\times\mathbf{B}}{|B|^2} - \frac{\nabla_{\mathbf{x}_\perp}(\widehat{n}\,\widehat{T}_\perp)\times\mathbf{B}}{\widehat{n}\,|B|^2}, \tag{22c}$$

the temperature $\widehat{T}$ is computed via $\widehat{T} = (\widehat{T}_\parallel + 2\,\widehat{T}_\perp)/3$, the macroscopic quantities $(\widehat{n},\,\widehat{u}_\parallel,\,\widehat{T}_\parallel)$ are defined through

$$\widehat{n} = \langle\widehat{g}\rangle_\parallel, \qquad \widehat{n}\,\widehat{u}_\parallel = \langle v_\parallel\widehat{g}\rangle_\parallel, \qquad \frac{1}{2}\widehat{n}\,\widehat{T}_\parallel = \frac{1}{2}\left\langle(v_\parallel-\widehat{u}_\parallel)^2\,\widehat{g}\right\rangle_\parallel, \tag{22d}$$

and the heat flux is given by

$$\mathbf{q} = \begin{pmatrix}\mathbf{q}_\perp\\ q_\times\end{pmatrix}, \qquad \mathbf{q}_\perp = -2\,\widehat{n}\,\widehat{T}_\perp\,\mathbb{D}_1\,\nabla_{\mathbf{x}_\perp}\widehat{T}_\perp, \qquad q_\times = -\frac{1}{2\,\nu_\perp}\,\widehat{n}\,\widehat{T}_\parallel\,\partial_z\widehat{T}_\perp. \tag{22e}$$

The quantity $\Lambda_{\widehat{g},\widehat{T}_\perp}$ *is a polynomial quantity in* $\mathbf{v}_\perp$ *entirely defined in terms of the quantities* $\widehat{g}, \widehat{T}_\perp$ *through*

$$\Lambda_{\widehat{g},\widehat{T}_\perp}(\mathbf{v}) := \frac{\widehat{\mathbf{u}}^{\mathrm{K}}_{\mathrm{drift}} \cdot \mathbf{v}_\perp}{\widehat{T}_\perp} - \left(\mathbb{D}_1 \frac{\nabla_{\mathbf{x}_\perp} \widehat{T}_\perp}{\widehat{T}_\perp}\right) \cdot \mathbf{v}_\perp \left[\frac{|\mathbf{v}_\perp|^2}{2\,\widehat{T}_\perp} - 2\right] + \frac{\partial_z \widehat{T}_\perp}{\widehat{T}_\perp} \frac{\widehat{u}_\parallel - v_\parallel}{2} \left[\frac{|\mathbf{v}_\perp|^2}{2\,\widehat{T}_\perp} - 1\right] - \left(\mathbb{D}_2 \frac{\nabla_{\mathbf{x}_\perp}(\widehat{g}/\widehat{n})}{(\widehat{g}/\widehat{n})}\right) \cdot \mathbf{v}_\perp, \tag{22f}$$

where the positive diffusion matrices are given by

$$\mathbb{D}_1 = \frac{1}{B^2 + 9\,\nu_\perp^2} \begin{bmatrix} 3\,\nu_\perp & B \\ -B & 3\,\nu_\perp \end{bmatrix}, \qquad \mathbb{D}_2 = \frac{\nu_\perp}{B} \frac{1}{B^2 + \nu_\perp^2} \begin{bmatrix} B & -\nu_\perp \\ \nu_\perp & B \end{bmatrix}. \tag{22g}$$

This system (22b) *is supplemented with the following well-prepared initial condition*

$$f^\varepsilon_{\mathrm{in}}(\mathbf{x}, \mathbf{v}) = g^\varepsilon_{\mathrm{in}}(\mathbf{x}, v_\parallel)\, \mathcal{M}_\perp^{T^\varepsilon_{\perp,\mathrm{in}}}(\mathbf{v}_\perp) \left(1 + \varepsilon\, \Lambda_{g^\varepsilon_{\mathrm{in}}, T^\varepsilon_{\perp,\mathrm{in}}}\right). \tag{22h}$$

Let us comment on the key aspects of Theorem 2:

- Setting $\varepsilon = 0$ in (22b) gives back the limit model from Theorem 1.
- The asymptotic form of $\widehat{f}^\varepsilon$ (22a) resembles (2) (which was given in [20]). In our case, however, there is the additional correction term $\Lambda_{\widehat{g},\widehat{T}}$, given in (22f), which destroys the product structure with respect to $(v_\parallel, \mathbf{v}_\perp)$ in the distribution function.
- The system of PDEs (22b) satisfied by $(\widehat{g}, \widehat{T}_\perp)$ has a higher dimensionality $3\mathrm{D}_x 1\mathrm{D}_v$ than the limit model ($1\mathrm{D}_x 1\mathrm{D}_v$). This is a) due to the perpendicular plasma drifts that occur at first order in ε in the present scaling, and b) due to perpendicular diffusion arising from collisions $Q_\perp$ at first order. Such diffusion terms are typical first-order corrections in fluid models - the Navier-Stokes equations being the prime example for first-order corrections to Euler equations.
- In the fluid equation for $\widehat{T}_\perp$, one observes the classical plasma drift $\widehat{\mathbf{u}}_{\mathrm{drift}}$, which is the sum of the $\mathbf{E} \times \mathbf{B}$ drift and the diamagnetic drift. Moreover, in the kinetic equation for $\widehat{g}$ appears the new "kinetic diamagnetic drift" term $\widehat{\mathbf{u}}^K_{\mathrm{drift}}$ which depends on $v_\parallel$ through $\widehat{g}$. The latter seems to be a quite unusual term when comparing for instance to standard guiding-center models for magnetized plasma. There, the diamagnetic drift appears only on the level of the moment equations, and not already on the kinetic level. In the model presented here, the fact that the diamagnetic drift is present in the kinetic equation suggest a sort of "hybrid character" of the model, due to the assumption of high collisionality in the perpendicular direction only.
- The temperature equation features a heat flux $\mathbf{q}$, given in (22e), of Braginskii type [15,37], composed of gyroviscous (antidiagonal) and viscous (diagonal) terms.
- One observes that there is, in the kinetic equation on $\widehat{g}$, a diffusion-type term in the $\mathbf{x}_\perp$ variable, coming from the combined effects of the magnetic field with the collision term. The diffusion frequency associated with this term scales as the matrix $\mathbb{D}_2$, which involves the two frequencies $\nu_\perp$ and qB/m. This term acts on a long time-scale, and is responsible for the homogenisation in the perpendicular plane of the macroscopic quantities. This

can be immediately seen by taking the moments of the kinetic equation:

$$\begin{cases} \partial_t \widehat{n} + \partial_z(\widehat{n}\,\widehat{u}_\parallel) + \varepsilon\,\nabla_{\mathbf{x}_\perp}\cdot(\widehat{n}\,\widehat{\mathbf{u}}_{\mathrm{drift}}) = 0\,, \\ \partial_t(\widehat{n}\,\widehat{u}_\parallel) + \partial_z(2\,\widehat{w}_\parallel) + \varepsilon\,\nabla_{\mathbf{x}_\perp}\cdot\left(\dfrac{\mathbf{E}_\perp\times\mathbf{B}}{|B|^2}\,\widehat{n}\,\widehat{u}_\parallel - \dfrac{\nabla_{\mathbf{x}_\perp}(\widehat{n}\,\widehat{u}_\parallel\,T_\perp)\times\mathbf{B}}{|B|^2}\right) - \widehat{n}\,E_\parallel \\ \qquad = \varepsilon\,\nabla_{\mathbf{x}_\perp}\cdot\left(\widehat{n}\,\widehat{T}_\perp\,\mathbb{D}_2\,\nabla_{\mathbf{x}_\perp}\,\widehat{u}_\parallel\right)\,, \\ \partial_t\widehat{w}_\parallel + \partial_z\left\langle \dfrac{{v_\parallel}^3}{2}\,\widehat{g}\right\rangle_\parallel + \varepsilon\,\nabla_{\mathbf{x}_\perp}\cdot\left(\dfrac{\mathbf{E}_\perp\times\mathbf{B}}{|B|^2}\,\widehat{w}_\parallel - \dfrac{\nabla_{\mathbf{x}_\perp}(\widehat{w}_\parallel\,T_\perp)\times\mathbf{B}}{|B|^2}\right) - \widehat{n}\,\widehat{u}_\parallel\,E_\parallel \\ \qquad = \dfrac{2}{3}\,\nu_{\mathrm{r}}\,\widehat{n}\,(\widehat{T}_\perp - \widehat{T}_\parallel) + \varepsilon\,\nabla_{\mathbf{x}_\perp}\cdot\left(\widehat{n}\,\widehat{T}_\perp\,\mathbb{D}_2\,\nabla_{\mathbf{x}_\perp}\dfrac{\widehat{w}_\parallel}{\widehat{n}}\right)\,, \end{cases} \tag{23a}$$

where we used the following notation for the second moment of $\widehat{g}$:

$$\widehat{w}_\parallel := \left\langle \frac{{v_\parallel}^2}{2}\,\widehat{g}\right\rangle_\parallel\,. \tag{24}$$

One notices that the diffusion term does not operate on n, but diffuses higher order moments in the perpendicular plane.

- Setting $\mathbf{E} = 0$ to not inject energy into the system, this first order correction model satisfies the following energy conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3_x}\left[\frac{3}{2}\,\widehat{n}\,\widehat{T} + \frac{1}{2}\,\widehat{n}\,\widehat{u}_\parallel^2\right]\mathrm{d}\mathbf{x} = 0\,.$$

One could be surprised by the fact that only the parallel kinetic energy $\frac{1}{2}\,\widehat{n}\,\widehat{u}_\parallel^2$ appears in this energy conservation. The reason why the perpendicular kinetic energy $\frac{1}{2}\,\widehat{n}\,|\varepsilon\,\widehat{\mathbf{u}}_{\mathrm{drift}}|^2$ does not appear is because it is of order $\mathcal{O}(\varepsilon^2)$. It is therefore within the range of error of the first order correction of the model.

Let us now turn to the strategy followed in this article to derive Theorems 1 and 2.

2.2. **Strategy: Hilbert hierarchy.** The proofs of Theorem 1 and 2 are based on a Hilbert expansion, which we shall present in this subsection. Let us denote by $\mathbb{A}$ the dominant operator in the full Vlasov equation (13), namely

$$\mathbb{A}(f) := (\mathbf{v}\times B\mathbf{e}_z)\cdot\nabla_{\mathbf{v}}f - \nu_\perp\,Q_\perp(f)\,, \tag{25}$$

where $Q_\perp$ is defined in (14). With this notation, equation (13) rewrites

$$\partial_t f^\varepsilon + \mathbf{v}\cdot\nabla_{\mathbf{x}}f^\varepsilon + \mathbf{E}\cdot\nabla_{\mathbf{v}}f^\varepsilon + \frac{1}{\varepsilon}\mathbb{A}(f^\varepsilon) = \nu_{\mathrm{r}}\,Q_{\mathrm{r}}(f^\varepsilon)\,. \tag{26}$$

Let us assume that the solution f^ε can be expanded in the following formal power series in ε:

$$f^\varepsilon = f^0 + \varepsilon\,f^1 + \varepsilon^2\,f^2 + \mathcal{O}(\varepsilon^3)\,.$$

Plugging this ansatz into (26) and comparing terms of the same order in ε leads to the following Hilbert hierarchy:

$$\mathbb{A}(f^0) = 0\,, \qquad : \mathcal{O}(\varepsilon^{-1}) \tag{27a}$$

$$\partial_t f^0 + \mathbf{v}\cdot\nabla_{\mathbf{x}}f^0 + \mathbf{E}\cdot\nabla_{\mathbf{v}}f^0 + \mathbb{A}^{\mathrm{lin}}_{f^0}(f^1) = \nu_{\mathrm{r}}\,Q_{\mathrm{r}}(f^0)\,, \qquad : \mathcal{O}(1) \tag{27b}$$

$$\partial_t f^1 + \mathbf{v}\cdot\nabla_{\mathbf{x}}f^1 + \mathbf{E}\cdot\nabla_{\mathbf{v}}f^1 + \mathbb{A}^{\mathrm{lin}}_{f^0}(f^2) - \nu_\perp\,\delta^2 Q_\perp(f^1) = \nu_{\mathrm{r}}\,\delta Q_{\mathrm{r}}[f^0](f^1)\,, \qquad : \mathcal{O}(\varepsilon) \tag{27c}$$

where we denoted by $\delta^2 Q_\perp(f^1)$ a remainder term, quadratic in the order one quantities (its expression is given later in (46) for the sake of shortness of the presentation), and by $\delta\mathbb{O}[f^0]$ the linearization of any nonlinear operator $\mathbb{O}$ around a function f^0, namely

$$\delta\mathbb{O}[f^0](\delta f) = \lim_{||\delta f||\to 0} \frac{\mathbb{O}(f^0+\delta f) - \mathbb{O}(f^0)}{||\delta f||} . \tag{28}$$

The linear part of the dominant operator (25) is hence

$$\mathbb{A}^{\text{lin}}_{f^0}(\delta f) := \delta\mathbb{A}[f^0](\delta f) = (\mathbf{v}\times B\mathbf{e}_z)\cdot\nabla_{\mathbf{v}}\delta f - \nu_\perp\, \delta Q_\perp[f^0](\delta f) . \tag{29}$$

The investigation of each one of the equation occurring in this hierarchy permits to get step by step information about the coefficients f^0, f^1. Section 3 deals thus with the detailed study of equations (27a)-(27b), yielding the limit model for f^0 and leading to the proof of Theorem 1. Section 4 focuses on equations (27b)-(27c), permitting to obtain f^1 and to conclude the proof of Theorem 2.

2.3. **Truncated system in physical units.** For physical and implementation purposes, we give here the first order correction model of Theorem 2 in physical units. The distribution function f is given in the considered regime by

$$f(t,\mathbf{x},\mathbf{v}) = g(t,\,\mathbf{x},\,v_\parallel)\ \mathcal{M}_\perp^{k_B\,T_\perp(t,\mathbf{x})/m}(\mathbf{v}_\perp)\,(1+\ \Lambda_{g,T_\perp})\ , \tag{30a}$$

where the Maxwellian distribution $\mathcal{M}_\perp^{k_B\,T_\perp(t,\mathbf{x})/m}$ is defined as

$$\mathcal{M}_\perp^{k_B\,T_\perp(t,\mathbf{x})/m}(\mathbf{v}_\perp) := \frac{m}{2\,\pi\,k_B\,T_\perp(t,\mathbf{x})}\, e^{-\frac{m\,|\mathbf{v}_\perp|^2}{2\,k_B\,T_\perp(t,\mathbf{x})}} .$$

The "reduced kinetic distribution g" and the perpendicular temperature $T_\perp$ satisfy the coupled system

$$\begin{cases} \partial_t g + v_\parallel\,\partial_z g + \dfrac{q}{m}E_\parallel\,\partial_{v_\parallel} g + \nabla_{\mathbf{x}_\perp}\cdot(\mathbf{u}^{\text{K}}_{\text{drift}}\,g) = \nu_{\text{r}}\,\partial_{v_\parallel}\left[(v_\parallel - u_\parallel)\,g + \dfrac{k_B\,T}{m}\partial_{v_\parallel} g\right] \\ \qquad\qquad + \nabla_{\mathbf{x}_\perp}\cdot\left(n\,\dfrac{k_B\,T_\perp}{m}\,\mathbb{D}_2\nabla_{\mathbf{x}_\perp}\dfrac{g}{n}\right), \\ \partial_t(n\,T_\perp) + \partial_z(n\,T_\perp\,u_\parallel) + \nabla_{\mathbf{x}_\perp}\cdot(2\,n\,T_\perp\,\mathbf{u}_{\text{drift}}) + \nabla_{\mathbf{x}}\cdot\mathbf{q} = n\,\dfrac{q}{k_B}\mathbf{u}_{\text{drift}}\cdot\mathbf{E}_\perp \\ \qquad\qquad + \dfrac{2}{3}\,\nu_{\text{r}}\,n\,(T_\parallel - T_\perp) . \end{cases} \tag{30b}$$

Here, the drift velocities are given by

$$\mathbf{u}^{\text{K}}_{\text{drift}} := \frac{\mathbf{E}_\perp\times\mathbf{B}}{|B|^2} - \frac{k_B}{q}\frac{\nabla_{\mathbf{x}_\perp}(g\,T_\perp)\times\mathbf{B}}{g\,|B|^2}, \qquad \mathbf{u}_{\text{drift}} := \frac{\mathbf{E}_\perp\times\mathbf{B}}{|B|^2} - \frac{k_B}{q}\frac{\nabla_{\mathbf{x}_\perp}(n\,T_\perp)\times\mathbf{B}}{n\,|B|^2}, \tag{30c}$$

the temperature T is computed through the directional temperatures via $T = (2\,T_\perp + T_\parallel)/3$, the macroscopic quantities $(n,\,u_\parallel,\,T_\parallel)$ are defined through

$$n = \langle g\rangle_\parallel, \qquad n\,u_\parallel = \langle v_\parallel g\rangle_\parallel, \qquad \frac{1}{2}n\,k_B\,T_\parallel = \frac{m}{2}\left\langle (v_\parallel - u_\parallel)^2\,g\right\rangle_\parallel , \tag{30d}$$

and the heat flux is given by

$$\mathbf{q} = \begin{pmatrix}\mathbf{q}_\perp \\ q_\times\end{pmatrix}, \qquad \mathbf{q}_\perp = -\frac{2\,k_B}{m}n\,T_\perp\,\mathbb{D}_1\,\nabla_{\mathbf{x}_\perp}T_\perp, \qquad q_\times = -\frac{k_B}{2\,m\,\nu_\perp}n\,T_\parallel\,\partial_z T_\perp. \tag{30e}$$

The quantity $\Lambda_{g,T_\perp}$ is a correction term, polynomial in $\mathbf{v}_\perp$, and entirely defined in terms of the quantities g and $T_\perp$, through

$$\Lambda_{g,T_\perp}(\mathbf{v}) := \frac{\mathbf{u}^{\mathrm{K}}_{\mathrm{drift}}\cdot \mathbf{v}_\perp}{T_\perp} - \left(\mathbb{D}_1 \frac{\nabla_{\mathbf{x}_\perp} T_\perp}{T_\perp}\right)\cdot \mathbf{v}_\perp \left[\frac{m\,|\mathbf{v}_\perp|^2}{2\,k_B\,T_\perp} - 2\right] + \frac{\partial_z T_\perp}{T_\perp}\frac{u_\parallel - v_\parallel}{2\,\nu_\perp}\left[\frac{m\,|\mathbf{v}_\perp|^2}{2\,k_B\,T_\perp} - 1\right]$$
$$- \left(\mathbb{D}_2 \frac{\nabla_{\mathbf{x}_\perp}(g/n)}{(g/n)}\right)\cdot \mathbf{v}_\perp\,. \tag{30f}$$

Finally, the positive diffusion matrices are given by

$$\mathbb{D}_1 = \frac{1}{(\frac{q\,B}{m})^2 + 9\,\nu_\perp^2}\begin{bmatrix} 3\,\nu_\perp & \frac{q\,B}{m} \\ -\frac{q\,B}{m} & 3\,\nu_\perp \end{bmatrix}, \qquad \mathbb{D}_2 = \frac{m\,\nu_\perp}{q\,B}\frac{1}{(\frac{q\,B}{m})^2 + \nu_\perp^2}\begin{bmatrix} \frac{q\,B}{m} & -\nu_\perp \\ \nu_\perp & \frac{q\,B}{m} \end{bmatrix}. \tag{30g}$$

3. Limit model

As expected from (27a), in order to fully characterize the limit distribution f^0, it is necessary to study in detail the kernel of $\mathbb{A}$. Using then the two equations (27a)-(27b) permits to get the limit model and prove Theorem 1. Let us underline that operators $Q_\perp$ and $\mathbb{A}$ act only on the velocity variable $\mathbf{v}$. Therefore in subsections 3.1 and 3.2, we shall deal with functions of $\mathbf{v}$ only. In subsection 3.3 however we shall consider the whole phase-space.

3.1. Properties of the collision operator $Q_\perp$.

The perpendicular Fokker-Planck collision operator $Q_\perp$ defined in (14) satisfies the following properties, which are easily checked:

- $Q_\perp$ can be expressed with the following alternative form

$$Q_\perp(f) = \nabla_{\mathbf{v}_\perp}\cdot\left[T_\perp\,\mathcal{M}_\perp^{\mathbf{u}_\perp,T_\perp}\,\nabla_{\mathbf{v}_\perp}\left(\frac{f}{\mathcal{M}_\perp^{\mathbf{u}_\perp,T_\perp}}\right)\right], \tag{31}$$

  where we denoted

$$\mathcal{M}_\perp^{\mathbf{u}_\perp,T_\perp}(\mathbf{v}_\perp) := \frac{1}{2\pi\,T_\perp}e^{-\frac{|\mathbf{v}_\perp - \mathbf{u}_\perp|^2}{2T_\perp}},$$

  and the macroscopic quantities $\mathbf{u}_\perp$, $T_\perp$, associated with f, are defined thanks to (6).
- Mass, momentum and energy conservation:

$$\int_{\mathbb{R}^3_v}\begin{pmatrix}1\\ \mathbf{v}_\perp \\ \frac{|\mathbf{v}_\perp|^2}{2}\end{pmatrix} Q_\perp(f)\,\mathrm{d}\mathbf{v} = 0\,, \qquad \text{and even} \qquad \int_{\mathbb{R}^2_v} Q_\perp(f)\,\mathrm{d}\mathbf{v}_\perp = 0\,. \tag{32}$$

- Entropy Decay *(H-Theorem)*:

$$\int_{\mathbb{R}^3_v} Q_\perp(f)\ln(f)\,\mathrm{d}\mathbf{v} = \int_{\mathbb{R}^3_v} Q_\perp(f)\ln\left(\frac{f}{\mathcal{M}_\perp^{\mathbf{u}_\perp,T_\perp}}\right)\mathrm{d}\mathbf{v}$$
$$= -\int_{\mathbb{R}^3_v} T_\perp\frac{(\mathcal{M}_\perp^{\mathbf{u}_\perp,T_\perp})^2}{f}\left|\nabla_{\mathbf{v}_\perp}\frac{f}{\mathcal{M}_\perp^{\mathbf{u}_\perp,T_\perp}}\right|^2\mathrm{d}\mathbf{v} \leqslant 0, \qquad \forall f > 0\,; \tag{33}$$

- Thermal equilibrium:

$$\int_{\mathbb{R}^3_v} Q_\perp(f)\ln(f)\,\mathrm{d}\mathbf{v} = 0 \quad \Leftrightarrow \quad f(\mathbf{v}) = g(v_\parallel)\,\mathcal{M}_\perp^{\mathbf{u}_\perp,T_\perp}(\mathbf{v}_\perp), \qquad \forall f > 0\,. \tag{34}$$

3.2. **Study of the dominant operator $\mathbb{A}$.** The main properties of $\mathbb{A}$ defined in (25) are regrouped in the following Proposition 1.

Proposition 1 (**Kernel of $\mathbb{A}$.**). *The kernel of the operator $\mathbb{A}$, defined in (25), namely, the set of positive functions such that $\mathbb{A}(f) = 0$, is given by*

$$\mathrm{Ker}\,\mathbb{A} = \{ f = g(v_\parallel)\mathcal{M}_\perp^{T_\perp}(\mathbf{v}_\perp)\,,\ g > 0\}\,, \tag{35}$$

where $T_\perp > 0$, and $\mathcal{M}_\perp^{T_\perp}$ is defined in (17b).

Proof. First, let us notice that if f is a function of the form

$$f(\mathbf{v}) = g(v_\parallel)\mathcal{M}_\perp^{T_\perp}(\mathbf{v}_\perp)\,,$$

then f is both

(1) radial in $\mathbf{v}_\perp$, cancelling the rotation term $(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f$,
(2) and Maxwellian in $\mathbf{v}_\perp$, leading to $Q_\perp(f) = 0$,

thus leading to $\mathbb{A}(f) = 0$.

Conversely, let us assume that $f \in \ker\mathbb{A}$, namely, assume that

$$(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f - \nu_\perp\, Q_\perp(f) \,= 0\,. \tag{36}$$

Testing (36) against $\ln(f)$ and integrating against $\,\mathrm{d}\mathbf{v}$ yields :

$$\nu_\perp \int_{\mathbb{R}^3_v} Q_\perp(f)\ln(f)\,\mathrm{d}\mathbf{v} = 0\,,$$

noticing that the term involving the magnetic field becomes zero. This implies that f is of the form

$$f(\mathbf{v}) = g(v_\parallel)\,\mathcal{M}_\perp^{\mathbf{u}_\perp, T_\perp}(\mathbf{v}_\perp)\,,$$

using (34). Plugging this expression of f in equality (36) cancels the collision term, and we are left with

$$B\, g\, \mathcal{M}_\perp^{\mathbf{u}_\perp, T_\perp}\left\{\frac{v_y\, u_x}{T_\perp} - \frac{v_x\, u_y}{T_\perp}\right\} = 0\,, \qquad \forall v_x, v_y \in \mathbb{R}_v\,,$$

where u_x, u_y are the components of $\mathbf{u}_\perp$. The identification of the coefficients of this polynomial expression in v_x, v_y gives $u_x = u_y = 0,$ leading to the required form of f.

□

Now that we characterized the kernel of $\mathbb{A}$, we are in capacity to deal with (27a). We still however need to study some properties of the linearized operator $\mathbb{A}^{\mathrm{lin}}_{f0}$ defined in (29), for the analysis of (27b). Let us state the following Lemma, which sums up the conservation properties of $\mathbb{A}^{\mathrm{lin}}_{f0}$:

Lemma 1. *The linearized version $\mathbb{A}^{\mathrm{lin}}_{f0}$ of the dominant operator satisfy the following conservation properties:*

$$\left\langle \mathbb{A}^{\mathrm{lin}}_{f0}(\xi)\right\rangle_\perp = 0\,, \qquad \left\langle |\mathbf{v}_\perp|^2 \mathbb{A}^{\mathrm{lin}}_{f0}(\xi)\right\rangle = 0\,, \tag{37}$$

for any arbitrary function ξ.

Proof of Lemma 1. The first step is to notice that these properties hold for the total operator $\mathbb{A}$ as a direct consequence of the conservations of $Q_\perp$ (32). But then $\mathbb{A}^{\mathrm{lin}}_{f0}$ also satisfies these conservation properties by linearity, integrating the definition (28) for $\mathbb{O} = \mathbb{A}$. □

3.3. **Proof of Theorem 1 (Limit model).** It this section, we shall keep in mind that the previous analysis was carried out for functions of the $\mathbf{v}$ variable only, while the solution f^ε of (26) depends on the parameters $(t, \mathbf{x})$. The proof is divided into several steps, based on the two first equations of the Hilbert hierarchy (27):

$$\mathbb{A}(f^0) = 0\,, \qquad : \mathcal{O}(\varepsilon^{-1}) \tag{38a}$$

$$\partial_t f^0 + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^0 + \mathbf{E}\cdot\nabla_{\mathbf{v}} f^0 + \mathbb{A}^{\mathrm{lin}}_{f^0}(f^1) = \nu_{\mathrm{r}}\, Q_{\mathrm{r}}(f^0)\,. \qquad : \mathcal{O}(1) \tag{38b}$$

Step 1: the limit distribution f^0. The first equation in the Hilbert hierarchy is

$$\mathbb{A}(f^0) = 0\,.$$

As a consequence, Proposition 1 yields the existence of two functions $g^0(t, \mathbf{x}, v_\parallel)$, $T^0_\perp(t, \mathbf{x})$ such that

$$f^0(t, \mathbf{x}, \mathbf{v}) = g^0(t, \mathbf{x}, v_\parallel)\, \mathcal{M}_\perp^{T^0_\perp}(\mathbf{v}_\perp)\,, \tag{39}$$

which is exactly (17a).

Step 2: Equation for the reduced distribution function g^0. We plug (39) into the second equation (38b) and then integrate with respect to $\mathbf{v}_\perp$. Using the conservation properties given in Lemma 1, one finds

$$\partial_t g^0 + v_\parallel\, \partial_z g^0 + \nabla_{\mathbf{x}_\perp}\cdot\left\langle \mathbf{v}_\perp\, f^0\right\rangle_\perp + E_\parallel \partial_{v_\parallel} g^0 = \nu_{\mathrm{r}}\, \partial_{v_\parallel}\left[(v_\parallel - u^0_\parallel)\, g^0 + T^0\, \partial_{v_\parallel} g^0_\parallel\right]\,. \tag{40}$$

It remains to compute the flux term, thanks to ansatz (39). We find that

$$\left\langle \mathbf{v}_\perp\, f^0\right\rangle_\perp = g^0 \int_{\mathbb{R}^2_v} \mathbf{v}_\perp\, \mathcal{M}_\perp^{T^0_\perp}\, \mathrm{d}\mathbf{v}_\perp = 0\,,$$

by imparity. Plugging this last equality into (40) yields exactly the kinetic equation on g^0.

Step 3: Equation for the perpendicular temperature $T^0_\perp$. Firstly, integrating (39) against $\frac{|\mathbf{v}_\perp|^2}{2}\,\mathrm{d}\mathbf{v}$ yields

$$n^0\, T^0_\perp = \left\langle \frac{|\mathbf{v}_\perp|^2}{2}\, f^0\right\rangle\,,$$

thanks to $n^0 = \langle g^0\rangle_\parallel$ (defined in (18)), and standard Gaussian moment computations. Now, let us integrate (38b) against $\frac{|\mathbf{v}_\perp|}{2}\,\mathrm{d}\mathbf{v}$. Using the conservation properties of $\mathbb{A}^{\mathrm{lin}}_{f^0}$ (given in Lemma 1) along with property (10) on Q_{r} ($\frac{k_B}{m}$ is now set to 1), one finds

$$\partial_t(n^0\, T^0_\perp) + \nabla_{\mathbf{x}_\perp}\cdot\mathbb{Q}^0_\perp + \partial_z \mathbb{Q}^0_\times = n^0\, \mathbf{u}^0_\perp\cdot\mathbf{E}_\perp + \frac{2}{3}\,\nu_{\mathrm{r}}\, n^0\, (T^0_\parallel - T^0_\perp)\,, \tag{41}$$

with the following definitions for the energy fluxes $\mathbb{Q}^0_\perp$ and $\mathbb{Q}^0_\times$

$$\mathbb{Q}^0_\perp := \left\langle \mathbf{v}_\perp\, \frac{|\mathbf{v}_\perp|^2}{2}\, f^0\right\rangle\,, \qquad \mathbb{Q}^0_\times := \left\langle v_\parallel\, \frac{|\mathbf{v}_\perp|^2}{2}\, f^0\right\rangle\,.$$

Let us now compute each of the terms in (41):

$$\begin{aligned}
\mathbb{Q}_\perp^0 &= \left\langle \mathbf{v}_\perp \frac{|\mathbf{v}_\perp|^2}{2} f^0 \right\rangle = \int_{\mathbb{R}_v} g^0 \,\mathrm{d}v_\parallel \int_{\mathbb{R}_v^2} \mathbf{v}_\perp \frac{|\mathbf{v}_\perp|^2}{2} \mathcal{M}_\perp^{T_\perp^0} \,\mathrm{d}\mathbf{v}_\perp = 0\,,\\
\mathbb{Q}_\times^0 &= \left\langle v_\parallel \frac{|\mathbf{v}_\perp|^2}{2} f^0 \right\rangle = \int_{\mathbb{R}_v} v_\parallel g^0 \,\mathrm{d}v_\parallel \int_{\mathbb{R}_v^2} \frac{|\mathbf{v}_\perp|^2}{2} \mathcal{M}_\perp^{T_\perp^0} \,\mathrm{d}\mathbf{v}_\perp = n^0\, u_\parallel^0\, T_\perp^0\,,\\
n^0\, \mathbf{u}_\perp^0 &= \langle \mathbf{v}_\perp f^0 \rangle = \int_{\mathbb{R}_v} g^0 \,\mathrm{d}v_\parallel \int_{\mathbb{R}_v^2} \mathbf{v}_\perp\, \mathcal{M}_\perp^{T_\perp^0} \,\mathrm{d}\mathbf{v}_\perp = 0\,,
\end{aligned}$$

where the first and third equalities come from imparity. We also used the definition of parallel moments (18) for the second equality. Plugging these equalities into (41) yields exactly the required equation on $T_\perp^0$, thus concluding the proof.

4. First order correction

This section contains the analysis for obtaining first-order corrections to the limit model of Theorem 1. After stating the problem in the prelminaries of section 4.1, we proceed with the analysis of the dominant operators in section 4.2; this will permit the computation of the distribution function f^1, carried out in sections 4.3 and 4.4. Eventually this will lead to the proof of Theorem 2 in section 4.7.

4.1. Preliminaries.

Let us now proceed to the computation of the first order correction model, investigating the following two equations of the Hilbert hierarchy (27):

$$O(1): \qquad \partial_t f^0 + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^0 + \mathbf{E}\cdot\nabla_{\mathbf{v}} f^0 + \mathbb{A}_{f^0}^{\mathrm{lin}}(f^1) = \nu_{\mathrm{r}}\, Q_{\mathrm{r}}(f^0)\,, \tag{42a}$$

$$O(\varepsilon): \qquad \partial_t f^1 + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^1 + \mathbf{E}\cdot\nabla_{\mathbf{v}} f^1 + \mathbb{A}_{f^0}^{\mathrm{lin}}(f^2) - \nu_\perp\, \delta^2 Q_\perp(f^1) = \nu_{\mathrm{r}}\, \delta Q_{\mathrm{r}}[f^0](f^1)\,. \tag{42b}$$

Here, $\mathbb{A}_{f^0}^{\mathrm{lin}}$ was given in (29) with the linearized operator $\delta Q_\perp[f^0]$ derived from the definition (28),

$$\delta Q_\perp[f^0](\delta f) = \nabla_{\mathbf{v}_\perp}\cdot\left[\mathbf{v}_\perp \delta f - \delta\mathbf{u}_\perp f^0 - \delta T_\perp \frac{\mathbf{v}_\perp}{T_\perp^0} f^0 + T_\perp^0 \nabla_{\mathbf{v}_\perp}\delta f\right]\,, \tag{43}$$

where we used $\mathbf{u}_\perp^0 = 0$ (as seen in the proof of Theorem 1), and the following definitions:

$$n^0\, \delta\mathbf{u}_\perp = \int_{\mathbb{R}_v^3} \mathbf{v}_\perp \delta f \,\mathrm{d}\mathbf{v}\,, \tag{44}$$

$$n^0\, \frac{\delta T_\perp}{T_\perp^0} = \int_{\mathbb{R}_v^3} \left(\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right) \delta f \,\mathrm{d}\mathbf{v}\,. \tag{45}$$

This last line can also be rewritten as

$$n^0\, \delta T_\perp + \delta n\, T_\perp^0 = \frac{1}{2}\int_{\mathbb{R}_v^3} |\mathbf{v}_\perp|^2\, \delta f \,\mathrm{d}\mathbf{v}\,, \qquad \text{where} \qquad \delta n := \int_{\mathbb{R}_v} \delta g \,\mathrm{d}v_\parallel = \int_{\mathbb{R}_v^3} \delta f \,\mathrm{d}\mathbf{v}\,.$$

The term $\delta^2 Q_\perp(f^1)$ is quadratic in f^1 and n^1, $\mathbf{u}_\perp^1, T_\perp^1$ defined as the first order terms in the expansions $X^\varepsilon = X^0 + \varepsilon\, X^1 + O(\varepsilon^2)$:

$$\delta^2 Q_\perp(f^1) = \nabla_{\mathbf{v}_\perp}\cdot\left[-\mathbf{u}_\perp^1 \left\{f^1 - \frac{n^1 f^0}{n^0}\right\} + T_\perp^1 \nabla_{\mathbf{v}_\perp}\left\{f^1 - \frac{n^1 f^0}{n^0}\right\} - \frac{1}{2}|\mathbf{u}_\perp^1|^2\, \nabla_{\mathbf{v}_\perp} f^0\right]\,. \tag{46}$$

This quantity inherits the following conservation properties from $Q_\perp$:

$$\int_{\mathbb{R}^3_v} \begin{pmatrix} 1 \\ \mathbf{v}_\perp \\ \frac{|\mathbf{v}_\perp|^2}{2} \end{pmatrix} \delta^2 Q_\perp(f^1)\, \mathrm{d}\mathbf{v} = 0\,, \qquad \text{and even} \qquad \int_{\mathbb{R}^2_v} \delta^2 Q_\perp(f^1)\, \mathrm{d}\mathbf{v}_\perp = 0\,. \tag{47}$$

To prove Theorem 2, we need firstly to characterize completely f^1. In order to explain more clearly how we shall do that, let us fix the functional setting in which we are going to work. In the following $(\mathbf{x}, t)$ are merely parameters, and thus shall be omitted, until subsection 4.3.

Let f^0 be the solution of the limit model (17). We define the Hilbert space

$$\mathcal{H} := \left\{ f \,:\, \mathbb{R}^3_v \to \mathbb{R}, \quad \int_{\mathbb{R}^3_v} |f|^2 \left(f^0\right)^{-1} \mathrm{d}\mathbf{v} < \infty \right\}, \tag{48}$$

which is associated with the following scalar product:

$$(f_1, f_2)_{\mathcal{H}} := \int_{\mathbb{R}^3_v} f_1 f_2 \frac{1}{f^0}\, \mathrm{d}\mathbf{v}\,, \qquad f_1, f_2 \in \mathcal{H}\,, \tag{49}$$

with norm denoted by $\|\cdot\|_{\mathcal{H}}$. We have $f^0 \in \mathcal{H}$ and

$$\|f^0\|^2_{\mathcal{H}} = \int_{\mathbb{R}^3_v} f^0\, \mathrm{d}\mathbf{v} = n^0\,.$$

In this space we shall define for any unbounded linear operator $\mathcal{O}$ its associated definition domain by:

$$\mathcal{D}(\mathcal{O}) := \{ f \in \mathcal{H}, \quad \mathcal{O} f \in \mathcal{H} \}\,. \tag{50}$$

Let us denote by $\Pi : \mathcal{H} \to \ker \mathbb{A}^{\text{lin}}_{f^0}$ the orthogonal projection onto the kernel of $\mathbb{A}^{\text{lin}}_{f^0}$. With this notation in mind, we shall decompose $f^1 \in \mathcal{D}(\mathbb{A}^{\text{lin}}_{f^0})$ in its macroscopic and microscopic parts, respectively:

$$f^1 = \bar{f}^1 + \widetilde{f}^1\,, \qquad \bar{f}^1 = \Pi f^1 \in \ker \mathbb{A}^{\text{lin}}_{f^0}\,, \qquad \widetilde{f}^1 = (\mathbb{I}\mathrm{d} - \Pi) f^1 \in \ker^\perp \mathbb{A}^{\text{lin}}_{f^0}\,. \tag{51}$$

Our goal is to compute both the macroscopic part $\bar{f}^1$ and the microscopic part $\widetilde{f}^1$. The macroscopic part $\bar{f}^1$ shall be characterized by taking the projection Π of equation (42b) (subsection 4.3). The microscopic part $\widetilde{f}^1$ will be characterized by taking $(\mathbb{I}\mathrm{d} - \Pi)$ of equation (42a) (subsection 4.4). The proof of Theorem 2 shall be concluded in subsection 4.7 from the complete characterization of f^0 and f^1.

4.2. Study of $\delta Q_\perp[f^0]$ and $\mathbb{A}^{\text{lin}}_{f^0}$.

In order to carry out this program let us firstly state several properties of $\delta Q_\perp[f^0]$ and $\mathbb{A}^{\text{lin}}_{f^0}$. All of these properties are proven in Appendix B. The normed space $\mathcal{H}$ defined in (48) is convenient for the study of the operator $\mathbb{A}^{\text{lin}}_{f^0}$. Indeed, $\mathbb{A}^{\text{lin}}_{f^0}$ is naturally decomposed as the sum of the skew-adjoint operator

$$((\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}}\, \chi, \chi)_{\mathcal{H}} = 0, \quad \forall\, \chi\,, \tag{52}$$

and the self-adjoint operator $\delta Q_\perp[f^0]$ given in (43), the properties of which are stated in the following proposition.

Proposition 2 (**Properties of** $\delta Q_\perp[f^0]$)**.** *In the space $\mathcal{H}$ endowed with the scalar product $(\cdot,\cdot)_{\mathcal{H}}$, the operator $\delta Q_\perp[f^0]$ given in (43) satisfies the following properties:*

- *It is self-adjoint, namely*

$$(-\delta Q_\perp[f^0]\xi,\chi)_{\mathcal{H}} = (\xi, -\delta Q_\perp[f^0]\chi)_{\mathcal{H}}, \quad \forall \xi,\, \chi\,, \tag{53}$$

and satisfies more specifically

$$\begin{aligned}(-\delta Q_\perp[f^0]\xi,\chi)_{\mathcal{H}} = &\int_{\mathbb{R}^3_v} T_\perp^0\, f^0\, \nabla_{\mathbf{v}_\perp}\left(\frac{\xi}{f^0}\right)\cdot \nabla_{\mathbf{v}_\perp}\left(\frac{\chi}{f^0}\right)\,\mathrm{d}\mathbf{v} \\ &- \frac{n^0}{T_\perp^0}\,\delta\mathbf{u}_{\perp,\xi}\cdot\delta\mathbf{u}_{\perp,\chi} - 2\,\frac{n^0}{(T_\perp^0)^2}\delta T_{\perp,\xi}\,\delta T_{\perp,\chi}, \quad \forall \xi,\chi\,,\end{aligned} \tag{54}$$

where we denoted by $\delta\mathbf{u}_{\perp,\xi}, \delta T_{\perp,\xi}$ (respectively $\delta\mathbf{u}_{\perp,\chi}, \delta T_{\perp,\chi}$) the bulk velocity and temperature defined in (44) *and* (45) *associated with the function ξ (respectively χ).*

- *It inherits from $Q_\perp$ the conservation properties* (32)*, namely*

$$\int_{\mathbb{R}^3_v}\begin{pmatrix}1\\ \mathbf{v}_\perp\\ \frac{|\mathbf{v}_\perp|^2}{2}\end{pmatrix}\,\delta Q_\perp[f^0](\xi)\,\mathrm{d}\mathbf{v} = 0 \qquad \text{and even} \qquad \int_{\mathbb{R}^2_v}\delta Q_\perp[f^0](\xi)\,\mathrm{d}\mathbf{v}_\perp = 0\,. \tag{55}$$

- *The kernel of $\delta Q_\perp[f^0]$ writes*

$$\begin{aligned}\ker(\delta Q_\perp[f^0]) = &\left\{\xi\in\mathcal{H},\quad \xi = \alpha(v_\parallel)\, f^0(\mathbf{v})\right\}\\ &\oplus\left\{\xi\in\mathcal{H},\quad \xi=\beta(\mathbf{v}_\perp)\,f^0(\mathbf{v}),\quad \beta(\mathbf{v}_\perp)\in\mathrm{Span}\,\{v_x,v_y,|\mathbf{v}_\perp|^2\}\right\}.\end{aligned} \tag{56}$$

Proof of Proposition 2. The two first points follow from a straightforward computation, while the third point is proven in Appendix B, subsection B.1. □

As a consequence of this characterization of $\ker\delta Q_\perp[f^0]$, one shall define an orthogonal projection onto that kernel.

Lemma 2. *The projection on* $\ker\delta Q_\perp[f^0]$ *reads*

$$\begin{aligned}\pi:\mathcal{H}\quad &\longrightarrow\quad \ker\delta Q_\perp[f^0]\\ \chi\quad &\mapsto\quad \left\langle\varphi^0\,\chi(v_\parallel)\right\rangle_\perp\,\Phi^0(\mathbf{v}) + \sum_{k=x,y}\left\langle\varphi^1_k\,\chi\right\rangle\,\Phi^1_k(\mathbf{v}) + \left\langle\varphi^2\,\chi\right\rangle\,\Phi^2(\mathbf{v})\,,\end{aligned} \tag{57}$$

where we introduce the following polynomial expressions $(\varphi^0,\varphi^1_x,\varphi^1_y,\varphi^2)$ in the velocity variable $\mathbf{v}_\perp$, and an orthonormal family $(\Phi^0,\Phi^1_x,\Phi^1_y,\Phi^2)$ of $\mathcal{H}$:

$$\begin{aligned}\varphi^0(v_\parallel)\ &:= \frac{\sqrt{n^0}}{g^0(v_\parallel)}\,, & \Phi^0(\mathbf{v})\ &:= \frac{1}{\sqrt{n^0}}\,f^0(\mathbf{v})\,,\\ \varphi^1_x(\mathbf{v}_\perp)\ &:= \frac{1}{\sqrt{n^0\,T_\perp^0}}\,v_x\,, & \Phi^1_x(\mathbf{v})\ &:= \varphi^1_x(\mathbf{v}_\perp)\,f^0(\mathbf{v})\,,\\ \varphi^1_y(\mathbf{v}_\perp)\ &:= \frac{1}{\sqrt{n^0\,T_\perp^0}}\,v_y\,, & \Phi^1_y(\mathbf{v})\ &:= \varphi^1_y(\mathbf{v}_\perp)\,f^0(\mathbf{v})\,,\\ \varphi^2(\mathbf{v}_\perp)\ &:= \frac{1}{\sqrt{n^0}}\left(\frac{|\mathbf{v}_\perp|^2}{2T_\perp^0}-1\right), & \Phi^2(\mathbf{v})\ &:= \varphi^2(\mathbf{v}_\perp)\,f^0(\mathbf{v})\,.\end{aligned} \tag{58}$$

One underlines that in the expression of π (57) the first term does not play exactly the same role as the others. Indeed, for instance $\langle\varphi^0\,\chi\rangle_\perp$ is a function of the variable $v_\parallel$, and is not a scalar.

The family $(\Phi^1_x,\, \Phi^1_y, \Phi^2)$ also permit to express the quantities $\delta \mathbf{u}_{\perp,\xi}$ and $\delta T_{\perp,\xi}$, defined in (44)-(45) and associated to an arbitrary function $\xi \in \mathcal{H}$. Indeed, one has for $\xi \in \mathcal{H}$:

$$(\xi,\, \Phi^1_x)_{\mathcal{H}} = \langle \varphi^1_x\, \xi \rangle = \sqrt{\frac{n^0}{T^0_\perp}}\, (\delta \mathbf{u}_{\perp,\xi})_x\,,$$

$$(\xi,\, \Phi^1_y)_{\mathcal{H}} = \langle \varphi^1_y\, \xi \rangle = \sqrt{\frac{n^0}{T^0_\perp}}\, (\delta \mathbf{u}_{\perp,\xi})_y\,,$$

$$(\xi,\, \Phi^2)_{\mathcal{H}} = \langle \varphi^2\, \xi \rangle = \sqrt{n^0}\, \frac{\delta T_{\perp,\xi}}{T^0_\perp}\,.$$

With these notations, $\ker \delta Q_\perp[f^0]$ rewrites

$$\ker(\delta Q_\perp[f^0]) = \left\{ \xi = \beta(v_\parallel)\, \Phi^0(\mathbf{v}), \quad \xi \in \mathcal{H} \right\} \oplus \mathrm{Span}\{\Phi^1_x,\, \Phi^1_y,\, \Phi^2\}\,.$$

Now, thanks to the definition of that projection π, one can write the following coercivity property of the collision operator.

Proposition 3. ***(Coercivity property of*** $\delta Q_\perp[f^0]$***).*** *The following coercivity estimate for the linearized collision operator holds:*

$$(-\delta Q_\perp[f^0]\chi, \chi)_{\mathcal{H}} = \int_{\mathbb{R}^3_v} T^0_\perp\, f^0 \left| \nabla_{\mathbf{v}_\perp} \left(\frac{\chi - \pi\chi}{f^0} \right) \right|^2 \mathrm{d}\mathbf{v} \geqslant \|\chi - \pi\chi\|^2_{\mathcal{H}} \qquad \forall \chi\,. \tag{59}$$

Proof. The proof can be found in Appendix B.2 □

This property guarantees the positivity of $-\delta Q_\perp[f^0]$, and is useful in the proof of the next property, where we characterize the kernel of $\mathbb{A}^{\mathrm{lin}}_{f^0}$ and state its closed range property.

Proposition 4 (**Properties of the operator** $\mathbb{A}^{\mathrm{lin}}_{f^0}$**).** *The operator* $\mathbb{A}^{\mathrm{lin}}_{f^0}$ *defined on* $\mathcal{H}$ *by* (29) *has the following properties:*

- *The kernel of* $\mathbb{A}^{\mathrm{lin}}_{f^0}$ *is given by*

$$\ker\, \mathbb{A}^{\mathrm{lin}}_{f^0} = \left\{ \xi \in \mathcal{H}, \quad \xi = \alpha(v_\parallel)\, f^0(\mathbf{v}) \right\} \oplus \{ \xi \in \mathcal{H}, \quad \xi = \beta |\mathbf{v}_\perp|^2\, f^0(\mathbf{v}), \quad \beta \in \mathbb{R} \}. \tag{60}$$

  As a consequence, the orthogonal of $\ker \mathbb{A}^{\mathrm{lin}}_{f^0}$ *in* $\mathcal{H}$ *is given by*

$$\ker^\perp \mathbb{A}^{\mathrm{lin}}_{f^0} = \left\{ \chi \in \mathcal{H}\,, \qquad \langle \chi \rangle_\perp = 0, \quad \left\langle \left(\frac{|\mathbf{v}_\perp|^2}{2\, T^0_\perp} - 1 \right) \chi \right\rangle = 0 \right\}\,. \tag{61}$$

- *The projection on* $\ker \mathbb{A}^{\mathrm{lin}}_{f^0}$ *is given by*

$$\begin{array}{rcl} \Pi : \mathcal{H} & \longrightarrow & \ker \mathbb{A}^{\mathrm{lin}}_{f^0} \\ \xi & \mapsto & \langle \varphi^0\, \xi \rangle_\perp\, (v_\parallel)\, \Phi^0(\mathbf{v}) + \langle \varphi^2\, \xi \rangle\; \Phi^2(\mathbf{v}) \end{array}\,. \tag{62}$$

- *The operator* $\mathbb{A}^{\mathrm{lin}}_{f^0}$ *satisfies the following Poincaré-type inequality: there exists an explicit constant* $C_A > 0$*, depending on* B *and* $\nu_\perp$ *only, such that*

$$\|\xi\|_{\mathcal{H}} \leqslant C_A \|\mathbb{A}^{\mathrm{lin}}_{f^0} \xi\|_{\mathcal{H}}, \qquad \forall \xi \in \ker^\perp \mathbb{A}^{\mathrm{lin}}_{f^0} \cap \mathcal{D}(\mathbb{A}^{\mathrm{lin}}_{f^0}). \tag{63}$$

  As a consequence, the operator $\mathbb{A}^{\mathrm{lin}}_{f^0}$ *has closed range, which implies* $\mathcal{R}(\mathbb{A}^{\mathrm{lin}}_{f^0}) = \ker^\perp \mathbb{A}^{\mathrm{lin}}_{f^0}$.

- *The operator* $\mathbb{A}^{\text{lin}}_{f^0}$ *can be restricted to the space* $\ker^{\perp}\mathbb{A}^{\text{lin}}_{f^0}$*, such that the restriction (which we still denote as* $\mathbb{A}^{\text{lin}}_{f^0}$*)*

$$\mathbb{A}^{\text{lin}}_{f^0} \ : \quad \ker^{\perp}\mathbb{A}^{\text{lin}}_{f^0} \to \ker^{\perp}\mathbb{A}^{\text{lin}}_{f^0},$$

is well defined, and is a bijection.

Let us now state a few remarks on this Proposition.

- As one can see, $\ker \mathbb{A}^{\text{lin}}_{f^0}$ is smaller than $\ker \delta Q_{\perp}[f^0]$, as it contains only functions that are invariant by rotation with respect to $\mathbf{v}_{\perp}$. This is due to the fact that $\mathbb{A}^{\text{lin}}_{f^0}$ contains the term $(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}}$. The projection Π reflects this fact, and its expression resembles that of π (equation (57)), but without the functions of type Φ^1_k, which are not invariant by rotation in $\mathbf{v}_{\perp}$.
- The closed range property comes from the Poincaré-type inequality (63) (see for reference [17]). The inequality (63) holds with an explicit constant $C_A = \frac{1}{\nu_{\perp}} + \frac{\sqrt{2}}{|B|}$.
- Let us recall that these properties are stated for a fixed set of parameters $(t,\mathbf{x})$ such that $\nu_{\perp} > 0$, $B(t,\mathbf{x}) > 0$. If one works in a time-space dependent setting, one can obtain for instance such a Poincaré-type inequality in $L^2((0,T)\times\mathbb{T}^3_x\,;\,\mathcal{H})$ by assuming a uniform lower bound condition on B, for instance assuming the existence of a constant γ such that

$$|B(t,\mathbf{x})| \geqslant \gamma > 0 \qquad \forall (t,\mathbf{x}) \in \mathbb{R}_+\times\mathbb{T}^3_x\,. \tag{64}$$

With that previous property in mind, we are now able to completely characterize f^1. Firstly, let us focus on its macroscopic part $\bar{f}^1$.

4.3. **Macroscopic part of f^1.** Using the form of the orthogonal projection Π onto $\ker\mathbb{A}^{\text{lin}}_{f^0}$, given in equation (62)), we know that the macroscopic part of f^1, namely $\bar{f}^1$, is given by the following equation:

$$\bar{f}^1 = \Pi f^1 = \langle \varphi^0 f^1\rangle_{\perp}\,\Phi^0 + \langle \varphi^2\, f^1\rangle\,\Phi^2 = g^1\,\mathcal{M}^{T^0_{\perp}}_{\perp} + \frac{r^1_{\perp}}{n^0}\left(\frac{|\mathbf{v}_{\perp}|^2}{2\,T^0_{\perp}} - 1\right) f^0\,, \tag{65}$$

where

$$g^1 := \left\langle f^1\right\rangle_{\perp} = \left\langle \bar{f}^1\right\rangle_{\perp}, \qquad \text{as} \quad \left\langle \widetilde{f}^1\right\rangle_{\perp} = 0\,, \tag{66}$$

$$r^1_{\perp} := n^0\,\frac{T^1_{\perp}}{T^0_{\perp}} = \left\langle\left(\frac{|\mathbf{v}_{\perp}|^2}{2\,T^0_{\perp}} - 1\right) f^1\right\rangle = \left\langle\left(\frac{|\mathbf{v}_{\perp}|^2}{2\,T^0_{\perp}} - 1\right) \bar{f}^1\right\rangle \qquad \text{as} \qquad \left\langle\left(\frac{|\mathbf{v}_{\perp}|^2}{2\,T^0_{\perp}} - 1\right) \widetilde{f}^1\right\rangle = 0\,. \tag{67}$$

The equation for g^1 can be derived integrating equation (42b) against $\mathrm{d}\mathbf{v}_{\perp}$, yielding thus

$$\partial_t g^1 + v_{\parallel}\partial_z g^1 + \nabla_{\mathbf{x}_{\perp}}\cdot\left\langle \mathbf{v}_{\perp}\,\widetilde{f}^1\right\rangle_{\perp} + E_{\parallel}\partial_{v_{\parallel}} g^1 = \nu_{\mathrm{r}}\,\partial_{v_{\parallel}}\left((v_{\parallel}-u^0_{\parallel})\,g^1 - u^1_{\parallel}\,g^0 + T^1\partial_{v_{\parallel}} g^0 + T^0\partial_{v_{\parallel}} g^1)\right)\,, \tag{68}$$

where the macroscopic quantities $u^1_{\parallel}$ and T^1 are depending on g^1 and $T^1_{\perp}$ only, and are the first order terms in the expansion of $u^{\varepsilon}_{\parallel}$ and T^{ε} defined in (15). We also used that

$$\left\langle \mathbf{v}_{\perp}\,f^1\right\rangle_{\perp} = \left\langle \mathbf{v}_{\perp}\,\widetilde{f}^1\right\rangle_{\perp}\,.$$

As one can see, this equation (68) is not closed, as the flux term still depends on the microscopic function $\widetilde{f}^1$.

Rather than computing the equation on $r_\perp^1$, in order to simplify the computations, we shall compute the equation on

$$w_\perp^1 := (n\,T_\perp)^1 = n^0\,T_\perp^1 + n^1\,T_\perp^0 = \left\langle \frac{|\mathbf{v}_\perp|^2}{2}\,f^1 \right\rangle ,$$

which is completely equivalent, with the knowledge of the equation on g^1. Integrating (42b) with respect to $\frac{|\mathbf{v}_\perp|^2}{2}\,\mathrm{d}\mathbf{v}$ yields

$$\partial_t w_\perp^1 + \nabla_{\mathbf{x}_\perp}\cdot\mathbb{Q}_\perp^1 + \partial_z\mathbb{Q}_\times^1 = n^0\mathbf{u}_\perp^1\cdot\mathbf{E}_\perp + \frac{2}{3}\,\nu_{\mathrm{r}}\,(n\,(T_\parallel - T_\perp))^1, \tag{69}$$

where we define

$$\mathbb{Q}_\perp^1 := \left\langle \mathbf{v}_\perp\frac{|\mathbf{v}_\perp|^2}{2}\,f^1 \right\rangle = \left\langle \mathbf{v}_\perp\frac{|\mathbf{v}_\perp|^2}{2}\,\widetilde{f}^1 \right\rangle , \qquad \mathbb{Q}_\times^1 := \left\langle v_\parallel\frac{|\mathbf{v}_\perp|}{2}\,f^1 \right\rangle .$$

This time again, this equation is not closed and we need more information of $\widetilde{f}^1$.

Altogether, we need to close equations (68)-(69) which depend on f^1, and in particular $\widetilde{f}^1$. In fact, we shall fully compute $\widetilde{f}^1$, and this is the purpose of the next subsection.

4.4. **Microscopic part of f^1.** In order to completely characterize the first order microscopic correction distribution $\widetilde{f}^1$, let us rearrange the terms of (42a), as

$$\mathbb{A}_{f^0}^{\mathrm{lin}}(\widetilde{f}^1) = \nu_{\mathrm{r}}\,Q_{\mathrm{r}}(f^0) - \partial_t f^0 - \mathbf{v}\cdot\nabla_{\mathbf{x}}f^0 - \mathbf{E}\cdot\nabla_{\mathbf{v}}f^0 =: \mathcal{R}^0 . \tag{70}$$

Using that $\mathcal{R}(\mathbb{A}_{f^0}^{\mathrm{lin}}) = \ker^\perp\mathbb{A}_{f^0}^{\mathrm{lin}}$, we deduce the following implicit definition for $\widetilde{f}^1$, namely

$$\widetilde{f}^1 = {\mathbb{A}_{f^0}^{\mathrm{lin}}}^{-1}(\mathcal{R}^0) . \tag{71}$$

There only remains to compute explicitly $\mathcal{R}^0$, and then to find its preimage. The following property sums up the result of this analysis.

Proposition 5. *The remainder term $\mathcal{R}^0$ defined in (70) belongs to $\mathcal{R}(\mathbb{A}_{f^0}^{\mathrm{lin}})$, and simplifies as*

$$\mathcal{R}^0 = \left\{ \frac{\mathbf{E}_\perp\cdot\mathbf{v}_\perp}{T_\perp^0} - \frac{(\mathbf{v}_\perp\cdot\nabla_{\mathbf{x}_\perp}T_\perp^0)}{T_\perp^0}\left[\frac{|\mathbf{v}_\perp|^2}{2T_\perp^0} - 1\right] + \frac{\partial_z T_\perp^0}{T_\perp^0}\,(u_\parallel^0 - v_\parallel)\left(\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right)\right\} f^0$$
$$- \mathbf{v}_\perp\cdot(\nabla_{\mathbf{x}_\perp}g^0)\mathcal{M}_\perp . \tag{72}$$

As a consequence, the microscopic density $\widetilde{f}^1$ defined through

$$\widetilde{f}^1 = {\mathbb{A}_{f^0}^{\mathrm{lin}}}^{-1}(\mathcal{R}^0) ,$$

reads

$$\begin{aligned}\widetilde{f}^1 = &\left(g^0\,\frac{\mathbf{E}_\perp\times\mathbf{B}}{|B|^2} - \frac{\nabla_{\mathbf{x}_\perp}(g^0\,T_\perp^0)\times\mathbf{B}}{|B|^2}\right)\cdot\frac{\mathbf{v}_\perp}{T_\perp^0}\mathcal{M}_\perp - \left(\mathbb{D}_1\frac{\nabla_{\mathbf{x}_\perp}T_\perp^0}{T_\perp^0}\right)\cdot\mathbf{v}_\perp\left[\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 2\right] f^0 \\ &+ \frac{\partial_z T_\perp^0}{T_\perp^0}\frac{u_\parallel^0 - v_\parallel}{2\,\nu_\perp}\left[\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right] f^0 - n^0\left\{\mathbb{D}_2\nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right)\right\}\cdot\mathbf{v}_\perp\mathcal{M}_\perp,\end{aligned} \tag{73}$$

where the definite positive matrices $\mathbb{D}_1$, $\mathbb{D}_2$ *are given by*

$$\mathbb{D}_1 = \frac{1}{B^2 + 9\,\nu_\perp^2}\begin{bmatrix} 3\,\nu_\perp & B \\ -B & 3\,\nu_\perp \end{bmatrix}, \qquad \mathbb{D}_2 = \frac{\nu_\perp}{B}\frac{1}{B^2+\nu_\perp^2}\begin{bmatrix} B & -\nu_\perp \\ \nu_\perp & B \end{bmatrix}. \tag{74}$$

Proof. The proof of this property is technical and is postponed in Appendix C. □

Thanks to this property, we can compute the moments of $\widetilde{f}^1$, which shall permit to close the macroscopic system (68)-(69).

4.5. **Closure of the macroscopic system.** Let us start by the closure of the equation on g^1 (68), in particular by giving an explicit form to the flux term. Using formula (73), one gets

$$\left\langle \mathbf{v}_\perp \widetilde{f}^1 \right\rangle_\perp = \frac{g^0\,\mathbf{E}_\perp \times \mathbf{B} - \nabla_{\mathbf{x}_\perp}(g^0\,T_\perp^0)\times \mathbf{B}}{|B|^2} - n^0\,T_\perp^0\,\mathbb{D}_2 \cdot \nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right). \tag{75}$$

As one can see, the right-hand side of (75) is decomposed into two terms playing different roles. Let us focus on the first one, and introduce the electric field drift

$$\mathbf{u}_E := \frac{\mathbf{E}_\perp \times \mathbf{B}}{|B|^2},$$

along with

$$\mathbf{u}_D^{\mathrm{K}} := -\frac{\nabla_{\mathbf{x}_\perp}(g^0\,T^0)\times \mathbf{B}}{g^0\,|B|^2}.$$

This quantity, which we call the "*kinetic diamagnetic drift*", is related to the classical diamagnetic drift

$$\mathbf{u}_D := -\frac{\nabla_{\mathbf{x}_\perp}(n^0\,T^0)\times \mathbf{B}}{n^0\,|B|^2},$$

through to the formula

$$\langle g^0\,\mathbf{u}_D^{\mathrm{K}}\rangle_\| = n^0\,\mathbf{u}_D\,.$$

Defining further

$$\mathbf{u}_{drift} := \mathbf{u}_E + \mathbf{u}_D\,, \qquad \mathbf{u}_{drift}^{\mathrm{K}} := \mathbf{u}_E + \mathbf{u}_D^{\mathrm{K}}\,,$$

the flux computed in (75) rewrites

$$\left\langle \mathbf{v}_\perp \widetilde{f}^1 \right\rangle_\perp = g^0\,\mathbf{u}_{drift}^{\mathrm{K}} - n^0\,T_\perp^0\,\mathbb{D}_2\cdot\nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right).$$

The second term on the right-hand side of (75) is a diffusion-type correction, acting in the perpendicular plane direction, and is a novelty.

With these computations in mind, one finds the following equation on g^1:

$$\partial_t g^1 + v_\|\partial_z g^1 + E_\|\partial_{v_\|} g^1 + \nabla_{\mathbf{x}_\perp}\cdot\left(g^0\,\mathbf{u}_{drift}^{\mathrm{K}}\right) - \nabla_{\mathbf{x}_\perp}\cdot\left(n^0\,T_\perp^0\,\mathbb{D}_2\cdot\nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right)\right)$$
$$= \nu_{\mathrm{r}}\,\partial_{v_\|}\left((v_\| - u_\|^0)\,g^1 - u_\|^1\,g^0 + T^1\partial_{v_\|}g^0 + T^0\partial_{v_\|}g^1\right),$$

which now does not depend on $\widetilde{f}^1$.

Let us now turn to the closure of equation (69). As we mentioned earlier, the first non-closed term in (69) is

$$\mathbb{Q}^1_\perp = \left\langle \mathbf{v}_\perp \frac{|\mathbf{v}_\perp|^2}{2} f^1 \right\rangle = \left\langle \mathbf{v}_\perp \frac{|\mathbf{v}_\perp|^2}{2} \widetilde{f}^1 \right\rangle . \tag{76}$$

Now, using Proposition 5 which gives the expression of $\widetilde{f}^1$, one computes

$$\mathbb{Q}^1_\perp = 2\, n^0\, T^0_\perp\, \mathbf{u}_{drift} - 2\, n^0\, T^0_\perp\, \mathbb{D}_1 \nabla_{\mathbf{x}_\perp} T^0_\perp . \tag{77}$$

In order to identify the role of each of these quantities, we shall use the following decomposition of $\mathbb{Q}^\varepsilon_\perp := \left\langle \mathbf{v}_\perp \frac{|\mathbf{v}_\perp|^2}{2} f^\varepsilon \right\rangle$:

$$\mathbb{Q}^\varepsilon_\perp = \mathbf{q}^\varepsilon_\perp + \mathbb{P}^\varepsilon_\perp \cdot \mathbf{u}^\varepsilon_\perp + p^\varepsilon_\perp\, \mathbf{u}^\varepsilon_\perp - \frac{1}{2}\, n^\varepsilon\, |\mathbf{u}^\varepsilon_\perp|^2\, \mathbf{u}^\varepsilon_\perp , \tag{78}$$

where the *perpendicular heat flux* $\mathbf{q}^\varepsilon_\perp$, the *perpendicular stress tensor* $\mathbb{P}^\varepsilon_\perp$ and the *perpendicular scalar pressure* $p^\varepsilon_\perp$ are defined through

$$\mathbf{q}^\varepsilon_\perp = \frac{1}{2} \int_{\mathbb{R}^3_v} |\mathbf{v}_\perp - \mathbf{u}^\varepsilon_\perp|^2\, (\mathbf{v}_\perp - \mathbf{u}^\varepsilon_\perp)\, f^\varepsilon\, \mathrm{d}\mathbf{v} , \qquad \mathbb{P}^\varepsilon_\perp := \int_{\mathbb{R}^3_v} (\mathbf{v}_\perp \otimes \mathbf{v}_\perp)\, f^\varepsilon\, \mathrm{d}\mathbf{v} , \qquad p^\varepsilon_\perp := n^\varepsilon\, T^\varepsilon_\perp . \tag{79}$$

At order one in ε, $\mathbb{Q}^\varepsilon_\perp$ writes

$$\mathcal{O}(\varepsilon) : \qquad \mathbb{Q}^1_\perp = \mathbf{q}^1_\perp + \mathbb{P}^0_\perp \cdot \mathbf{u}^1_\perp + p^0_\perp\, \mathbf{u}^1_\perp = \mathbf{q}^1_\perp + 2\, n^0\, T^0_\perp\, \mathbf{u}^1_\perp . \tag{80}$$

Comparing this last equation with (77) permits to find the following expression for the order one perpendicular heat flux

$$\mathbf{q}^1_\perp = -2\, n^0\, T^0_\perp\, \mathbb{D}_1 \nabla_{\mathbf{x}_\perp} T^0_\perp . \tag{81}$$

This is a Fourier law of Bragiinski-type, with gyroviscous (antidiagonal) and viscous (diagonal) terms [15, 37].

Now, we compute the second non-closed term in (69). It can be decomposed as follows into two parts, coming respectively from the microscopic and macroscopic part of f^1:

$$\mathbb{Q}^1_\times = \left\langle v_\parallel \frac{|\mathbf{v}_\perp|^2}{2} f^1 \right\rangle = \left\langle v_\parallel \frac{|\mathbf{v}_\perp|^2}{2} \widetilde{f}^1 \right\rangle + \left\langle v_\parallel \frac{|\mathbf{v}_\perp|^2}{2} \bar{f}^1 \right\rangle \tag{82}$$

$$= -\frac{1}{2\,\nu_\perp} n^0\, T^0_\parallel\, \partial_z T^0_\perp + n^0\, u^0_\parallel\, T^1_\perp + (n\, u_\parallel)^1\, T^0_\perp \tag{83}$$

$$= -\frac{1}{2\,\nu_\perp} n^0\, T^0_\parallel\, \partial_z T^0_\perp + (n\, u_\parallel\, T_\perp)^1 . \tag{84}$$

This time again, the different terms carry some physical meaning. Defining the "*heat flux in the parallel direction*" $q^\varepsilon_\times$ as follows

$$q^\varepsilon_\times := \frac{1}{2} \int_{\mathbb{R}^3_v} (v_\parallel - u^\varepsilon_\parallel)\, |\mathbf{v}_\perp - \mathbf{u}^\varepsilon_\perp|^2 f^\varepsilon\, \mathrm{d}\mathbf{v} , \tag{85}$$

permits to show, similarly as before, the following Fourier law

$$q^1_\times = -\frac{1}{2\,\nu_\perp}\, n^0\, T^0_\parallel\, \partial_z T^0_\perp . \tag{86}$$

4.6. **Recap: Order one Hilbert expansion of** (26)**.** Let us summarize in this paragraph what we have proven in subsections 4.3, 4.4 and 4.5. Assuming our distribution function f^ε has the following Hilbert decomposition:

$$f^\varepsilon = f^0 + \varepsilon f^1 + \mathcal{O}(\varepsilon^2),$$

the distribution f^0 is given by the limit model (17). The first order correction $f^1 \in \mathcal{D}(\mathbb{A}^{\text{lin}}_{f^0})$ writes:

$$f^1 = \widetilde{f}^1 + \bar{f}^1, \tag{87}$$

where the microscopic part $\widetilde{f}^1 \in \ker^\perp \mathbb{A}^{\text{lin}}_{f^0}$ is written (see Proposition 5):

$$\begin{aligned}\widetilde{f}^1 = {} & \frac{\mathbf{u}^{\text{K}}_{drift} \cdot \mathbf{v}_\perp}{T^0_\perp} f^0 - \left(\mathbb{D}_1 \frac{\nabla_{\mathbf{x}_\perp} T^0_\perp}{T^0_\perp}\right) \cdot \mathbf{v}_\perp \left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 2\right] f^0 \\ & + \frac{\partial_z T^0_\perp}{T^0_\perp} \frac{u^0_\parallel - v_\parallel}{2\,\nu_\perp} \left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 1\right] f^0 - n^0 \left\{\mathbb{D}_2 \nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right)\right\} \cdot \mathbf{v}_\perp \mathcal{M}_\perp,\end{aligned} \tag{88}$$

with $\mathbb{D}_1$, $\mathbb{D}_2$ given by

$$\mathbb{D}_1 = \frac{1}{B^2 + 9\,\nu_\perp^2}\begin{bmatrix} 3\,\nu_\perp & B \\ -B & 3\,\nu_\perp\end{bmatrix}, \qquad \mathbb{D}_2 = \frac{\nu_\perp}{B}\frac{1}{B^2+\nu_\perp^2}\begin{bmatrix} B & -\nu_\perp \\ \nu_\perp & B\end{bmatrix}. \tag{89}$$

The macroscopic part $\bar{f}^1 \in \ker \mathbb{A}^{\text{lin}}_{f^0}$ writes (see (65)) as follows

$$\bar{f}^1 = g^1 \mathcal{M}^{T^0_\perp}_\perp + \frac{T^1_\perp}{T^0_\perp}\left(\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 1\right) f^0, \tag{90}$$

with the order one quantities $(g^1, T^1_\perp)$ given by (see subsection 4.3 for the equations and 4.5 for the closure):

$$\left\{\begin{aligned} & \partial_t g^1 + v_\parallel \partial_z g^1 + E_\parallel \partial_{v_\parallel} g^1 + \nabla_{\mathbf{x}_\perp} \cdot \left(g^0\, \mathbf{u}^{\text{K}}_{drift}\right) - \nabla_{\mathbf{x}_\perp} \cdot \left(n^0\, T^0_\perp\, \mathbb{D}_2 \cdot \nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right)\right) \\ & \qquad = \nu_{\text{r}}\, \partial_{v_\parallel}\left(\left(v_\parallel - u^0_\parallel\right) g^1 - u^1_\parallel\, g^0 + T^1 \partial_{v_\parallel} g^0 + T^0 \partial_{v_\parallel} g^1\right), \\ & \partial_t (n\,T_\perp)^1 + \partial_z (n\, u_\parallel\, T_\perp)^1 + \nabla_{\mathbf{x}_\perp} \cdot \left(2\, n^0\, T^0_\perp\, \mathbf{u}_{drift}\right) + \nabla_{\mathbf{x}_\perp} \cdot \mathbf{q}^1_\perp + \partial_z q^1_\times \\ & \qquad = n^0\, \mathbf{u}_{drift} \cdot \mathbf{E}_\perp + \frac{2}{3}\nu_{\text{r}} (n(T_\parallel - T_\perp))^1, \end{aligned}\right. \tag{91}$$

where

$$\mathbf{q}^1_\perp = -2\, n^0\, T^0_\perp\, \mathbb{D}_1\, \nabla_{\mathbf{x}_\perp} T^0_\perp, \qquad q^1_\times = -\frac{1}{2\,\nu_\perp} n^0\, T^0_\parallel\, \partial_z T^0_\perp. \tag{92}$$

With this in mind, we are now ready to deal with the proof of Theorem 2.

4.7. **Proof of Theorem 2.** In this proof, we shall denote

$$\check{f} = f^0 + \varepsilon f^1,$$

and all quantities with a check symbol $\check{}$ on top shall be associated with $f^0 + \varepsilon f^1$. The proof is based on the fact that, due to the Hilbert expansion, one has

$$f^\varepsilon - \check{f} = \mathcal{O}(\varepsilon^2), \qquad \text{as } \varepsilon \to 0.$$

As a consequence, it is enough to show that the function $\widehat{f}$, constructed in Theorem 2 and solution of our truncated model, is such that

$$\widehat{f} - \breve{f} = \mathcal{O}(\varepsilon^2)\,, \qquad \text{as } \varepsilon \to 0\,. \tag{93}$$

Firstly, we show that $\breve{f}$ satisfies the following development.

$$\breve{f} = \breve{g}\, \mathcal{M}_{\perp}^{\breve{T}_\perp}(1 + \varepsilon\, \Lambda_{\breve{g},\breve{T}_\perp}) + \mathcal{O}(\varepsilon^2)\,, \tag{94}$$

where Λ is defined in (22f). We recognize a decomposition analogous to that of $\widehat{f}$ (22a), namely

$$\widehat{f} = \widehat{g}\, \mathcal{M}_{\perp}^{\widehat{T}_\perp}(1 + \varepsilon\, \Lambda_{\widehat{g},\widehat{T}_\perp})\,. \tag{95}$$

As a consequence, we shall prove in a second time that

$$(\widehat{g},\, \widehat{T}_\perp) - (\breve{g},\, \breve{T}_\perp) = \mathcal{O}(\varepsilon^2)\,, \qquad \text{as } \varepsilon \to 0\,. \tag{96}$$

This shall finish the proof, as injecting expansion (96) into (94) leads to (93).

Step 1: Form of $\breve{f}$. Let us start with the decomposition

$$\breve{f} = (f^0 + \varepsilon\, \bar{f}^1) + \varepsilon\, \widetilde{f}^1.$$

Developing in powers of ε the term $\breve{g}\, \mathcal{M}_{\perp}^{\breve{T}_\perp}$ permits to find, with the help of (90), the first term of the previous decomposition:

$$f^0 + \varepsilon\, \bar{f}^1 = \breve{g}\, \mathcal{M}_{\perp}^{\breve{T}_\perp} + \mathcal{O}(\varepsilon^2)\,, \qquad \text{as } \varepsilon \to 0\,. \tag{97}$$

Then, one can focus on rewriting $\widetilde{f}^1$ given in (88): thanks to definition of Λ in (22f), one finds

$$\widetilde{f}^1 = g^0\, \mathcal{M}_{\perp}^{T_\perp^0}\, \Lambda_{g^0,T_\perp^0} = \breve{g}\, \mathcal{M}_{\perp}^{\breve{T}_\perp}\, \Lambda_{\breve{g},\breve{T}_\perp} + \mathcal{O}(\varepsilon)\,, \qquad \text{as } \varepsilon \to 0\,. \tag{98}$$

Taking (97) + ε (98) yields (94).

Step 2: First order PDE model. Summing the limit PDE model (17c) for $(g^0,\, T_\perp^0)$ with the system of equation (91) giving $\varepsilon\,(g^1,\, T_\perp^1)$ yields, after grouping terms of the same nature

$$\left\{\begin{aligned}
&\partial_t \breve{g} + v_\| \partial_z \breve{g} + E_\| \partial_{v_\|} \breve{g} + \varepsilon\, \nabla_{\mathbf{x}_\perp} \cdot \left(g^0\, \mathbf{u}^{\mathrm{K}}_{drift}\right) - \varepsilon\, \nabla_{\mathbf{x}_\perp} \cdot \left(n^0\, T_\perp^0\, \mathbb{D}_2 \cdot \nabla_{\mathbf{x}_\perp} \left(\frac{g^0}{n^0}\right)\right) \\
&\qquad = \nu_{\mathrm{r}}\, \partial_{v_\|} \left(\left(v_\| - u_\|^0\right) g^0 + T^0 \partial_{v_\|} g^0 + \varepsilon\, \left(\left(v_\| - u_\|^0\right) g^1 - u_\|^1\, g^0 + T^1 \partial_{v_\|} g^0 + T^0 \partial_{v_\|} g^1\right)\right)\,, \\
&\partial_t \left\{(n\, T_\perp)^0 + \varepsilon\, (n\, T_\perp)^1\right\} + \partial_z \left\{(n\, u_\|\, T_\perp)^0 + \varepsilon\, (n\, u_\|\, T_\perp)^1\right\} + \varepsilon\, \nabla_{\mathbf{x}_\perp} \cdot (2\, n^0\, T_\perp^0\, \mathbf{u}_{drift}) + \varepsilon\, \nabla_{\mathbf{x}} \cdot \breve{\mathbf{q}} \\
&\qquad = \varepsilon\, n^0\, \mathbf{u}_{drift} \cdot \mathbf{E}_\perp + \frac{2}{3}\, \nu_{\mathrm{r}} \left\{(n(T_\| - T_\perp))^0 + \varepsilon\, (n(T_\| - T_\perp))^1\right\}\,.
\end{aligned}\right.$$

Finally, adding several terms of order $\mathcal{O}(\varepsilon^2)$ permitxs to simplify the previous system:

$$\left\{\begin{aligned}
&\partial_t \breve{g} + v_\|\, \partial_z \breve{g} + E_\|\, \partial_{v_\|} \breve{g} + \varepsilon\, \nabla_{\mathbf{x}_\perp} \cdot (\breve{\mathbf{u}}^{\mathrm{K}}_{\mathrm{drift}}\, \breve{g}) = \nu_{\mathrm{r}}\, \partial_{v_\|} \left[(v_\| - \breve{u}_\|)\, \breve{g} + \breve{T}\, \partial_{v_\|} \breve{g}\right] \\
&\qquad\qquad + \varepsilon\, \nabla_{\mathbf{x}_\perp} \cdot \left(\breve{n}\, \breve{T}_\perp\, \mathbb{D}_2 \nabla_{\mathbf{x}_\perp} \frac{\breve{g}}{\breve{n}}\right) + \mathcal{O}(\varepsilon^2)\,, \\
&\partial_t(\breve{n}\, \breve{T}_\perp) + \partial_z(\breve{n}\, \breve{T}_\perp\, \breve{u}_\|) + \varepsilon\, \nabla_{\mathbf{x}_\perp} \cdot (2\, \breve{n}\, \breve{T}_\perp\, \breve{\mathbf{u}}_{\mathrm{drift}}) + \varepsilon\, \nabla_{\mathbf{x}} \cdot \breve{\mathbf{q}} = \varepsilon\, \breve{n}\, \breve{\mathbf{u}}_{\mathrm{drift}} \cdot \mathbf{E}_\perp \\
&\qquad\qquad + \frac{2}{3}\, \nu_{\mathrm{r}}\, \breve{n}\, (\breve{T}_\| - \breve{T}_\perp) + \mathcal{O}(\varepsilon^2)\,,
\end{aligned}\right.$$

$$\text{as } \varepsilon \to 0\,.$$

Therefore, in finite time, (96) holds. This concludes the proof.

5. Discussion and Conclusion

The main purpose of this work was to derive a reduced description of a plasma undergoing anisotropic collisions in a strong magnetic field. We started from a normalized kinetic equation featuring a small parameter $\varepsilon \in (0,1)$ and performed a formal analysis, leading in the asymptotic regime $\varepsilon \ll 1$ to the factorization of the distribution function into a kinetic part in the parallel direction, and a macroscopic part in the plane perpendicular to the magnetic field. This new plasma model is an enhancement of the one used in [20], as it includes plasma drifts and perpendicular diffusion terms. Classical Bragiinski-type closure terms were found for the perpendicular temperature, while new, fluid-like terms were discovered in the kinetic-parallel description. In particular we found in this kinetic equation a diamagnetic drift term. Such a term is usually not present in reduced kinetic models, for instance in gyrokinetic models, and points to the hybrid character of the newly derived model. This hybrid character of course arises because of the assumption of high collisionality perpendicular to $\mathbf{B}$. Moreover, a diffusion term occurs in the direction perpendicular to the magnetic field, which is responsible for the homogenisation of parallel moments in the perpendicular direction.

One can build upon this work in several directions. Firstly, one can investigate on the restrictions we made for this study: one can study the same regime in the context of a more complex geometry, with a curved magnetic field. We conjecture that such a modification will add several other terms to the drift velocity, such as a grad-B drift and a curved-B drift. It is also possible to remove the assumption of periodic boundary conditions in $\mathbf{x}$, to add the effects of multiple species, or to solve the electromagnetic fields in a self-consistent manner. Secondly, it would be interesting to consider numerical discretizations of the new model. Due to the fact that the dimensionality (4D) is lower than for instance in gyrokinetic descriptions (5D), a significant gain in performance can be expected. Thirdly, the range of validity of the new model should be investigated. For instance, one could try to reproduce the numerical experiments conducted in [20] and then study the effect of the additional drift-/diffusion terms of the new model.

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Appendix A. Scaling assumptions and renormalization

In this section, we detail the scaling assumptions leading to the renormalized equation (13). We start from (4)-(6), and express dependent variables in terms of a characteristic unit (denoted with a "bar") and a rescaled function (denoted with a "prime"), for instance

$$f = \bar{f}\, f' \,, \qquad \mathbf{E} = \bar{E}\,\mathbf{E}' \,, \qquad \mathbf{B} = \bar{B}\,\mathbf{B}' \,.$$

We normalize equations (4)-(6) by assuming characteristic scales for time and phase space,

$$t = \bar{t}\, t' \,, \qquad \mathbf{x} = \bar{x}\,\mathbf{x}' \,, \qquad \mathbf{v} = v_{\mathrm{th}}\mathbf{v}' \,,$$

where the thermal speed v_{th} is defined as follows:

$$v_{th} := \sqrt{\frac{k_B \bar{T}}{m}},$$

with $\bar{T} = \bar{T}_\perp$ the characteristic temperature scale associated to f. We assume the relation $\bar{x} = \bar{t}\, v_{\rm th}$, as well as

$$\bar{n} = \bar{f} v_{\rm th}^3 , \qquad \bar{u} = v_{\rm th} , \qquad k_{\rm B} \bar{T} = m\, v_{\rm th}^2 = q\, \bar{\phi} , \qquad \bar{E} = \frac{\bar{\phi}}{\bar{x}} .$$

We also introduce the observation frequency, and the cyclotron frequency:

$$\bar{\omega} = \frac{1}{\bar{t}} , \qquad \Omega_c = \frac{q\, \bar{B}}{m} .$$

This leads to the normalized VFP model

$$\partial_{t'} f' + \mathbf{v}' \cdot \nabla_{\mathbf{x}'} f' + \frac{\Omega_{\rm c}}{\bar{\omega}} \left(\frac{\bar{E}}{v_{\rm th} \bar{B}} \mathbf{E}' + \mathbf{v}' \times B' \mathbf{e}_z \right) \cdot \nabla_{\mathbf{v}'} f' = \frac{\bar{\nu}_\perp}{\bar{\omega}}\, \nu'_\perp Q'_\perp(f') + \frac{\bar{\nu}_{\rm r}}{\bar{\omega}}\, \nu'_{\rm r} Q'_{\rm r}(f') , \tag{99}$$

where $Q'_\perp$ and $Q'_{\rm r}$ are defined as in (5)-(7), but setting the constants k_B and m to 1. The physical regime is now determined by four quantities:

i) $\Omega_{\rm c}/\bar{\omega}$, the ratio between the cyclotron frequency and the chosen frequency scale,

ii) $\bar{E}/(v_{\rm th}\bar{B})$, the ratio between the $E \times B$ drift velocity and the thermal velocity,

iii) $\bar{\nu}_\perp/\bar{\omega}$, the ratio between the perpendicular collision frequency and the chosen frequency scale,

iv) $\bar{\nu}_{\rm r}/\bar{\nu}_\perp$, the anisotropic collision parameter, defined as the ratio between the two collision frequencies.

One assumes firstly that the cyclotron frequency is much larger than the chosen frequency scale, due to the strong magnetic field. This assumption can be formulated as follows:

$$\frac{\Omega_{\rm c}}{\bar{\omega}} = \frac{1}{\varepsilon} \gg 1 , \tag{100}$$

where ε is the small asymptotic parameter. This choice constraints the next quantity ii):

$$\frac{\bar{E}}{v_{\rm th}\bar{B}} = \frac{q\, \varepsilon\, \bar{E}}{m\, v_{\rm th}\, \bar{\omega}} = \frac{q\, \varepsilon\, \bar{\phi}}{m\, v_{\rm th}\, \bar{\omega}\, \bar{x}} = \frac{q\, \varepsilon\, \bar{\phi}}{m\, v_{\rm th}^2} = \frac{\varepsilon\, k_B\, \bar{T}}{m\, v_{\rm th}^2} = \varepsilon . \tag{101}$$

Next, we assume strong collisions in the perpendicular direction. In particular, we assume that the $\mathbf{v} \times \mathbf{B} \cdot \nabla_{\mathbf{v}}$ operator and the collision operator appear on the same order in the Vlasov equation, meaning

$$\frac{\bar{\nu}_\perp}{\bar{\omega}} = \frac{1}{\varepsilon} \gg 1 . \tag{102}$$

The main novelty in this work is the assumption of anisotropic collisions, namely

$$\bar{\nu}_{\rm r}/\bar{\nu}_\perp = \varepsilon \ll 1 . \tag{103}$$

One can now rewrite our rescaled Vlasov equation as follows, (the primes were omitted, for simplicity)

$$\partial_t f^\varepsilon + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^\varepsilon + \mathbf{E} \cdot \nabla_{\mathbf{v}} f^\varepsilon + \frac{1}{\varepsilon} (\mathbf{v} \times B \mathbf{e}_z) \cdot \nabla_{\mathbf{v}} f^\varepsilon = \frac{\nu_\perp}{\varepsilon} Q_\perp(f^\varepsilon) + \nu_{\rm r}\, Q_{\rm r}(f^\varepsilon) , \tag{104}$$

where $Q_\perp$ and $Q_{\rm r}$ are defined in (14).

Appendix B. Proof of the properties stated in Subsection 4.2

B.1. Proof of Proposition 2.

In this subsection, we give a proof of the last point of Proposition 2, namely of (56). The first step is to state one of the inclusion of (56) in the following Lemma.

Lemma 3. *Let $\delta Q_\perp[f^0]$ be the operator defined in (43) and define the set*

$$\mathcal{S}_1 := \left\{\xi = \alpha(v_\parallel)\, f^0(\mathbf{v}),\, \xi \in \mathcal{H}\right\} \oplus \left\{\xi = \beta(\mathbf{v}_\perp)\, f^0(\mathbf{v}),\, \beta \in \mathrm{Span}\{v_x, v_y, |\mathbf{v}_\perp|^2\}\right\} .$$

Then one has

$$\mathcal{S}_1 \subset \ker(\delta Q_\perp[f^0]) \,. \tag{105}$$

Proof of Lemma 3. By direct computation (apply $\delta Q_\perp[f^0]$ to an element of $\mathcal{S}_1$). □

Let us show now the reciprocal inclusion, using the coercivity inequality (59). For this, let us investigate the mapping π

$$\begin{aligned} \pi : \mathcal{H} \quad &\longrightarrow \quad \mathcal{H} \\ \chi \quad &\mapsto \quad \left\langle \varphi^0\, \chi(v_\parallel)\right\rangle_\perp \, \Phi^0(\mathbf{v}) \, + \sum_{k=x,y} \left\langle \varphi^1_k\, \chi\right\rangle \, \Phi^1_k(\mathbf{v}) + \left\langle \varphi^2\, \chi\right\rangle \, \Phi^2(\mathbf{v}) \,, \end{aligned} \tag{106}$$

defined in (57). Let us show that it is well defined. For this, it is enough to notice that if $\chi \in \mathcal{H}$, then $\langle \chi\varphi^0\rangle_\perp\, \Phi^0 \in \mathcal{H}$. Indeed,

$$\begin{aligned} \left\| \langle \chi\varphi^0\rangle_\perp\, \Phi^0 \right\|^2_{\mathcal{H}} = \left\| \langle \chi\rangle_\perp\, \mathcal{M}_\perp^{T_\perp^0} \right\|^2_{\mathcal{H}} &= \int_{\mathbb{R}_v} \langle\chi\rangle_\perp^2 \frac{1}{g^0}\, \mathrm{d}v_\parallel \\ &= \int_{\mathbb{R}_v} \left(\int_{\mathbb{R}^2_v} \frac{\chi}{\sqrt{f^0}} \sqrt{f^0}\, \mathrm{d}\mathbf{v}_\perp \right)^2 \frac{1}{g^0}\, \mathrm{d}v_\parallel \\ &\leqslant \int_{\mathbb{R}_v} \left(\int_{\mathbb{R}^2_v} \frac{\chi^2}{f^0}\, \mathrm{d}\mathbf{v}_\perp \right) \underbrace{\left(\int_{\mathbb{R}^2_v} f^0\, \mathrm{d}\mathbf{v}_\perp \right)}_{=g^0} \frac{1}{g^0}\, \mathrm{d}v_\parallel \\ &= \|\chi\|^2_{\mathcal{H}} < \infty, \end{aligned}$$

using Cauchy-Schwarz inequality, leading to the well-definition.

Observe that $\mathcal{S}_1 = \mathcal{R}(\pi)$, where $\mathcal{R}(\pi)$ is the range of π. As a consequence, Lemma 3 shows

$$\mathcal{R}(\pi) \subset \ker \delta Q_\perp[f^0] \,. \tag{107}$$

Proving (56) is exactly proving that (107) is an equality. But the latter fact is a direct consequence of the coercivity inequality[1] (59). Indeed, taking $\chi \in \ker \delta Q_\perp[f^0]$ we obtain

$$0 = (-\delta Q_\perp[f^0]\chi, \chi)_{\mathcal{H}} \geqslant \|\chi - \pi\chi\|^2_{\mathcal{H}} \,. \tag{108}$$

We quickly sketch the proof of this inequality. It shall finish the proof of Proposition 2.

[1] At this point we do not know that the mapping π is the orthogonal projection onto $\ker \delta Q_\perp[f^0]$. However, the coercivity relation from Proposition 3 still holds, which can be verified by direct computation, or from the proof of Proposition 3 in B.2, where only (107) is used.

B.2. **Proof of Proposition 3.** Let us recall firstly that $(\Phi^1_x,\, \Phi^1_y, \Phi^2)$ permit to express the quantities $\delta\mathbf{u}_{\perp,\xi}$ and $\delta T_{\perp,\xi}$, defined in (44)-(45) and associated to a function $\xi \in \mathcal{H}$. Indeed, one has

$$(\xi,\, \Phi^1_x)_{\mathcal{H}} = \langle \varphi^1_x\, \xi\rangle = \sqrt{\frac{n^0}{T^0_\perp}}(\delta\mathbf{u}_{\perp,\xi})_x\,,$$
$$(\xi,\, \Phi^1_y)_{\mathcal{H}} = \langle \varphi^1_y\, \xi\rangle = \sqrt{\frac{n^0}{T^0_\perp}}(\delta\mathbf{u}_{\perp,\xi})_y\,,$$
$$(\xi,\, \Phi^2)_{\mathcal{H}} = \langle \varphi^2\, \xi\rangle = \sqrt{n^0}\,\frac{\delta T_{\perp,\xi}}{T^0_\perp}\,.$$

With this in mind, we state that

$$(-\delta Q_\perp[f^0]\chi,\chi)_{\mathcal{H}} = (-\delta Q_\perp[f^0](\chi-\pi\chi),\chi-\pi\chi)_{\mathcal{H}} = \int_{\mathbb{R}^3_v} T^0_\perp\, f^0 \left|\nabla_{\mathbf{v}_\perp}\left(\frac{\chi-\pi\chi}{f^0}\right)\right|^2 \mathrm{d}\mathbf{v} + 0\,.$$

For the first equality, we used inclusion (107) and the self-adjointness of $\delta Q_\perp[f^0]$. For the second one, we used (54), along with the fact that

$$\delta\mathbf{u}_{\perp,\chi-\pi\chi} = \sqrt{\frac{T^0_\perp}{n^0}}\left(\chi-\pi\chi,\,\begin{pmatrix}\Phi^1_x\\ \Phi^1_y\end{pmatrix}\right)_{\mathcal{H}} = 0\,,\qquad \delta T_{\perp,\chi-\pi\chi} = \frac{T^0_\perp}{\sqrt{n^0}}(\chi-\pi\chi,\Phi^2)_{\mathcal{H}} = 0\,,$$

which can be seen using the orthonormality of the family $(\Phi^0,\, \Phi^1_x,\, \Phi^1_y,\, \Phi^2)$. Finally, the proof is ended by the following inequality

$$(-\delta Q_\perp[f^0]\chi,\chi)_{\mathcal{H}} = \int_{\mathbb{R}^3_v} T^0_\perp\, f^0 \left|\nabla_{\mathbf{v}_\perp}\left(\frac{\chi-\pi\chi}{f^0}\right)\right|^2 \mathrm{d}\mathbf{v} \geqslant \|\chi-\pi\chi\|^2_{\mathcal{H}}\,,$$

which is a consequence of the Gaussian Poincaré inequality. Such an inequality can be proven using well scaled Hermite functions (see for instance [11]).

B.3. **Proof of Proposition 4.** Let us turn to the proof of Proposition 4. Let us firstly give a proof of the first point, giving the form of $\ker \mathbb{A}^{\mathrm{lin}}_{f^0}$. In other terms we prove the following lemma.

Lemma 4. *Let $\mathbb{A}^{\mathrm{lin}}_{f^0}$ be the operator defined in (29) and define the set*

$$\mathcal{S}_2 := \left\{\xi\in\mathcal{H},\quad \xi = \alpha(v_\parallel)\, f^0(\mathbf{v})\right\} \oplus \left\{\xi\in\mathcal{H},\quad \xi = \beta|\mathbf{v}_\perp|^2\, f^0(\mathbf{v}),\quad \beta\in\mathbb{R}\right\}. \tag{109}$$

Then one has

$$\mathcal{S}_2 = \ker\, \mathbb{A}^{\mathrm{lin}}_{f^0}\,. \tag{110}$$

Proof of Lemma 4. Let us prove it by double inclusion. Firstly, every function of $\mathcal{S}_2$ is rotation invariant with respect to $\mathbf{v}_\perp$, and $\mathcal{S}_2 \subset \mathcal{S}_1 = \ker \delta Q_\perp[f^0]$. Therefore

$$\mathcal{S}_2 \subset \ker \delta Q_\perp[f^0] \cap \ker(\mathbf{v}\times\mathbf{B}\cdot\nabla_{\mathbf{v}}) \subset \ker \mathbb{A}^{\mathrm{lin}}_{f^0}\,.$$

Let us now focus on the reciprocal inclusion of (110). To prove it, we are going to prove that

$$\ker \mathbb{A}^{\mathrm{lin}}_{f^0} \cap \mathcal{S}_2^\perp = \{0\}\,, \tag{111}$$

where $\mathcal{S}_2^\perp$ is the orthogonal of $\mathcal{S}_2$ in $\mathcal{H}$. This fact shall prove that $\mathcal{S}_2 = \ker \mathbb{A}^{\mathrm{lin}}_{f^0}$, as $\mathcal{S}_2$ is closed in $\mathcal{H}$.

In order to prove (111) we follow three steps. Firstly we show that if $\chi \in \ker \mathbb{A}^{\mathrm{lin}}_{f^0}$, then

$$\langle \varphi^1_x\, \chi\rangle_\perp = \langle \varphi^1_y\, \chi\rangle_\perp = 0\,,$$

directly from the definition of $\mathbb{A}^{\text{lin}}_{f^0}$. Then, using the coercivity property (59), we shall show that $\chi = \pi\chi$. Finally assuming further that $\chi \in \ker \mathbb{A}^{\text{lin}}_{f^0} \cap \mathcal{S}_2^{\perp}$, we shall prove that $\chi = 0$.

- Let $\chi \in \ker \mathbb{A}^{\text{lin}}_{f^0}$. Firstly, integrating against $\mathbf{v}_\perp \, \mathrm{d}\mathbf{v}$ the equation
$$\mathbb{A}^{\text{lin}}_{f^0}(\chi) = 0\,, \tag{112}$$
yields, after integrating by parts
$$-\langle \mathbf{v}_\perp \, \chi \rangle \times B\, \mathbf{e}_z = 0\,.$$
As a consequence, $\langle \mathbf{v}_\perp \, \chi \rangle = 0$, which reformulates as
$$\langle \varphi^1_x \, \chi \rangle = \langle \varphi^1_y \, \chi \rangle = 0\,. \tag{113}$$
- Then, testing (112) against χ in $\mathcal{H}$ yields, using the skew-symmetry of the magnetic transport term and the coercivity (59):
$$0 = (\mathbb{A}^{\text{lin}}_{f^0}\chi, \chi)_{\mathcal{H}} = (-\nu_\perp \, \delta Q_\perp[f^0]\chi, \chi)_{\mathcal{H}} \geqslant \nu_\perp \, \|\chi - \pi\chi\|^2_{\mathcal{H}}\,.$$
Therefore $\chi = \pi\chi$.
- Finally, assume further that $\chi \in \ker \mathbb{A}^{\text{lin}}_{f^0} \cap \mathcal{S}_2^{\perp}$. Let us denote by Π the orthogonal projection on the space $\mathcal{S}_2$[2]. This projection writes
$$\Pi\, \chi := \left\langle \varphi^0 \chi \right\rangle_\perp \, \Phi^0 + \left\langle \varphi^2 \, \chi \right\rangle \, \Phi^2(\mathbf{v})\,.$$
Since $\chi \in \mathcal{S}_2^{\perp}$, we have that $\Pi\chi = 0$, yielding
$$\pi\, \chi = \Pi\, \chi + \sum_{k=x,y} \langle \varphi^1_k \, \chi \rangle \, \Phi^1_k = 0 + \sum_{k=x,y} \langle \varphi^1_k \, \chi \rangle \, \Phi^1_k = 0\,, \tag{114}$$
using further (113) for the last equality.
We therefore conclude that $\chi = \pi\chi = 0$, thus concluding the proof.

□

The second point of Proposition 4 is an easy consequence of the first one, as Π is the orthogonal projector on $\mathcal{S}_2$, and thus onto $\ker \mathbb{A}^{\text{lin}}_{f^0}$. Let us now prove the third point of Proposition 4, namely the Poincaré-type inequality. From there, the closed range property (point 4) is an immediate consequence (see [17] for instance).

Lemma 5. *The operator $\mathbb{A}^{\text{lin}}_{f^0}$ defined in (29) on $\mathcal{H}$ satisfies a Poincaré-type inequality. In other terms, there exists a constant $C_A > 0$, such that*
$$\|\xi\|_{\mathcal{H}} \leqslant C_A \|\mathbb{A}^{\text{lin}}_{f^0}\xi\|_{\mathcal{H}}, \qquad \forall \xi \in \ker^\perp \mathbb{A}^{\text{lin}}_{f^0} \cap \mathcal{D}(\mathbb{A}^{\text{lin}}_{f^0})\,. \tag{115}$$

Proof. Let us fix $\xi \in \ker^\perp \mathbb{A}^{\text{lin}}_{f^0}$, and decompose the left-hand side of (115) as follows,
$$\|\xi\|^2_{\mathcal{H}} = \|\xi - \pi\xi\|^2_{\mathcal{H}} + \|\pi\xi\|^2_{\mathcal{H}} \tag{116}$$
$$= \|\xi - \pi\xi\|^2_{\mathcal{H}} + \left|\left\langle \varphi^1 \, \xi \right\rangle\right|^2\,, \tag{117}$$
with
$$\varphi^1(\mathbf{v}_\perp) = \begin{pmatrix} \varphi^1_x \\ \varphi^1_y \end{pmatrix}, \quad \Phi^1 = \varphi^1 \, f^0.$$

[2] At this stage, we still do not know that the projection Π defined in (62) is the projection onto $\ker \mathbb{A}^{\text{lin}}_{f^0}$, but it is clear that it is the projection onto $\mathcal{S}_2$.

Indeed, the projection (57), together with $\xi \in \ker^{\perp}\mathbb{A}^{\rm lin}_{f^0}$, yields

$$\pi(\xi) = \Pi\xi + \langle \xi\,\varphi^1\rangle \cdot \Phi^1 = \langle \xi\,\varphi^1\rangle \cdot \Phi^1 \,. \tag{118}$$

Let us now estimate the two terms arising in (117). To simplify notation, let us denote

$$\chi := \mathbb{A}^{\rm lin}_{f^0}(\xi)\,. \tag{119}$$

- For the first term in (117), we test (119) against ξ. Using the coercivity inequality (59) and the skew-adjointness of the $(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}}$ operator yields

$$\nu_\perp\,\|\xi-\pi\xi\|^2_{\mathcal{H}} \leqslant \big(-\nu_\perp\,\delta Q_\perp[f^0]\xi,\xi\big)_{\mathcal{H}} = \big(\mathbb{A}^{\rm lin}_{f^0}\xi,\xi\big)_{\mathcal{H}} = (\chi,\xi)_{\mathcal{H}} \leqslant \|\chi\|_{\mathcal{H}}\,\|\xi\|_{\mathcal{H}}. \tag{120}$$

- Now, to control the second term of (117), we test (119) against $-\frac{1}{B}\left\langle\varphi^1\xi\right\rangle^\top\cdot(\Phi^1)$, and get on the one hand the following sequence of inequalities

$$\begin{aligned}
\left|\left(\mathbb{A}^{\rm lin}_{f^0}(\xi), \frac{1}{B}\left\langle\varphi^1\xi\right\rangle^\top\cdot(\Phi^1)\right)_{\mathcal{H}}\right| &= \left|\frac{1}{B}\left(\chi, \left\langle\varphi^1\xi\right\rangle^\top\cdot(\Phi^1)\right)_{\mathcal{H}}\right| \\
&\leqslant \frac{1}{B}\left(\left|\left(\chi,\left\langle\varphi^1_x\xi\right\rangle(\Phi^1_y)\right)_{\mathcal{H}}\right| + \left|\left(\chi,\left\langle\varphi^1_y\xi\right\rangle(\Phi^1_x)\right)_{\mathcal{H}}\right|\right) \\
&\leqslant \frac{1}{|B|}\|\chi\|_{\mathcal{H}}\left\{\left|\left\langle\varphi^1_x\xi\right\rangle\right| + \left|\left\langle\varphi^1_y\xi\right\rangle\right|\right\} \\
&\leqslant \frac{\sqrt{2}}{|B|}\|\chi\|_{\mathcal{H}}\sqrt{\left|\left\langle\varphi^1_x\xi\right\rangle\right|^2 + \left|\left\langle\varphi^1_y\xi\right\rangle\right|^2} \\
&\leqslant \frac{\sqrt{2}}{|B|}\|\chi\|_{\mathcal{H}}\,\|\pi\xi\|_{\mathcal{H}} \\
&\leqslant \frac{\sqrt{2}}{|B|}\|\chi\|_{\mathcal{H}}\,\|\xi\|_{\mathcal{H}}\,,
\end{aligned}$$

where we used Cauchy-Schwarz for the third line and Parceval's theorem for the fifth line. On the other hand, using that Φ^1_x, $\Phi^1_y \in \ker\delta Q_\perp[f^0] = \mathcal{R}(\delta Q_\perp[f^0])^\perp$, one computes

$$\begin{aligned}
\left(\mathbb{A}^{\rm lin}_{f^0}(\xi), -\frac{1}{B}\left\langle\varphi^1\xi\right\rangle^\top\cdot(\Phi^1)\right)_{\mathcal{H}} &= -\left(B(\mathbf{v}\times\mathbf{e}_z)\cdot\nabla_{\mathbf{v}}\xi, \frac{1}{B}\left\langle\varphi^1\xi\right\rangle^\top\cdot(\Phi^1)\right)_{\mathcal{H}} + 0 \\
&= -\int_{\mathbb{R}^3_v}\left[\nabla_{\mathbf{v}_\perp}\cdot((\mathbf{v}_\perp)^\top\xi)\right]\,\langle\varphi^1\,\xi\rangle^\top\cdot(\varphi^1)\,\mathrm{d}\mathbf{v} \\
&= \int_{\mathbb{R}^3_v}\left[(\mathbf{v}_\perp)^\top\xi\right]\cdot\nabla_{\mathbf{v}_\perp}\left[\langle\varphi^1\,\xi\rangle^\top\cdot(\varphi^1)\right]\,\mathrm{d}\mathbf{v} \\
&= \int_{\mathbb{R}^3_v}\frac{\left[(\mathbf{v}_\perp)^\top\xi\right]\cdot\nabla_{\mathbf{v}_\perp}\left[\langle\varphi^1\,\xi\rangle^\top\cdot(\mathbf{v}_\perp)\right]}{\sqrt{n^0\,T^0_\perp}}\,\mathrm{d}\mathbf{v} \\
&= \int_{\mathbb{R}^3_v}\left[(\varphi^1)^\top\xi\right]\cdot\nabla_{\mathbf{v}_\perp}\left[\langle\varphi^1\,\xi\rangle^\top\cdot(\mathbf{v}_\perp)\right]\,\mathrm{d}\mathbf{v} \\
&= \int_{\mathbb{R}^3_v}\left[(\varphi^1)^\top\xi\right]\,\mathrm{d}\mathbf{v}\cdot\langle\varphi^1\,\xi\rangle^\top \\
&= \left|\langle\varphi^1\xi\rangle\right|^2.
\end{aligned}$$

Assembling the two previous sequence of equalities/inequalities, we get

$$\left|\langle\varphi^1\xi\rangle\right|^2 \leqslant \frac{\sqrt{2}}{|B|}\|\chi\|_{\mathcal{H}}\|\xi\|_{\mathcal{H}}. \tag{121}$$

The result follows, summing (120) with (121), and taking $C_A = \frac{1}{\nu_\perp} + \frac{\sqrt{2}}{|B|}$. □

Appendix C. Proof of the properties stated during the Hilbert expansion

C.1. **Proof of Proposition 5.** The goal of this subsection is to prove Proposition 5 characterizing the microscopic density $\widetilde{f}^1$. The first part of the proof will be dedicated to the rearrangement and computation of the term $\mathcal{R}^0$, using the limit model (17). The second step is dedicated to the computation of the preimage of this term $\mathcal{R}^0$.

Step 1: Computation of $\mathcal{R}^0$. Let us firstly recall the equations on the quantities $(g^0, T_\perp^0)$ (17):

$$\begin{cases} \partial_t g^0 + v_\parallel\,\partial_z g^0 + E_\parallel\,\partial_{v_\parallel} g^0 = \nu_{\rm r}\,\partial_{v_\parallel}\left[(v_\parallel - u_\parallel^0)g^0 + T^0\,\partial_{v_\parallel} g^0\right], \\ \partial_t T_\perp^0 + u_\parallel^0\,\partial_z T_\perp^0 = \dfrac{2}{3}\,\nu_{\rm r}\,\left(T_\parallel^0 - T_\perp^0\right). \end{cases}$$

The computation of $\mathcal{R}^0$ shall be a consequence of the previous system of equations. One decomposes as follows each of the terms composing $\mathcal{R}^0$:

$$\begin{aligned} \partial_t f^0 &= \partial_t g^0\,\mathcal{M}_\perp^{T_\perp^0} + g^0\,\partial_t\mathcal{M}_\perp^{T_\perp^0}, \\ \mathbf{v}\cdot\nabla_{\mathbf{x}} f^0 &= \mathbf{v}_\perp\cdot(\nabla_{\mathbf{x}_\perp} g^0)\mathcal{M}_\perp^{T_\perp^0} + \mathbf{v}_\perp\cdot(\nabla_{\mathbf{x}_\perp}\mathcal{M}_\perp^{T_\perp^0})g^0 + v_\parallel(\partial_z\mathcal{M}_\perp^{T_\perp^0})g^0 + (v_\parallel\partial_z g^0)\,\mathcal{M}_\perp^{T_\perp^0}, \\ \mathbf{E}\cdot\nabla_{\mathbf{v}} f^0 &= \mathbf{E}_\perp\cdot\nabla_{\mathbf{v}_\perp} f^0 + (E_\parallel\partial_{v_\parallel} g^0)\mathcal{M}_\perp^{T_\perp^0}, \\ Q_{\rm r}(f^0) &= \partial_{v_\parallel}\left[(v_\parallel - u_\parallel^0)g^0 + T^0\,\partial_{v_\parallel} g^0\right]\mathcal{M}_\perp^{T_\perp^0} + \nabla_{\mathbf{v}_\perp}\cdot\left[\mathbf{v}_\perp f^0 + T^0\,\nabla_{\mathbf{v}_\perp} f^0\right], \\ &= \partial_{v_\parallel}\left[(v_\parallel - u_\parallel^0)g^0 + T^0\,\partial_{v_\parallel} g^0\right]\mathcal{M}_\perp^{T_\perp^0} + \frac{2}{3\,T_\perp^0}\,(T_\parallel^0 - T_\perp^0)\left[\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right] f^0. \end{aligned}$$

Using those decompositions yields, after a rearrangement of the terms of $\mathcal{R}^0$,

$$\begin{aligned} \mathcal{R}^0 = &\left[-\mathbf{v}_\perp\cdot(\nabla_{\mathbf{x}_\perp} g^0)\mathcal{M}_\perp^{T_\perp^0} - \mathbf{v}_\perp\cdot(\nabla_{\mathbf{x}_\perp}\mathcal{M}_\perp^{T_\perp^0})g^0 - \mathbf{E}_\perp\cdot\nabla_{\mathbf{v}_\perp} f^0\right] \\ &- g^0\partial_t\mathcal{M}_\perp^{T_\perp^0} - v_\parallel(\partial_z\mathcal{M}_\perp^{T_\perp^0})g^0 + \frac{2}{3\,T_\perp^0}\,\nu_{\rm r}\left(T_\parallel^0 - T_\perp^0\right)\left[\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right] f^0 \\ &- \left[\partial_t g^0 + v_\parallel\,\partial_z g^0 + E_\parallel\,\partial_{v_\parallel} g^0 - \nu_{\rm r}\,\partial_{v_\parallel}\left[(v_\parallel - u_\parallel^0)g^0 + T^0\,\partial_{v_\parallel} g^0\right]\right]\mathcal{M}_\perp^{T_\perp^0}. \end{aligned}$$

The third line is zero, in view of the limit model equation on g^0 (17c). Let us now simplify the other terms.

- $$\left(-\mathbf{v}_\perp \cdot (\nabla_{\mathbf{x}_\perp} \mathcal{M}_\perp^{T_\perp^0})\, g^0\right) = -\frac{(\mathbf{v}_\perp \cdot \nabla_{\mathbf{x}_\perp} T_\perp^0)}{T_\perp^0}\left[\frac{|\mathbf{v}_\perp|^2}{2T_\perp^0} - 1\right] f^0,$$
- $$\left(-\mathbf{E}_\perp \cdot \nabla_{\mathbf{v}_\perp} f^0\right) = \mathbf{E}_\perp \cdot \frac{\mathbf{v}_\perp}{T_\perp^0} f^0,$$
- $$\begin{aligned} -g^0 \partial_t \mathcal{M}_\perp^{T_\perp^0} - v_\parallel (\partial_z \mathcal{M}_\perp^{T_\perp^0}) g^0 &= \frac{-\partial_t T_\perp^0 - v_\parallel \partial_z T_\perp^0}{T_\perp^0}\left[\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right] f^0 \\ &= \frac{(u_\parallel^0 - v_\parallel)\, \partial_z T_\perp^0}{T_\perp^0}\left[\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right] f^0 \\ &\quad - \frac{2}{3\,T_\perp^0}\, \nu_{\mathrm{r}}\, (T_\parallel^0 - T_\perp^0)\left[\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right] f^0, \end{aligned}$$

the very last equality being given thanks to the equation on the temperature $T_\perp^0$ (20). In view of the last three equalities, $\mathcal{R}^0$ rewrites in the following way

$$\begin{aligned} \mathcal{R}^0 = &\left\{ \frac{\mathbf{E}_\perp \cdot \mathbf{v}_\perp}{T_\perp^0} - \frac{(\mathbf{v}_\perp \cdot \nabla_{\mathbf{x}_\perp} T_\perp^0)}{T_\perp^0}\left[\frac{|\mathbf{v}_\perp|^2}{2T_\perp^0} - 1\right] + \frac{\partial_z T_\perp^0}{T_\perp^0}\,(u_\parallel^0 - v_\parallel)\left[\frac{|\mathbf{v}_\perp|^2}{2\,T_\perp^0} - 1\right]\right\} f^0 \\ &- \mathbf{v}_\perp \cdot \left(\nabla_{\mathbf{x}_\perp} g^0\right) \mathcal{M}_\perp^{T_\perp^0}. \end{aligned}$$

One notices that $\Pi\, \mathcal{R}^0 = 0$, thus yielding that $\mathcal{R}^0 \in \ker^\perp \mathbb{A}_{f^0}^{\mathrm{lin}} = \mathcal{R}(\mathbb{A}_{f^0}^{\mathrm{lin}})$. It therefore makes sense to solve $\widetilde{f}^1 = {\mathbb{A}_{f^0}^{\mathrm{lin}}}^{-1}(\mathcal{R}^0)$. This ends the first part of the proof. Let us now turn to the computation of the microscopic density $\widetilde{f}^1$.

Step 2: Computation of $\widetilde{f}^1$. One can separate this equality into several key terms:

$$\begin{aligned} \mathcal{R}^0 = &\left\{ \left(\mathbf{E}_\perp + \nabla_{\mathbf{x}_\perp} T_\perp^0\right) \cdot \frac{\mathbf{v}_\perp}{T_\perp^0} - \frac{\nabla_{\mathbf{x}_\perp} T_\perp^0}{2\,(T_\perp^0)^2} \cdot \left(\mathbf{v}_\perp |\mathbf{v}_\perp|^2\right) + \frac{\partial_z T_\perp^0}{T_\perp^0}\,(u_\parallel^0 - v_\parallel)\left[\frac{|\mathbf{v}_\perp|^2}{2T_\perp^0} - 1\right]\right\} f^0 \\ &- \mathbf{v}_\perp \cdot \left(\nabla_{\mathbf{x}_\perp} g^0\right) \mathcal{M}_\perp^{T_\perp^0}. \end{aligned}$$

It is therefore enough to find a preimage by $\mathbb{A}_{f^0}^{\mathrm{lin}}$ for each of those terms, by linearity. One however needs to notice that $\nabla_{\mathbf{x}_\perp} g^0$ depends on $v_\parallel$, therefore the associated term needs a special treatment.

The goal of the following Lemma is to compute those preimages. This Lemma is technical, so we shall firstly admit it to finish the computation, its proof is postponed in the next subsection of the Appendix.

Lemma 6. *One gathers in this lemma the preimages in* $\ker^\perp \mathbb{A}^{\text{lin}}_{f^0}$ *of some specific functions.*

$$\mathbb{A}^{\text{lin}}_{f^0}\left(\frac{-(\mathbf{v}_\perp)^\top}{B}\, f^0\right) = \mathbf{v}_\perp\, f^0, \tag{122}$$

$$\mathbb{A}^{\text{lin}}_{f^0}\left(\frac{u^0_\parallel - v_\parallel}{2\,\nu_\perp}\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 1\right] f^0\right) = (u^0_\parallel - v_\parallel)\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 1\right] f^0, \tag{123}$$

$$\mathbb{A}^{\text{lin}}_{f^0}\left(n^0\left[\mathbb{D}_2\nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right)\right]\cdot \mathbf{v}_\perp \mathcal{M}^{T^0_\perp}_\perp - \frac{1}{B}\nabla_{\mathbf{x}_\perp} g^0\cdot(\mathbf{v}_\perp)^\top \mathcal{M}^{T^0_\perp}_\perp\right) = \mathbf{v}_\perp\cdot[\nabla_{\mathbf{x}_\perp} g^0]\mathcal{M}^{T^0_\perp}_\perp, \tag{124}$$

$$\mathbb{A}^{\text{lin}}_{f^0}\left(2T^0_\perp\,\mathbb{D}^t_1\,\mathbf{v}_\perp\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 2\right] f^0 - 4\,T^0_\perp\,\frac{(\mathbf{v}_\perp)^\top}{B}\, f^0\right) = \mathbf{v}_\perp\,|\mathbf{v}_\perp|^2\, f^0. \tag{125}$$

where the exponent t *is the transposition of matrices, and* $\mathbb{D}_1$, $\mathbb{D}_2$ *are given by*

$$\mathbb{D}_1 = \frac{1}{B^2 + 9\,\nu_\perp^2}\begin{bmatrix} 3\,\nu_\perp & B \\ -B & 3\,\nu_\perp \end{bmatrix}, \qquad \mathbb{D}_2 = \frac{\nu_\perp}{B}\frac{1}{B^2 + \nu_\perp^2}\begin{bmatrix} B & -\nu_\perp \\ \nu_\perp & B \end{bmatrix}. \tag{126}$$

One checks easily that the preimages are in $\ker^\perp \mathbb{A}^{\text{lin}}_{f^0}$. The proof of this Lemma is quite technical, and is delayed in the next subsection, for the sake of clarity.

Using this Lemma permits thus to explicit the microscopic part of the first order correction: using repeatedly that $X\cdot Y^\top = -X^\top\cdot Y$, we find

$$\widetilde{f}^1 = (\mathbb{A}^{\text{lin}}_{f^0})^{-1}(\mathcal{R}^0) \tag{127}$$

$$= \left(g^0\,\mathbf{E}^\top_\perp + g^0\,\nabla^\top_{\mathbf{x}_\perp} T^0_\perp\right)\cdot\frac{\mathbf{v}_\perp}{B\,T^0_\perp}\mathcal{M}^{T^0_\perp}_\perp \tag{128}$$

$$-\left(\mathbb{D}_1\frac{\nabla_{\mathbf{x}_\perp} T^0_\perp}{T^0_\perp}\right)\cdot\mathbf{v}_\perp\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 2\right] f^0 - 2\,\nabla^\top_{\mathbf{x}_\perp} T^0_\perp\cdot\frac{\mathbf{v}_\perp}{B\,T^0_\perp} f^0 \tag{129}$$

$$+\frac{\partial_z T^0_\perp}{T^0_\perp}\frac{u^0_\parallel - v_\parallel}{2\,\nu_\perp}\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 1\right] f^0 \tag{130}$$

$$-\,n^0\left\{\mathbb{D}_2\nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right)\right\}\cdot\mathbf{v}_\perp\mathcal{M}^{T^0_\perp}_\perp - \frac{1}{B}\nabla^\top_{\mathbf{x}_\perp} g^0\cdot\mathbf{v}_\perp\,\mathcal{M}^{T^0_\perp}_\perp\,. \tag{131}$$

Arranging the last terms in the first, second, and last line in the right-hand side of the previous equality gives,

$$\begin{aligned}\widetilde{f}^1 &= \left(g^0\,\mathbf{E}^\top_\perp - \nabla^\top_{\mathbf{x}_\perp}(g^0\,T^0_\perp)\right)\cdot\frac{\mathbf{v}_\perp}{B\,T^0_\perp}\mathcal{M}_\perp \\ &\quad-\left(\mathbb{D}_1\frac{\nabla_{\mathbf{x}_\perp} T^0_\perp}{T^0_\perp}\right)\cdot\mathbf{v}_\perp\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 2\right] f^0 \\ &\quad+\frac{\partial_z T^0_\perp}{T^0_\perp}\frac{u^0_\parallel - v_\parallel}{2\,\nu_\perp}\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 1\right] f^0 \\ &\quad-\,n^0\left\{\mathbb{D}_2\nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right)\right\}\cdot\mathbf{v}_\perp\mathcal{M}_\perp\,,\end{aligned}$$

which is exactly (73). Therefore, it remains only to prove Lemma 6.

C.2. **Proof of Lemma 6.** The first equality (122) follows from a simple computation, and we skip it for the sake of shortness.

Proof of (123)**.** One firstly notices that

$$\mathbb{A}^{\text{lin}}_{f^0}\left(\frac{u^0_\parallel - v_\parallel}{2}\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 1\right] f^0\right) = -\mathbb{A}^{\text{lin}}_{f^0}\left(v_\parallel \frac{|\mathbf{v}_\perp|^2}{4\,T^0_\perp} f^0\right). \tag{132}$$

Then, the computation goes as follows:

$$\begin{aligned}
\mathbb{A}^{\text{lin}}_{f^0}\left(\frac{u^0_\parallel - v_\parallel}{2}\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 1\right] f^0\right) &= -\mathbb{A}^{\text{lin}}_{f^0}\left(v_\parallel \frac{|\mathbf{v}_\perp|^2}{4\,T^0_\perp} f^0\right)\\
&= \nu_\perp\, \delta Q_\perp[f^0]\left(v_\parallel \frac{|\mathbf{v}_\perp|^2}{4\,T^0_\perp} f^0\right)\\
&= \nu_\perp\, \nabla_{\mathbf{v}_\perp}\cdot\left[v_\parallel\, \mathbf{v}_\perp \frac{|\mathbf{v}_\perp|^2}{4\,T^0_\perp} f^0 - \frac{1}{2}u^0_\parallel\, \mathbf{v}_\perp\, f^0 + T^0_\perp\, \nabla_{\mathbf{v}_\perp}\left(v_\parallel \frac{|\mathbf{v}_\perp|^2}{4\,T^0_\perp} f^0\right)\right]\\
&= \nu_\perp \frac{(v_\parallel - u^0_\parallel)}{2}\nabla_{\mathbf{v}_\perp}\cdot\left[\mathbf{v}_\perp\, f^0\right]\\
&= \nu_\perp \frac{(v_\parallel - u^0_\parallel)}{2}\left(2 - \frac{|\mathbf{v}_\perp|^2}{T^0_\perp}\right) f^0\\
&= \nu_\perp\,(u^0_\parallel - v_\parallel)\left(\frac{|\mathbf{v}_\perp|^2}{2T^0_\perp} - 1\right) f^0.
\end{aligned}$$

Proof of (124)**.** In this proof, $\chi(v_\parallel)$ will denote a function of the parallel variable $v_\parallel$ only. One computes the following equality:

$$\mathbb{A}^{\text{lin}}_{f^0}(\mathbf{v}_\perp\,\chi(v_\parallel)\,\mathcal{M}^{T^0_\perp}_\perp) = \begin{pmatrix}\nu_\perp & B\\ -B & \nu_\perp\end{pmatrix}\mathbf{v}_\perp\,\chi(v_\parallel)\,\mathcal{M}^{T^0_\perp}_\perp - \nu_\perp\frac{\langle\chi\rangle_\parallel}{n^0}\mathbf{v}_\perp\, f^0. \tag{133}$$

Thus, multiplying by the inverse matrix

$$\begin{pmatrix}\nu_\perp & B\\ -B & \nu_\perp\end{pmatrix}^{-1} = \frac{1}{B^2+\nu_\perp^2}\begin{pmatrix}\nu_\perp & -B\\ B & \nu_\perp\end{pmatrix}, \tag{134}$$

using equation (122), and reordering the terms, one isolates $\mathbf{v}_\perp\,\chi(v_\parallel)\,\mathcal{M}^{T^0_\perp}_\perp$ and we find

$$\frac{1}{B^2+\nu_\perp^2}\mathbb{A}^{\text{lin}}_{f^0}\left(\nu_\perp\,\mathbf{v}_\perp\chi\mathcal{M}^{T^0_\perp}_\perp - B\,(\mathbf{v}_\perp)^\top\chi\mathcal{M}^{T^0_\perp}_\perp - \nu_\perp^2\frac{\langle\chi\rangle_\parallel}{n^0}\frac{(\mathbf{v}_\perp)^\top}{B} f^0 - \nu_\perp\frac{\langle\chi\rangle_\parallel}{n^0}\mathbf{v}_\perp\, f^0\right) = \mathbf{v}_\perp\chi\mathcal{M}^{T^0_\perp}_\perp.$$

Now, let us take $\chi = \partial_x g^0$ and look at the first coordinate of the previous equality. Then take $\chi = \partial_y g^0$, and look at the second coordinate. Summing these two observations yields

$$\begin{aligned}
\frac{1}{B^2+\nu_\perp^2}\mathbb{A}^{\text{lin}}_{f^0}\Big(\nu_\perp\,\mathbf{v}_\perp\cdot[\nabla_{\mathbf{x}_\perp}g^0]\mathcal{M}^{T^0_\perp}_\perp - B\,(\mathbf{v}_\perp)^\top\cdot[\nabla_{\mathbf{x}_\perp}g^0]\mathcal{M}^{T^0_\perp}_\perp&\\
-\nu_\perp^2\frac{\langle\nabla_{\mathbf{x}_\perp}g^0\rangle_\parallel}{n^0}\cdot\frac{(\mathbf{v}_\perp)^\top}{B} f^0 - \nu_\perp\frac{\langle\nabla_{\mathbf{x}_\perp}g^0\rangle_\parallel}{n^0}\cdot\mathbf{v}_\perp\, f^0\Big) &= \mathbf{v}_\perp\cdot[\nabla_{\mathbf{x}_\perp}g^0]\mathcal{M}^{T^0_\perp}_\perp.
\end{aligned}$$

Now, getting the result is just a matter of presentation. Using that $\langle g^0\rangle_\parallel = n^0$, and performing a simple computation, the last equality rewrites

$$\mathbb{A}^{\text{lin}}_{f^0}\left(n^0\left[\mathbb{D}_2\,\nabla_{\mathbf{x}_\perp}\left(\frac{g^0}{n^0}\right)\right]\cdot\mathbf{v}_\perp\mathcal{M}^{T^0_\perp}_\perp - \frac{1}{B}\nabla_{\mathbf{x}_\perp}g^0\cdot(\mathbf{v}_\perp)^\top\mathcal{M}^{T^0_\perp}_\perp\right) = \mathbf{v}_\perp\cdot[\nabla_{\mathbf{x}_\perp}g^0]\mathcal{M}^{T^0_\perp}_\perp,$$

thus finishing the proof.

Proof of (125)**.** One computes:

$$\mathbb{A}^{\text{lin}}_{f^0}\left(\mathbf{v}_\perp|\mathbf{v}_\perp|^2 f^0\right) = M_3^{\mathbb{A}^{\text{lin}}_{f^0}}\mathbf{v}_\perp|\mathbf{v}_\perp|^2 f^0 \; - 12\,\nu_\perp\,T^0_\perp\mathbf{v}_\perp\,f^0, \tag{135}$$

where we denoted by $M_3^{\mathbb{A}^{\text{lin}}_{f^0}}$ the matrix

$$M_3^{\mathbb{A}^{\text{lin}}_{f^0}} := \begin{pmatrix} 3\,\nu_\perp & B \\ -B & 3\,\nu_\perp \end{pmatrix},$$

which is invertible with inverse

$$(M_3^{\mathbb{A}^{\text{lin}}_{f^0}})^{-1} = \frac{1}{B^2+9\,\nu_\perp^2}\begin{pmatrix} 3\,\nu_\perp & -B \\ B & 3\,\nu_\perp \end{pmatrix} = \mathbb{D}^t_1\,.$$

Therefore one gets, after multiplication by $\mathbb{D}^t_1$, and reordering the terms

$$\begin{aligned}
\mathbf{v}_\perp\,|\mathbf{v}_\perp|^2\,f^0 &= \frac{36\,T^0_\perp\,\nu_\perp^2}{B^2+9\,\nu_\perp^2}\,\mathbf{v}_\perp\,f^0 - \frac{12\,T^0_\perp\,B\,\nu_\perp}{B^2+9\,\nu_\perp^2}\,(\mathbf{v}_\perp)^\top\,f^0 + \mathbb{A}^{\text{lin}}_{f^0}\left(\mathbb{D}^t_1\,\mathbf{v}_\perp\,|\mathbf{v}_\perp|^2\,f^0\right) \\
&= \mathbb{A}^{\text{lin}}_{f^0}\left(-\frac{36\,T^0_\perp\,\nu_\perp^2}{B^3+9\,B\,\nu_\perp^2}\,(\mathbf{v}_\perp)^\top\,f^0 - \frac{12\,T^0_\perp\,\nu_\perp}{B^2+9\,\nu_\perp^2}\,(\mathbf{v}_\perp)\,f^0 + \mathbb{D}^t_1\,\mathbf{v}_\perp\,|\mathbf{v}_\perp|^2\,f^0\right) \\
&= \mathbb{A}^{\text{lin}}_{f^0}\left(-4\frac{T^0_\perp}{B}\,(\mathbf{v}_\perp)^\top\,f^0 + 2\,T^0_\perp\,\mathbb{D}^t_1\mathbf{v}_\perp\,\left[\frac{|\mathbf{v}_\perp|^2}{2\,T^0_\perp} - 2\right]\,f^0\right).
\end{aligned}$$

For the second equality, we used equality (122). The last equality comes after simple computation using the definition of $\mathbb{D}^t_1$.

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