FOKKER-PLANCK EQUATION FOR ENERGETIC PARTICLES. THE κ-DISTRIBUTION FUNCTION.

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Abstract. The first concern of the present paper is the mathematical study of a specific Fokker-Planck equation describing the dynamics of energetic particles (runaway electrons for example) occurring in thermonuclear fusion plasmas or astrophysics. In the long-time limit, the velocity distribution function of these particles tends towards a (thermal) non-equilibrium distribution, namely a κ-distribution function which is a steady-state of the considered Fokker-Planck equation. Our aim is firstly to investigate the convergence rate of the velocity distributions towards these stationary states as $t \to \infty$, as well as in a second step to design an efficient spectral scheme, permitting to cope with this long-time asymptotics, without too much numerical costs. Non-equilibrium (or non-Maxwellian) distributions need to be considered properly enough in fusion or astrophysical plasmas in order to accurately reproduce the plasma dynamics and to understand the impact of the runaway electrons on the whole thermal plasma bulk as well as for issues like astonaut safety in astrophysics.

Keywords: Plasma modelling, Fokker-Planck kinetic equation, energetic particles, kappa-distribution function, (thermal) non-equilibrium steady-states, asymptotic analysis, spectral methods, numerical scheme.

Contents

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1. Introduction

Stationary states out of thermal equilibrium, are often observed in a large number of fields, such as laboratory (tokamak) plasmas [37], astrophysical plasmas [6, 47, 59], molecular dynamics [9], economy [23] and so on. In magnetically confined fusion plasmas a lot of mechanisms keep the particle distribution function far from a thermal equilibrium (Maxwellian distribution), mechanisms such as radio-frequency heating and neutral beam injection used for ignition, production of highly energetic α-particles through fusion reactions, non-thermal acceleration processes via electric fields (runaway electrons) and so on. The physical reason for the departure from Maxwellian distributions is linked to the fact that while low-energy particles are often collisional, faster ones are generally not any more (or less collisional), being thus easily driven out of the equilibrium and escaping to large distances. This comes from the fact that fast particles spend less time in the neighbourhood of each particle it collides with, when compared to slower particles. The question is now, how to model such energetic particle distribution functions (in the velocity space), keeping the Maxwellian distributions as a limiting case.
It is important to understand that an accurate description of the energetic particle dynamics is of crucial importance for fusion reactor performances, in particular to predict confinement. Indeed, the success of magnetically confined fusion reactors relies upon a proper confinement of the energetic fusion products (*α-particles*), and this for sufficiently long times, such that they can transfer enough energy to the fuel ions in order to permit the fusion reaction to take place in a self-sustained manner. Furthermore, *runaway electrons* can have an essential impact on the global stability of the fusion bulk plasma, their control being thus an essential point. In astrophysics a good understanding of runaway dynamics is essential for issues like astonaut safety or for more fundamental reasons, such as the understanding of solar flares. For more physical details about all these phenomena we refer to [13, 38].

Different mathematical approaches can be now considered to describe a plasma gas. A fully kinetic description of the whole electron-ion plasma is a very precise approach, however for the moment still numerically out of reach in the full coupled $6D$ phase-space. Fluid approximations become questionable when the velocity distribution functions have suprathermal tails, as they assume the plasma to be in a local thermodynamic equilibrium. Thus, different strategies have been introduced in literature, trying to keep a balance between physical precision and computational costs. One of these approaches is based on the introduction of analytic non-Maxwellian distribution functions, permitting to reduce the degrees of freedom of the full kinetic distribution $f(t,\mathbf{x},\mathbf{v})$ to the computation of a small amount of macroscopic quantities, such as the particle density $n(t,\mathbf{x})$, the mean velocity $\mathbf{u}(t,\mathbf{x})$, the temperature $T(t,\mathbf{x})$ *etc.* For example *beam-distributions* of the type

$$f(t,\mathbf{x},\mathbf{v}) := n_b \left(\frac{m_b}{2\,\pi\,k_B\,T_b}\right)^{3/2} e^{-m_b \frac{|\mathbf{v}-\mathbf{u}_b|^2}{2\,k_B\,T_b}} + n_f \left(\frac{m_f}{2\,\pi\,k_B\,T_f}\right)^{3/2} e^{-m_f \frac{|\mathbf{v}-\mathbf{u}_f|^2}{2\,k_B\,T_f}}\,, \tag{1.1}$$

are a good approximation for plasmas comprising also fast particles, and this in the case there is no interaction between the bulk plasma (main Gaussian bump, described by $(n_b,\mathbf{u}_b,T_b)$) and the fast particles (secondary Gaussian bump, described by $(n_f,\mathbf{u}_f,T_f)$). The generalization to a sum of more than two Maxwellians is also possible.
κ-distribution functions are commonly encountered in astrophysics to describe particle distributions having a core Maxwellian and a power-law (and not exponential) decrease in the large velocity ranges. This means, κ-distributions describe an over-population of particles at high speeds, when compared to Maxwellian distributions, however the fast particles are

not any more localized around some specific velocity as in the bump-distribution case (1.1). Several definitions of κ-distributions exist in literature, characterizing the power-law nature of the energetic tails in different manners. The κ-distribution of first kind is given by the formula

$$(1.2) \qquad f_\kappa(t,\mathbf{x},\mathbf{v}) = A_\kappa\, n(t,\mathbf{x}) \left(1+\frac{|\mathbf{v}|^2}{\kappa\, v_{th}^2}\right)^{-\kappa}, \qquad A_\kappa := \frac{1}{(\pi\,\kappa\, v_{th}^2)^{3/2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-3/2)},$$

where the thermal speed $v_{th} := \sqrt{\frac{2\,k_B\,T}{m}}$ and the particle density n are fixed by the associated Maxwellian $\mathcal{M}$, given by

$$\mathcal{M}(t,\mathbf{x},\mathbf{v}) := \frac{n(t,\mathbf{x})}{(\pi\, v_{th}^2)^{3/2}}\, e^{-\frac{|\mathbf{v}|^2}{v_{th}^2}}, \qquad n := \int_{\mathbb{R}^3} \mathcal{M}\, d\mathbf{v} = \int_{\mathbb{R}^3} f_\kappa\, d\mathbf{v}, \quad \frac{3}{2}\, n\, k_B\, T := \frac{m}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\, \mathcal{M}\, d\mathbf{v}.$$

Let us recall here for simplicity the Euler Gamma function Γ, defined for $r \in \mathbb{R}_*^+$ by

$$\Gamma(r) := \int_0^\infty s^{r-1}\, e^{-s}\, ds, \qquad \Gamma(1) = 1, \ \ \Gamma(1/2) = \sqrt{\pi}, \ \ \Gamma(r+1) = r\,\Gamma(r),$$

and satisfying $\Gamma(r+\alpha) \sim_{r\to\infty} \Gamma(r)\, r^\alpha$ for all $\alpha \in \mathbb{R}^+$, where the symbol $\sim$ stands for the asymptotic equivalence. Remark now that f_κ contains three parameters, v_{th} and n which are linked to the associated Maxwellian, and $3/2 < \kappa < \infty$ which is the only free parameter and determines the distance away from the Maxwellian, in particular the lower the κ, the more pronounced are the tails. One can introduce a temperature T_κ (for $\kappa > 5/2$) corresponding to this κ-distribution function

$$\frac{3}{2}\, n\, k_B\, T_\kappa(t,\mathbf{x}) := \frac{m}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\, f_\kappa(t,\mathbf{x},\mathbf{v}) d\mathbf{v},$$

which is however different from the Maxwellian temperature T. Note that temperature definitions are appropriate for thermal equilibrium (Maxwellian) distributions, thus not rigorously valid for a κ-distribution function, but there are practical reasons for using such kinetic temperature definitions.

We represented on the left of Figure 1 a comparison between the different κ-distributions and the associated Maxwellian. What one observes is that the κ-distribution function has an asymptotic power law decrease for large v and joints smoothly with a Maxwellian core at low speeds. Furthermore, remark that one recovers the associated Maxwellian distribution function for $\kappa \to \infty$, in particular one has $T_\kappa \to_{\kappa\to\infty} T$. All this can be simply shown, using

$$e^{-\xi} = \lim_{\kappa\to\infty}\left(1+\frac{\xi}{\kappa}\right)^{-\kappa}, \quad \lim_{\kappa\to\infty} A_\kappa = \frac{1}{(\pi\, v_{th}^2)^{3/2}} \qquad \Rightarrow \qquad f_\kappa(\mathbf{v}) \to_{\kappa\to\infty} \frac{1}{(\pi\, v_{th}^2)^{3/2}}\, e^{-\frac{|\mathbf{v}|^2}{v_{th}^2}},$$

as well as

$$f_\kappa(\mathbf{v}) \propto_{|\mathbf{v}|\ll 1} \left(1-\frac{|\mathbf{v}|^2}{v_{th}^2}\right), \qquad f_\kappa(\mathbf{v}) \propto_{|\mathbf{v}|\gg 1} |\mathbf{v}|^{-2\,\kappa},$$

where the symbol $\propto$ shall stand in the following for the proportionality relation, to be distinguished from the asymptotic equivalence symbol $\sim$. Finally, let us finish with mentioning that the moments of the κ-distribution function are bounded only for a finite number of $l \in \mathbb{N}$, as the distributions are not exponentially decaying in $|\mathbf{v}|$, but polynomially. One can show indeed that

$$(1.3) \qquad \int_{\mathbb{R}^3} |\mathbf{v}|^l\, f_\kappa(t,\mathbf{x},\mathbf{v}) d\mathbf{v} = \frac{n(t,\mathbf{x})}{\pi^{3/2}} \left(\kappa\, v_{th}^2\right)^{l/2} \frac{\Gamma\left(\frac{l+3}{2}\right)\Gamma\left(\kappa-\frac{l+3}{2}\right)}{\Gamma\left(\kappa-\frac{3}{2}\right)}, \quad \forall 0 \le l < 2\,\kappa - 3.$$

The fact that higher moments are unbounded excludes the design of spectral methods based on orthogonal polynomials as basis-sets (orthogonal with respect to the weight-function f_κ), such as the Hermite-polynomials for the Maxwellian weights. Finally, integrating (1.2) over two velocity space dimensions yields the 1D κ-distribution function of first kind

$$f_\kappa^{1D}(t,\mathbf{x},v) = A_\kappa^{1D}\, n(t,\mathbf{x}) \left(1+\frac{v^2}{\kappa\, v_{th}^2}\right)^{-(\kappa-1)}, \qquad A_\kappa^{1D} := \frac{1}{(\pi\,\kappa\, v_{th}^2)^{1/2}}\,\frac{\Gamma(\kappa-1)}{\Gamma(\kappa-3/2)}. \tag{1.4}$$

Remark 1. *The more conventional form of the κ-distribution function is the so-called κ-distribution of second kind (see Fig. 1, for a comparison with the first kind)*

$$g_k(t,\mathbf{x},\mathbf{v}) = N_k \left(1+\frac{|\mathbf{v}|^2}{k\, v_{th,k}^2}\right)^{-(k+1)}, \qquad N_k := \frac{n(t,\mathbf{x})}{(\pi\, k\, v_{th,k}^2)^{3/2}}\,\frac{\Gamma(k+1)}{\Gamma(k-1/2)}, \tag{1.5}$$

with $k>3/2$ and $v_{th,k}$ the k-dependent effective thermal velocity $v_{th,k} := \sqrt{\frac{(k-3/2)}{k}}\sqrt{\frac{2\,k_B\,T}{m}}$. The particle density and temperature of this distribution function

$$n(t,\mathbf{x}) := \int_{\mathbb{R}^3} g_k(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \qquad \frac{3}{2}\,k_B\, n\, T(t,\mathbf{x}) := \frac{m}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\, g_k(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,,$$

are the same as for the associated Maxwellian

$$\mathcal{M}(t,\mathbf{x},\mathbf{v}) := \frac{n(t,\mathbf{x})}{(\pi\, v_{th}^2)^{3/2}}\, e^{-\frac{|\mathbf{v}|^2}{v_{th}^2}}, \qquad \text{where} \quad v_{th} := \sqrt{\frac{2\,k_B\,T}{m}}.$$

Remark that this time, the effective thermal velocity $v_{th,k}$ is a reference speed and not the usual thermal speed of the Maxwellian, but linked to it through $v_{th,k} \to_{k\to\infty} v_{th} := \sqrt{\frac{2\,k_B\,T}{m}}$.

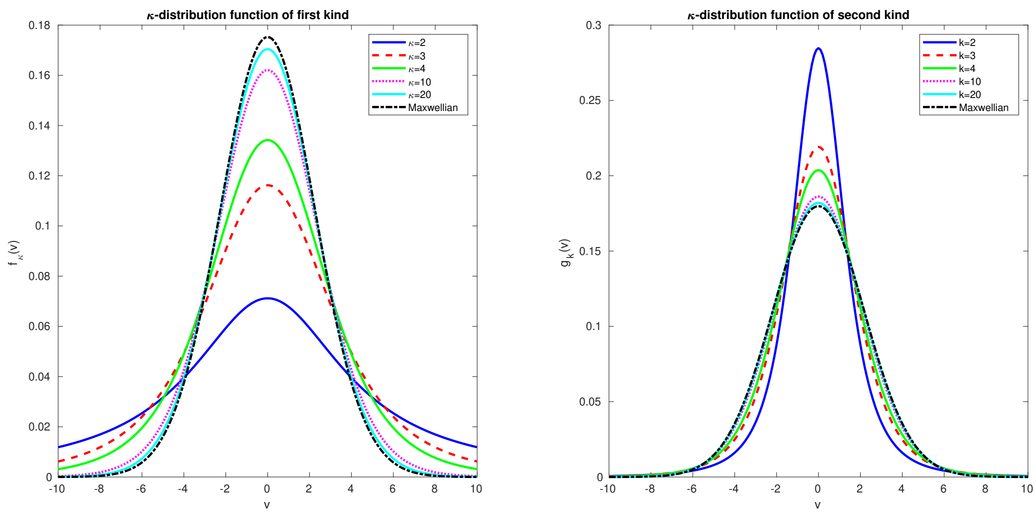

FIGURE 1. 1D κ-distribution functions for several κ-values and associated Maxwellian distributions, recovered for $\kappa\to\infty$. Left κ-distribution of first kind, given in (1.4). Right: κ-distribution of second kind, given by integrating (1.5) over 2 velocity degrees of freedom.

1.1. **Physical motivation.** Let us say some words in this subsection, about the manner how κ-distributions arise. We shall restrict here, for simplicity reasons, to the one-dimensional case $v \in \mathbb{R}$. κ-distribution functions are steady state solutions of specific Fokker-Planck equations, where the drift and diffusion coefficients are velocity dependent and are linked through a specific ratio-relation. It is a non-equilibrium effect which leads to κ-distributions, where the driving (drift) and the compensation via diffusion give rise to a stationary state that does not satisfy the fluctuation-dissipation theorem (Einstein's relation is violated).

Let us precise this in more details and consider first the general one-dimensional Fokker-Planck equation

$$\partial_t f = \nu\, \partial_v \left[D(v)\, (U'(v)\, f + \partial_v f)\right] , \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R}\,, \tag{1.6}$$

where $\nu > 0$ is the collisional frequency, $D(v) \geq 0$ is the velocity dependent diffusion coefficient and $U(v)$ a potential function describing the drift mechanism. Remark that we have omitted in this paper the usual transport term in the kinetic equation, in order to concentrate only on the Fokker-Planck collision operator. In plasma physics the diffusion coefficient is usually decaying for $v \to \pm\infty$, for example we have the proportionality relation $D(v) \propto_{v \gg 1} v^{-3}$ in the case of Coulomb collisions. In the work [18] of one of the authors, it is more precisely explained that one has indeed $D(v) := \frac{G(v/v_{th})}{v/v_{th}}$, with $v_{th} := \sqrt{\frac{2\,k_B\,T}{m}}$ the thermal speed and where the Chandrasekhar function G is defined via the error function ϕ as follows

$$G(x) := \frac{\phi(x) - x\,\phi'(x)}{2\,x^2}\,, \qquad \phi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\,dy\,. \tag{1.7}$$

The stationary solutions of (1.6) are given by

$$f_\infty(v) := \alpha\, e^{-U(v)}\,, \qquad \forall v \in \mathbb{R}\,,$$

with $\alpha > 0$ determined by the initial condition f_{in}. The first observation is that a quadratic potential $U(v) := \frac{v^2}{v_{th}^2}$ yields the standard Fokker-Planck equation

$$\partial_t f = \nu\, \partial_v \left[\frac{2\,D(v)}{v_{th}^2} \left(v\, f + \frac{v_{th}^2}{2}\, \partial_v f\right)\right] , \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R}\,,$$

with the usual Maxwellian equilibrium $\frac{n}{\sqrt{\pi\, v_{th}^2}}\, e^{-v^2/v_{th}^2}$.

The question now is how κ-distributions can arise. Consider thus in a second step the following specific Fokker-Planck equation

$$\partial_t f = \nu\, \partial_v \left\{\gamma(v) \left[v\, f + \left(\frac{v_{th}^2}{2} + D_{turb}(v)\right) \partial_v f\right]\right\} , \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R}\,, \tag{1.8}$$

which includes the combined effects of Coulomb collisions with a dense plasma background and a turbulent acceleration mechanism described by $D_{turb}(v)$. Here $v_{th} := \sqrt{\frac{2\,k_B\,T}{m}}$ is again the thermal speed and $\gamma(v)$ is the friction coefficient. One can put this Fokker-Planck equation under the form (1.6) with $D(v) = \gamma(v) \left(\frac{v_{th}^2}{2} + D_{turb}(v)\right)$ and $U'(v) = v \left(\frac{v_{th}^2}{2} + D_{turb}(v)\right)^{-1}$. Hence, one observes immediately that for $D_{turb}(v) \equiv cst.$, meaning $\gamma(v)$ proportional to $D(v)$ (Einstein's relation), one gets standard Maxwellian steady states. However for large $D_{turb}(v)$, for example if $D_{turb}(v) \propto v^2$, one gets a potential of logarithmic type. In particular, taking

$D_{turb}(v) = D_0\, v^2$, which leads to $U(v) = \kappa \ln\left(1 + \frac{|v|^2}{\kappa\, v_{th}^2}\right)$, yields the following κ-distribution function of first kind as a steady-state

$$
f_\kappa(v) = C_\kappa \left(1 + \frac{v^2}{\kappa\, v_{th}^2}\right)^{-\kappa}, \qquad \kappa := \frac{1}{2\, D_0}, \quad v_{th} := \sqrt{\frac{2\, k_B\, T}{m}}, \tag{1.9}
$$

with $C_\kappa > 0$ a normalization factor. More generally, the shape of the distribution function is obtained by examining the exponent in the ratio between the friction coefficient and the diffusion coefficient $\gamma(v)/D(v) = \left(\frac{v_{th}^2}{2} + D_{turb}(v)\right)^{-1}$ in the high-velocity regime. In particular, if this ratio behaves for large velocities as $v^{\beta-1}$ with some $\beta \in (-1, 1)$, then we have to cope with different types of suprathermal tails. Einstein's relation is recovered for $\beta = 1$ and the κ-distributions are recovered for $\beta = -1$. The addition of the turbulent term D_{turb} is only empirical. This diffusion term describes on a mesoscopic level the effect on the particles of force-fields which arise from turbulent phenomena not including the usual stochastic Brownian motion. This extra term injects energy into the system, which combined with the friction force, determines the steady-state distribution function.

1.2. **Outline of this paper and main results.** The relaxation of an initial distribution function towards the steady-state solution of the Fokker-Planck equation is a classical problem in plasma kinetic theory. The knowledge of the asymptotic behaviour of the solution as well as of the convergence rate towards this steady-state function is of crucial importance for experiments and numerical simulations. Several studies have been performed for the standard (constant coefficient) Fokker-Planck collision operator, investigating via coercivity techniques the convergence rate towards Maxwellian distribution functions [3, 7, 21]. The first aim of the present work is to adapt these techniques to the more general Fokker-Planck collision operator of the type (1.6) (with velocity dependent drift and diffusion coefficients), in order to study the convergence towards the corresponding κ-distribution function. The particularity of this operator is that it features a gapless, continuous spectrum, thus the usual exponential decay in time is now replaced by an algebraic time-decay rate towards the steady state. Secondly Hermite spectral numerical schemes have been proposed in literature [27] to efficiently discretize standard Fokker-Planck equations in the velocity space, permitting among others to get, without too much numerical burden, the limiting Maxwellian distribution function as $t \to \infty$. Our second aim in this work is hence to propose an efficient numerical, scheme adapted to the here treated specific Fokker-Planck operators and their κ-distribution steady-states. The problem is now more challenging than in the standard Fokker-Planck case, the gapless continuous spectrum leading to numerical difficulties. Spectral techniques will be used in this paper to render an accurate treatment of the small λ-modes (small energy-modes) of the solution, which are essential for a correct handling of the long-time asymptotic behaviour of the solutions. Given these characteristics of our scheme, we shall call it *Low Energy Accurate Scheme* (LEAS). Finally, we would like to refer here to the related works [4, 12, 48, 49], which deal with the diffusion limit of Vlasov-Lévy-Fokker-Planck or linear Boltzmann equations. The equilibria in these equations are also heavy-tail distribution functions, the long-time/mean-free path limit leading to fractional diffusion equations. A more recent work [29] studies functional inequalities for such type of heavy-tail distributions.

The outline of this paper is the following. Section 2 reviews the mathematical and numerical results of the standard Fokker-Planck operator. Section 3 and 4 focus on the mathematical study of the more general Fokker-Planck operator (1.6), which can be also rewritten, with a well-defined equilibrium function $f_{eq} = \alpha\, e^{-U}$, $\alpha \in \mathbb{R}^+$, as

$$\partial_t f = \nu\, \partial_v \left[D(v)\, f_{eq}\, \partial_v \left(\frac{f}{f_{eq}} \right) \right], \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{1.10}$$

Section 3 investigates the long-time asymptotics and the (algebraic) convergence rate of the solutions to (1.10) towards the stationary states (Theorems 1, 2 and 3). Section 4 focuses on the spectral analysis of these Fokker-Planck operators, in particular we are interested to express the solution in terms of generalized eigenfunctions and density functions (Proposition 5), in order to prepare the design of the LEAS scheme. Based on these mathematical results, Section 5 presents an efficient numerical scheme (LEAS), permitting to compute without too much numerical burden, the solutions of the corresponding Fokker-Planck equation, even for long simulation times. In order not to render the lecture too heavy, we preferred to postpone to the Appendix the proof of the continuous spectral theorem.

2. Study of the Fokker-Planck collision operator

The main objective of this paper is the mathematical and numerical study of the following $1D$ evolution problem

$$\partial_t f = \partial_v \left[D(v)\, f_{eq}\, \partial_v \left(\frac{f}{f_{eq}} \right) \right], \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R}\,, \tag{2.1}$$

associated with some initial condition f_{in}, and we shall focus on four particular cases, namely

- (I) $D(v) \equiv 1$ and $f_{eq} = \mathcal{M}(v)$;
- (II) $D(v) \equiv 1$ and $f_{eq} = f_\kappa(v)$;
- (III) $D(v) = G(v)/v$ and $f_{eq} = \mathcal{M}(v)$;
- (IV) $D(v) = G(v)/v$ and $f_{eq} = f_\kappa(v)$.

Here the Maxwellian equilibrium is defined as

$$\mathcal{M}(v) := \frac{1}{\sqrt{2\,\pi}}\, e^{-v^2/2}, \qquad \forall v \in \mathbb{R}\,, \tag{2.2}$$

whereas the κ-equlibria are given for $\kappa > 1/2$ and $v \in \mathbb{R}$ by the formula

$$f_\kappa(v) := \alpha_\kappa \left(1 + \frac{v^2}{\kappa} \right)^{-\kappa}, \qquad \text{where} \quad \alpha_\kappa := \frac{1}{(\pi\,\kappa)^{1/2}}\, \frac{\Gamma(\kappa)}{\Gamma(\kappa - 1/2)}\,.$$

Let us recall here the definitions of the error function $\phi \in C^\infty(\mathbb{R})$ and the Chandrasekhar function $G \in C^\infty(\mathbb{R})$

$$\phi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy\,, \qquad G(x) := \frac{\phi(x) - x\,\phi'(x)}{2\,x^2} \quad \text{and} \quad G(0) := 0\,, \tag{2.3}$$

as well as their asymptotic developments for small and large arguments, given by

$$\phi(x) \sim_{x \ll 1} \frac{2x}{\sqrt{\pi}} \left(1 - \frac{x^2}{3} + \frac{x^4}{10} \cdots \right) \quad ; \quad \phi(x) \sim_{x \gg 1} 1 - \frac{e^{-x^2}}{\sqrt{\pi}\, x} \left(1 - \frac{1}{2x^2} + \frac{3}{4\,x^4} \cdots \right),$$

$$G(x) \sim_{x \ll 1} \frac{2x}{3\sqrt{\pi}} - \frac{2x^3}{5\sqrt{\pi}} \cdots \quad ; \quad G(x) \sim_{x \gg 1} \frac{1}{2x^2} - \frac{e^{-x^2}}{\sqrt{\pi}\, x} \left(1 + \frac{1}{2x^2} - \frac{1}{4\,x^4} \cdots \right).$$

The degeneracy of the Fokker-Planck collision operator (2.1) can be understood, when taking a look at Figure 2, where $D(\cdot) \in C^\infty(\mathbb{R})$ is plotted. Observe thus that the collision operator becomes negligible for high velocities of the energetic particles, due to $D(v) \to_{v\to\infty} 0$.

Our aim is to investigate in more details the convergence of the solution f of (2.1) towards the corresponding stationary solution f_∞ and to understand the influence of the diffusion

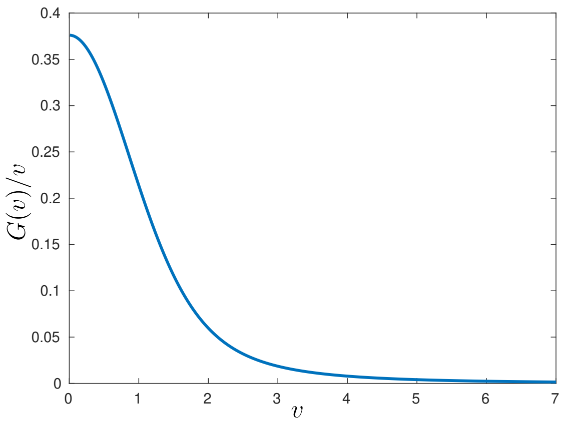

FIGURE 2. Plot of the symmetric, smooth diffusion term $D(v) := G(v)/v$.

coefficient $D(v)$ on the convergence rate. This shall be done via spectral analysis, with the specificity that cases (II)-(IV) are not standard, featuring a gapless, continuous spectrum.

When considering the time decay of the solutions towards the corresponding stationary states, we shall adopt in each of these four cases the following functional transformation $f = h\, f_{eq}$, leading to the evolution problem

$$\partial_t h = \frac{1}{f_{eq}(v)}\, \partial_v \left[D(v)\, f_{eq}(v)\, \partial_v\, h \right] , \tag{2.4}$$

associated with an initial condition h_{in} and with the corresponding stationary state $h_\infty \equiv \bar{h}_{in} = \int_{\mathbb{R}} h_{in}(v)\, f_{eq}(v)\, \mathrm{d}v$, which is obtained from the conservation of mass property of the equation, namely $\partial_t \bar{h} = 0$.

Proposition 1. (Existence/uniqueness of a solution) *Let us denote the second order differential operator occurring in* (2.4) *by*

$$\mathcal{L}_{D,eq}(h) := -\frac{1}{f_{eq}}\, \partial_v \left[D\, f_{eq}\, \partial_v\, h \right] .$$

Then one can show that $\mathcal{L}_{D,eq}(h) : \mathcal{D}(\mathcal{L}_{D,eq}) \subset L^2_{eq} \to L^2_{eq}$ is a linear, unbounded, self-adjoint and positive operator on the Hilbert-space

$$L^2_{eq} := \left\{ h : \mathbb{R} \to \mathbb{R} \ \textit{measurable}, \quad \int_{\mathbb{R}} |h|^2\, f_{eq}\, \mathrm{d}v < \infty \right\}, \qquad ||h||^2_{L^2_{eq}} := \int_{\mathbb{R}} |h|^2\, f_{eq}\, \mathrm{d}v\,, \tag{2.5}$$

and with domain

$$\mathcal{D}(\mathcal{L}_{D,eq}) = \{ h \in L^2_{eq}, \quad \mathcal{L}_{D,eq}(h) \in L^2_{eq} \} . \tag{2.6}$$

Furthermore, for each $h_{in} \in \mathcal{D}(\mathcal{L}_{D,eq})$ there exists a unique solution $h \in C^1([0,\infty); L^2_{eq}) \cap C([0,\infty); \mathcal{D}(\mathcal{L}_{D,eq}))$ of the Fokker-Planck equation (2.4). *For less regular initial conditions $h_{in} \in L^2_{eq}$ and each $T > 0$, one has nonetheless a unique weak solution $h \in W^1_2(0,T; H^1_{eq}, L^2_{eq}) \subset C([0,T]; L^2_{eq})$, where*

$$H^1_{eq} := \{ u \in L^2_{eq} \ / \ \sqrt{D(v)}\, \partial_v u \in L^2_{eq} \} .$$

Finally one has also a maximum principle, meaning that if furthermore $h_{in} \in L^\infty(\mathbb{R}_v)$, then

$$||h(t,\cdot)||_{L^\infty(\mathbb{R}_v)} \le ||h_{in}||_{L^\infty(\mathbb{R}_v)} , \quad \forall t \ge 0 .$$

Proof. The proof is a simple consequence of Weyl's theory for second order differential operators [17], the unbounded operator $\mathcal{L}_{D,eq}$ being in the limit-point case in $v = \pm\infty$, and hence it is not necessary to implement additional boundary conditions at infinity. Furthermore standard arguments, such as Hille Yosida theorem respectively Lions theorem, permit to show the existence and uniqueness of a solution to the Fokker-Planck equation (2.4). $\square$

We shall denote in the following the Hilbert-space (2.5) simply by L^2_κ in cases (II) and (IV), with $\|\cdot\|_\kappa$ its associated norm, and $L^2_{\mathcal{M}}$ in cases (I) and (III), with $\|\cdot\|_{\mathcal{M}}$ the associated norm.

The aim of the rest of this paper will be to investigate the decay of the solutions to (2.4) towards the stationary state h_∞, to say more about the spectrum of $\mathcal{L}_{D,eq}$ and the spectral representation of the solutions, all this in the aim to design an efficient numerical scheme for the resolution of (2.4). Let us however start with the standard case.

2.1. **Spectral analysis of case (I) with $D \equiv 1$ and $f_{eq} = \mathcal{M}$.** This case is very well documented, we recall here the results only for the sake of completeness. In this situation, the space L^2_{eq} rewrites

$$L^2_{\mathcal{M}} := \left\{ h : \mathbb{R} \to \mathbb{R} \ \text{measurable}\,, \quad \int_{\mathbb{R}} |h|^2\, \mathcal{M}(v)\,\mathrm{d}v < \infty \right\}, \tag{2.7}$$

while the operator is the standard linear Fokker-Planck operator

$$\mathcal{L}_{1,\mathcal{M}}(h) = -\frac{1}{\mathcal{M}}\partial_v[\mathcal{M}\,\partial_v h] = -\partial_{vv}h + v\partial_v h\,. \tag{2.8}$$

This operator is well studied, and one has the following standard result.

Proposition 2. [58] **(Spectrum of $\mathcal{L}_{1,\mathcal{M}}$)** *The operator* (2.8) *is self-adjoint, positive and with compact resolvant, such that its spectrum is discrete, real, positive and consists of a sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ with $\lambda_k \to \infty$ as $k \to \infty$. In particular one has for all $k \in \mathbb{N}$*

$$\mathcal{L}_{1,\mathcal{M}}\, H_k(v) = \lambda_k\, H_k(v)\,, \qquad \forall v \in \mathbb{R}\,, \quad \text{with eigenvalue } \lambda_k := k\,,$$

and the associated eigenvectors are the Hermite polynomials, defined recursively via $H_0 \equiv 1\,,\ H_1(v) \equiv v$ and for all $k \geqslant 1$ by the formulae

$$\sqrt{k+1}\, H_{k+1}(v) = v\, H_k(v) - \sqrt{k}\, H_{k-1}(v)\,, \quad \forall v \in \mathbb{R}\,. \tag{2.9}$$

Remark that $H_k'(v) = \sqrt{k}\, H_{k-1}(v)$ and that $\{H_k\}_{k\in\mathbb{N}}$ form a complete orthonormal basis set of the space $L^2_{\mathcal{M}}$.

This property is useful both from an analytic point of view, permitting to study the decay rate towards the equilibrium, but also from a numerical perspective, permitting the construction of a spectral method. We shall illustrate this fact in the two following subsections.

2.1.1. *Time decay.* The first consequence of the discrete spectrum of the operator $\mathcal{L}_{1,\mathcal{M}}$ is the existence of a spectral gap between the smallest two eigenvalues $\lambda_0 = 0$ and $\lambda_1 = 1$. This leads to the following Poincaré-Wirtinger inequality (with constant $C = \lambda_1 = 1$)

$$\int_{\mathbb{R}} |h(v) - \bar{h}|^2\, \mathcal{M}(v)\,\mathrm{d}v \leqslant \int_{\mathbb{R}} |\partial_v h(v)|^2 \mathcal{M}(v)\,\mathrm{d}v\,, \qquad \forall h \in H^1(\mathbb{R}, \mathcal{M}dv)\,, \tag{2.10}$$

where $\bar{h} := \int_{\mathbb{R}} h(v)\, \mathcal{M}(v)\, \mathrm{d}v$. This equality permits to show the convergence for $t \to \infty$ of the solution of the evolution problem

$$
\text{(2.11)} \qquad \begin{cases} \partial_t h = \partial_{vv} h - v \partial_v h = \frac{1}{\mathcal{M}} \partial_v [\mathcal{M}\, \partial_v h]\,, & \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R} \\ h(0,\cdot) = h_{in}\,, \end{cases}
$$

towards its stationary solution $h_\infty \equiv \bar{h}_{in}$. Indeed, multiplying (2.11) by $h\,\mathcal{M}$ and integrating in v, remarking besides that $\partial_t \bar{h} = 0$ and using Poincaré-Wirtinger's inequality (2.10), yields
(2.12)

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|h(t) - \bar{h}\|^2_{\mathcal{M}} = -\int_{\mathbb{R}} |\partial_v h(t,v)|^2 \mathcal{M}(v)\, \mathrm{d}v \leqslant -\int_{\mathbb{R}} |h(t,v) - \bar{h}|^2\, \mathcal{M}(v)\, \mathrm{d}v = -\|h(t) - \bar{h}\|^2_{\mathcal{M}}\,.
$$

Gronwall's inequality implies then immediately the following exponential decay of the solution towards its equilibrium

$$
\text{(2.13)} \qquad \|h(t) - h_\infty\|_{\mathcal{M}} \leqslant \|h_{in} - \bar{h}_{in}\|_{\mathcal{M}}\, e^{-t}\,, \quad \forall t \geq 0\,.
$$

The rate of convergence is directly given by the spectral gap between the eigenvalue $\lambda_0 = 0$ and the eigenvalue $\lambda_1 = 1$. The spectral gap is indeed an essential quantity which gives important physical information about the system under study, in particular it determines the low-energy physics.

2.1.2. *Spectral representation.* Now, one can make use of the orthonormal basis of eigenvectors $\{H_k\}_{k\in\mathbb{N}} \subset L^2_{\mathcal{M}}$ to expand the solution of the evolution problem (2.11).

Proposition 3. (Spectral representation in the discrete case) *The solution of* (2.11) *can be expressed in terms of the Hermite polynomials introduced in Proposition 2, i.e.*

$$
\text{(2.14)} \quad h(t,v) = \sum_{k=0}^{\infty} \alpha_k(t)\, H_k(v) \qquad \textit{with} \quad \alpha_k(t) = (h(t), H_k)_{\mathcal{M}} = \int_{\mathbb{R}} h(t,v)\, H_k(v)\, \mathcal{M}(v)\, \mathrm{d}v\,.
$$

Inserting this expression in (2.11) *yields an equation to be solved for the expansion coefficients $\alpha_k(t)$, leading for all $t \geq 0$ and $k \in \mathbb{N}$ to*

$$
\text{(2.15)} \quad \alpha'_k(t) + \lambda_k\, \alpha_k(t) = 0 \quad \Rightarrow \quad \alpha_k(t) = e^{-kt}\, \alpha_{\mathrm{in},k}\,, \quad \alpha_{\mathrm{in},k} := \int_{\mathbb{R}} h_{in}(v)\, H_k(v)\, \mathcal{M}(v)\, \mathrm{d}v\,.
$$

This proposition is the starting point of numerical spectral methods. To prepare the continuous case (Proposition 5), let us remark that the expansion (2.14) may be written also in the equivalent Riemann-Stieltjes form
(2.16)

$$
h(t,v) = \int_{-\infty}^{\infty} \alpha(t,\lambda)\, H_\lambda(v)\ \mathrm{d}\rho_d(\lambda)\,, \qquad \alpha(t,\lambda) = (h(t), H_\lambda)_{\mathcal{M}} = \int_{\mathbb{R}} h(t,v)\, H_\lambda(v)\, \mathcal{M}(v)\, \mathrm{d}v\,,
$$

where ρ_d is the spectral function defined as

$$
\rho_d(\lambda) := \sum_{\substack{k=0 \\ k\leq\lambda}}^{\infty} \frac{1}{||H_k||^2_{\mathcal{M}}}\,, \qquad \forall \lambda \in \mathbb{R}\,,
$$

and where H_λ are defined only for $\lambda = \lambda_k$, thus almost everywhere for the measure $\mathrm{d}\rho_d(\lambda)$. The spectral function ρ_d is a step-function (*"d"* standing for discontinuous or discrete), monotonically increasing and right-continuous at the eigenvalues $\lambda_k = k$ (see Figure 5 for an example). The magnitude of the jumps is fixed by the normalization of the eigenvectors. The spectral function ρ_d encapsulates somehow the scaling factors of the eigenvectors, when λ is an eigenvalue.

2.2. **Some numerical observations on the Hermite spectral scheme for case (I).** In order to construct a spectral method based on the spectral representation (2.14), it is obvious that one would like to have a basis set which, apart of being orthogonal and easy to compute, also yields a rapid convergence. Thus, firstly one needs to transform the Hermite polynomials in Hermite functions, as the Hermite polynomials are not convenient in practice due to their non-vanishing asymptotic behaviour at infinity. Hermite functions, defined as $\psi_k(v) := \frac{1}{\sqrt{2\,\pi}}\, H_k(v)\, e^{-v^2/2}$, are therefore used in numerical schemes, forming an orthonormal basis in $L^2(\mathbb{R}; \mathcal{M}^{-1}\, dv)$. Secondly, the rapid convergence can be handled with, via the introduction of a scaling parameter $\alpha \in \mathbb{R}$, which is closely related to the physical phenomenon one is studying. It is usually chosen such that in the long-time limit $t \to \infty$ lesser and lesser Hermite-basis functions are necessary in the spectral decomposition, leading in this manner to an acceleration of the numerical scheme.

After having considered these last two points, the Hermite-spectral scheme for the resolution of the evolution problem (2.11) is very efficient, due mainly to two reasons. Firstly, the higher the mode in (2.14), the faster the decay (see (2.15)), justifying thus a truncation at a reasonable index $N \in \mathbb{N}$ (higher precision can be of course achieved by taking a bigger N). Secondly, the associated numerical decay rate matches the theoretical decay given in (2.13). In other words, if h^N is given by the truncated expansion of (2.14) up to an order $N \in \mathbb{N}$, then one can show, remarking that $\bar{H}_k = 0$ for all $k \neq 0$, that

$$\|h^N(t) - h_\infty\|^2_{\mathcal{M}} = \|h^N(t) - \alpha_{\text{in},0}\|^2_{\mathcal{M}} = \sum_{k=1}^{N} |\alpha_{\text{in},k}|^2\, e^{-2\,kt} \leqslant e^{-2\,t}\, \|h^N_{\text{in}} - \bar{h}_{\text{in}}\|^2_{\mathcal{M}}\,. \tag{2.17}$$

This method is therefore very accurate for approximating the evolution problem (2.11), especially in the long-time asymptotics.

The second advantage of this spectral method, namely the correspondence of the theoretical and the numerical (exponential) decay rate, relies on the existence of a spectral gap. We shall see in the following examples that in the cases (II)-(IV), the operator $\mathcal{L}_{D,eq}$ features a continuous spectrum and that there is no spectral gap, leading to a more challenging problem.

3. Time decay of the cases (II) to (IV)

Sections 3 and 4 will be now concerned with the study of the Fokker-Planck collision operator in the cases (II)-(IV), which have the particularity of possessing a continuous spectrum, with no spectral gap. Gapless systems exhibit very particular behaviours and are very challenging from a mathematical as well as numerical point of view. Before passing to the spectral analysis, let us start in this section with the investigation of the rate of convergence of the solution to the evolution problem

$$\begin{cases} \partial_t h = \dfrac{1}{f_{eq}}\, \partial_v \left[D(v)\, f_{eq}\, \partial_v\, h\right] =: -\mathcal{L}_{D,eq}(h) \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R} \\ h(0,\cdot) = h_{in}\,, \end{cases} \tag{3.1}$$

towards the corresponding stationary solution $h_\infty = \bar{h} := \int_{\mathbb{R}_v} h\, f_{eq}\, \mathrm{d}v$, and this in the remaining three cases (II)-(IV). For simplicity reasons, let us introduce in the rest of this paper the following notation

$$\langle v \rangle_\kappa = \sqrt{1 + v^2/\kappa}\,, \qquad \langle v \rangle_{\mathcal{M}} = \sqrt{1 + v^2}\,.$$

3.1. **Study of the operator in the case (II) with $D = 1$ and $f_{eq} = f_\kappa$.** We focus now on the following evolution problem

$$\partial_t h(t,v) = \frac{1}{f_\kappa}\,\partial_v\,\left[\,f_\kappa\,\partial_v h\,\right]\,, \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R}\,, \tag{3.2}$$

and the decay of h, as $t \to \infty$, towards the stationary solution h_∞, which is indeed a quantity independent on time, due to the conservation of mass property of the evolution problem (3.2). The considered functional space is

$$L^2_\kappa := \left\{ h : \mathbb{R} \to \mathbb{R} \text{ measurable}\,, \quad \int_{\mathbb{R}} |h|^2\,f_\kappa\,\mathrm{d}v < \infty \right\}\,, \tag{3.3}$$

with associated norm $\|\cdot\|_\kappa$.

Theorem 1. [10] **(Time decay for case (II))** *Let h be a solution of the evolution problem (3.2), with an initial condition $h_{in} \in L^\infty(\mathbb{R}_v) \subset L^2_\kappa$. Then for all $0 < p < 2\,\kappa - 1$, with $\kappa > 1$, the following estimate holds*

$$\|h(t) - h_\infty\|^2_\kappa \leq \left[\|h_{in} - \bar{h}_{in}\|_\kappa^{-4/p} + K^{II}_{p,\kappa}\,\frac{4t}{p} \right]^{-p/2}\,, \qquad \forall t \geq 0\,, \tag{3.4}$$

with $h_\infty = \bar{h} := \int_{\mathbb{R}_v} h\,f_\kappa\,\mathrm{d}v = \bar{h}_{in}$ and $K^{II}_{p,\kappa} > 0$ a constant given in (3.11).

The proof of this theorem follows [10], and is especially based on the following Hardy-Poincaré inequality.

Lemma 1. [7] **(Hardy-Poincaré)** *Let us introduce the function $\Theta_\beta(v) := (1 + |v|^2)^{-\beta}$ for some $\beta > 1$ and define furthermore the two measures*

$$\mathrm{d}\nu := \Theta_\beta\,dv\,, \qquad \mathrm{d}\mu = \Theta_{\beta+1}\,\mathrm{d}v = \frac{1}{1+|v|^2}\,\mathrm{d}\nu\,.$$

Then there exists a positive constant $C_\beta > 0$ such that

$$\int_{\mathbb{R}} |g - \tilde{g}|^2\,\mathrm{d}\mu \leq C_\beta \int_{\mathbb{R}} |\partial_v g|^2\,\mathrm{d}\nu\,, \qquad \forall g \in H^1(\mathbb{R},\,\mathrm{d}\nu)\,, \quad \tilde{g} := \int_{\mathbb{R}} g\,\mathrm{d}\mu\,. \tag{3.5}$$

Hardy-Poincaré inequalities can be seen as "weak Poincaré inequalities" and occur often for unbounded domains, for measures that are "not confining enough". Adapting Lemma 1 for our case, yields the following inequality for all $r \geq 0$

$$\int_{\mathbb{R}} |h(v) - \tilde{h}_r|^2\,\langle v\rangle_\kappa^{-r-2}\,f_\kappa(v)\,\mathrm{d}v \leq C_{r,\kappa} \int_{\mathbb{R}} |\partial_v h(v)|^2\,\langle v\rangle_\kappa^{-r}\,f_\kappa(v)\,\mathrm{d}v\ . \tag{3.6}$$

with $\tilde{h}_r := \int_{\mathbb{R}} h\,\langle v\rangle_\kappa^{-r-2}\,f_\kappa(v)\,\mathrm{d}v$ and $C_{r,\kappa} > 0$ a constant. Indeed, we used Lemma 1 with $\beta := \kappa + r/2$ and $\kappa > 1$, $\Theta_\beta(v) := \langle v\rangle_\kappa^{-r}\,f_\kappa$, $\mathrm{d}\nu := \Theta_\beta(v)\,\mathrm{d}v$ and $\mathrm{d}\mu = \langle v\rangle_\kappa^{-2}\,\mathrm{d}\nu$, after a change of variables.

Proof of Theorem 1. Integrating equation (3.2) against $h(t,v)\,f_\kappa(v)\,\mathrm{d}v$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |h - \bar{h}|^2\,f_\kappa\,\mathrm{d}v = -\int_{\mathbb{R}} |\partial_v h|^2 f_\kappa\,\mathrm{d}v\,, \quad \forall t \geq 0\,. \tag{3.7}$$

Thanks to Hardy-Poincaré's inequality (3.6) with $r = 0$, we can estimate (Nash-type inequality [55])

$$
\begin{aligned}
\int_{\mathbb{R}} |h-\bar h|^2 f_\kappa \,\mathrm{d}v &= \inf_{c\in\mathbb{R}} \int_{\mathbb{R}} |h-c|^2 f_\kappa \,\mathrm{d}v\,, && \bar h := \int_{\mathbb{R}} h\, f_\kappa \,\mathrm{d}v\\
&\leqslant \int_{\mathbb{R}} |h-\tilde h_0|^2 f_\kappa \,\mathrm{d}v\,, && \tilde h_0(t) := \int_{\mathbb{R}} h(t)\,\langle v\rangle_\kappa^{-2} f_\kappa(v)\,\mathrm{d}v\\
&\overset{Hoelder}{\leqslant} \left(\int_{\mathbb{R}} |h-\tilde h_0|^2 \langle v\rangle_\kappa^{-2} f_\kappa \,\mathrm{d}v\right)^{\frac{p}{p+2}} \left(\int_{\mathbb{R}} |h-\tilde h_0|^2 \langle v\rangle_\kappa^{p} f_\kappa \,\mathrm{d}v\right)^{\frac{2}{p+2}}\\
&\overset{(3.6)}{\leq} C_{0,\kappa}^{\frac{p}{p+2}} \left(\int_{\mathbb{R}} |\partial_v h(v)|^2 f_\kappa(v) \,\mathrm{d}v\right)^{\frac{p}{p+2}} \left(\int_{\mathbb{R}} |h-\tilde h_0|^2 \langle v\rangle_\kappa^{p} f_\kappa \,\mathrm{d}v\right)^{\frac{2}{p+2}} .
\end{aligned}
$$

Plugging this into (3.7) yields

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |h-\bar h|^2 f_\kappa \,\mathrm{d}v \le -\frac{1}{C_{0,\kappa}} \left(\int_{\mathbb{R}} |h-\bar h|^2 f_\kappa \,\mathrm{d}v\right)^{\frac{p+2}{p}} \left(\int_{\mathbb{R}} |h-\tilde h_0|^2 \langle v\rangle_\kappa^{p} f_\kappa \,\mathrm{d}v\right)^{-\frac{2}{p}} . \tag{3.8}
$$

In order to conclude, it is enough to find one $p > 0$ such that

$$
\int_{\mathbb{R}} |h-\tilde h_0(t)|^2 \langle v\rangle_\kappa^{p} f_\kappa \,\mathrm{d}v \le K_{p,\kappa}, \tag{3.9}
$$

with some constant $K_{p,\kappa} > 0$ to be determined. This would lead to

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |h-\bar h|^2 f_\kappa \,\mathrm{d}v \le -K^{II}_{p,\kappa} \left(\int_{\mathbb{R}} |h-\bar h|^2 f_\kappa \,\mathrm{d}v\right)^{\frac{p+2}{p}} , \tag{3.10}
$$

with $K^{II}_{p,\kappa} = \frac{1}{C_{0,\kappa}} K_{p,\kappa}^{-2/p}$ and we can finally establish (3.4) thanks to Gronwall's inequality.

One way to show (3.9) is to prove some propagation of moments estimate, namely assume that the initial condition h_{in} has bounded velocity moments, and prove then that these moments stay bounded for all times (see [10] for more details). Here, we impose a much stronger assumption on the initial data h_{in}, namely $h_{in} \in L^\infty(\mathbb{R}_v)$. Starting thus from

$$
\int_{\mathbb{R}} |h-\tilde h_0(t)|^2 \langle v\rangle_\kappa^{p} f_\kappa \,\mathrm{d}v \le \left(\int_{\mathbb{R}} \langle v\rangle_\kappa^{p} f_\kappa \,\mathrm{d}v\right) \|h-\tilde h_0(t)\|_\infty^2 \,,
$$

we remark first that the maximum principle for diffusion equations implies $||h(t)||_\infty \le ||h_{in}||_\infty$ for all $t > 0$, yielding

$$
\|h-\tilde h_0(t)\|_\infty \le \left(1+\int_{\mathbb{R}} \langle v\rangle_\kappa^{-2} f_\kappa \,\mathrm{d}v\right) ||h_{in}||_\infty \,.
$$

Secondly, denoting the moment by

$$
M_{p,\kappa} := \int_{\mathbb{R}} \langle v\rangle_\kappa^{p} f_\kappa \,\mathrm{d}v\,,
$$

we remark that it is bounded for $0 < p < 2\kappa - 1$. Altogether we have (3.9) with

$$
K_{p,\kappa} := M_{p,\kappa} \left(1+\int_{\mathbb{R}} \langle v\rangle_\kappa^{-2} f_\kappa \,\mathrm{d}v\right)^2 ||h_{in}||_\infty^2 \quad \text{and} \quad K^{II}_{p,\kappa} = \frac{1}{C_{0,\kappa}} K_{p,\kappa}^{-2/p} . \tag{3.11}
$$

□

3.2. **Convergence rate in the case (III) with $D = G(v)/v$ and $f_{eq} = \mathcal{M}$.** This section focuses on the evolution problem

$$\partial_t h = \frac{1}{\mathcal{M}(v)} \partial_v \left[D(v)\, \mathcal{M}(v)\, \partial_v h \right], \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R}, \tag{3.12}$$

with a diffusion coefficient $D(v) = G(v)/v$ which is this time vanishing as $v \to \infty$, and with a Maxwellian equilibrium. Remark however that this diffusion coefficient is such that there exists a constant $C_d > 0$ s.t. $D(v) \geq C_d \langle v \rangle_{\mathcal{M}}^{-3}$ for large v.

Theorem 2. (Time decay for case (III)) *Let h be a solution of the evolution problem* (3.12)*, with an initial condition $h_{in} \in L^\infty(\mathbb{R}_v) \subset L^2_{\mathcal{M}}$. Then the following estimate holds*

$$\|h(t) - h_\infty\|^2_{\mathcal{M}} \leq \left[\|h_{in} - \bar{h}_{in}\|_{\mathcal{M}}^{-6/p} + K_p^{III} \frac{6\,t}{p} \right]^{-p/3}, \quad \forall t \geq 0, \tag{3.13}$$

for all $p > 0$ and where $h_\infty = \bar{h} = \int_{\mathbb{R}} h\mathcal{M}\, \mathrm{d}v = \bar{h}_{in}$ and $K_p^{III} > 0$ a constant given in (3.17).

We have this time only a "super-algebraic decay" rate, and not an exponential one as in the standard Fokker-Planck case (see (2.13)), because although the functional space we are working with enjoys a Poincaré-Wirtinger inequality, the operator on the right-hand side of (3.12) features a vanishing diffusion coefficient, such that we need a Nash-type inequality. Let us start with the following Poincaré inequality (weighted version of (2.10))

Lemma 2. (Poincaré-Wirtinger inequality) *The following Poincaré inequality holds*

$$\int_{\mathbb{R}} |h - \tilde{h}|^2 \langle v \rangle_{\mathcal{M}}^{-3} \mathcal{M}\, \mathrm{d}v \leq C_P \int_{\mathbb{R}} |\partial_v h(v)|^2 \langle v \rangle_{\mathcal{M}}^{-3} \mathcal{M}\, \mathrm{d}v, \qquad \forall h \in H^1(\mathbb{R}; \mathcal{M}dv), \tag{3.14}$$

where $\tilde{h} := \int_{\mathbb{R}} h \langle v \rangle_{\mathcal{M}}^{-3} \mathcal{M}\, \mathrm{d}v$ and $C_P > 0$ is the Poincaré constant.

Proof. To prove this Lemma, we use [3], Corollary 1.6, with $V(v) := \frac{1}{2}|v|^2 + 3 \ln\langle v \rangle_{\mathcal{M}} + \ln(\sqrt{2\,\pi})$. □

Proof of Theorem 2. The general idea is still the same, starting with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h(t) - \bar{h}\|^2_{\mathcal{M}} = -\int_{\mathbb{R}} D(v)(\partial_v h)^2 \mathcal{M}\, \mathrm{d}v \leq -C_d \int_{\mathbb{R}} |\partial_v h|^2 \langle v \rangle_{\mathcal{M}}^{-3} \mathcal{M}\, \mathrm{d}v. \tag{3.15}$$

Now, we conclude in the same fashion as in the previous subsection, namely via

$$\begin{aligned}
\int_{\mathbb{R}} |h(t) - \bar{h}|^2 \mathcal{M}\, \mathrm{d}v &= \inf_{c \in \mathbb{R}} \int_{\mathbb{R}} |h - c|^2 \mathcal{M}\, \mathrm{d}v, \qquad\qquad \bar{h} := \int_{\mathbb{R}} h\, \mathcal{M}\, \mathrm{d}v \\
&\leqslant \int_{\mathbb{R}} |h - \tilde{h}|^2 \mathcal{M}\, \mathrm{d}v, \qquad\qquad \tilde{h}(t) := \int_{\mathbb{R}} h(t)\, \langle v \rangle_{\mathcal{M}}^{-3} \mathcal{M}\, \mathrm{d}v \\
&\overset{Hoelder}{\leqslant} \left(\int_{\mathbb{R}} |h - \tilde{h}|^2 \langle v \rangle_{\mathcal{M}}^{-3} \mathcal{M}\, \mathrm{d}v \right)^{\frac{p}{p+3}} \left(\int_{\mathbb{R}} |h - \tilde{h}|^2 \langle v \rangle_{\mathcal{M}}^{p} \mathcal{M}\, \mathrm{d}v \right)^{\frac{3}{p+3}} \\
&\overset{(3.14)}{\leq} C_P^{\frac{p}{p+3}} \left(\int_{\mathbb{R}} |\partial_v h(v)|^2 \langle v \rangle_{\mathcal{M}}^{-3} \mathcal{M}\, \mathrm{d}v \right)^{\frac{p}{p+3}} \left(\int_{\mathbb{R}} |h - \tilde{h}|^2 \langle v \rangle_{\mathcal{M}}^{p} \mathcal{M}\, \mathrm{d}v \right)^{\frac{3}{p+3}}.
\end{aligned}$$

Now, since this time we have bounded moments for all $p > 0$ (advantage of the Maxwellian equilibria)

$$M_p := \int_{\mathbb{R}} \langle v \rangle_{\mathcal{M}}^p \mathcal{M}\, \mathrm{d}v < \infty, \tag{3.16}$$

one gets for all $p > 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h-\bar h\|^2_{\mathcal M} \le -K_p^{III}\left(\int_{\mathbb R} |h-\bar h|^2\,\mathcal M\,\mathrm{d}v\right)^{\frac{p+3}{p}},$$

with

$$K_p^{III} = \frac{C_d}{C_P}\,K_p^{-3/p}\,, \qquad K_p := M_p\,\left(1+\int_{\mathbb R}\langle v\rangle_{\mathcal M}^{-3}\,\mathcal M\,\mathrm{d}v\right)^2\,||h_{in}||^2_\infty\,. \tag{3.17}$$

We can conclude the proof with Gronwall's lemma. Let us remark that one can probably improve this result, by choosing an adequate $p > 0$, in order to recover a better rate of convergence (like for instance e^{-t^σ} for some $\sigma > 0$). □

3.3. **Convergence rate in the case (IV) with $D = G(v)/v$ and $f_{eq} = f_\kappa$.** Finally let us consider now the more physical evolution problem

$$\partial_t h = \frac{1}{f_\kappa}\,\partial_v\left[D(v)\,f_\kappa\,\partial_v h\right]\,, \qquad \forall (t,v)\in\mathbb R^+\times\mathbb R\,, \tag{3.18}$$

where $D(v) = G(v)/v$ and a κ-equilibrium. Again we shall use the fact that this diffusion coefficient is such that there exists a constant $C_{d,\kappa} > 0$ s.t. $D(v) \ge C_{d,\kappa}\,\langle v\rangle_\kappa^{-3}$ for large v.

Theorem 3. (Time decay for case (IV)) *Let h be a solution of the evolution problem (3.18), with an initial condition $h_{in}\in L^\infty(\mathbb R_v)\subset L^2_\kappa$. Then for all $0 < p < 2\,\kappa-1$, with $\kappa > 1/2$, the following estimate holds*

$$\|h(t)-h_\infty\|^2_\kappa \le \left[\|h_{in}-\bar h_{in}\|_\kappa^{-10/p} + K^{IV}_{p,\kappa}\frac{10\,t}{p}\right]^{-p/5}\,, \quad \forall t\ge 0\,, \tag{3.19}$$

where $h_\infty = \bar h := \int_{\mathbb R} h\,f_\kappa\,\mathrm{d}v = \bar h_{in}$ and $K^{IV}_{p,\kappa} > 0$ a constant given in (3.21).

Proof of Theorem 3. The proof is given by the same technique as before. Starting again from the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h-\bar h\|^2_\kappa = -\int_{\mathbb R} D(v)(\partial_v h)^2\,f_\kappa\,\mathrm{d}v \le -C_{d,\kappa}\int_{\mathbb R}|\partial_v h|^2\,\langle v\rangle_\kappa^{-3}\,f_\kappa\,\mathrm{d}v\,, \tag{3.20}$$

and using Hardy-Poincare's inequality (3.6) with $r = 3$ ($\kappa > 1/2$ possible here), permits to conclude in the same fashion as in the previous subsection, via

$$\begin{aligned}
\int_{\mathbb R}|h-\bar h|^2\,f_\kappa\,\mathrm{d}v &= \inf_{c\in\mathbb R}\int_{\mathbb R}|h-c|^2\,f_\kappa\,\mathrm{d}v\,, && \bar h := \int_{\mathbb R} h\,f_\kappa\,\mathrm{d}v\\
&\leqslant \int_{\mathbb R}|h-\tilde h_3|^2\,f_\kappa\,\mathrm{d}v\,, && \tilde h_3(t) := \int_{\mathbb R} h(t)\,\langle v\rangle_\kappa^{-5}\,f_\kappa(v)\,\mathrm{d}v\\
&\overset{Hoelder}{\leqslant} \left(\int_{\mathbb R}|h-\tilde h_3|^2\,\langle v\rangle_\kappa^{-5}\,f_\kappa\,\mathrm{d}v\right)^{\frac{p}{p+5}}\left(\int_{\mathbb R}|h-\tilde h_3|^2\,\langle v\rangle_\kappa^{p}\,f_\kappa\,\mathrm{d}v\right)^{\frac{5}{p+5}} &&\\
&\overset{(3.6)}{\le} C_{3,\kappa}^{\frac{p}{p+5}}\left(\int_{\mathbb R}|\partial_v h(v)|^2\,\langle v\rangle_\kappa^{-3}\,f_\kappa(v)\,\mathrm{d}v\right)^{\frac{p}{p+5}}\left(\int_{\mathbb R}|h-\tilde h_3|^2\,\langle v\rangle_\kappa^{p}\,f_\kappa\,\mathrm{d}v\right)^{\frac{5}{p+5}}. &&
\end{aligned}$$

Thus one gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h-\bar h\|^2_\kappa \le -K^{IV}_{p,\kappa}\left(\int_{\mathbb R}|h-\bar h|^2\,f_\kappa\,\mathrm{d}v\right)^{\frac{p+5}{p}},$$

with

$$K^{IV}_{p,\kappa} = \frac{C_{d,\kappa}}{C_{3,\kappa}}\, K^{-5/p}_{p,\kappa}\,, \qquad K_{p,\kappa} := M_{p,\kappa}\, \left(1 + \int_{\mathbb{R}} \langle v\rangle_\kappa^{-5}\, f_\kappa\, \mathrm{d}v\right)^2 \, ||h_{in}||^2_\infty\,. \tag{3.21}$$

Gronwall's lemma permits then to get the desired decay rate. □

4. Spectral analysis of the cases (II) to (IV)

In this section, we show that the spectrum of the operator $\mathcal{L}_{D,eq}$ is continuous, with eigenvalue $\lambda = 0$ as an accumulation point. The spectral investigation is based on the transformation of the Fokker-Planck operator into an associated Schrödinger operator through a change of variable, so-called *Liouville transformation*. This transformation permits to reduce the number of parameters from two, namely the diffusion-coefficient and the drift-potential (D, U), to only one parameter, namely the potential Q. We deduce then the spectral properties of our operator, along with the associated spectral representation formula, from well-known Schrödinger-operator theory [15, 45, 54].

4.1. **Liouville transformation and Schrödinger form of the Fokker-Planck operator.**

Let us rewrite the equilibria for the four cases (I)-(IV) via a well-chosen potential U as follows

$$f_{eq} := \alpha\, e^{-U(v)}\,, \qquad \forall v \in \mathbb{R}\,,$$

with $\alpha > 0$ a normalization constant. Then the Fokker-Planck eigenvalue problem

$$-\partial_v \left[D(v)\, f_{eq}\, \partial_v \left(\frac{f}{f_{eq}}\right)\right] = \lambda\, f\,, \qquad \lambda \in \mathbb{C}\,, \qquad \forall v \in \mathbb{R}\,, \tag{4.1}$$

can be transformed into the following Schrödinger eigenvalue problem

$$-\partial_{ss} g + Q(v(s))\, g = \lambda\, g\,, \qquad \lambda \in \mathbb{C}\,, \qquad \forall s \in \mathbb{R}\,. \tag{4.2}$$

To do this, we performed the change of variable $v \leftrightarrow s$

$$v'(s) = \frac{\mathrm{d}}{\mathrm{d}s} v(s) = \sqrt{D(v(s))}\,, \tag{4.3}$$

along with the functional transformation $g(s) := \wp(s)\, h(v(s)) = \wp(s)\, f(v(s))/f_{eq}(v(s))$, where

$$\wp(s) := \{D(v(s))\}^{1/4}\, \{f_{eq}(v(s))\}^{1/2}\,, \quad \forall s \in \mathbb{R}\,. \tag{4.4}$$

The potential $Q(v(s)) := \frac{\wp''(s)}{\wp(s)}$ occurring in the Schrödinger eigenvalue problem (4.2) is given by

$$Q(v) := \frac{D''(v)}{4} - \frac{(D'(v))^2}{16\, D(v)} - \frac{1}{2} D'(v)\, U'(v) + D(v) \left[\frac{1}{4}\,(U'(v))^2 - \frac{1}{2}\, U''(v)\right]\,. \tag{4.5}$$

The next subsections summarize these transformations in the particular cases (II) to (IV).

4.1.1. *Schrödinger form in case (II).* In the case $D \equiv 1$ and $f_{eq} = f_\kappa$, the previous transformations writes $f = g\sqrt{f_\kappa}$, where

$$U(v) = \kappa \ln\left(1 + \frac{|v|^2}{\kappa}\right), \tag{4.6}$$

$$D(v) = 1\,, \qquad v(s) = s\,. \tag{4.7}$$

The potential $Q(v)$ is given by

$$Q(v) := \frac{|U'(v)|^2}{4} - \frac{1}{2}U''(v) = \frac{1}{\langle v\rangle^4}\left(\left[\frac{1}{\kappa}+1\right]v^2 - 1\right). \tag{4.8}$$

As a consequence one has the asymptotic behaviour, where we recall that $\sim$ stands for the asymptotic equivalence

$$Q(v) \sim \kappa^2\,\frac{(\frac{1}{\kappa}+1)}{v^2}\,, \qquad \text{as } v \to \pm\infty\,.$$

4.1.2. *Schrödinger form in case (III).* In the case $D(v) = G(v)/v$ and $f_{eq} = \mathcal{M}$, the previous procedure leads to

$$U(v) = v^2/2\,, \tag{4.9}$$

$$v'(s) = \frac{\mathrm{d}}{\mathrm{d}s}v(s) = \sqrt{D(v(s))}\,, \qquad v(s) \sim \left(\frac{5}{2\sqrt{2}}\right)^{2/5} s^{2/5} \qquad \text{as } s \to +\infty\,. \tag{4.10}$$

Here the potential $Q(v)$ is given by

$$Q(v) := \frac{D''(v)}{4} - \frac{(D'(v))^2}{16\,D(v)} - \frac{1}{2}D'(v)\,v + D(v)\left[\frac{1}{4}v^2 - \frac{1}{2}\right], \tag{4.11}$$

and as a consequence, we have asymptotically

$$Q(v) \sim \frac{1}{8\,|v|}\,, \qquad \text{as } v \to \pm\infty\,,$$

and we plotted $Q(v(s))$ on Figure 3 to visualize the potential occurring in the Schrödinger operator.

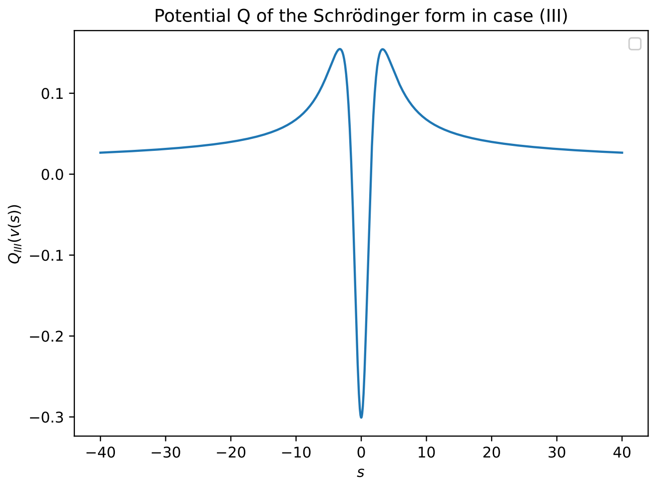

FIGURE 3. Plot of the potential $Q(v(s))$ in case (III), with respect to the variable s.

4.1.3. *Schrödinger form in case (IV).* In the case $D(v) = G(v)/v$ and $f_{eq} = f_\kappa$, the Liouville transformation leads to

$$U(v) = \kappa \ln\left(1 + \frac{|v|^2}{\kappa}\right), \tag{4.12}$$

$$v'(s) = \frac{\mathrm{d}}{\mathrm{d}s} v(s) = \sqrt{D(v(s))}, \qquad v(s) \sim \left(\frac{5}{2\sqrt{2}}\right)^{2/5} s^{2/5}, \qquad \text{as } s \to +\infty. \tag{4.13}$$

Here the potential $Q(v)$ is given by

$$Q(v) := \frac{D''(v)}{4} - \frac{(D'(v))^2}{16\, D(v)} - \frac{v\, D'(v)}{\langle v\rangle^2} + \frac{D}{\langle v\rangle^4}\left(\left[\frac{1}{\kappa}+1\right] v^2 - 1\right). \tag{4.14}$$

As a consequence,

$$Q(v) \sim \frac{1}{|v|^5}\left[\frac{\kappa^2}{2}\left(\frac{1}{\kappa}+1\right) + \frac{3\,\kappa}{2} + \frac{39}{32}\right], \qquad \text{as } v \to \pm\infty,$$

with again a representation of $Q(v(s))$ given on Figure 4.

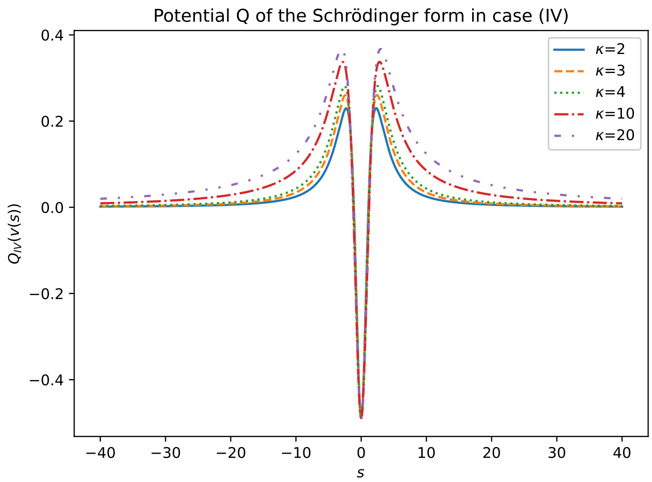

FIGURE 4. Plot of the potential $Q(v(s))$ in case (IV), with respect to the variable s, and for several values of κ.

4.2. **Spectrum and spectral representation.** From this reduction to a Schrödinger form, one can deduce now information on the spectrum of the Fokker-Planck operator, applying the spectral theory of Quantum Mechanics. This shall permit in a second step to study in more details the evolution problem

$$\begin{cases} \partial_t h = \frac{1}{f_{eq}}\, \partial_v \left[D(v)\, f_{eq}\, \partial_v\, h\right] = -\mathcal{L}_{D,eq}(h) & \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R} \\ h(0,\cdot) = h_{in}\,. \end{cases} \tag{4.15}$$

Taking a look at the potentials plotted in Figures 3-4, one expects that the Fokker-Planck operator possesses a continuous spectrum $[0,\infty)$, as shown in the next proposition.

Proposition 4. (Spectrum of $\mathcal{L}_{D,eq}$) *In each of the cases (II)-(IV), the operator $\mathcal{L}_{D,eq}$ is self-adjoint, positive and its spectrum consists of the set*

$$\sigma(\mathcal{L}_{D,eq}) = \sigma_{ess}(\mathcal{L}_{D,eq}) = [0,+\infty). \tag{4.16}$$

Furthermore, it comprises an eigenvalue $\lambda = 0$ *and an absolute continuous part*

$$\sigma_{pp}(\mathcal{L}_{D,eq}) = \{0\}\,, \qquad \sigma(\mathcal{L}_{D,eq}) = \sigma_{ac}(\mathcal{L}_{D,eq}) \cup \sigma_{pp}(\mathcal{L}_{D,eq}) = [0,+\infty)\,. \tag{4.17}$$

Proof. The proof of this theorem is based on standard arguments, scattered a little bit in literature. For completeness, we detailed this proof, but postponed it to the Appendix, in order not to interrupt the flow of the lecture. □

The spectral representation of the solution to (4.15) is more intricate in cases (II)-(IV) than the one of the standard case (I), (2.14), given the fact that the operators posses now a continuous spectrum. Points in the continuous spectrum can be associated with generalized eigenfunctions, which solve the corresponding Sturm-Liouville eigenvalue problem, however do not belong anymore to the considered Hilbert-space L^2_{eq}. However, similar to the Fourier-transform, one can give a spectral representation of solutions to (4.15) in terms of these generalized eigenfunctions and so-called *spectral functions* $\rho(\lambda)$, as mentioned in the next proposition. The interested reader can refer to [17, 52] for a complete introduction to this topic.

Proposition 5. [17] **(Spectral representation in the continuous cases)** *The solution of the evolution problem* (4.15) *can be expanded for all* $(t,v) \in \mathbb{R}^+ \times \mathbb{R}$ *as follows*

$$h(t,v) = \sum_{l,j=0}^{1} \int_{[0,\infty)} \mathfrak{A}_l(t,\lambda)\, \varphi_j(v,\lambda)\, \mathrm{d}\rho_{lj}(\lambda)\,, \qquad \mathfrak{A}_l(t,\lambda) := \int_{\mathbb{R}} h(t,v)\, \varphi_l(v,\lambda)\, f_{eq}(v)\, \mathrm{d}v\,, \tag{4.18}$$

where $\{\varphi_l(\cdot,\lambda)\}_{l=0}^1$ *are two linearly independent, generalized eigenfunctions of* $\mathcal{L}_{D,eq}$ *(see* (4.19)-(4.20)*), belonging to* $C^\infty(\mathbb{R} \times \mathbb{R}^+)$ *and corresponding to the generalized eigenvalue* $\lambda \in \mathbb{R}^+$*, and* $\{\rho_{lj}(\lambda)\}_{l,j=0}^1$ *is a positive, symmetric spectral matrix, necessarily linked to the chosen generalized eigenfunctions. The coefficients* $\mathfrak{A}(t,\lambda) := (\mathfrak{A}_0, \mathfrak{A}_1)^T(t,\lambda)$ *are said to be the spectral transform of* h*, i.e.*

$$h(t,\cdot) \in L^2_{eq} \longrightarrow \mathfrak{A}(t,\cdot) \in \mathcal{H}\,, \quad \forall t \geq 0\,,$$

where $\mathcal{H}$ *is the Hilbert-space of vector-valued functions, measurable with respect to the Lebesgue-Stieltjies measure defined by* ρ*, and is given by*

$$\mathcal{H} := \{(\zeta_1,\zeta_2) : \mathbb{R}^+ \to \mathbb{R}^2 \ \ / \ \ \sum_{l,j=0}^{1} \int_{[0,\infty)} \zeta_l(\lambda)\, \zeta_j(\lambda)\, \mathrm{d}\rho_{lj}(\lambda) < \infty\}\,.$$

This last Proposition deserves several remarks.

Remark 2. *Let us underline that the integral in the decomposition of* $h(t,v)$ *in* (4.18) *has to be understood as a Lebesgue-Stieltjes integral, involving a Stieltjes measure* $\mathrm{d}\rho$ *which is obtained from an increasing, right-continuous function* ρ *which assigns to a half-open interval* $(a,b]$ *the measure* $\mathrm{d}\rho((a,b]) = \rho(b) - \rho(a)$*. The use of half-open integrals is significant here, because a Lebesgue-Stieltjes measure may allocate non-zero values to single points. Furthermore, let us also observe that in the case* ρ *is an absolute-continuous function, one has* $\mathrm{d}\rho(\lambda) = \rho'(\lambda)\, \mathrm{d}\lambda$*.*

Remark 3. *The decomposition* (4.18) *is very similar to a Fourier transformation. When compared to the discrete spectral decomposition* (2.14)*, one remarks that the summation has been now replaced by an integration over the continuous spectrum, the eigenfunctions are now replaced by generalized eigenfunctions, which are not any more belonging to the Hilbert-space* L^2_{eq} *(in the Fourier case, these functions are trigonometric functions). Probably the* *main difference comes from the occurrence of the spectral density matrix, which accounts somehow for the normalization of the generalized eigenfunctions, and contains four terms, due to the fact that we are working on the whole velocity-space, thus with two singular end-points $v = \pm\infty$.*

Remark 4. *The expression of the spectral coefficients $\mathfrak{A}_l$ in* (4.18) *is simply obtained by introducing this decomposition into the evolution equation and solving the obtained equations for $\mathfrak{A}_l$, leading thus for all $t > 0$ and $\lambda \geq 0$ to (compare with* (2.15)*)*

$$\mathfrak{A}_l(t,\lambda) = e^{-\lambda t}\,\mathfrak{A}_{in,l}(\lambda)\,, \qquad \mathfrak{A}_{in,l}(\lambda) = \int_{\mathbb{R}} h_{in}(v)\,\varphi_l(v,\lambda)\,f_{eq}(v)\,\mathrm{d}v\,.$$

The most difficult part in computing the solution $h(t,v)$ of (4.15) via the spectral representation (4.18), is the determination of the spectral density measures $\{\,\mathrm{d}\rho_{lj}(\lambda)\}_{l,j=0}^1$ and the corresponding generalized eigenfunctions $\{\varphi_l(\cdot,\lambda)\}_{l=0}^1$. Let us thus say now some more words about the delicate mathematical choice or numerical computation of these quantities. The following next two sections are rather technical, however important to understand the main idea of the LEAS numerical method proposed in this paper, Section 5.

4.2.1. *Titchmarsh-Weyl approach.* [17] The Titchmarsh-Weyl strategy is often used to evaluate the spectral function via a correct normalization of the generalized eigenfunctions. This is done in the following manner. Let us introduce two functions $\varphi_0(\cdot,\eta), \varphi_1(\cdot,\eta) \in C^\infty(\mathbb{R})$ as being the fundamental solution set of the following Sturm-Liouville problem

$$\mathcal{L}_{D,eq}\,\varphi = \eta\,\varphi\,, \qquad \forall \eta \in \mathbb{C}\,, \tag{4.19}$$

associated with the following conditions in $v = 0$

$$\begin{cases} \varphi_0(0,\eta) & = 1 \\ D(0)\,\varphi_0'(0,\eta) & = 0 \end{cases}, \qquad \begin{cases} \varphi_1(0,\eta) & = 0 \\ D(0)\,\varphi_1'(0,\eta) & = 1 \end{cases}, \quad \forall \eta \in \mathbb{C}\,. \tag{4.20}$$

We underline that these two linearly independent solutions are fixed through the specific condition in $v = 0$ and can be considered as weak or generalized eigenfunctions of the Sturm-Liouville problem.

Starting from these two functions, one can compute the so-called *Titchmarsh-Weyl m-function* of the problem at the infinite boundaries, namely for $v = \pm\infty$, as follows

$$m_{\pm\infty}(\lambda + i\varepsilon) := -\lim_{v\to\pm\infty} \frac{\varphi_0(v,\lambda + i\,\varepsilon)}{\varphi_1(v,\lambda + i\,\varepsilon)}, \qquad \forall \lambda \in \mathbb{R}\,, \quad \forall \varepsilon \in \mathbb{R}^*\,. \tag{4.21}$$

We want to emphasize that the previous definition only holds for complex, non-real numbers and that this limit exists because $\mathcal{L}_{D,eq}$ is in the limit-point case in $v = \pm\infty$ [17].

These complex coefficients $m_{\pm\infty}(\lambda + i\varepsilon)$ and their boundary values on the real axis (as $\varepsilon \to 0+$) are the main building blocks in the computation of the spectral matrix $\{\rho_{lj}(\lambda)\}_{l,j=0}^1$ occurring in Proposition 5. To see this, let us define the matrix $M(\eta) := \{M_{lj}(\eta)\}_{l,j\in\{0,1\}}$ in terms of $m_{\pm\infty}$ as follows

(4.22)

$$M(\eta) := \frac{1}{m_{-\infty}(\eta) - m_{+\infty}(\eta)} \begin{pmatrix} 1 & \dfrac{m_{-\infty}(\eta) + m_{+\infty}(\eta)}{2} \\ \frac{m_{-\infty}(\eta)+m_{+\infty}(\eta)}{2} & m_{-\infty}(\eta)\,m_{+\infty}(\eta) \end{pmatrix}, \qquad \forall \eta \in \mathbb{C}\setminus\mathbb{R}\,.$$

This matrix enables finally to define 4 real-valued Stieltjes measures $\mathrm{d}\rho_{lj}$, generated by some functions $\{\rho_{lj}\}_{l,j=0}^1$, which are right-continuous, defined up to an additive constant and of bounded total variation on every finite interval of $\mathbb{R}$ [17]. To be more precise, at all continuity points λ of ρ_{lj} (meaning λ is not eigenvalue of $\mathcal{L}_{D,eq}$), and for $\lambda > \mu$ one has

$$\mathrm{d}\rho_{lj}((\mu,\lambda]) = \rho_{lj}(\lambda) - \rho_{lj}(\mu) = \frac{1}{\pi} \lim_{\varepsilon\to 0+} \int_\mu^\lambda \mathrm{Im}\,(M_{lj}(\tau + i\,\varepsilon))\ \mathrm{d}\tau\,, \quad \forall l,j = 0,1\,. \tag{4.23}$$

At eigenvalues $\lambda = \lambda_k$, the density function $\rho_{lj}(\lambda)$ is discontinuous, with a jump defined by

$$\mathrm{d}\rho_{lj}(\{\lambda_k\}) = \rho_{lj}(\lambda_k) - \rho_{lj}(\lambda_k^-) = r_{lk}\, r_{jk}\,, \qquad \forall l,j = 0,1\,,$$

where $\{r_{lk}\}_{k\in\mathbb{N},l\in\{0,1\}}$ are the coefficients of the decomposition of the eigenfunction $\phi_{\lambda_k}(v)$ in the fundamental basis, namely

$$\phi_{\lambda_k}(v) = r_{0k}\,\varphi_0(v,\lambda_k) + r_{1k}\,\varphi_1(v,\lambda_k)\,, \qquad \forall v\in\mathbb{R}\,.$$

Notice that $\mathrm{d}\rho_{0,1} = \mathrm{d}\rho_{1,0}$ because of the symmetry of the matrix M.

Remark 5. *Since, in our case, both $D(v)$ and $f_{eq}(v)$ are even functions, we conclude that φ_0 is even and φ_1 is odd. Therefore, using (4.21), the following equality holds true*

$$m_{-\infty}(\eta) = -m_{+\infty}(\eta)\,, \qquad \forall \eta\in\mathbb{C}\setminus\mathbb{R}\,. \tag{4.24}$$

This results in a diagonal matrix M, thus simplifying the computations. However, since we develop a general method for any Fokker-Planck equation (a priori with non-even coefficients), we do not assume M to be diagonal in the rest of the paper.

To separate the eigenvalue $\lambda = 0$ from the absolute continuous spectrum in (4.18), we recall that the Lebesgue decomposition theorem [15, 54] permits to separate the Stieltjes measure $\mathrm{d}\rho_{lj}$ in a unique way as

$$\mathrm{d}\rho_{lj} = \mathrm{d}\rho_{lj,pp} + \mathrm{d}\rho_{lj,ac} + \mathrm{d}\rho_{lj,sc}\,,$$

these three measures corresponding to the pure point, absolute continuous and singular continuous spectrum. Now we know that:

- $\mathcal{L}_{D,eq}$ has no singular continuous spectrum (See Proposition 4);
- The pure point spectrum of $\mathcal{L}_{D,eq}$ contains the only eigenvalue $\lambda = 0$, associated with the eigenfunction $\varphi_0(0,v) \equiv 1$, which is of norm 1 in L^2_{eq}. Therefore, the pure point component of the measure $\mathrm{d}\rho_{lj}$ is given by

$$\mathrm{d}\rho_{lj,pp} = \begin{cases} \mathrm{d}\delta_0\,, & \text{if } l = j = 0 \\ 0\,, & \text{else} \end{cases}\,, \qquad \mathrm{d}\delta_0 \text{ being the Dirac measure in } \lambda = 0\,; \tag{4.25}$$

- Concerning the absolute continuous spectrum, we know from [52] that for almost every $\lambda\in\mathbb{R}$, the limit $\lim_{\varepsilon\to 0^+} M(\lambda + i\varepsilon)$ exists and that the functions $\rho_{lj,ac}$ are differentiable at λ. Thus one has $\mathrm{d}\rho_{lj,ac} = \rho'_{lj,ac}\,\mathrm{d}\lambda$ and we can define the spectral density matrix

$$P(\lambda) := (\rho'_{lj}(\lambda))_{l,j\in\{0,1\}} = \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \mathrm{Im}\, M(\lambda + i\,\varepsilon)\,, \quad \text{for a.e } \lambda > 0\,. \tag{4.26}$$

With all these informations, let us introduce now the following vectors and matrices

$$\Phi(v,\lambda) = \begin{pmatrix} \varphi_0(v,\lambda) \\ \varphi_1(v,\lambda) \end{pmatrix}, \qquad \mathfrak{A}(t,\lambda) := \begin{pmatrix} \mathfrak{A}_0(t,\lambda) \\ \mathfrak{A}_1(t,\lambda) \end{pmatrix}, \qquad P(\lambda) := \left(\rho'_{lj}(\lambda)\right)_{l,j=0}^1\,, \tag{4.27}$$

which shall permit to write the spectral representation (4.18) in a more concise manner

$$h(t,v) = \mathfrak{A}_0(t,0) + \int_0^{+\infty} \mathfrak{A}(t,\lambda)^T\, P(\lambda)\,\Phi(v,\lambda)\,\mathrm{d}\lambda\,, \tag{4.28}$$

where again for all $t > 0$ and $\lambda \geq 0$ one has

$$\mathfrak{A}_l(t,\lambda) = e^{-\lambda t}\,\mathfrak{A}_{in,l}(\lambda)\,, \qquad \mathfrak{A}_{in,l}(\lambda) = \int_{\mathbb{R}} h_{in}(v)\,\varphi_l(v,\lambda)\,f_{eq}(v)\,\mathrm{d}v\,. \tag{4.29}$$

Let us underline that the first term on the right-hand side in (4.28) corresponds to the only eigenvalue $\lambda = 0$, whereas the integral term browses over the whole continuous spectrum. Notice that this integral term vanishes as $t \to \infty$, thus emphasizing the convergence of h towards the steady state $\bar{h}$. If there were a spectral gap, it would be apparent from (4.28)-(4.29) that this convergence of h is exponential as $t \to +\infty$, however this does not hold true in the cases (II)-(IV). The time decay (for $t \to \infty$) of the evolution semigroup is intimately connected to the asymptotics of $P(\lambda)$ in the limit $\lambda \to 0^+$, asymptotics which is widely studied in literature and shall be exploited in Section 5, in order to develop a numerical method that preserves the spectral structure of the Fokker-Planck operator in this $\lambda \to 0^+$ regime.

In practice it is rare to find explicit expressions for the fundamental solutions φ_0, φ_1 as well as for the spectral densities $\rho'_{lj}(\lambda)$ in order to use the spectral representation (4.18) in a simple manner, similar to a Fourier transform. There exist however papers in literature which use this Titchmarsh-Weyl theory in conjunction with a numerical computation of these quantities $(\varphi_0, \varphi_1, \rho'_{lj})$, see [62]. Our approach is different, similar to [53] and based rather on the truncation of the velocity domain and the introduction of a subsequent correction term.

4.2.2. *Numerical approach.* A numerical approach for the computation of the spectral function $\rho_{lj}(\lambda)$ and the subsequent resolution of (4.15) is based on the idea of truncating the domain at $v = \pm L$ for large $L \gg 1$, representing then the solution in terms of the corresponding orthonormal eigenfunctions (discrete spectral theorem), computing the step spectral functions $\rho^L_{lj}(\lambda)$ and passing finally to the limit $L \to \infty$. The passage to the limit is not so trivial, and is possible if the generalized eigenfunctions are chosen and normalized in a specific manner, given by Titchmarsh-Weyl's approach, presented above.

Let us first truncate the velocity domain and consider the following evolution problem

$$\begin{cases} \partial_t h^L = -\mathcal{L}_{D,eq}(h^L)\,, & \forall (t,v) \in \mathbb{R}^+ \times (-L,L)\,, \\ \partial_v h^L(t,-L) = \partial_v h^L(t,L) = 0 & \\ h^L(0,\cdot) = h^L_{in}\,, & \end{cases} \tag{4.30}$$

where h^L_{in} is a regularization of the initial condition, such that it satisfies the homogeneous Neumann boundary conditions and that $h^L_{in} \to_{L\to\infty} h_{in}$ in a certain sense. Domain truncation regularizes the Sturm-Liouville problem (the resolvant of the self-adjoint operator becoming now compact) and renders the spectrum discrete. The discreteness of the spectrum enables then a simple expansion of the solution in the eigenfunction basis.

Let thus $\{\lambda^L_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+$ be the increasing sequence of eigenvalues of the associated positive, self-adjoint Sturm-Liouville problem, considered in the Hilbert-space $L^2((-L,L); f_{eq}\,\mathrm{d}v)$

$$\begin{cases} \mathcal{L}_{D,eq}(\phi) = \lambda\,\phi\,, & \forall v \in (-L,L)\,, \\ \phi'(-L) = \phi'(L) = 0\,, & \end{cases} \tag{4.31}$$

and $\{\phi^L_k\}_{k\in\mathbb{N}} \subset L^2((-L,L); f_{eq}\,\mathrm{d}v))$ be the associated sequence of orthonormal eigenfunctions. Notice that the boundary conditions have been chosen so that $\lambda^L_0 = 0$ is indeed an eigenvalue, associated with a constant solution ϕ_0^L of norm 1 in $L^2((-L,L); f_{eq}\,\mathrm{d}v))$. The general solution h^L of (4.30) can thus be represented via the discrete spectral theorem (compare with (2.14)) for all $(t,v) \in \mathbb{R}^+ \times \mathbb{R}$ as

(4.32)

$$h^L(t,v) = \alpha_0^L\, \phi_0^L + \sum_{k=1}^{\infty} \alpha_k^L(t)\, \phi_k^L(v)\,, \qquad \alpha_k^L(t) = e^{-\lambda_k^L t}\, \alpha_{\mathrm{in},k}^L\,, \quad \alpha_{\mathrm{in},k}^L := \int_{-L}^{L} h_{in}^L(v)\, \phi_k^L(v)\, f_{eq}(v)\,\mathrm{d}v\,.$$

This representation is however not adapted to pass to the limit $L \to \infty$, as the normalized eigenfunctions $\phi_k^L(v)$ could not converge towards the non-normalized generalized eigenfunctions. To overcome this difficulty, one considers the fundamental basis functions $\varphi_0(\cdot,\lambda)$, $\varphi_1(\cdot,\lambda) \in C^\infty(\mathbb{R})$ as defined in (4.19)-(4.20). Due to their linear independence, we decompose the eigenfunctions $\phi_k^L(v)$ in this basis set and find scalars $r_{0,k}^L$, $r_{1,k}^L \in \mathbb{R}$ verifying

$$\phi_k^L(\cdot) = r_{0,k}^L\, \varphi_0(\cdot,\lambda_k) + r_{1,k}^L\, \varphi_1(\cdot,\lambda_k)\,, \qquad \forall k \in \mathbb{N}, \quad \forall L > 0\,. \tag{4.33}$$

Inserting this expression of ϕ_k^L into (4.32) yields the following representation

$$h^L(t,v) = \alpha_0^L\, \phi_0^L + \sum_{l,j=0}^{1} \sum_{k=1}^{\infty} \mathfrak{A}_l^L(t,\lambda_k)\, \varphi_j(v,\lambda_k)\, r_{l,k}^L\, r_{j,k}^L\,,$$

$$\mathfrak{A}_l^L(t,\lambda) := \int_{-L}^{L} h^L(t,v)\, \varphi_l(v,\lambda)\, f_{eq}(v)\,\mathrm{d}v\,.$$

This expansion can be reformulated as follows

$$h^L(t,v) = \alpha_0^L\, \phi_0^L + \sum_{l,j=0}^{1} \int_{(0,+\infty)} \mathfrak{A}_l^L(t,\lambda)\, \varphi_j(v,\lambda)\,\mathrm{d}\rho_{lj}^L(\lambda)\,, \tag{4.34}$$

where $\rho_{lj}^L(\lambda)$ is simply defined by

$$\rho_{lj}^L(\lambda) := \sum_{\substack{k\in\mathbb{N}\\ \lambda_k \leqslant \lambda}} r_{l,k}^L\, r_{j,k}^L\,, \qquad \forall l,j = 0,1\,, \tag{4.35}$$

and is a right-continuous step-function, constant in the intervals $[\lambda_{k-1},\lambda_k)$ and with a jump discontinuity at the eigenvalues λ_k.

The fact that we decomposed $h^L(t,v)$ in the non-normalized eigenfunctions $\{\varphi_0(\cdot,\lambda),\, \varphi_1(\cdot,\lambda)\}$, rather than in the more natural normalized basis-functions $\{\phi_k^L(v)\}_{k\in\mathbb{N}}$ shall permit, via Titchmarsh-Weyl's theory, to pass to the limit $L \to \infty$ and to derive the continuous representation framework. Indeed one can show the following convergence at points of continuity of $\rho_{lj}(\lambda)$ [17]

$$\rho_{lj}^L(\lambda) \longrightarrow_{L\to+\infty} \rho_{lj}(\lambda)\,, \qquad \forall l,j = 0,1\,, \tag{4.36}$$

where $\rho_{lj}(\lambda)$ was defined in (4.23). Sketches of $\rho_{0,0}$ and the approaching step function $\rho_{0,0}^L$ were represented in Figure 5. The discrete spectral representation (4.32) is a good approximation of the continuous spectral decomposition (4.18), for large domain truncations $L \gg 1$. However domain truncation always introduces a spectral gap, which governs the long-time behaviour of the solution and besides, from a practical point of view, taking too large $L \gg 1$ can be very time and memory consuming. The aim of the next section is to show how to cope with this new difficulty.

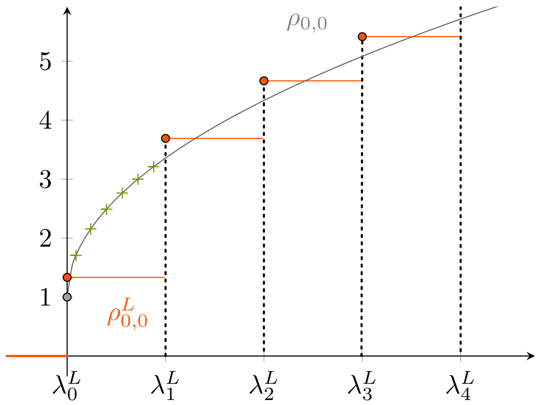

FIGURE 5. Sketch of the behaviour of the spectral function $\rho_{0,0}$ (in grey) as a function of λ. The approached density $\rho^L_{0,0}$ (in orange) is right-continuous. In green (+) is the part of $\rho_{0,0}$ that is misrepresented, leading to a numerical spectral gap.

5. Low Energy Accurate numerical Scheme

We start this section by briefly reviewing the existent numerical methods for the resolution of an evolution problem of the type

$$\begin{cases} \partial_t h = \frac{1}{f_{eq}}\,\partial_v\left[D(v)\, f_{eq}\,\partial_v\, h\right] = -\mathcal{L}_{D,eq}(h)\,, & \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R}\,, \\ h(0,\cdot) = h_{in}\,, \end{cases} \tag{5.1}$$

where the operator $\mathcal{L}_{D,eq}$ possesses a gapless, continuous spectrum. One of our main interests is to explore how standard methods fail to correctly describe the long-time behaviour of such problems. Indeed, when one is interested in accurately describing the long-time asymptotics, the behaviour of the small energy-modes is very important, as they dominate this long-time regime. Indeed, due to the fact that for long times the modes decay as $e^{-\lambda t}$, they are quickly damped for intermediate to large $\lambda > 0$. The aim of subsection 5.2 is thus to propose the addition of a new correction term to standard numerical methods, in order to accurately take into account for the small λ-modes and catch the right rate of relaxation towards the equilibrium (algebraic decay in our case).

5.1. **Discussions on standard discretizations.** Let us discuss first the difficulties encountered when trying to solve (5.1). The first immediately visible difficulty is related to the unboundedness of the velocity-domain, which leads to challenging numerical complications. Several possibilities are available to treat such problems, for example:

(a) *Spectral method*, based on basis functions intrinsic to unbounded domains, such as Hermite-basis functions or Sinc-functions;
(b) *Domain truncation*, approximating the unbounded interval $(-\infty,\infty)$ by a truncated one $(-L,L)$ with $L \gg 1$, and solving then the problem on the bounded domain, with appropriate artificial or transparent boundary conditions at $v = \pm L$;
(c) *Mapping techniques*, meaning mapping the unbounded interval $(-\infty,\infty)$ via a well-chosen change of variable (such as for ex. the algebraic mapping $v = \frac{w/a}{\sqrt{1-(w/a)^2}}$) to a bounded domain $(-a,a)$, and solving then the new problem on the bounded domain.

All these methods have their advantages and disadvantages and are well-adapted for specific situations. For example, Hermite spectral methods are usually used for solving Fokker-Planck equations of the type (2.1), case (I), which posses a discrete spectrum. These methods are however not appropriate for the cases (II)-(IV), which feature continuous spectra. Secondly, domain truncation with artificial boundary conditions is very well suited for exponentially decaying functions as $v \to \pm\infty$, but not for a slower decay in the velocity variable as in the case of energetic particles (κ-distributions). Constructing open (exact or transparent) boundary conditions or a well-suited mapping (c) is a very good idea in such situations, of gapless, continuous spectrum, however usually a very complex procedure from a mathematical and practical (numerical) point of view.

The approach we propose in this paper is based on the domain truncation procedure, supplied with artificial boundary conditions. The evolution problem on the truncated velocity domain (4.32) can be solved via standard methods, such as finite-differences, finite-elements or finite-volumes, we shall focus here rather on a spectral resolution, due to its bigger accuracy as compared to standard discretizations. As artificial boundary conditions introduce an artificial gap into the problem, we shall introduce a correction term in our scheme, permitting to take into account more precisely for the small λ-modes, such that our numerical scheme shall reveal an algebraic time-decay rate as the theoretical results suggest (Theorems 1, 2, 3). This correction term shall be obtained via perturbation techniques.

5.2. **Correction term.** To understand how to redress the appearance of the artificial spectral gap, let us start from the spectral representation (4.28) of the solution, namely

$$h(t,v) = \mathfrak{A}_0(t,0) + \int_0^{+\infty} \mathfrak{A}(t,\lambda)^T \, P(\lambda) \, \Phi(v,\lambda) \, \mathrm{d}\lambda \,, \tag{5.2}$$

where the spectral coefficients $\mathfrak{A}(t,\lambda)$ are defined in (4.29). Recent works as [62] employed such a spectral representation to compute solutions for analogous Fokker-Planck type equations, by simply discretizing the integral term (hence the spectrum), leading finally to the introduction of a spectral gap. Such procedures are intimately linked to the truncation of the velocity domain at $v = \pm L$, which is the starting point of classical discretization methods, such as finite-differences, finite-elements or finite-volumes. In each of these discretization methods a spectral gap is artificially introduced into the problem and one only accurately approximates the term

$$h^{\varepsilon}_{st}(t,v) = \mathfrak{A}_0(t,0) + \int_{\varepsilon}^{+\infty} \mathfrak{A}(t,\lambda)^T \, P(\lambda) \, \Phi(v,\lambda) \, \mathrm{d}\lambda \,, \tag{5.3}$$

for some small ε, omitting the part

$$\mathcal{C}_{\varepsilon}(t,v) := \int_0^{\varepsilon} \mathfrak{A}(t,\lambda)^T \, P(\lambda) \, \Phi(v,\lambda) \, \mathrm{d}\lambda \,, \qquad \forall (t,v) \in \mathbb{R}^+ \times \mathbb{R} \,. \tag{5.4}$$

This last part is however the one carrying the information in the long-time limit and gives rise to the algebraic time-decay rate. We propose thus to carefully evaluate this small-energy term (5.4) (using perturbation techniques) and to introduce it in standard discretization methods as a correction term, as follows

$$h(t,v) = h^{\varepsilon}_{st}(t,v) + \mathcal{C}_{\varepsilon}(t,v) \,.$$

For the computation of $\mathcal{C}_{\varepsilon}$ we shall make use of the specific asymptotic behaviour close to $\lambda = 0$ of the functions $\mathfrak{A}(t,\lambda)$, $P(\lambda)$ and $\Phi(v,\lambda)$. More specifically, let us observe the following:

- *Spectral coefficients.* The regularity of the vector-valued function $\mathfrak{A}_{in}(\lambda)$ depends on the decay of the initial condition $h_{in}(v)$ as $v \to \pm\infty$, as it happens with Fourier coefficients. In particular, we shall assume $h_{in}(v)$ sufficiently decaying, such that the following asymptotic behaviour of the spectral coefficients holds

$$\mathfrak{A}(t,\lambda) = e^{-\lambda t}\,\mathfrak{A}_{in}(\lambda)\,, \qquad \mathfrak{A}_{in}(\lambda) = \mathfrak{A}^0_{in} + \lambda\,\tilde{\mathfrak{A}}_{in,\lambda}\,, \qquad \forall (t,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+\,, \tag{5.5}$$

where the 2×1 vector $\tilde{\mathfrak{A}}_{in,\lambda}$ is supposed to be bounded near $\lambda = 0$.

- *Spectral density.* We shall also assume the following development of the spectral density matrix

$$P(\lambda) = \lambda^\alpha P_0 + \lambda^{\alpha+\beta}\,\tilde{P}_\lambda\,, \qquad \forall \lambda > 0\,, \tag{5.6}$$

with $\beta > 0$, $\alpha > -1$ and $\tilde{P}_\lambda$ a bounded matrix near $\lambda = 0^+$.

It is common for spectral density matrices to take the specific form (5.6). It is especially the case when the potential in the Schrödinger eigenvalue problem (4.2) behaves like $o(1/s^2)$, for $s \to \pm\infty$, leading to $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$, as discussed in works like [16, 40, 57].

Concerning the cases considered in the present paper, due to the fact that the potential is decaying much slower than the one of the previous remark, being only of order $\mathcal{O}(1/s^2)$ (or even $\mathcal{O}(1/s^{2/5})$ for case (III)), one cannot apply the theory mentioned above. However, even if (5.6) is still not proven for the cases we are studying, there are indications suggesting that this Ansatz may hold true for case (II) and (IV). In [41, 42] the analysis is conducted solely on the half line $\mathbb{R}^+$, for potentials behaving like $\sim \frac{c}{s^2}$, with a constant $c \in \mathbb{R}$, and (5.6) is proven to be valid in this case. In [50] the case of the whole line $\mathbb{R}$ is treated, with the specificity that the presence of the zero eigenvalue complicates the analysis; the author show that in this case one has $P(\lambda) = \mathcal{O}(\lambda^r)$ for some $r > 0$. Concerning case (III), the literature is very sparse, but there are some results suggesting a more rapid decay of the spectral density function (as $\lambda \to 0$) than (5.6) could suggest (see [63]). Our method could therefore be also adapted to this case in the same fashion as [63]. There are also hints of a very quickly decaying spectral density in this case when looking at [62] (Figure 3.1) .

- *Eigenfunctions.* Perturbation theory [5] permits to show finally that one has the development

$$\Phi(v,\lambda) = \Phi(v,0) + \lambda\,\tilde{\Phi}(v,\lambda)\,, \quad \forall (v,\lambda) \in \mathbb{R} \times \mathbb{R}^+\,, \tag{5.7}$$

with $\tilde{\Phi}(\cdot,\lambda)$ bounded near $\lambda = 0$, in the sense given later in (5.14).

Retaining now the main parts of the developments in (5.5), (5.6) and (5.7), we can approximate the low-energy term (5.4) for all $(t,v) \in \mathbb{R}^+ \times \mathbb{R}$ as follows

$$\mathcal{C}_\varepsilon(t,v) = \int_0^\varepsilon e^{-\lambda t}\,\mathfrak{A}^T_{in}(\lambda)\,P(\lambda)\,\Phi(v,\lambda)\,\mathrm{d}\lambda \simeq \tilde{\mathcal{C}}_{\varepsilon,\alpha}(t,v) := \left(\int_0^\varepsilon e^{-\lambda t}\,\lambda^\alpha\;\mathrm{d}\lambda\right)(\mathfrak{A}^0_{in})^T\,P_0\,\Phi(v,0)\,. \tag{5.8}$$

For simplicity reasons let us denote in the following the time-integral occurring in the expression of $\tilde{\mathcal{C}}_{\varepsilon,\alpha}(t,v)$ simply by

$$\theta^\alpha_\varepsilon(t) := \int_0^\varepsilon e^{-\lambda t}\lambda^\alpha\;\mathrm{d}\lambda\,. \tag{5.9}$$

Let us comment a little bit on this correction term and its approximation (5.8). Firstly, the correction term $\mathcal{C}_\varepsilon(t,v)$ brings the algebraic time-decay rate in our solution, and this rate is linked to the asymptotics of the spectral matrix $P(\lambda)$, as $\lambda \to 0^+$. Secondly, the approximation we propose for this term (5.8), is not destroying this decay rate, as the error done is smaller than the term itself, as shown in the next theorem.

Theorem 4. (Low energy correction term) *Let Φ be computed as in* (4.19),(4.20),(4.27). *Assume furthermore that we can write for all $\lambda > 0$*

$$\mathfrak{A}_{in}(\lambda) = \mathfrak{A}_{in}^0 + \lambda\,\tilde{\mathfrak{A}}_{in,\lambda}\,, \qquad P(\lambda) = \lambda^\alpha P_0 + \lambda^{\alpha+\beta}\,\tilde{P}_\lambda\,, \tag{5.10}$$

where $\beta > 0$, $\alpha > -1$, and $\tilde{\mathfrak{A}}_{in,\lambda}\,, \tilde{P}_\lambda$ are assumed to be bounded near $\lambda = 0^+$. Now, for each compact subset $K \subset \mathbb{R}$ and each $0 < \varepsilon < 1$, there exists a constant $C_{\varepsilon,K} > 0$ (bounded with respect to ε) such that

$$\begin{aligned}\mathfrak{E}_\varepsilon(t) &:= \sup_{v\in K}\left|\mathcal{C}_\varepsilon(t,v) - \tilde{\mathcal{C}}_{\varepsilon,\alpha}(t,v)\right| \\ &= \sup_{v\in K}\left|\int_0^\varepsilon e^{-\lambda t}\,\mathfrak{A}_{in}^T(\lambda)\,P(\lambda)\,\Phi(v,\lambda)\,\mathrm{d}\lambda - \left(\int_0^\varepsilon e^{-\lambda t}\,\lambda^\alpha\ \mathrm{d}\lambda\right)\,(\mathfrak{A}_{in}^0)^T\,P_0\,\Phi(v,0)\right| \\ &\leqslant C_{\varepsilon,K}\ \int_0^\varepsilon e^{-\lambda t}\,\lambda^{\alpha+\min\{1,\beta\}}\,\mathrm{d}\lambda\,.\end{aligned} \tag{5.11}$$

Remark 6. *Let us make now some observations about the error term $\mathfrak{E}_\varepsilon(t)$ estimated in this theorem, in particular let us investigate its behaviour in two limiting regimes, namely:*

- *The case of fixed $t > 0$ and $\varepsilon \to 0$: Due to the simple estimate*

$$\int_0^\varepsilon e^{-\lambda t}\,\lambda^{\alpha+\min\{1,\beta\}}\,\mathrm{d}\lambda \le \varepsilon^{\min\{1,\beta\}}\int_0^\varepsilon e^{-\lambda t}\,\lambda^\alpha\,\mathrm{d}\lambda\,,$$

  one concludes firstly that $\mathfrak{E}_\varepsilon(t) \to_{\varepsilon\to 0} 0$ and secondly that the error done, $\mathfrak{E}_\varepsilon(t)$, is smaller by a factor of $\varepsilon^{\min\{1,\beta\}}$ than the approximated correction term $\tilde{\mathcal{C}}_{\varepsilon,\alpha}(t,v)$, justifying thus that $\tilde{\mathcal{C}}_{\varepsilon,\alpha}(t,v)$ is a good approximation for $\varepsilon \ll 1$.
- *The case of fixed $\varepsilon > 0$ and $t \to \infty$: One observes firstly that the change of variable $\eta = \lambda\,t$ leads to*

$$\int_0^\varepsilon e^{-\lambda t}\,\lambda^\alpha\ \mathrm{d}\lambda = t^{-(\alpha+1)}\left(\int_0^{\varepsilon\,t} e^{-\eta}\,\eta^\alpha\ \mathrm{d}\eta\right) \sim_{t\to\infty} t^{-(\alpha+1)}\,\Gamma(\alpha+1) = \mathfrak{O}(t^{-(\alpha+1)})\,. \tag{5.12}$$

  On the other hand the same change of variables yields

$$\begin{aligned}\int_0^\varepsilon e^{-\lambda t}\,\lambda^{\alpha+\min\{1,\beta\}}\,\mathrm{d}\lambda &= t^{-(\min\{1,\beta\}+\alpha+1)}\int_0^{\varepsilon\,t} e^{-\eta}\,\eta^{\alpha+\min\{1,\beta\}}\,\mathrm{d}\eta \\ &\sim_{t\to\infty} t^{-(\min\{1,\beta\}+\alpha+1)}\,\Gamma(\min\{1,\beta\}+\alpha+1) = \mathfrak{o}(t^{-(\alpha+1)})\,.\end{aligned}$$

  Hence, the error term $\mathfrak{E}_\varepsilon(t)$ vanishes faster than the approximated correction term $\tilde{\mathcal{C}}_{\varepsilon,\alpha}(t,v)$ by a factor of $t^{-\min\{1,\beta\}}$. This permits to understand that the incorporation of the correction term $\mathcal{C}_\varepsilon(t,v)$ (via its approximation $\tilde{\mathcal{C}}_\varepsilon(t,v)$) accurately renders the algebraic convergence rate of the evolution semigroup $e^{-t\mathcal{L}_{D,eq}}$, which is proportional to $t^{-(1+\alpha)}$ (compare with Theorems 1, 2, 3), instead of an artificial exponential decay rate, as obtained with standard methods.

Proof of Theorem 4. Let $K \subset \mathbb{R}$ be a compact set. Then the following development holds true [5]

$$\Phi(v,\lambda) = \Phi(v,0) + \lambda\,\tilde{\Phi}(v,\lambda)\,, \quad \forall (v,\lambda) \in \mathbb{R}\times\mathbb{R}^+\,, \tag{5.13}$$

where the vector valued function $\tilde{\Phi}(v,\lambda)$ is uniformly bounded in $(v,\lambda)\in K\times[0,\varepsilon]$:

$$\sup_{(v,\lambda)\in K\times[0,\varepsilon]} \|\tilde{\Phi}(v,\lambda)\|_2 < \infty\,. \tag{5.14}$$

Developments (5.10) and (5.13) yield the following decomposition of the error term

$$\begin{aligned}
&\int_0^\varepsilon e^{-\lambda t}\,\mathfrak{A}_{in}^T(\lambda)\,P(\lambda)\,\Phi(v,\lambda)\,\mathrm{d}\lambda - \left(\int_0^\varepsilon e^{-\lambda t}\,\lambda^\alpha\;\;\mathrm{d}\lambda\right)\,(\mathfrak{A}_{in}^0)^T\,P_0\,\Phi(v,0)\\
&= \int_0^\varepsilon e^{-\lambda t}\,\lambda^{1+\alpha}\,\tilde{\mathfrak{A}}_{in,\lambda}^T\,(\lambda^{-\alpha}P(\lambda))\,\Phi(v,\lambda)\,\mathrm{d}\lambda\\
&\quad + \int_0^\varepsilon e^{-\lambda t}\lambda^{\alpha+\beta}\,\mathfrak{A}_{in}^T(\lambda)\,\tilde{P}_\lambda\,\Phi(v,\lambda)\,\mathrm{d}\lambda\\
&\quad + \int_0^\varepsilon e^{-\lambda t}\,\lambda^{1+\alpha}\,\mathfrak{A}_{in}^T(\lambda)\,(\lambda^{-\alpha}P(\lambda))\,\tilde{\Phi}(v,\lambda)\,\mathrm{d}\lambda\,.
\end{aligned}$$

Because of the boundedness of the first order terms $\tilde{\mathfrak{A}}_{in,\lambda}$, $\tilde{P}_\lambda$ and $\tilde{\Phi}(\cdot,\lambda)$ with respect to $\lambda>0$, one can find constants $C_{\varepsilon,K}^i>0$, $i\in\{1,2,3\}$, all bounded for ε close to 0^+, such that

$$\begin{aligned}
&\sup_{(v,\lambda)\in K\times[0,\varepsilon]} |\tilde{\mathfrak{A}}_{in,\lambda}^T\,(\lambda^{-\alpha}P(\lambda))\,\Phi(v,\lambda)| &&\leqslant C_{\varepsilon,K}^1\,,\\
&\sup_{(v,\lambda)\in K\times[0,\varepsilon]} |\mathfrak{A}_{in}^T(\lambda)\,\tilde{P}_\lambda\,\Phi(v,\lambda)| &&\leqslant C_{\varepsilon,K}^2\,,\\
&\sup_{(v,\lambda)\in K\times[0,\varepsilon]} |\mathfrak{A}_{in}^T(\lambda)\,(\lambda^{-\alpha}P(\lambda))\,\tilde{\Phi}(v,\lambda)| &&\leqslant C_{\varepsilon,K}^3\,.
\end{aligned}$$

As a consequence, the error term is bounded as follows

$$\sup_{v\in K}\left|\int_0^\varepsilon e^{-\lambda t}\,\mathfrak{A}_{in}^T(\lambda)\,P(\lambda)\,\Phi(v,\lambda)\,\mathrm{d}\lambda - \left(\int_0^\varepsilon e^{-\lambda t}\,\lambda^\alpha\;\;\mathrm{d}\lambda\right)\,(\mathfrak{A}_{in}^0)^T\,P_0\,\Phi(v,0)\right| \tag{5.15}$$

$$\leqslant (C_{\varepsilon,K}^1 + C_{\varepsilon,K}^3)\int_0^\varepsilon e^{-\lambda t}\,\lambda^{1+\alpha}\,\mathrm{d}\lambda + C_{\varepsilon,K}^2\int_0^\varepsilon e^{-\lambda t}\,\lambda^{\alpha+\beta}\,\mathrm{d}\lambda \tag{5.16}$$

$$\leqslant \left(C_{\varepsilon,K}^1 + C_{\varepsilon,K}^2 + C_{\varepsilon,K}^3\right)\int_0^\varepsilon e^{-\lambda t}\,\lambda^{\alpha+\min\{1,\beta\}}\,\mathrm{d}\lambda\,,\qquad (\text{since } \varepsilon<1)\,. \tag{5.17}$$

The result follows immediately after taking $C_{\varepsilon,K} = C_{\varepsilon,K}^1 + C_{\varepsilon,K}^2 + C_{\varepsilon,K}^3$. □

5.3. **LEAS numerical method.** Let us summarize now the method we propose in this paper for the resolution of the Fokker-Planck equation (5.1), cases (II)-(IV). Remark however that this method can be applied to any parabolic evolution problem with gapless, continuous spectrum. Furthermore, remark also that a simplified version of this method can be designed for time evolution problems on the half-line $\mathbb{R}^+$, by simply changing the spectral matrix $(\rho'_{lj})_{l,j\in\{0,1\}}$ into a scalar valued spectral function $\rho'(\lambda)$.

We shall assume that the low energy asymptotics of the corresponding spectral density matrix $P(\lambda)$ is known and of the form

$$P(\lambda)\sim\lambda^\alpha\,P_0\,,\quad \text{when }\ \lambda\to 0^+\,, \tag{5.18}$$

for some real symmetric spectral matrix $P_0=\{p_{l,j}\}_{l,j=0,1}$. This form can be obtained from theoretical results [16, 40, 42, 50], however it is also possible to determine numerically this asymptotic form through linear regressions, when a theoretical result has not yet been found.

<u>*LEAS* algorithm:</u>

(a) *Truncation:* First, let us truncate the velocity domain at $v = \pm L$ with fixed $L \gg 1$. The continuous problem (5.1) is hence approximated by

$$\begin{cases} \partial_t h^L = -\mathcal{L}_{D,eq}(h^L)\,, & \forall v \in (-L, L)\,, \\ \partial_v h^L(t,-L) = \partial_v h^L(t,L) = 0 \\ h^L(0,\cdot) = h^L_{in}\,. \end{cases} \tag{5.19}$$

This problem can be now solved either with classical schemes, such as finite-differences for example, leading to an approximation of $h^L(t_i, v_j)$, and one can pass then directly to step (c). We shall here focus rather on a spectral resolution, the corresponding spectral representation of the solution being given by (4.32), namely

$$h^L(t,v) = \alpha^L_0\,\phi^L_0 + \sum_{k=1}^{\infty} \alpha^L_k(t)\,\phi^L_k(v)\,, \qquad \alpha^L_k(t) := \int_{-L}^{L} h^L(t,v)\,\phi^L_k(v)\,f_{eq}(v)\,\mathrm{d}v\,. \tag{5.20}$$

(b) *Eigenvalue/function computation:* Compute then some approximations of a finite number N of eigenfunctions $\{\phi^L_k\}_{k=0}^{N-1}$ and eigenvalues $\{\lambda^L_k\}_{k=0}^{N-1}$. This permits to truncate the sum in (5.20) at $k = N-1$. Let us remark at this point the relations between the velocity space $v \in \mathbb{R}$ (truncation size L; discretization step $\mathrm{d}v$) and the spectral space $\lambda \in \mathbb{R}^+$ (discretization step $\mathrm{d}\lambda$; truncation index N).

(c) *Correction term:* Now, let us set

$$\varepsilon := \lambda^L_1\,,$$

and compute the correction term obtained in (5.8) via

$$\tilde{\mathcal{C}}_{\lambda^L_1,\alpha}(t,v) = \left(\int_0^{\varepsilon} e^{-\lambda t}\,\lambda^{\alpha}\;\mathrm{d}\lambda\right)\,(\mathfrak{A}^0_{in})^T\,P_0\,\Phi(v,0)\,, \tag{5.21}$$

where we recall

$$\Phi(v,0) = \begin{pmatrix} \varphi_0(v,0) \\ \varphi_1(v,0) \end{pmatrix}\,, \qquad \mathfrak{A}^0_{in} = \int_{-\infty}^{+\infty} h^L_{in}(v)\,\Phi(v,0)\,f_{eq}(v)\,\mathrm{d}v\,, \tag{5.22}$$

with $\varphi_0(\cdot,\lambda)\,,\varphi_1(\cdot,\lambda)$ the *generalized eigenfunctions* defined in (4.19)-(4.20). Recalling (5.9), we remark that past a certain threshold-value for t (to be determined), it is better (for numerical reasons) to use the following formulae

$$\theta^{\alpha}_{\varepsilon}(t) = t^{-(\alpha+1)}\left(\int_0^{\varepsilon\,t} e^{-\eta}\,\eta^{\alpha}\;\mathrm{d}\eta\right)\,, \tag{5.23}$$

obtaiend via the change of variable $\eta = t\,\lambda$.

(d) *LEAS scheme:* As a result, the solution of the evolution problem (5.1) is well approximated, for $L \gg 1$ large enough, by the following eigenfunction expansion:

$$h^L(t,v) = \alpha^L_0\,\phi^L_0 + \theta^{\alpha}_{\varepsilon}(t)\,\sum_{i=0}^{1}\left[p_{0,i}\,\mathfrak{A}^0_{in,0} + p_{1,i}\,\mathfrak{A}^0_{in,1}\right]\,\varphi_i(v,0) + \sum_{k=1}^{N-1}\alpha^L_k(t)\,\phi^L_k(v)\,, \tag{5.24}$$

where

$$\alpha^L_k(t) = e^{-\lambda^L_k\,t}\,\alpha^L_{\mathrm{in},k}\,, \quad \alpha^L_{\mathrm{in},k} := \int_{-L}^{L} h^L_{in}(v)\,\phi^L_k(v)\,f_{eq}(v)\,\mathrm{d}v\,, \qquad \forall k = 0,\cdots,N-1\,,\ \forall t > 0\,,$$

and with $P_0 = \{p_{l,j}\}_{l,j=0,1}$ the main term in the asymptotic form of the spectral density, given in (5.18).

The second term on the right hand side of (5.24) is the correction term we propose in this paper in order to take into account for the specificities of the here considered Fokker-Planck equation, possessing a gapless, continuous spectrum which leads to an algebraic relaxation towards the steady states. A forthcoming paper shall compare this scheme with standard numerical methods in order to evaluate the importance of the introduction of the correction term in a real physical situation.

At the end we would like to remark that the big advantage of the LEAS-scheme, as compared to methods such as [53, 62], is its simplicity (see (5.24)), and in particular the fact that it introduces a correction term, permitting to get the right algebraic time-decay rate for $t \to \infty$, and this in any standard discretization scheme, namely finite-difference, finite-element, finite-volume or spectral schemes.

6. Concluding remarks and perspectives

Let us conclude this paper by summarizing what was achieved in this work and what remains still to be done in future works. The main part of this work was concerned with the mathematical study of a specific Fokker-Planck equation, whose stationary states are given by κ-distribution functions, which are (thermal) non-equilibrium distributions and describe the energetic particle population in a fusion plasma gas. In particular we studied the (algebraic) decay rate as $t \to \infty$ of the velocity distribution function towards these steady-states and prepared the mathematical framework for the design of an efficient numerical method, based on the spectral representation theorem of the solutions to this Fokker-Planck operator. The here treated problem differs from standard Fokker-Planck equations, which posses Maxwellian steady-states (thus thermal equilibria) and feature an exponential relaxation rate in time. As the long-time behaviour of the solutions of our Fokker-Planck equation is dominated by the low-energy modes, a precise description of this $\lambda \ll 1$ modes is the basic ingredient of our scheme, called *Low Energy Accurate Scheme* (LEAS). For lengthy reasons of this paper, we postponed to a second paper the implementation of this LEAS-method in a physical realistic framework for fusion plasmas and the comparison of the results with those obtained via standard methods.

7. Appendix

We postponed to this Appendix the proof of the Proposition 4 about the spectrum of the Fokker-Planck operator $\mathcal{L}_{D,eq}$ in cases (II)-(IV).

Proof. Before starting, let us recall that the spectrum of any self-adjoint operator is a subset of $\mathbb{R}$.

- *Proof of* $\sigma_{pp}(\mathcal{L}_{D,eq}) = \{0\}$. Let us first show that the only eigenvalue of $\mathcal{L}_{D,eq}$ is $\lambda = 0$. First, $\lambda = 0$ is indeed an eigenvalue of $\mathcal{L}_{D,eq}$, since constant solutions are in $\ker \mathcal{L}_{D,eq}$, in particular in L^2_{eq}. Second, $\mathcal{L}_{D,eq}$ has no eigenvalue in $\mathbb{R} \setminus \mathbb{R}^+$ because it is a positive, self-adjoint linear differential operator of order 2 [17]. Finally, we claim that there is no eigenvalue in $\mathbb{R}^+_*$. Let us show this with the associated Schrödinger form of the eigenproblem stated in (4.2). Equation (4.2) has a set of two linearly independent and complex conjugate solutions g_+, g_- [51] verifying, if $\lambda \in \mathbb{R}_*^+$, that

$$g_\pm(s) \sim e^{\pm i\sqrt{\lambda}\, s}\,, \qquad s \to +\infty\,. \tag{7.1}$$

Therefore, any nonzero real solution of (4.2) behaves as

$$A\,\cos(\sqrt{\lambda}\, s - \theta) + o(1)\,, \qquad \text{when } s \to +\infty\,,$$

for some $A \in \mathbb{R}^*, \theta \in \mathbb{R}$. Such a real solution thus cannot be in $L^2(\mathbb{R})$. Therefore, using the Liouville transformation, we have proven that for all $\lambda > 0$ there is no solution $h \in L^2_{eq}$ of

$$\mathcal{L}_{D,eq}(h) = \lambda\, h\,,$$

thus $\sigma_{pp}(\mathcal{L}_{D,eq}) = \{0\}$.

- *Proof of* (4.16). As we can see, in each of the cases (II)-(IV), the potential Q is such that $s \to \partial_s(Q(v(s)))$ is bounded on $\mathbb{R}$ and $\lim_{s\to\pm\infty} Q(v(s)) = 0$. Therefore the following equality holds ([15] chapter 6):

$$\sigma_{ess}(\mathcal{L}_{D,eq}) = [0, +\infty)\,. \tag{7.2}$$

To prove (4.16), it therefore remains to show that the standard decomposition

$$\sigma(\mathcal{L}_{D,eq}) = \sigma_{Fred}(\mathcal{L}_{D,eq}) \cup \sigma_{ess}(\mathcal{L}_{D,eq})\,, \tag{7.3}$$

features no Fredholm spectrum $\sigma_{Fred}(\mathcal{L}_{D,eq})$. Since the union in (7.3) is disjoint (and since $\sigma(\mathcal{L}_{D,eq}) \subset \mathbb{R}$), from (7.2) one deduces that

$$\sigma_{Fred}(\mathcal{L}_{D,eq}) \subset \mathbb{R}_*^-\,.$$

Additionally, using Lemmas 6.15 and 6.16 from [15], we know that every element of $\sigma_{Fred}(\mathcal{L}_{D,eq})$ must be an eigenvalue. But as we have shown, there is no eigenvalue in $\mathbb{R}_*^-$, thus leading to

$$\sigma_{Fred}(\mathcal{L}_{D,eq}) = \emptyset\,, \qquad \sigma(\mathcal{L}_{D,eq}) = \sigma_{Fred}(\mathcal{L}_{D,eq}) \cup \sigma_{ess}(\mathcal{L}_{D,eq}) = \sigma_{ess}(\mathcal{L}_{D,eq})\,. \tag{7.4}$$

- *Proof of* (4.17). Let us remark first that Q is *vanishing with little oscillation* [45]. This means that
  - its derivative decays fast enough as $s \to \pm\infty$, namely:

    $$\partial_s(Q(v(s))) = o(|s|^{-1-\varepsilon})\,, \qquad \text{as } s \to \pm\infty\,,$$

    for some $\varepsilon > 0$,
  - and

    $$\lim_{s\to\pm\infty} Q(v(s)) = 0\,.$$

Therefore ([45] Theorem 1 (a)), one can show that there is no singular continuous spectrum, such that the following usual decomposition of the spectrum [54]

$$\sigma(\mathcal{L}_{D,eq}) = \overline{\sigma_{pp}(\mathcal{L}_{D,eq})} \cup \sigma_{ac}(\mathcal{L}_{D,eq}) \cup \sigma_{sc}(\mathcal{L}_{D,eq})\,, \tag{7.5}$$

reduces to

$$\sigma(\mathcal{L}_{D,eq}) = \sigma_{pp}(\mathcal{L}_{D,eq}) \cup \sigma_{ac}(\mathcal{L}_{D,eq})\,.$$

□

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