Hybrid modelling of plasmas in a multiscale framework.

Electr/ion low mass ratio regime.

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Motivation

Aim: obtention of a hybrid model (kinetic resp. hydrodyn. ions / adiabatic electrons) for the description of strongly confined plasmas with mass disparate and energetic particles.

Phys. context: study of tokamak fusion plasmas (ITER reactor)

Characteristics:

- high temperatures $\Rightarrow$ instable medium
- strong magn. fields $\Rightarrow$ anisotropic medium
- diff. electron/ion masses $\Rightarrow$ multi-scale dynamics

Models:

- particle description (very precise, unpracticable)
- kinetic description (precise, too costly)
- fluid description (not accurate enough, reasonable costs)

Motivation

Starting model: Fokker-Planck equations for the two species (ions, electrons)

$$\begin{cases} \partial_t f_i + \mathbf{v} \cdot \nabla_x f_i + \dfrac{e}{m_i} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_i = Q_{ii}(f_i) + Q_{ie}(f_i, f_e) \\ \partial_t f_e + \mathbf{v} \cdot \nabla_x f_e - \dfrac{e}{m_e} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_e = Q_{ee}(f_e) + Q_{ei}(f_e, f_i)\,, \end{cases}$$

coupled to Poisson's equation for the electrostatic potentiel

$$-\Delta \phi = \frac{e}{\varepsilon_0}(n_i - n_e)\,, \quad \mathbf{E} = -\nabla \phi\,.$$

Collision operator: Fokker-Planck or Fokker-Planck-Landau type

$$Q_{\alpha\beta}(f_\alpha, f_\beta) := \nu_{\alpha\beta} \nabla_v \cdot \left((\mathbf{v} - \mathbf{u}_{\alpha\beta}) f_\alpha + \frac{k_B T_{\alpha\beta}}{m_\alpha} \nabla_v f_\alpha \right)\,.$$

Aim: Design of an effective numerical scheme for the adiabatic plasma regime.

- introduction of multi-scale Fokker-Planck collision operator (modelling);
- find right scaling, leading to the electron Boltzmann relation in the low mass limit;
- obtain hybrid model via an asymptotic study, design an efficient numerical scheme (Hermite spectral approach).

Hybrid modelling of fusion plasmas in the low mass ratio regime

Main features/small parameter:

- Study electron/ion dynamics on ion scales
- $\varepsilon := m_e/m_i \ll 1$
- Electrons will thermalize quickly

Kinetic ion/electron model

Starting model: Fokker-Planck equations for the two species (ions, electrons)

$$(V)_\varepsilon \quad \begin{cases} \partial_t f_i + v\,\partial_x f_i + \dfrac{e}{m_i}\,E\,\partial_v f_i = Q_{ii}(f_i) + Q_{ie}(f_i, f_e) \\ \partial_t f_e + v\,\partial_x f_e - \dfrac{e}{m_e}\,E\,\partial_v f_e = Q_{ee}(f_e) + Q_{ei}(f_e, f_i) \end{cases} , \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$$

coupled to Poisson's equation for the electrostatic potentiel

$$-\partial_{xx}\phi = \frac{e}{\varepsilon_0}(n_i - n_e)\,, \quad E = -\partial_x \phi\,.$$

Collision operator:

$$Q_{\alpha\beta}(f_\alpha, f_\beta) := \nu_{\alpha\beta}\partial_v\left((v - u_{\alpha\beta})f_\alpha + \frac{k_B T_{\alpha\beta}}{m_\alpha}\partial_v f_\alpha\right)\,. \qquad \langle \xi \rangle := \int_{\mathbb{R}} \xi(v)\,dv$$

$$n_\alpha(t,x) := \langle f_\alpha \rangle\,, \quad n_\alpha u_\alpha := \langle v\, f_\alpha \rangle\,, \quad \frac{1}{2}\,k_B n_\alpha T_\alpha := \frac{m_\alpha}{2}\,\langle |v - u_\alpha|^2\, f_\alpha \rangle\,,$$

$$u_{ei} = u_{ie} := \frac{u_e + u_i}{2}\,, \qquad T_{ei} = T_{ie} := \frac{m_i T_e + m_e T_i}{m_e + m_i} + \frac{m_i m_e}{m_i + m_e}\,\frac{|u_e - u_i|^2}{2k_B}\,.$$

Properties: Mass, momentum, energy conservation; entropy decay; thermal equilibrium; H-theorem.

Physical scaling

- Definition of characteristic quantities and regime of interest (4 indep. parameters):

$$\varepsilon^2 := \frac{m_e}{m_i} \quad \text{(disparate masses)}\,; \qquad T_\alpha = \bar{T}\, T'_\alpha\,; \qquad \gamma := \frac{e\,\bar{\phi}}{k_B\,\bar{T}}$$

$$\bar{v}_i := v_{th,i} = \sqrt{\frac{k_B \bar{T}}{m_i}}\,, \quad \bar{v}_e := v_{th,e} = \sqrt{\frac{k_B \bar{T}}{m_e}} = \frac{1}{\varepsilon}\,\bar{v}_i \quad \text{(micro. velocity scales)}$$

$$\tau_c := \tau_{ii}\,, \quad l_c := \bar{v}_i\,\tau_c\,, \quad \text{(micro. time/space scales} \to \text{collisons)}$$

$$u_\alpha = \bar{u}_\alpha\, u'_\alpha\,, \quad \bar{u}_\alpha = \bar{v}_\alpha \quad \text{(macro. velocity scales)}$$

$$\bar{x}\,, \quad \bar{t} := \frac{\bar{x}}{\bar{u}_i}\,, \quad \tau := \frac{\tau_c}{\bar{t}} \quad \text{(macro. time/space scales)}\,, \quad \lambda := \frac{\lambda_D}{\bar{x}}\,, \quad \lambda_D := \sqrt{\frac{\varepsilon_0 k_B \bar{T}}{e^2\,\bar{n}}}$$

$$\bar{Q}_{\alpha\beta} = \bar{\nu}_{\alpha\beta}\bar{f}_\alpha\,, \quad \bar{f}_\alpha = \frac{\bar{n}}{\bar{v}_\alpha}\,, \qquad \bar{\nu}_{ie} : \bar{\nu}_{ii} : \bar{\nu}_{ee} : \bar{\nu}_{ei} = \varepsilon^2 : \varepsilon : 1 : 1$$

- Non-dimensional kinetic system: $(\gamma \equiv 1)$

$$\begin{cases} \partial_t f_i + v\,\partial_x f_i + E\,\partial_v f_i = \dfrac{1}{\tau}\,[Q_{ii}(f_i) + \varepsilon\, Q_{ie}(f_i, f_e)] \\[2ex] \partial_t f_e + \dfrac{1}{\varepsilon}\, v\,\partial_x f_e - \dfrac{1}{\varepsilon}\, E\,\partial_v f_e = \dfrac{1}{\varepsilon\,\tau}\,[Q_{ee}(f_e) + Q_{ei}(f_e, f_i)] \end{cases}$$

Aim: Describing plasma phenomena (instabilities) occurring on ion scales

$\Rightarrow$ Electrons will be approx. by an adiabatic model, obtained via adequate asympt. limit $\varepsilon \to 0$.

- Starting point: Electron kinetic model: $(\tau = \lambda = \gamma = 1)$

$$\begin{cases} \partial_t f_e^\varepsilon + \frac{1}{\varepsilon}\, v\, \partial_x f_e^\varepsilon - \frac{1}{\varepsilon}\, E^\varepsilon\, \partial_v f_e^\varepsilon = \frac{1}{\varepsilon}\left[Q_{ee}(f_e^\varepsilon) + Q_{ei}(f_e^\varepsilon, f_i^\varepsilon)\right], \\ -\partial_{xx}\phi^\varepsilon = n_i^\varepsilon - n_e^\varepsilon\,, \qquad E^\varepsilon = -\partial_x \phi^\varepsilon \end{cases}$$

- Fokker-Planck collision operator:

$$Q_{ee} = \nu_{ee}\, \partial_v \left[(v - u_e^\varepsilon) f_e^\varepsilon + T_e^\varepsilon\, \partial_v f_e^\varepsilon\right], \quad Q_{ei} = \nu_{ei}\, \partial_v \left[(v - u_{ei}^\varepsilon) f_e^\varepsilon + T_{ei}^\varepsilon\, \partial_v f_e^\varepsilon\right]$$

- Inter-species macro-quantities:

$$u_{ei}^\varepsilon = \frac{u_e^\varepsilon + \varepsilon u_i^\varepsilon}{2} = u_{ie}^\varepsilon$$

$$T_{ei}{}^\varepsilon = T_{ie}{}^\varepsilon = \frac{1}{1+\varepsilon^2}\left(T_e + \varepsilon^2 T_i + \frac{|u_e - \varepsilon u_i|^2}{2}\right)$$

- Macroscopic quantities:

$$
\begin{aligned}
n_e^\varepsilon(t,x) &:= \int_{\mathbb{R}} f_e^\varepsilon(t,x,v)dv\,,\\
n_e^\varepsilon(t,x)u_e^\varepsilon(t,x) &:= \int_{\mathbb{R}} v f_e^\varepsilon(t,x,v)dv\,,\\
w_e^\varepsilon(t,x) &:= \tfrac{1}{2}\int_{\mathbb{R}} |v|^2 f_e^\varepsilon(t,x,v)dv = \tfrac{1}{2}\, n_e^\varepsilon T_e^\varepsilon + \tfrac{1}{2}\, n_e^\varepsilon\,|u_e^\varepsilon|^2\,,\\
\tfrac{1}{2}\, n_e^\varepsilon(t,x)T_e^\varepsilon(t,x) &:= \tfrac{1}{2}\int_{\mathbb{R}} |v-u_e^\varepsilon|^2 f_e^\varepsilon(t,x,v)dv\,,\\
q_e^\varepsilon(t,x) &:= \tfrac{1}{2}\int_{\mathbb{R}} (v-u_e^\varepsilon)^3 f_e^\varepsilon(t,x,v)dv\,,
\end{aligned}
$$

- Corresponding (not closed) fluid model:

$$
\begin{cases}
\varepsilon\partial_t n_e^\varepsilon + \partial_x(n_e^\varepsilon\, u_e^\varepsilon) = 0\,,\\[2ex]
\varepsilon\partial_t(n_e^\varepsilon\, u_e^\varepsilon) + \partial_x\left[n_e^\varepsilon\,|u_e^\varepsilon|^2 + n_e^\varepsilon\, T_e^\varepsilon\right] + n_e^\varepsilon\, E^\varepsilon = -\nu_{ei}\, n_e^\varepsilon\, \dfrac{u_e^\varepsilon - \varepsilon u_i^\varepsilon}{2}\,,\\[2ex]
\varepsilon\partial_t w_e^\varepsilon + \partial_x\left[w_e^\varepsilon\, u_e^\varepsilon + n_e^\varepsilon\, T_e^\varepsilon\, u_e^\varepsilon + q_e^\varepsilon\right] + n_e^\varepsilon\, u_e^\varepsilon\, E^\varepsilon = -\nu_{ei}\, n_e^\varepsilon\left[T_e^\varepsilon - T_{ei}^\varepsilon + u_e^\varepsilon\,\dfrac{u_e^\varepsilon - \varepsilon u_i^\varepsilon}{2}\right].
\end{cases}
$$

- Starting rescaled Fokker-Planck equation:

$$\partial_t f_e^\varepsilon + \frac{1}{\varepsilon}\, v\, \partial_x f_e^\varepsilon - \frac{1}{\varepsilon}\, E^\varepsilon\, \partial_v f_e^\varepsilon = \frac{1}{\varepsilon}\, [Q_{ee}(f_e^\varepsilon) + Q_{ei}(f_e^\varepsilon, f_i^\varepsilon)]\ ,$$

- H-theorem: Entropy $\mathcal{H}_\alpha^\varepsilon(t) := \int\int_{\mathbb{T}\times\mathbb{R}} f_\alpha^\varepsilon \ln(f_\alpha^\varepsilon)\, dv\, dx\,,\quad h_{\alpha\beta}^\varepsilon := \frac{f_\alpha^\varepsilon}{\mathcal{M}_{\alpha\beta}^\varepsilon}$

$$\begin{aligned}\frac{d}{dt}\left(\mathcal{H}_e^\varepsilon(t) + \mathcal{H}_i^\varepsilon(t)\right) \quad = \quad & -\frac{1}{\varepsilon}\int\int_{\mathbb{T}\times\mathbb{R}} \left[\nu_{ee}\, T_{ee}^\varepsilon\, \frac{\mathcal{M}_e^\varepsilon}{h_e^\varepsilon}\, |\partial_v h_e^\varepsilon|^2 + \nu_{ei}\, T_{ei}^\varepsilon\, \frac{\mathcal{M}_{ei}^\varepsilon}{h_{ei}^\varepsilon}\, \left|\partial_v h_{ei}^\varepsilon\right|^2\right] dv\, dx \\ & - \int\int_{\mathbb{T}\times\mathbb{R}} \left[\nu_{ii}\, T_{ii}^\varepsilon\, \frac{\mathcal{M}_i^\varepsilon}{h_i^\varepsilon}\, \left|\partial_v h_i^\varepsilon\right|^2 + \varepsilon\, \nu_{ie}\, T_{ie}^\varepsilon\, \frac{\mathcal{M}_{ie}^\varepsilon}{h_{ie}^\varepsilon}\, \left|\partial_v h_{ie}^\varepsilon\right|^2\right] dv\, dx\,.\end{aligned}$$

$$\begin{cases} \int_{\mathbb{T}}\int_{\mathbb{R}} Q_{ee}(f_e^0) \ln(f_e^0) dv dx = 0 \quad \Rightarrow \quad f_e^0 = \mathcal{M}_{n_e^0, u_e^0, T_e^0} \\ \int_{\mathbb{T}}\int_{\mathbb{R}} Q_{ei}(f_e^0, f_i^0) \ln(f_e^0) dv dx = 0 \quad \Rightarrow \quad f_e^0 = \mathcal{M}_{n_e^0, 0, T_e^0} \end{cases}$$

- Kinetic hierarchy:

$$v\, \partial_x f_e^0 - E^0\, \partial_v f_e^0 = Q_{ee}^0(f_e^0) + Q_{ei}^0(f_e^0, f_i^0)$$

- Electron Boltzmann relation: Inserting $f_e^0 = \mathcal{M}_{n_e^0,0,T_e^0}$ in the first eq. of the kinetic hierarchy

$$v\, \partial_x \mathcal{M}_{n_e^0,0,T_e^0} - E^0\, \partial_v \mathcal{M}_{n_e^0,0,T_e^0} = 0 \quad \Leftrightarrow \quad v \left[\frac{\partial_x n_e}{n_e} + \left(\frac{v^2}{2\, T_e} - \frac{1}{2}\right) \frac{\partial_x T_e}{T_e}\right] - v\, \frac{\partial_x \phi}{T_e} = 0$$

- $\varepsilon \to 0$ electron asymptotics: $\quad [\chi] := \int_{\mathbb{T}} \chi(x)\, dx$

$$f_e^0(t,x,v) = \frac{n_e^0(t,x)}{\sqrt{2\,\pi\, T_e^0(t)}}\, e^{-\frac{v^2}{2T_e^0(t)}}\,, \quad n_e^0(t,x) = c(t)\, e^{\frac{\phi^0(t,x)}{T_e^0(t)}}\,, \quad c(t) = \mathfrak{m}/\left[e^{\frac{\phi^0}{T_e^0}}\right]$$

- Limit-model:

$$(L) \quad \begin{cases} \partial_t f_i^0 + v\partial_x f_i^0 + E^0 \partial_v f_i^0 = \nu_{ii}\partial_v \left((v - u_i^0) f_i^0 + T_i^0 \partial_v f_i^0\right) \\ -\partial_{xx}\phi^0 + c(t)\, e^{\frac{\phi^0(t,x)}{T_e^0(t)}} = n_i^0 \\ \left[w_i^0\right](t) + \dfrac{\mathfrak{m}}{2} T_e(t) + \dfrac{1}{2}\left[\,\left|\partial_x \phi^0\right|^2\right](t) = \mathfrak{E}_{ini} \end{cases}$$

- Initial conditions:

➠ total mass: $\mathfrak{m} := \int_{\mathbb{T}} \int_{\mathbb{R}} f_{e,i}^0(t=0,x,v)\, dvdx$

➠ total energy: $\mathfrak{E}_{ini} := \left[w_i^0(t=0)\right] + \left[w_e^0(t=0)\right] + \frac{1}{2}\left[\,\left|\partial_x \phi^0(t=0)\right|^2\right]$

Existence/Uniqueness of a solution

Fix $(n_i^0, w_i^0) \in L^\infty(\mathbb{R}^+; L^1(\mathbb{T}); \mathbb{R}^+)$, $\mathfrak{m} > 0$, $\mathfrak{E}_{ini} > 0$

- Search unique solution (ϕ, T) of

$$-\partial_{xx}\phi + c(t)e^{\phi/T} = n_i^0 \,, \quad \text{period. BC.}$$

associated to the constraints $\int_{\mathbb{T}} \phi(t,x)\, dx = 0$ and

$$c(t) \int_{\mathbb{T}} e^{\phi/T} dx = \int_{\mathbb{T}} n_i^0 dx = \mathfrak{m}\,,$$

$$\mathfrak{E}(T) := \frac{\mathfrak{m}}{2} T + \frac{1}{2} \int_{\mathbb{T}} |\partial_x \phi|^2 \, dx + \int_{\mathbb{T}} w_i^0 dx = \mathfrak{E}_{ini}\,.$$

- For each $T > 0$ there exists a unique $\phi_T \in \mathcal{H} := \{g \in H^1(\mathbb{T}) \;/\; \int_{\mathbb{T}} g\, dx = 0\}$ sol. to the nonlin. elliptic eq. (minimization of a strictly convexe functional)
- The function $\mathfrak{E}(T)$ is strictly increasing in T and satisfies

$$\mathfrak{E}(T) \to_{T\to\infty} \infty\,, \quad \mathfrak{E}(T) \to_{T\to 0} \int_{\mathbb{T}} w_i^0 \, dx \le \mathfrak{E}_{ini}$$

- There exists a unique $T_\star > 0$ s.t. $\mathfrak{E}(T_\star) = \mathfrak{E}_{ini}$ and hence a unique associated $\phi_{T_\star} \in \mathcal{H}$.

Conclusions mathematical part

- Achieved:
  - introduction of a mixed Fokker-Planck collision operator, satisfying the three conservation laws and the entropy decay;
  - found the right kinetic scaling, leading to the electron Boltzmann relation (adiabatic electrons);
  - Electron Limit-model obtained for $\varepsilon \to 0$.
- Next step :
  - design an efficient scheme, describing the ion/electron mixture;
  - ideally design of an AP-scheme, permitting an ε-independent mesh.

Numerical simulation of fusion plasmas in the low mass ratio regime

Main features:

- Hermite spectral approach in the velocity variable;
- Standard FD/Fourier/Galerkin approach in the space variable;
- Crank-Nicolson (implicit) time-discretization;
- Investigation of an AP-property (ε-independent mesh).

Hermite spectral approach

- Study of the electron dynamics (given ion dyn.):

$$\begin{cases} \varepsilon\, \partial_t f^\varepsilon + v\, \partial_x f^\varepsilon - E^\varepsilon\, \partial_v f^\varepsilon = \mathcal{Q}_{ee}(f^\varepsilon) + \mathcal{Q}_{ei}(f^\varepsilon)\,, \\ -\partial_{xx}\phi^\varepsilon = n_i^\varepsilon - n^\varepsilon, \qquad E^\varepsilon = -\partial_x \phi^\varepsilon\,. \end{cases}$$

- Standard Hermite polynomials $\{J_k\}_{k\in\mathbb{N}}$, orth. basis of $L^2(\mathcal{M}\, dv)$:

$$\sqrt{k+1}\, J_{k+1} = v\, J_k(v) - \sqrt{k}\, J_{k-1}\,, \quad J_0 \equiv 1, \;\; J_1 \equiv v\,.$$

- Fokker-Planck specific Hermite basis $\{\psi_k(t,v)\}_{k\in\mathbb{N}}$:

$$\psi_k(t,v) := \frac{1}{v_{th}(t)}\, J_k\left(\frac{v}{v_{th}(t)}\right) \mathcal{M}\left(\frac{v}{v_{th}(t)}\right)\,,$$

$$\mathcal{M}_{v_{th}(t)}(v) := \frac{1}{\sqrt{2\pi}\, v_{th}(t)} \exp\left(-\frac{v^2}{2v_{th}^2(t)}\right)\,, \qquad v_{th}(t) := \sqrt{T^0(t)}\,,$$

$$\sqrt{k+1}\, \psi_{k+1}(t,v) = \frac{v}{v_{th}}\, \psi_k(t,v) - \sqrt{k}\, \psi_{k-1}(t,v)\,, \quad \psi_0 \equiv \mathcal{M}_{v_{th}}\,, \;\; \psi_1 \equiv v\mathcal{M}_{v_{th}}/v_{th}\,.$$

Hermite spectral approach

- Decomposition of f^ε in the orth. basis of $L^2(\mathcal{M}_{v_{th}}^{-1}\, dv)$:

$$f^\varepsilon(t,x,v) := \sum_{k=0}^{\infty} \alpha_k^\varepsilon(t,x)\, \psi_k(t,v)$$

- Hermite coef. $\{\alpha_k^\varepsilon\}_{k\in\mathbb{N}}$ sol. of infinite, coupled, nonlin. PDE syst.:

$$\left\{ \begin{array}{l} \varepsilon \left(\partial_t \alpha_k^\varepsilon + \frac{v'_{th}}{v_{th}} \left[k\, \alpha_k^\varepsilon + \sqrt{(k-1)k}\, \alpha_{k-2}^\varepsilon \right] \right) + v_{th}\, \partial_x \left(\sqrt{k}\, \alpha_{k-1}^\varepsilon + \sqrt{k+1}\, \alpha_{k+1}^\varepsilon \right) \\ + \sqrt{k}\, \frac{E^\varepsilon}{v_{th}}\, \alpha_{k-1}^\varepsilon + \nu_{ee} \left(k\, \alpha_k^\varepsilon - \sqrt{k} \frac{u^\varepsilon}{v_{th}} \alpha_{k-1}^\varepsilon + \left[1 - \frac{T^\varepsilon}{v_{th}^2} \right] \sqrt{(k-1)\, k}\, \alpha_{k-2}^\varepsilon \right) \\ + \nu_{ei} \left(k\, \alpha_k^\varepsilon - \sqrt{k} \frac{u_{ei}^\varepsilon}{v_{th}} \alpha_{k-1}^\varepsilon + \left[1 - \frac{T_{ei}^\varepsilon}{v_{th}^2} \right] \sqrt{(k-1)\, k}\, \alpha_{k-2}^\varepsilon \right) = 0 \\ -\partial_{xx}\phi^\varepsilon = n_i^\varepsilon - \alpha_0^\varepsilon\,, \qquad E^\varepsilon = -\partial_x \phi^\varepsilon\,. \end{array} \right.$$

- Observe that:

$$n^\varepsilon = \int_{\mathbb{R}} f^\varepsilon(t,x,v)\, dv = \alpha_0^\varepsilon\,, \quad \alpha_1^\varepsilon = \frac{n^\varepsilon\, u^\varepsilon}{v_{th}}\,, \quad T^\varepsilon = v_{th}^2 \left[1 + \sqrt{2}\, \frac{\alpha_2^\varepsilon}{\alpha_0^\varepsilon} - \left(\frac{\alpha_1^\varepsilon}{\alpha_0^\varepsilon} \right)^2 \right]\,.$$

Hermite spectral approach

- Advantages of Hermite spectral approach:
  - ➠ basis fct. $\{\psi_k\}_{k\in\mathbb{N}}$ are well-adapted to approach Maxwellian-like distribution fct.;
  - ➠ lower order coef. are related to the lower order moments of f^ε;
  - ➠ permits to make the link between kinetic and fluid world, due to $\alpha^\varepsilon_{k\neq 0} \to_{\varepsilon\to 0} 0$.
- Asymptotic limit $\varepsilon \to 0$:

$$\left\{\begin{array}{l} v_{th}\,\partial_x\left(\sqrt{k}\,\alpha^0_{k-1} + \sqrt{k+1}\,\alpha^0_{k+1}\right) + \sqrt{k}\,\dfrac{E^0}{v_{th}}\,\alpha^0_{k-1} \\ +\nu_{ee}\left(k\,\alpha^0_k - \sqrt{k}\,\dfrac{\alpha^0_1}{\alpha^0_0}\,\alpha^0_{k-1} - \left[\sqrt{2}\,\dfrac{\alpha^0_2}{\alpha^0_0} - \left(\dfrac{\alpha^0_1}{\alpha^0_0}\right)^2\right]\sqrt{(k-1)\,k}\,\alpha^0_{k-2}\right) \\ +\nu_{ei}\left(k\,\alpha^0_k - \sqrt{k}\,\dfrac{\alpha^0_1}{2\,\alpha^0_0}\,\alpha^0_{k-1} - \left[\sqrt{2}\,\dfrac{\alpha^0_2}{\alpha^0_0} - \dfrac{1}{2}\left(\dfrac{\alpha^0_1}{\alpha^0_0}\right)^2\right]\sqrt{(k-1)\,k}\,\alpha^0_{k-2}\right) = 0\,, \\ \displaystyle\int_{\mathbb{T}}\alpha^0_0\,dx = \mathfrak{m}\,, \qquad \frac{1}{2}\int_{\mathbb{T}}\left[v^2_{th}\left(\alpha^0_0 + \sqrt{2}\,\alpha^0_2\right) + |\partial_x\phi^0|^2 + 2\,w^0_i\right]dx = \mathfrak{E} \\ -\partial_{xx}\phi^0 = n^0_i - \alpha^0_0\,, \qquad E^0 = -\partial_x\phi^0\,. \end{array}\right.$$

- Unique sol. of this PDE syst: Electron Boltzmann relation

$$v_{th}\,\partial_x\alpha^0_0 + \frac{E^0}{v_{th}}\,\alpha^0_0 = 0 \quad\Rightarrow\quad \alpha^0_0(t,x) = c(t)\,\exp\left(\frac{\phi^0(t,x)}{T^0(t)}\right).$$

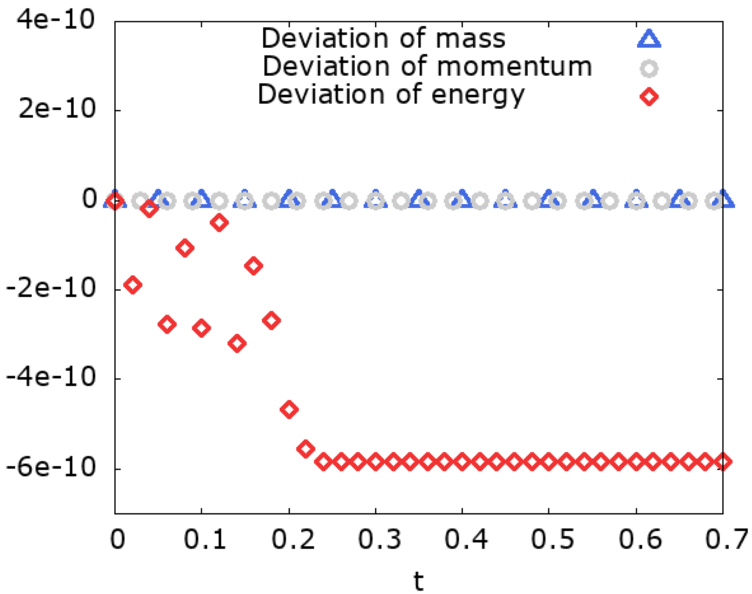
Numerical simulations: One species test case

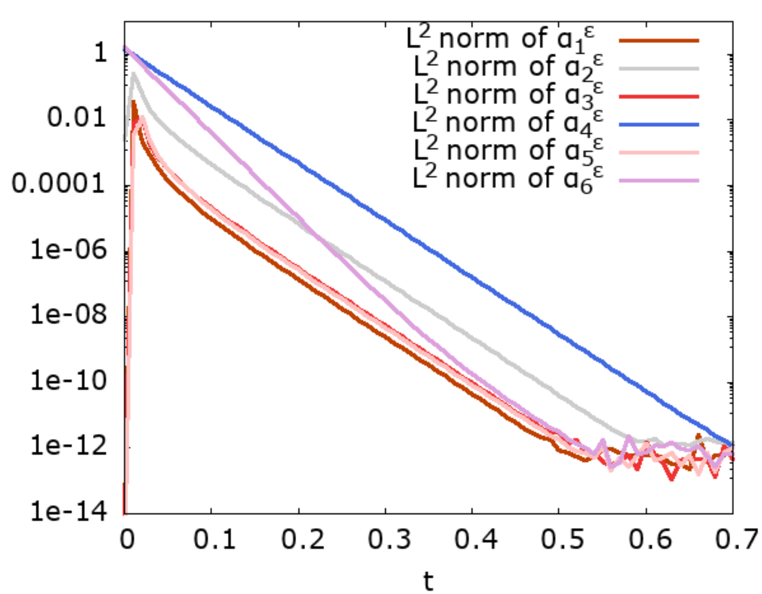
- Initial condition: $\mathbb{T} := [0, L]\,, \quad L = 12\,, \quad k = \frac{2\pi}{L}\,, \quad \kappa = 0.04$

$$f_e(t=0,x,v) \;=\; \frac{1}{7\sqrt{2\pi}}\,(2+5\,v^2)\,\exp\left(-\frac{v^2}{2}\right)\,,$$

$$n_i(x) \;=\; 1+\kappa\left[\frac{\cos(2k\,x)+\cos(3k\,x)}{1.2}+\cos(k\,x)\right]\,,$$

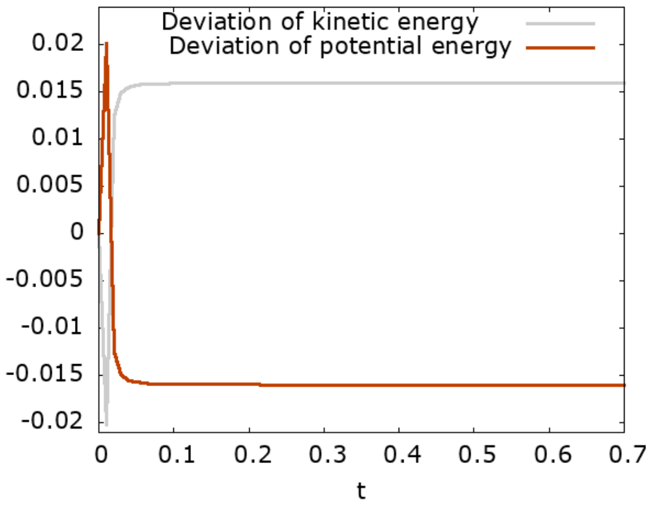
- Left: Conservations; Right: L^2-norm ev. of Hermite coef; $\varepsilon = 10^{-3}$

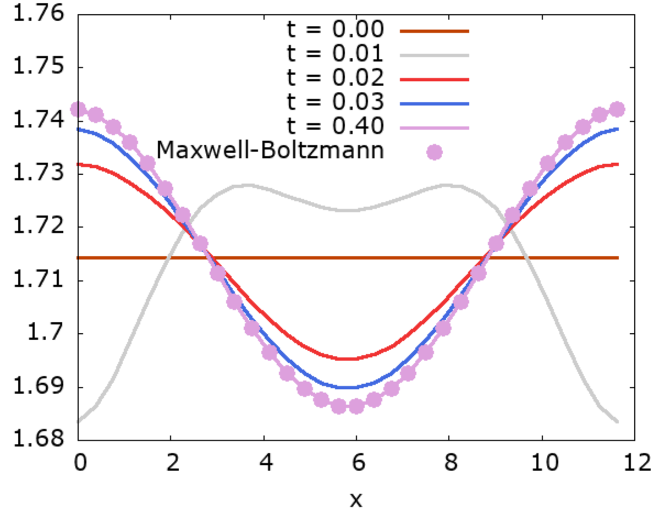




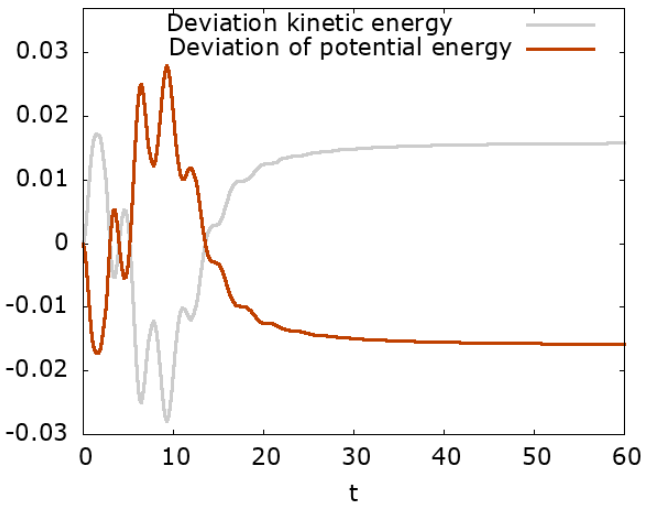
Numerical simulations: One species test case

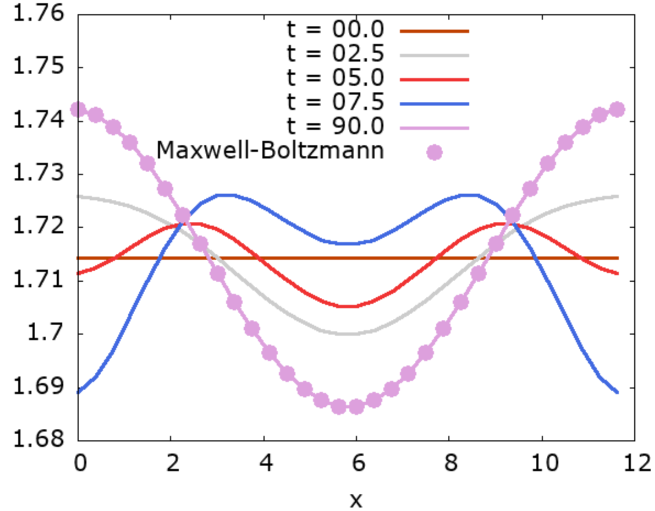
- Left: Pot. & Kin. energy evol.; Right: Density $n_e^\varepsilon(t,\cdot)$ at diff. t; $\varepsilon = 10^{-3}$





- Left: Pot. & Kin. energy evol.; Right: Density $n_e^\varepsilon(t,\cdot)$ at diff. t; $\varepsilon = 1$





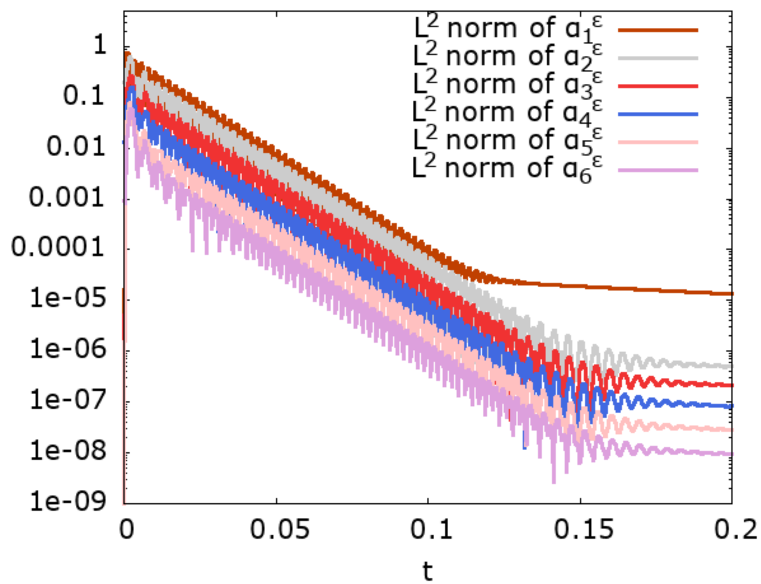
Numerical simulations: Two species test case

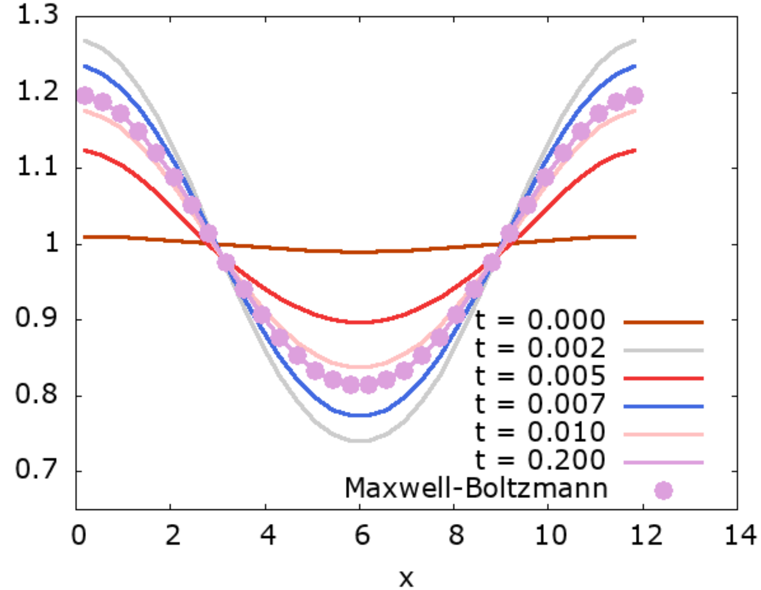
- Initial condition $\mathbb{T} := [0, L]\,, \quad L = 12\,, \quad k = \frac{2\pi}{L}\,, \quad \kappa = 0.01$

$$f_e^\varepsilon(0,x,v) \;=\; \frac{1}{\sqrt{2\,\pi}}\,\exp\left(-\frac{v^2}{2}\right)\,(1+\kappa\cos(k\,x))\,,$$

$$n_i^\varepsilon(x) \;=\; 1\,+\,0.2\,\cos(k\,x)\,,$$

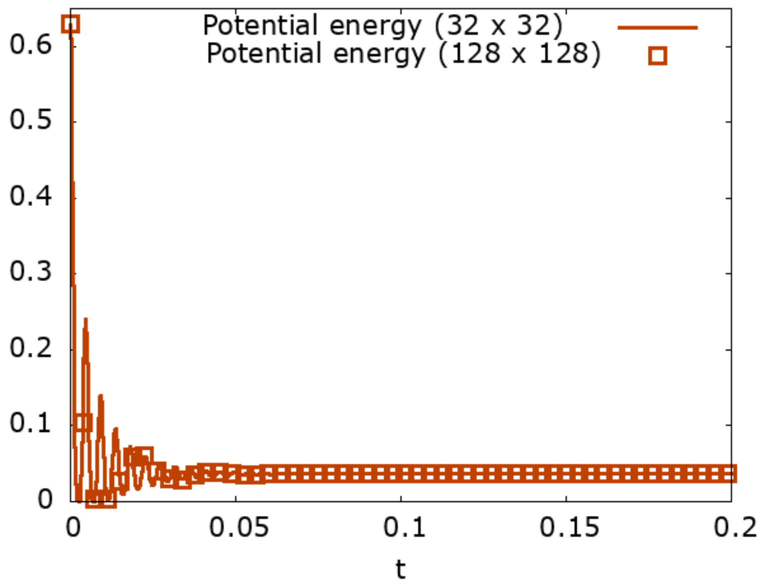
- Left: L^2-norm ev. of Hermite coef; Right: Density $n_e^\varepsilon(t,\cdot)$ at diff. t; $\varepsilon = 10^{-3}$

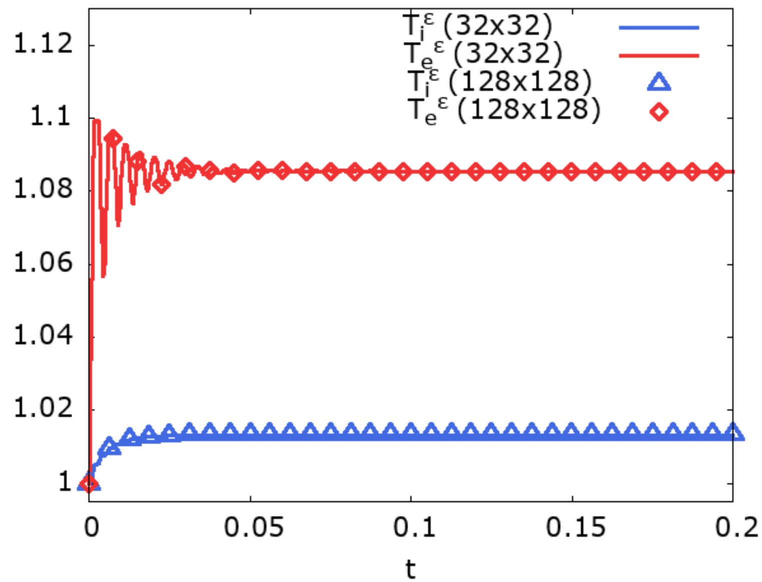




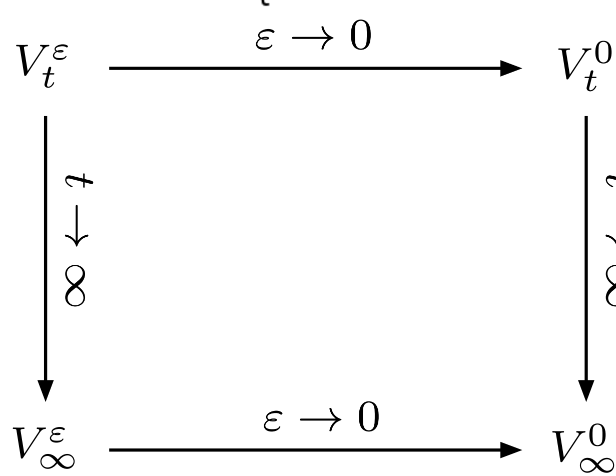
Numerical simulations: Two species test case

- Left: Pot. energy evol.;   Right: Global temp. evol.; $\varepsilon = 10^{-3}$



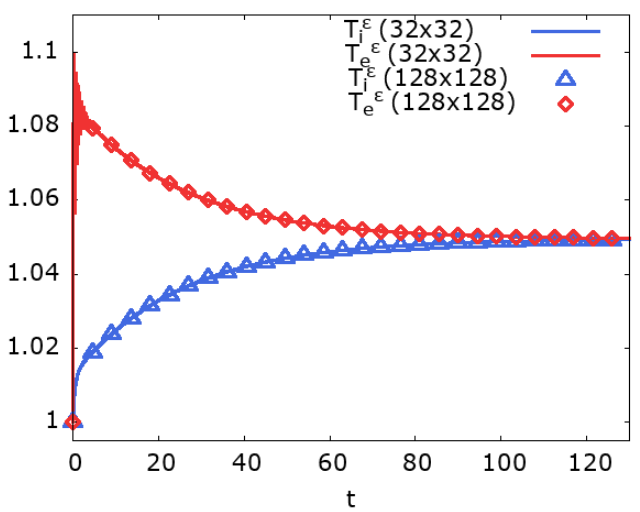


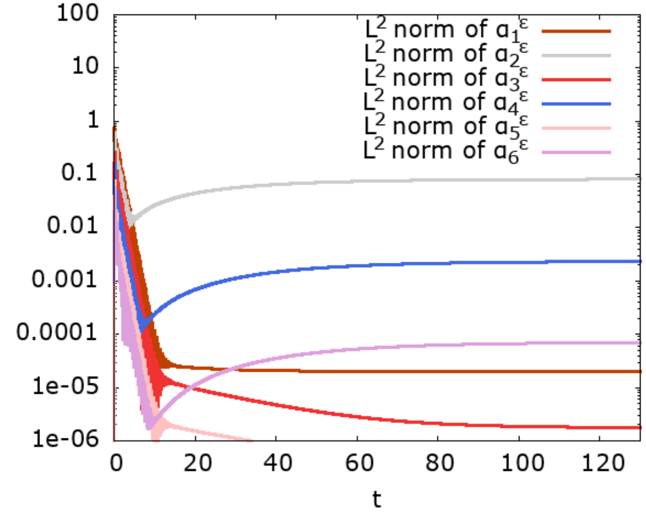
- Different $\varepsilon \to 0$ and $t \to \infty$ asymptotics



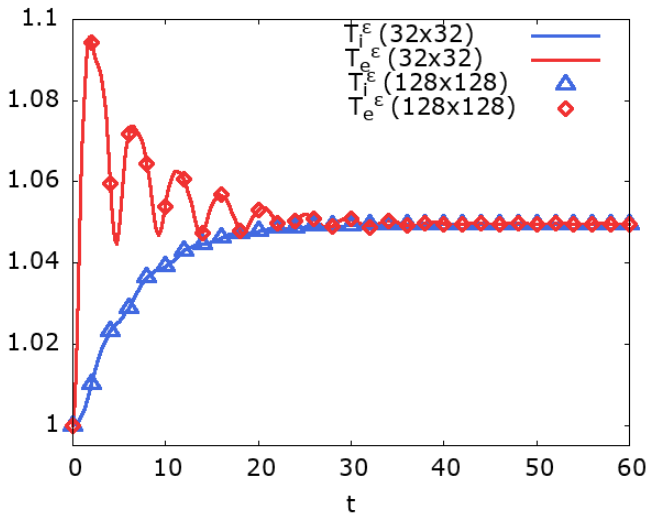
Numerical simulations: Two species test case

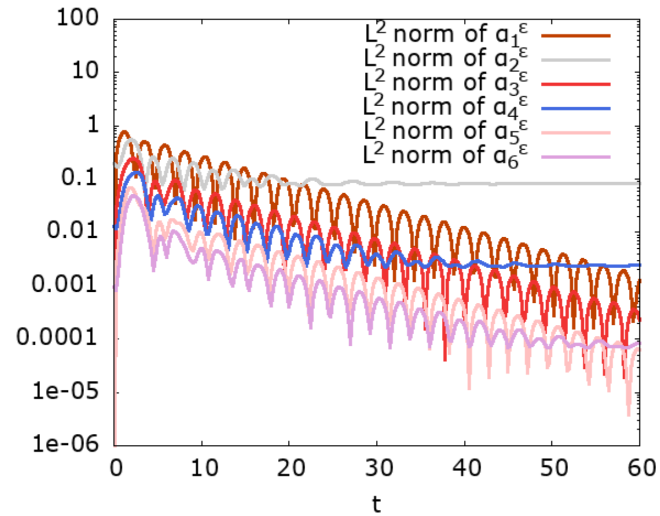
- Left: Global temp. evol.; Right: L^2-norm ev. of Hermite coef; $\varepsilon = 10^{-1}$





- Left: Global temp. evol.; Right: L^2-norm ev. of Hermite coef; $\varepsilon = 1$





Conclusions numerical part

- Achieved:
  - ➠ the advantage of a Hermite spectral approach for small ε-regimes was shown;
  - ➠ asymptotic regime was recovered for small ε-values, however still with an ε-dep. mesh. (for the moment not AP).
- To be done :
  - ➠ *Asymptotic-Preserving* approach, permitting to choose ε-independent Δt-meshes;
  - ➠ tricky discretization of the Poisson-Boltzmann nonlin, elliptic eq. (L-model);
  - ➠ design an efficient scheme, describing the full ion/electron dyn.

Thank you for your attention