Les différents modèles utilisés en neutronique: une revue et quelques résultats mathématiques

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Contents

The basic initial model: Boltzmann's equation

Linear Boltzmann equation with a linear collision kernel

$$\partial_t f(t,x,v) + v.\nabla_x f(t,x,v) = \int K(t,x,v,v')f(t,x,v')dv'. \quad (1)$$

Another vision of this kernel, which decomposes into two nonnegative parts

$$K(t,x,v,v') = \nu K_f(t,x,v,v') - K_a(t,x,v,v') \quad (2)$$

$$\frac{d}{dt}\mathcal{A}(t,x) = A_p(t,x) - A_a(t,x)$$

($\mathcal{A}$ stands for the number of neutrons in a volume (Lagrangian)),
$A_p(t,x)$ is the number of neutrons produced by unit of time,
$A_a(t,x)$ is the number of neutrons absorbed by unit of time,
$A_p(t,x)$ is produced by fission.
Based on $m_1'v_1' + m_2'v_2' = m_1v_1 + m_2v_2 + \tilde{d}$, where particles 1 and 2 interact, and produce neutrons of impulse $\tilde{d}$.
Remark: $\frac{1}{2}m_1'(v_1')^2 + \frac{1}{2}m_2'(v_2')^2 < \frac{1}{2}m_1(v_1)^2 + \frac{1}{2}m_2(v_2)^2$.

Multi group neutron diffusion equation

Based on $v = ||v||\Omega$, where Ω is of norm 1, and selects a direction.

Recall $v = \frac{p}{m\gamma(p)}$, $\gamma(p) = \sqrt{1 + \frac{p^2}{m^2c^2}}$ ($p = ||p||\Omega$ as well).

$$[\frac{1}{||v||}\partial_t f + \Omega.\nabla_x f](t, x, ||v||\Omega) = \int_{\mathbf{R}^3} \frac{K(t,x,||v||\Omega,v')}{||v||} f(t, x, v')dv'$$

$$= \int_0^{+\infty} \frac{||p'||^2}{||v||} d||p'|| \int_{S^2} K(t, x, ||v||\Omega, ||v'||\Omega') f(t, x, \frac{||p'||}{m\gamma(||p'||)}\Omega')d\Omega'.$$

As $p' \in [0, +\infty)$, $v' \in [0, c)$, replace $\int_0^{+\infty}$ by a discrete sum.

$$\int_0^{+\infty} \varphi(p')d||p'|| = \int_0^c \tilde{\varphi}(v')\rho(||v'||)d||v'|| \to \sum_{g=1}^{K} l_g \varphi(v'_g)$$

where v'_g is the 'mean' of $||v'||$ in the interval I_g of length l_g.

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After discretization of the integral,

$$\frac{1}{||v||}\partial_t f + \Omega.\nabla_x f + \sum_{g'} \int_{S^2} k_a(.,.,||v||\Omega, v_{g'}\Omega'))(v_{g'})^2 l_{g'} f(.,.,v_{g'}\Omega')d\Omega'$$
$$= \sum_{g'} \int_{S^2} \nu k_f(.,.,v_g\Omega, v_{g'}\Omega') f(.,.,v_{g'}|\Omega')(v_{g'})^2 l_{g'} d\Omega'.$$

Same with v: for each v_g in the list $g \in \{1,..,K\}$

$$[\frac{1}{v_g}\partial_t f + \Omega.\nabla_x f] + \sum_{g'} \int_{S^2} (v_{g'})^2 l_{g'} k_a(.,.,v_g\Omega, v_{g'}\Omega')) f(.,.,v_{g'}\Omega')d\Omega'$$
$$= \sum_{g'} \int_{S^2} (v_{g'})^2 l_{g'} \nu k_f(.,.,v_g\Omega, v_{g'}\Omega') f(.,.,v_{g'}\Omega')d\Omega'.$$

Called multi-group transport equation for neutrons.

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More precisely, with $\Phi_g(t,x,\Omega) = \int_{I_g} f(t,x,||v||\Omega)||v||^2 d||v||$ $(K_{g,g'} = S^2 \times I_g \times I_{g'})$:

$$\begin{aligned}
&[\tfrac{1}{v_g}\partial_t \Phi_g + \Omega.\nabla_x \Phi_g] \\
&+ \textstyle\sum_{g'} \int_{K_{g,g'}} (v_g)^2 (v_{g'})^2 k_a(.,.,v_g\Omega, v_{g'}\Omega')) f(.,.,v_{g'}\Omega') d\Omega' \\
&= \textstyle\sum_{g'} \int_{K_{g,g'}} (v_g)^2 (v_{g'})^2 \nu k_f(.,.,v_g\Omega, v_{g'}\Omega') f(.,.,v_{g'}\Omega') d\Omega'.
\end{aligned}$$

Modeled by using $\Phi_{g'}$ as well:

$$\sum_{g'} \int_{K_{g,g'}} (v_g)^2 (v_{g'})^2 \nu k_f(.,.,v_g\Omega, v_{g'}\Omega') f(.,.,v_{g'}\Omega') d\Omega'$$

$$\rightarrow$$

$$\sum_{g'} \int_{S^2} \nu K_f(.,.,v_g\Omega, v_{g'}\Omega') \Phi_{g'}(.,.,\Omega') d\Omega'.$$

Final model

Multi-group ($\Phi_g \geq 0$ on $\mathcal{O}$):

$$\frac{1}{v_g}\partial_t \Phi_g + \Omega.\nabla_x \Phi_g + \sum_{g'} \int_{\Omega'} (\Sigma_a)_g^{g'}(.,.,\Omega,\Omega')\Phi_{g'}(.,.,\Omega')d\Omega' \\ = \\ \sum_{g'} \int_{\Omega'} \nu(\Sigma_f)_g^{g'}(.,.,\Omega,\Omega')\Phi_{g'}(.,.,\Omega,\Omega')d\Omega'. \tag{3}$$

One group ($\Phi \geq 0$ on K):

$$\frac{1}{v}\partial_t \Phi + \Omega.\nabla_x \Phi + \int_{\Omega'} \Sigma_a(.,.,\Omega,\Omega')\Phi_{(.,.,\Omega')}d\Omega' \\ = \\ \int_{\Omega'} \nu\Sigma_f(.,.,\Omega,\Omega')\Phi(.,.,\Omega')d\Omega'. \tag{4}$$

What is found in neutronics books

Standard representation of the previous equation (other formulation) ($\varphi(t,x,E)=\int_{S^2}\psi(t,x,E,\Omega)d\Omega$, $\chi(E)$: spectrum of enery of fission neutrons):

$$\frac{1}{v(E)}\partial_t\psi(t,x,E,\Omega)+\Omega.\nabla_x\psi(t,x,E,\Omega)+\Sigma_t(t,E,x)\psi(t,x,E,\Omega)$$
$$=$$
$$\int_{E_{min}}^{E_{max}}\int_{S^2}(\Sigma_s)_{E',\Omega'}^{E,\Omega}(t,x)\psi(t,x,E,\Omega')d\Omega'$$
$$+\frac{\chi(E)}{4\pi}\int_{E_{min}}^{E_{max}}\nu\Sigma_f(E')\varphi(t,x,E')dE'$$
$$+\text{ other terms}$$

other terms: all isotopes and the delayed neutrons.

Critical stationary transport equation

$$\Omega.\nabla_x\psi(t,x,E,\Omega)+\Sigma_t(t,E,x)\psi(x,E,\Omega)$$
$$=$$
$$\int_{E_{min}}^{E_{max}}\int_{S^2}(\Sigma_s)_{E',\Omega'}^{E,\Omega}(t,x)\psi(x,E,\Omega')d\Omega'+\frac{\chi(E)}{4\pi k_{eff}}\int_{E_{min}}^{E_{max}}\nu\Sigma_f(E')\varphi(x,E')dE'$$

Classical approximations

Approximation of transport by diffusion (analogous of what financial mathematicians do), $\varphi \geq 0$

$$\frac{1}{v}\partial_t\varphi - \mathrm{div}(d(t,x)\nabla_x\varphi) + \sigma_a(t,x)\varphi(t,x) = \nu\sigma_f(t,x)\varphi(t,x). \quad (5)$$

Boundary conditions: $\Phi_g = \Phi = \varphi = 0$ on $\partial\mathcal{O}$, $\mathcal{O}$ bounded.

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Criticality condition

Start in the case of one group, coefficients independent on t, bounded below by strictly positive constants on K:

$$\frac{1}{v}\partial_t\varphi - \mathrm{div}(d(x)\nabla_x\varphi) + \sigma_a(x)\varphi(t,x) = \nu\sigma_f(x)\varphi(t,x).$$

Subcase: d, σ_a, σ_f independent on x. Solutions of the form $k(t)\varphi(x)$. Separation of variables $\frac{1}{v}k'(t)\varphi(x) = k(t)[\mathrm{div}(d(x)\nabla\varphi) - \sigma_a(x)\varphi(x) + \nu\sigma_f(x)\varphi(x)]$. Obtain $\frac{k'(t)}{k(t)} = \tau^{-1}$ and

$$-\mathrm{div}(d(x)\nabla_x\varphi(x)) + \sigma_a(x)\varphi(x) = (\nu\sigma_f(x) - \frac{\tau}{v})\varphi(x).$$

This problem is $A\varphi = B(\tau)\varphi$ on $H^1_0(K)$, where

$$A : \varphi \to -\mathrm{div}(d(x)\nabla_x\varphi(x)) + \sigma_a(x)\varphi(x), B : \varphi \to (\nu\sigma_f(x) - \frac{\tau}{v})\varphi$$

Mathematical result 1

Proposition

1. *For* $\frac{\tau}{v} < min_{\mathcal{O}} \nu \sigma_f(x)$*, the problem* $A\varphi = B(\tau)\varphi$ *on* $H_0^1(\mathcal{O}), \varphi|_{\partial\mathcal{O}} = 0$ *is equivalent to*

$$A^{-\frac{1}{2}} B(\tau) A^{-\frac{1}{2}} \psi = \psi, \psi \in H_0^1(\mathcal{O}).$$

2. *The operator* $A^{-\frac{1}{2}} B(\tau) A^{-\frac{1}{2}}$ *is a self-adjoint compact operator, hence admits an infinite countable decreasing sequence* $\mu_n(\tau)$ *of eigenvalues.*
3. *The unique nonnegative solution of* $A\varphi = B(\tau)\varphi$ *in* $H_0^1(\mathcal{O})$ *is obtained for* $\mu_1(\tau) = 1$.

Proof: A is a symmetric self-adjoint coercive operator hence by Lax Milgram lemma, is invertible, and $A^{-\frac{1}{2}}$ is well defined. Denote by ψ such that $A^{-\frac{1}{2}}\psi = \varphi$. Deduce $A\varphi = A^{\frac{1}{2}}\psi = B(\tau)A^{-\frac{1}{2}}\psi$, which is the desired equation. The second item follows (because $B(\tau)$ is an operator on $H_0^1(\mathcal{O})$). The third item is a consequence that the first eigenvector of a positive self-adjoint compact operator is of constant sign, and all others are orthogonal in $L^2(\mathcal{O})$ to this one.

Now, solve $\mu_1(\tau) = 1$.
If $\tau > 0$, unique family of growing in time solution (surcritical).
If $\tau < 0$, unique family of decaying in time solution (subcritical).
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Simple classical application

Further reducing: constant coefficients, $\mathcal{O} = [0, L]$: Spectrum of A: $\tilde{\mu}_n = n^2 \frac{\pi^2}{L^2} d + \sigma_a$.
Eigenvectors $x \to \sin \frac{n\pi x}{L}$.
Spectrum of $A^{-\frac{1}{2}} B(\tau) A^{-\frac{1}{2}}$:

$$\mu_n(\tau) = \frac{\nu\sigma_f - \frac{\tau}{v}}{\frac{n^2\pi^2}{L^2} d + \sigma_a}.$$

Resolution of $\mu_1(\tau) = 1$:

$$\frac{\tau}{v} = \nu\sigma_f - \sigma_a - \frac{\pi^2}{L^2} d.$$

Criticality condition

$$\nu\sigma_f = \sigma_a + \frac{\pi^2}{L^2} d \tag{6}$$

New formulation of the eigenvalue problem

Replace $\nu\sigma_f - \frac{\tau}{v}$ by $\frac{\nu\sigma_f}{k_{eff}}$

$$-\text{div}(d(x)\nabla\varphi(x)) + \sigma_a(x)\varphi(x) = \frac{\nu\sigma_f(x)}{k_{eff}}\varphi, \varphi_{\partial\mathcal{O}} = 0.$$

Means k_{eff} is an eigenvalue of $A^{-\frac{1}{2}}(\nu\sigma_f(.).)A^{-\frac{1}{2}}$.

A lot different, except for $\nu\sigma_f$ constant or $k_{eff} = 1$.

In particular: nothing can be said if $k_{eff} \neq 1$ for a value of τ.

Only information: if the spectrum of $A^{-\frac{1}{2}}(\nu\sigma_f(.).)A^{-\frac{1}{2}}$ lies in $(0,1)$, the spectrum of $\nu\sigma_f - A$ lies in $(-\infty, 0)$, always subcritical.

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Multi-group presentation of the eigenvalue problem

$$\frac{1}{v_g}\partial_t\Phi_g - \mathrm{div}(D_g\nabla\Phi_g) + \sum_{g'}\Sigma_a^{g'->g}\Phi_{g'} = \sum_{g'}\nu\Sigma_f^{g'->g}\Phi_{g'}$$

Critical value equation

$$-\mathrm{div}(D_g\nabla\Phi_g)+\sum_{g'}\Sigma_a^{g'->g}\Phi_{g'} = \sum_{g'}\nu\Sigma_f^{g'->g}\Phi_{g'}-\frac{\tau}{v_g}\Phi_g,\ \text{for all } g$$

MODEL (Allaire-Capdebosc, 1998)

$$-\mathrm{div}(\sum_{g'}D_{g'->g}\nabla\Phi_{g'}) + \sum_{g'}\Sigma_a^{g'->g}\Phi_{g'} = \frac{1}{k_{eff}}\sum_{g'}\nu\Sigma_f^{g'->g}\Phi_{g'}, \tag{7}$$

with $\Phi_g|_{\partial K} = 0$.

Mathematical result on the multi-group problem

Hypotheses: $D_{g'->g} = D_g$ (block-diagonal) and $D_g(x).\xi.\xi \geq \alpha\xi.\xi$ (coerciveness),
Exists $C > 0$ such that $\Sigma_a^{g'->g} \leq 0$ when $g' \neq g$, $\Sigma_a^{g->g} \geq C$,
$-\Sigma_a^{g'->g} \geq C$ for $0 < |g' - g| \leq 1$,
$\Sigma_f^{g'->g} \geq 0$ for all g, g', $\Sigma_f^{1->K} \geq C$,
$\sum_{g'=1}^K \Sigma_a^{g'->g} \geq C\nu \sum_{g'=1}^K \Sigma_f^{g'->g}$.

Proposition

Problem (7) has at least one, and at most a countable number of eigenvalues. The first eigenvalue is real and simple, and a corresponding eigenvector can be chosen with all positive components.

Habetler-Martino 1958.

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Homogeneization result: Allaire-Bal: transport equation

Equation (4):

$$\begin{aligned} v.\nabla_x \varphi + \Sigma_a(x,v)\varphi &= \frac{1}{k_{eff}} \int_V \nu \Sigma_f(x,v',v)\varphi(x,v')dv' \\ &+ \int_V f(x,v',v)\varphi(x,v')dv' \end{aligned}$$

Σ_a: absorbing, f scattering, Σ_f: fission cross-sections. k_{eff}^{-1} smallest eigenvalue, φ associated eigenvector.
Assumption: K is $\epsilon-$periodic. $\Sigma^\epsilon_{f,a}(x,v) := \Sigma_{f,a}(\frac{x}{\epsilon},v)$, $f^\epsilon(x,v',v) = f(\frac{x}{\epsilon},v',v)$.

$$\begin{aligned} v.\nabla_x \varphi + \Sigma^\epsilon_a(x,v)\varphi &= \lambda^\epsilon \int_V \nu \Sigma^\epsilon_f(x,v',v)\varphi(x,v')dv' \\ &+ \int_V f^\epsilon(x,v',v)\varphi(x,v')dv' \end{aligned}$$

Hypotheses and general result (Dautray-Lions)

- Velocity space is compact, K is compact.
- $\Sigma_f(x,v',v) \geq C$,
- $\Sigma_a(x,v) - \int_V f(x,v',v)dv' \geq C$,

$\Sigma_a(x,v) - \int_V f(x,v,v')dv' \geq C$,
Introduce $W^2(\mathcal{O} \times V) = \{u \in L^2(\mathcal{O} \times V), v.\nabla_x u \in L^2(\mathcal{O} \times V)\}$.

Proposition

The problem has at most a countable number of eigenvalues. The eigenvalue of smallest modulus is real positive, associated with a positive eigenvector.

The cell problem (notations)

Let λ_∞, ψ the first eigenvalue-eigenvector of the cell problem (periodicity condition)

$$v.\nabla_x \psi + \Sigma_a(y,v)\psi = \int_V [\lambda_\infty \nu \Sigma_f(y,v',v) + f(y,v',v)]\psi(x,v')dv'$$

Let ψ^* solve (periodicity condition)

$$-v.\nabla_x \psi^* + \Sigma_a(y,v)\psi^* = \int_V [\lambda_\infty \nu \Sigma_f(y,v,v') + f(y,v,v')]\psi^*(x,v')dv'$$

Assume the drift flux is zero

$$\int_{\text{cell}\times V} v\psi(y,v).\psi^*(y,v)dv = 0.$$

The general problem: theorem (Allaire-Bal)

Let $\bar\Sigma$ given by

$$\bar\Sigma = \int_{\text{cell}\times V^2} \Sigma_f(y,v',v)\psi^*(y,v')\psi(y,v)dydvdv'.$$

Let D identified as $D_{ij} = -\int_{\text{cell}\times V} v_j\psi(y,v)\psi^*(y,v)\theta^i(y,v)dydv$.

Let θ^i solves a cell problem

$$v.\nabla\theta^i + Q(\theta^i) = -v_i,$$

where Q is a scattering operator constructed from the cell problem.

Theorem

Let λ_k^ϵ be the $k-$th eigenvalue of (4) in the periodic case. Denote by φ_k^ϵ a normalized eigenvector.

1. *This eigenvalue satisfies $\lambda_k^\epsilon = \lambda_\infty + \epsilon^2[\nu^k + o(1)]$*
2. *The associated eigenvector satisfies $\varphi_k^\epsilon(x,v) \simeq \psi(\frac{x}{\epsilon},v)u^k(x)$,*
3. *The couple (u^k,ν^k) satisfies*

$$-div(D\nabla u^k) = \nu^k\bar\Sigma u^k(\Omega), u^k|_{\partial\Omega} = 0.$$

For the model with diffusion (Allaire-Capdeboscq)

- One group: already mentioned,
- multi-group

$$R\partial_t\varphi - \mathrm{div}(A(x)\nabla\varphi) + \Sigma_a\varphi = \nu\Sigma_f\varphi.$$

$\varphi : \mathbf{R}^3 \times \mathbf{R}_+ \to \mathbf{R}^K$ (K groups of energy),
R diagonal matrix of all v_g^{-1},
A is a $3 \times 3 \times K$ matrix (block-diagonal), symmetric in 3×3,
Σ_a, Σ_f are $K \times K$ matrices,
Separation of variables:

$$-\mathrm{div}(A(x)\nabla\varphi) + \Sigma_a\varphi = (\nu\Sigma_f - \tau R)\varphi,$$

Replaced by

$$-\mathrm{div}(A(x)\nabla\varphi) + \Sigma_a\varphi = \frac{\nu\Sigma_f}{k_{eff}}\varphi,$$

with same solution if $k_{eff} = 1$ or $\tau = 0$ equivalently.

Rescaled problem and theorem (Allaire-Capdebosc)

$$-\epsilon^2 \text{div}(A(\frac{x}{\epsilon})\nabla\varphi^\epsilon) + \Sigma_a(\frac{x}{\epsilon})\varphi^\epsilon = \mu^\epsilon \nu \Sigma_f(\frac{x}{\epsilon})\varphi^\epsilon, \tag{8}$$

A, Σ_a, Σ_f are periodic functions on $[0,1]^3$, system with K equations (number of groups), Σ_a and Σ_f are $K \times K$ matrices, and φ^ϵ is a vector of K components.

Hyp: A is a block-diagonal tensor (coupling only with 0th order terms).

Theorem

Let μ^∞, ψ the smallest eigenvalue of the cell problem, and ψ an associated eigenvector. Let $\mu^{\epsilon,m}$ the $m-$th eigenvalue of (8), and $\varphi^{\epsilon,m}$ and associated eigenvector. Then one has

$$\varphi^{\epsilon,m}(x) = u^m(x)\psi(\tfrac{x}{\epsilon}), \mu^{\epsilon,m} = \mu^\infty + \epsilon^2\nu^m + o(\epsilon^2),$$

where ν^m is the $m-$th eigenvalue and u^m an associated scalar eigenvector for the problem with Dirichlet boundary conditions in Ω

$$-div(\bar{D}\nabla u) = \nu\bar{\Sigma}u.$$

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Systems of equations (I)

Neutronics hypothesis: One group. Transport → diffusion:

$$\begin{cases} \frac{1}{V}\partial_t\varphi - \mathrm{div}[D(H)\nabla\varphi] + [\Sigma_a(H) - \nu\Sigma_f(H)(1-\beta)]\varphi - \lambda c = 0, \\ \partial_t c + \lambda c - \nu\Sigma_f(H)\beta\varphi = 0. \end{cases} \tag{9}$$

Thanks to E.Jamelot and F. Madiot, c converges rapidly to its equilibrium ($\frac{\lambda V}{L} \simeq 10^{-2}$), hence $\lambda c = \nu\Sigma_f(H)\beta\varphi$ and

$$\frac{1}{V}\partial_t\varphi - \mathrm{div}[D(H)\nabla\varphi] + [\Sigma_a(H) - \frac{\nu\Sigma_f(H)}{k_{eff}}]\varphi = 0. \tag{10}$$

Seek k_{eff}: multiplication factor. Ideal: $k_{eff} = 1$.
Boundary conditions are needed for φ.
Natural ones: $\varphi \geq 0$ on the domain and $\varphi|_{\partial\Omega} = 0$.

Systèmes (III): hydrodynamics equations

($\rho, \vec{u}, E$ usual unknowns of density, velocity, energy (that is enthalpy or temperature), p given by an equation of state

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \vec{u}) = 0, & \text{(a)} \\ \partial_t(\rho \vec{u}) + \mathrm{div}(\rho \vec{u} \otimes \vec{u}) + \nabla[p(\rho, H)] = \rho \vec{g}, & \text{(b)} \\ \partial_t(\rho E) + \mathrm{div}(\rho \vec{u} E) = \mathcal{S}_{H,\varphi}(t,x) + \rho \vec{g}.\vec{u} & \text{(c)} \end{cases} \tag{11}$$

Source term of energy: $\mathcal{S}_{H,\varphi}(t,x) := \mathbb{E}\Sigma_f(H)\varphi(t,x)$, associated with the power of energy produced $\mathcal{P}(t) = \int_\Omega \mathcal{S}_{H,\varphi}(t,x)dx$. The last equation rewrites

$$\partial_t(\rho E) + \nabla \cdot (\rho \vec{u} E) = \mathcal{P}(t) \times \frac{\mathbb{E}\Sigma_f(H)\varphi(t,x)}{\displaystyle\int_\Omega \mathbb{E}\Sigma_f(H)\varphi(t,x)dx} + \rho \vec{g}.\vec{u}. \tag{12}$$

Note that we assume no diffusion in the heat equation.

Systèmes (IV): the main assumption on the problem: low Mach number (Dellacherie 2011)

The model studied fits into a category called the **quasi-isobaric models**.

Indeed, in a **low Mach number approximation**, in (11)(b), the pressure (given by the e.o.s) is replaced by what S. D et al call a 'dynamic pressure' π, and in the last equation p is considered as **approximately constant**:

$$p(\rho, H) \to p + M^2\pi(t, x)$$

Equation of state $p = P_0 := p(\rho, H) \leftrightarrow \rho := \rho(H)$ thanks to $\partial_\rho p < 0$. The equation (11)(c) is replaced by the equation for the enthalpy $\rho(\partial_t H + \vec{u}.\nabla H) = \partial_t p + \vec{u} \cdot \nabla p + \mathbb{E}\Sigma_f(H)\varphi(t, x)$, hence

$$\rho(H)(\partial_t H + \vec{u}.\nabla H) = \mathbb{E}\Sigma_f(H)\varphi(t, x). \qquad (13)$$

1d set-up

Coefficients $D(h), \Sigma_a(h), \nu\Sigma_f(h)$ are given functions of h (h replaces H and is linked with T through $h = c_p\rho(h)T$).
Result: exact computation of **the unique solution** on $[0, L]$ of the coupled problem.
Finding k_{eff} such that the system has a stationary solution.
Boundary conditions $\varphi(0) = \varphi(L) = 0$, $h(0) = h_e$ and $h(L) = h_s$ given, $D_e = \rho(h(0))u(0)$ (incoming flux):
One too many conditions if we impose $k_{eff} = 1$*!!!*
Historical remark: the equation on φ only is a generalized eigenvalue problem, and has been solved, with success.

Coupled 1D model

System:

$$\begin{cases} -\frac{d}{dz}(D(h)\frac{d\varphi}{dz}) + \Sigma_a(h)\varphi = \frac{\nu\Sigma_f(h)}{k_{eff}}\varphi \\ \frac{d}{dz}(\rho(h(z))u(z)h(z)) = \mathbb{E}\nu\Sigma_f(h)\varphi(z) \\ \frac{d}{dz}(\rho(h(z))u(z)) = 0 \end{cases} \tag{14}$$

Non linear system of ODEs of order three with one parameter and four boundary conditions.

Auxiliary functions

Define X et Y the two C^2 vanishing at h_e and h_s functions such that

$$(D(h)\nu\Sigma_f(h)X'(h))' = \frac{\Sigma_a(h)}{\nu\Sigma_f(h)}, (D(h)\nu\Sigma_f(h)Y'(h))' = 1. \quad (15)$$

Remark: if $\frac{\Sigma_a(h)}{\nu\Sigma_f(h)}$ and $D(h)\nu\Sigma_f(h)$ are interpolation polynomials, X and Y are known explicitly.
Remark: with a change of unknown $x(h)$, reduce to solve $\frac{d^2}{dx^2}(X(h(x))) = (D\Sigma_a)(h(x))$, $\frac{d^2}{dx^2}(Y(h(x))) = (D\nu\Sigma_f)(h(x))$.

Main theoretical result

Theorem

Assume $\Sigma_a(h) \geq \Sigma_a > 0$, $\nu\Sigma_f(h) \geq \Sigma_* > 0$, $D(h) \geq D_0 > 0$, *continuous functions.*

1. *For any* L*, the equation*

$$\int_{h_e}^{h_s} \frac{dh}{\nu\Sigma_f(h)\sqrt{2(X(h) - \frac{1}{k}Y(h))}} = L \tag{16}$$

   has a unique solution, denoted by k_{eff}*.*

2. *Let* h *be the solution of the implicit equality* $\int_{h_e}^{h} \frac{dh}{\nu\Sigma_f(h)\sqrt{2(X(h)-\frac{1}{k_{eff}}Y(h))}} = z$, $\varphi = h'$, $\rho(h(z))u(z) = D_e$. *The set* (φ, h, ρ, u) *is the unique solution of (14) satisfying the boundary conditions* $\varphi(0) = \varphi(L) = 0$, $h(0) = h_e$ *and* $h(L) = h_s$*.*

Calculus

Le débit D_e est donc constant, et après manipulations

$$-\frac{\mathbb{E}}{D_e}\frac{d}{dz}(D(h)\frac{d\varphi}{dh}) + \frac{\Sigma_a(h)}{\nu\Sigma_f(h)}\frac{dh}{dz} = \frac{1}{k_{eff}}\frac{dh}{dz},$$

que l'on intègre exactement ($S'(h) = \frac{\Sigma_a(h)}{\nu\Sigma_f(h)}$):

$$-\frac{\mathbb{E}}{D_e}D(h)\frac{d\varphi}{dz} + S(h(z)) = \frac{1}{k_{eff}}h + C_0.$$

En multipliant par $\frac{\varphi(z)}{D(h)} = \frac{D_e}{\mathbb{E}}\frac{h'(z)}{\nu\Sigma_f(h)D(h)}$ on obtient

$$X(h) - \frac{1}{k_{eff}}Y(h) = \frac{1}{2}(\frac{\mathbb{E}}{D_e})^2\varphi^2 = \frac{1}{2}\frac{(h')^2}{(\nu\Sigma_f(h))^2}.$$

Proof of item (i): Finding k_{eff}

As $\varphi \geq 0$, $h'(z) = \nu\Sigma_f(h)\sqrt{2(X(h) - \frac{1}{k_{eff}}Y(h))}$, with k_{eff} such that, for all $h \in [h_e, h_s]$, $X(h) - \frac{1}{k_{eff}}Y(h) \geq 0$. Necessarily, $k_{eff} \in (0, \min_{[h_e,h_s]} \frac{Y}{X})$. Equation for k_{eff}:

$$L = \int_{h_e}^{h_s} \frac{dh}{\nu\Sigma_f(h)\sqrt{2(X(h) - \frac{1}{k_{eff}}Y(h))}}.$$

The function $L(k_{eff})$ is strictly increasing, the limit of $\int_{h_e}^{h_s} \frac{dh}{\nu\Sigma_f(h)\sqrt{2(X(h) - \frac{1}{k_{eff}}Y(h))}}$ when k_{eff} converges to 0 is 0, the limit of $\int_{h_e}^{h_s} \frac{dh}{\nu\Sigma_f(h)\sqrt{2(X(h) - \frac{1}{k_{eff}}Y(h))}}$ when k_{eff} goes to $\min_{[h_e,h_s]}$ is $+\infty$, hence a unique solution to this equation.

This problem is nonlinear.

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Numerical integration

$\lambda := \frac{1}{k}$. $\psi_\lambda(h) = X(h) - \lambda Y(h)$. Equation (16) to solve:

$$\int_0^L \frac{h'}{\sqrt{2\psi_\lambda(h)}} = 1.$$

$L = 1$ for the sequel. Integration of the system:

Meshing $0 = z_0 < z_1 < ... < z_{N-1} < z_N = 1$,

Numerical integration of h (Crank-Nicolson):

$$\frac{h_{j+1} - h_j}{\Delta z_{j+\frac{1}{2}}} = \frac{\varphi_j + \varphi_{j+1}}{2} \quad \Delta z_{j+\frac{1}{2}} = z_{j+1} - z_j \tag{17}$$

$$\varphi_j = \sqrt{\psi_\lambda(h_j)}, \varphi_{j+1} = \sqrt{\psi_\lambda(h_{j+1})},$$

Equation

$$1 = \sum \Delta z_{j+\frac{1}{2}} \Leftrightarrow 2 \sum_{j=0}^{N-1} \frac{h_{j+1} - h_j}{\sqrt{\psi_\lambda(h_j)} + \sqrt{\psi_\lambda(h_{j+1})}} = 1. \tag{18}$$

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Explicit result

Model case $(\sigma_0, \sigma_{\frac{1}{2}}, \sigma_1) = (14.864, 11.148, 5.574)$ yields λ: $1.01722, 1.07373, 1.0043, 1.01991, 1, 01507$. Representations of φ are almost not distinguishable, though precision on λ is at most 10^{-2}.

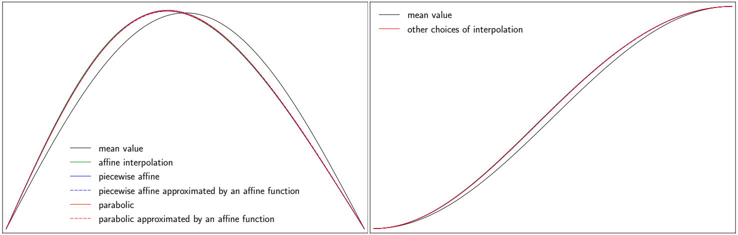

Figure: Functions $z \longmapsto \varphi(z)$ on the left and $z \longmapsto h(z)$ on the right.