CLOSURE OF THE STRONGLY-MAGNETIZED ELECTRON FLUID EQUATIONS IN THE ADIABATIC REGIME

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Abstract. We derive closure relations for a plasma fluid model, issued from the BGK equation for electrons in a strong magnetic-field. Our scaling of the BGK equation leads in the asymptotic limit $\varepsilon \to 0$ towards the adiabatic electron regime, where ε embodies the scaled Larmor radius as well as a low Mach number. In this regime the electron density adjusts instantaneously to perturbations of the electric potential via a Boltzmann relation, i.e. $n = ce^{\phi/T}$. The fluid closures are obtained in the small-ε regime from a Hilbert ansatz of the distribution function; the ensuing hierarchy of kinetic equations is solved exactly up to the desired order. Different closures emerge depending on the importance of the ratio ν/ω_c between the electron-electron collision frequency ν and the cyclotron frequency ω_c. Anisotropy in the transport coefficients is found when $\nu/\omega_c \ll 1$. Moreover, bringing into play the obtained closures we present a drift-fluid model valid for $\varepsilon \ll 1$, and identify the correct limit model as $\varepsilon \to 0$. The limit model can be used to avoid the crude approximations $c = const.$ and $T = const.$ in the electron Boltzmann relation, frequently used for plasma simulations.

Keywords: Plasma modelling, kinetic equations, fluid equations, Braginskii closure, Boltzmann electrons, finite Larmor radius effects, Hilbert expansion, asymptotic limit.

1. Introduction

The goal of this work is to derive closure relations for a plasma fluid model describing the evolution of electrons close to an adiabatic regime, the so-called Boltzmann regime. The model we are investigating is significant for electrons in strong magnetic fields, for example those in a Tokamak fusion plasma [11,19,22]. In such strongly-magnetized plasmas the electron dynamics occur on a variety of time scales. On the longest time scale of interest, i.e. the time scale of the macroscopic plasma drifts (guiding center motion), one observes a quasi-instantaneous adjustment of the electron density n to any kind of potential perturbation ϕ. This behaviour is commonly expressed via the Boltzmann relation:

$$n(t, \mathbf{x}) = c(t, \mathbf{x}_\perp) \exp\left(\frac{e\phi(t, \mathbf{x})}{k_B T(t, \mathbf{x}_\perp)}\right) , \quad \mathbf{x} = (\mathbf{x}_\perp, x_\parallel) \in \mathbb{R}^3 , \ \ t \in \mathbb{R} . \tag{1}$$

Here, k_B stands for the Boltzmann constant, e denotes the elementary charge, T is the electron temperature and c is a function to be determined. The coordinates $x_\parallel$ and $\mathbf{x}_\perp$ for $\mathbf{x} \in \mathbb{R}^3$ are defined with respect to the magnetic field $\mathbf{B}$, with unit vector $\mathbf{b} := \mathbf{B}/|\mathbf{B}|$, *i.e.*

$$(\mathbf{x}_\perp)_1 := [(\mathbb{I} - \mathbf{b} \otimes \mathbf{b})\mathbf{x}]_1 , \qquad (\mathbf{x}_\perp)_2 := [(\mathbb{I} - \mathbf{b} \otimes \mathbf{b})\mathbf{x}]_2 , \qquad x_\parallel := \mathbf{x} \cdot \mathbf{b} . \tag{2}$$

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The Boltzmann relation is frequently employed for electrons in numerical simulations of ion turbulence in strongly-magnetized plasmas [9, 12, 18, 24, 28], in order to avoid a time step restriction due to the fast electron dynamics. However, there are situations where this approximation is not adapted, for example in the sheath and pre-sheath regions of a Tokamak, such that alternative models have to be introduced. The question we address in this work concerns hence the validity of the electron Boltzmann relation and the derivation of more precise (fluid) models, leading in a certain asymptotic limit to this Boltzmann relation.

We consider the electrons in a regime characterized by a low Mach number, a strong magnetic field and a collision frequency small compared to the cyclotron frequency. On the kinetic level, the equation describing this specific situation is given by the following scaled Boltzmann-BGK-equation:

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_x f^\varepsilon - \frac{1}{\varepsilon}\left(\mathbf{E} + \frac{1}{\varepsilon}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f^\varepsilon = \frac{\eta(\varepsilon)}{\varepsilon^2}Q^\varepsilon_{BGK}(f^\varepsilon)\,. \tag{3}$$

Here, ε and $\eta(\varepsilon)$ are small parameters with $0 < \varepsilon \leq \eta(\varepsilon) < 1$. We suppose in this work that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are given, assuming also[1] $\mathbf{E} = -\nabla\phi$. The scaling, the meaning of ε and η as well as the detailed physical context is the subject of Section 2. The operator $Q^\varepsilon_{BGK}(\cdot)$ denotes a collision operator of BGK-type,

$$Q^\varepsilon_{BGK}(f^\varepsilon) := \mathcal{M}_{n^\varepsilon,\varepsilon\mathbf{u}^\varepsilon,T^\varepsilon} - f^\varepsilon\,, \qquad \mathcal{M}_{n^\varepsilon,\varepsilon\mathbf{u}^\varepsilon,T^\varepsilon} = \frac{n^\varepsilon}{(2\pi T^\varepsilon)^{3/2}}\exp\left(-\frac{|\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon|^2}{2T^\varepsilon}\right)\,, \tag{4}$$

where $\mathcal{M}_{n^\varepsilon,\varepsilon\mathbf{u}^\varepsilon,T^\varepsilon}$ stands for the Maxwellian function with the same moments as the distribution function f^ε, in particular we have

$$n^\varepsilon(t,\mathbf{x}) := \int_{\mathbb{R}^3} f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \tag{5}$$

$$\varepsilon\, n^\varepsilon(t,\mathbf{x})\mathbf{u}^\varepsilon(t,\mathbf{x}) := \int_{\mathbb{R}^3} \mathbf{v} f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \tag{6}$$

$$w^\varepsilon(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v} = \frac{3}{2}\,n^\varepsilon T^\varepsilon + \varepsilon^2\,\frac{n^\varepsilon|\mathbf{u}^\varepsilon|^2}{2}\,, \tag{7}$$

$$\frac{3}{2}\,n^\varepsilon(t,\mathbf{x})T^\varepsilon(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon|^2 f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,. \tag{8}$$

The kinetic model (3) is very precise for describing the physical situation we are interested in, however it contains too much information and has some inherent difficulties, preventing its use for numerical simulations. Indeed, it is a 6D model in phase-space and furthermore contains multiple scales, represented by the small perturbation parameters $\varepsilon \ll 1$ and $\eta < 1$ (which can be ε-dependent). The idea is now to find approximate models of the BGK equation (3), valid in the small $\varepsilon \ll 1$ regime and better suited for numerical simulations, for example a closed set of fluid equations. Ideally, the asymptotic limit $\varepsilon \to 0$ of these

[1] Remark that we shall omit the index "x" in the Nabla-operator ∇ if it is clear that we deal with space derivatives.

approximate models should lead to the Boltzmann relation (1), motivating thus their use in regions where this relation is not perfectly valid.

The fluid model associated to the kinetic equation (3) is obtained by taking the moments of the Boltzmann equation and reads

$$\left\{\begin{aligned} &\partial_t n^\varepsilon + \nabla\cdot(n^\varepsilon \mathbf{u}^\varepsilon) = 0\,,\\ &\partial_t(n^\varepsilon \mathbf{u}^\varepsilon) + \nabla\cdot(n^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \frac{1}{\varepsilon^2}\nabla\cdot\mathbb{P}^\varepsilon + \frac{1}{\varepsilon^2}\, n^\varepsilon(\mathbf{E} + \mathbf{u}^\varepsilon\times\mathbf{B}) = 0\,,\\ &\partial_t w^\varepsilon + \nabla\cdot(w^\varepsilon\,\mathbf{u}^\varepsilon + \mathbb{P}^\varepsilon\cdot\mathbf{u}^\varepsilon) + \frac{1}{\varepsilon}\nabla\cdot\mathbf{q}^\varepsilon + n^\varepsilon\mathbf{E}\cdot\mathbf{u}^\varepsilon = 0\,. \end{aligned}\right. \tag{9}$$

The collision operator $Q^\varepsilon_{BGK}(\cdot)$ conserves mass, momentum and energy and therefore does not appear in the moment equations (9). The fluid equations are not closed because the stress tensor $\mathbb{P}^\varepsilon$ and the heat flux $\mathbf{q}^\varepsilon$ depend on the full distribution function f^ε via

$$\mathbb{P}^\varepsilon(t,\mathbf{x}) := \int_{\mathbb{R}^3}(\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon)\otimes(\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon) f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \tag{10}$$

$$\mathbf{q}^\varepsilon(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^3}|\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon|^2(\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon) f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,. \tag{11}$$

Finding adequate closure relations for the stress tensor $\mathbb{P}^\varepsilon$ and the heat flux $\mathbf{q}^\varepsilon$ means expressing them in terms of the fluid variables n^ε, $\mathbf{u}^\varepsilon$ and T^ε. A systematic way to obtain these asymptotic closure relations is to expand the distribution function f^ε via a Hilbert ansatz, truncate this expansion at a limited number of terms, and use the approximated distribution function to evaluate the two moments (10)-(11). The thus obtained set of closed fluid equations provides an accurate approximation of the kinetic model (3) in the case $\varepsilon \ll 1$, which turns out to lead in the asymptotic limit $\varepsilon \to 0$ towards the electron Boltzmann relation (1), as will be demonstrated in what follows.

The closure relations for $\mathbb{P}^\varepsilon$ and $\mathbf{q}^\varepsilon$ depend on the choice of the parameter η, symbolizing the importance of electron-electron collisions in the plasma. Three cases are studied in this paper:

$$(i)\quad \eta = const.\,,\qquad (ii)\quad \eta=\sqrt{\varepsilon}\,,\qquad (iii)\quad \eta=\varepsilon\,. \tag{12}$$

Case (i) corresponds to the so-called "general drift ordering", which is frequently encountered in Plasma Physics literature, see for example [10,23,29], and has also been studied in the Mathematics community [1,13,14]. Case (ii) is commonly referred to as the collisional regime of neoclassical transport theory, or Pfirsch-Schlüter regime [21,26]. The authors are unaware of existing closures for the case (iii) so far. An asymptotic study is carried out for a similar case (however shorter time-scales) on the kinetic level in [2,7]. Other works on asymptotic regimes for strongly magnetized plasmas where collisions are taken into account are mentioned here for completeness [5,6,20]. The collision-less case ($\eta = 0$) has been studied quite extensively, for example in [3,4,16,17].

Let us briefly outline the tasks we wish to accomplish in this work:

a) Identify the right scaling (choice of space-, time- and velocity- scales, physical regime) of the kinetic electron equation that leads in an asymptotic limit $\varepsilon \to 0$ towards the Boltzmann relation (1) (c.f. Section 2).

b) Derive closures for the fluid equations (9) for three different strengths of the collision operator Q^{ε}_{BGK}, represented in (12) via the parameter η (c.f. Sections 4-6). Our aim is to confirm fluid models appearing in the plasma physics literature by means of the Hilbert expansion technique, and render them accessible for a mathematics-oriented community, for future studies (c.f. Section 3.3).

c) Find the well-posed limit model corresponding to the kinetic equation (3) in the asymptotics $\varepsilon \to 0$ (c.f. Section 3.4).

d) Prepare the foundation (reference work) for future numerical studies of strongly-magnetized plasmas within our group. In particular, asymptotic-preserving schemes dealing with the asymptotic limit either from the kinetic level to the Boltzmann regime (1D simulations) or from the derived approximated fluid models to the Boltzmann relation (3D simulations) are in progress.

The paper is organized as follows: In Section 2 we shall present the physical context of this work and introduce the scaling leading to (3). Section 3 summarizes the main results of this work and comments on them. Sections 4, 5 and 6 contain finally the three different truncation procedures related to the cases (12) and leading to closure relations for (9)-(11). An Appendix assembles some cumbersome computations.

2. Physical context and scaling

The Boltzmann equation for electrons in an electromagnetic field with BGK collision term reads

$$\partial_t f + \mathbf{v} \cdot \nabla_x f - \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = \nu(\mathcal{M}_{n,\mathbf{u},T} - f)\,. \tag{13a}$$

Here, e stands for the (positive) elementary charge, m is the electron mass and $\nu > 0$ denotes the collision frequency. The Maxwellian $\mathcal{M}_{n,\mathbf{u},T}$ corresponding to f is given by

$$\mathcal{M}_{n,\mathbf{u},T}(t,\mathbf{x},\mathbf{v}) := n(t,\mathbf{x}) \left(\frac{m}{2\pi k_B T(t,\mathbf{x})} \right)^{3/2} \exp\left(-\frac{m|\mathbf{v} - \mathbf{u}(t,\mathbf{x})|^2}{2k_B T(t,\mathbf{x})} \right), \tag{13b}$$

where k_B denotes the Boltzmann constant and n, $\mathbf{u}$ and T are related to the first three moments of the distribution function $f(t,\mathbf{x},\mathbf{v})$, via

$$n(t,\mathbf{x}) := \int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}\,, \tag{13c}$$

$$n(t,\mathbf{x})\,\mathbf{u}(t,\mathbf{x}) := \int_{\mathbb{R}^3} \mathbf{v} f(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}\,, \tag{13d}$$

$$\frac{3}{2}\, n(t,\mathbf{x})\, k_B\, T(t,\mathbf{x}) := \frac{m}{2}\int_{\mathbb{R}^3} |\mathbf{v}-\mathbf{u}(t,\mathbf{x})|^2 f(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}\,. \tag{13e}$$

It is easily checked that the energy w satisfies

$$w(t,\mathbf{x}) := \frac{m}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f(t,\mathbf{x},\mathbf{v}) d\mathbf{v} = \frac{3}{2} n k_B T + n\,\frac{m|\mathbf{u}|^2}{2}\,. \tag{13f}$$

We shall now write the Boltzmann equation (13a) in dimensionless form. This procedure permits to identify small parameters, which separate the different time scales and thus help in describing different plasma regimes. We employ the notation $n = \hat{n}\, n'$, where $\hat{n}$ is a constant denoting the density scale (characteristic density of the plasma under consideration) and $n' = n'(t,\mathbf{x})$ is of order one. The same is done for the other unknowns and for the independent variables: $t = \hat{t}\, t'$, $\mathbf{x} = \hat{x}\,\mathbf{x}'$ and $\mathbf{v} = \hat{v}\,\mathbf{v}'$, where $\hat{t}$ denotes the observation time scale, $\hat{x}$ the characteristic space scale and $\hat{v}$ the microscopic velocity scale. The Boltzmann equation (13a) is thus written in the dimensionless variables as follows:

$$\partial_{t'} f' + \frac{\hat{v}\hat{t}}{\hat{x}}\, \mathbf{v}'\cdot\nabla_{x'} f' - \frac{e\hat{B}\hat{t}}{m}\left(\frac{\hat{E}}{\hat{v}\hat{B}}\,\mathbf{E}' + \mathbf{v}'\times\mathbf{B}'\right)\cdot\nabla_{v'} f' = \nu\hat{t}\left[\frac{\hat{n}}{\hat{f}}\left(\frac{m}{k_B\hat{T}}\right)^{3/2}\mathcal{M}' - f'\right], \tag{14}$$

where $\mathcal{M}'$ is given by

$$\mathcal{M}' = n'\left(\frac{1}{2\pi T'}\right)^{3/2} \exp\left(-\frac{m\hat{v}^2}{k_B\hat{T}}\,\frac{|\mathbf{v}-\frac{\hat{u}}{\hat{v}}\mathbf{u}|^2}{2T'}\right)\,. \tag{15}$$

We denote by L a macroscopic length scale, for example the large radius of a Tokamak, and introduce the following relevant quantities:

$$v_{th} := \sqrt{\frac{k_B\hat{T}}{m}}\,,\qquad \omega_c := \frac{e\hat{B}}{m}\,,\qquad \rho_{th} := \frac{v_{th}}{\omega_c}\,. \tag{16}$$

Here, v_{th} stands for the thermal velocity of the electrons, ω_c is their characteristic cyclotron frequency and ρ_{th} denotes the electron Larmor radius related to their thermal velocity. Our choice of scales for the independent variables is the following:

$$\hat{x} := L\,,\qquad \hat{t} := \frac{L^2}{v_{th}\rho_{th}}\,,\qquad \hat{v} := v_{th}\,. \tag{17}$$

The time scale $\hat{t}$ is also known as "Bohm time". We now introduce the small parameter ε as the ratio between the electron Larmor radius and the macroscopic length scale L,

$$\varepsilon := \frac{\rho_{th}}{L} \ll 1\,, \tag{18}$$

signifying that we are interested in strongly magnetized plasmas. Remark that the chosen Bohm time scale is much longer than the time scale ω_c^{-1} of the cyclotron waves, as well as the time scale $\omega_{th}^{-1} := L/v_{th}$ of the sound waves, as evidenced by the relations

$$\hat{t}\omega_c = \frac{1}{\varepsilon^2}\,, \qquad \hat{t}\omega_{th} = \frac{1}{\varepsilon}\,. \tag{19}$$

Indeed, the time scale $\hat{t}$ is well-suited for observing the macroscopic plasma drifts perpendicular to the magnetic field lines. We now relate the characteristic macroscopic velocity $\hat{u}$ to the chosen space and time scales, respectively, leading to

$$\hat{u} := \frac{\hat{x}}{\hat{t}} \qquad \Longrightarrow \qquad \frac{\hat{u}}{v_{th}} = \varepsilon \ll 1 \qquad \text{(low Mach regime)}\,. \tag{20}$$

It becomes clear that the flow velocities we aim to observe are small compared to the thermal velocity; hence we are interested in the subsonic regime, characterized by a small Mach number. In order to complete the scaling, we suppose

$$\hat{f} = \frac{\hat{n}}{\hat{v}^3}\,, \qquad \hat{E} = \hat{u}\hat{B}\,, \qquad k_B\hat{T} = e\hat{\phi}\,. \tag{21}$$

Inserting the assumptions (20)-(21) into the Boltzmann equation (14) yields the scaled equation (3) with the scaled collision operator (4), and

$$\eta := \frac{\nu}{\omega_c}\,. \tag{22}$$

The parameter η symbolizes the opposition of two competing "forces" in the magnetized plasma: on one hand, the strong magnetic field tends to induce a strong anisotropy in the plasma dynamics, the transport being much faster in the parallel (toroidal) direction than in the perpendicular (poloidal) direction. On the other hand, collisions tend to homogenize the plasma, breaking-down the one-directional effect of the **B**-field. Case (i) in Eq. (12) ($\eta = cst.$) corresponds to a situation where these two antagonist forces are of the same order of magnitude, leading in the regime $\varepsilon \ll 1$ to a more or less isotropic situation. Conversely, cases (ii) ($\eta = \sqrt{\varepsilon}$) and (iii) ($\eta = \varepsilon$) in (12) correspond to situations where the collision frequency ν is much smaller than the cyclotron frequency ω_c; thus, the electrons perform many gyrations around the magnetic field lines between two subsequent collisions. As a result, a strong anisotropy occurs in the macroscopic transport coefficients, such as the heat diffusivity, as will be seen from the closure relations below.

3. Notation and main results

For clarity reasons let us introduce here some notation as well as the main results obtained in this work. The detailed calculations leading to the closure relations are presented in the sections 4-6.

3.1. **Notation.** We introduce the scalar pressure p^ε, the heat flux h_i^ε and the Poisson brackets $\{\cdot,\cdot\}_{i,j}$ via

$$p^\varepsilon := n^\varepsilon T^\varepsilon\,, \qquad h_i^\varepsilon := -\frac{5}{2} n^\varepsilon T^\varepsilon \partial_i T^\varepsilon\,, \qquad \mathbf{h} := \begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix}\,, \tag{23a}$$

$$\{n^\varepsilon, T^\varepsilon\}_{i,j} := \partial_i n^\varepsilon \partial_j T^\varepsilon - \partial_j n^\varepsilon \partial_i T^\varepsilon\,, \qquad i,j \in \{x,y,z\}\,. \tag{23b}$$

Note that

$$T^\varepsilon \{n^\varepsilon, T^\varepsilon\}_{i,j} = -\frac{2}{5}\left(\partial_i h_j^\varepsilon - \partial_j h_i^\varepsilon\right). \tag{24}$$

Let us moreover define the rate-of-strain tensor $\mathbf{W}_u$ by

$$\mathbf{W}_u := \nabla \mathbf{u} + (\nabla \mathbf{u})^t - \frac{2}{3}(\nabla \cdot \mathbf{u})\,\mathbb{I}\,, \tag{25a}$$

and introduce the gyro-viscosity [22]

$$\Pi_\wedge^{u^\varepsilon} := \frac{p^\varepsilon}{2} \cdot \frac{1}{2}\left\{(\mathbf{b} \times \mathbf{W}_{u^\varepsilon}) \cdot (\mathbb{I} + 3\mathbf{b} \otimes \mathbf{b}) + [(\mathbf{b} \times \mathbf{W}_{u^\varepsilon}) \cdot (\mathbb{I} + 3\mathbf{b} \otimes \mathbf{b})]^t\right\}\,, \tag{25b}$$

as well as the parallel viscosity [22]

$$\Pi_{||}^{u^\varepsilon} := \frac{p^\varepsilon}{2}(\mathbf{b} \cdot \mathbf{W}_{u^\varepsilon} \cdot \mathbf{b})\,(\mathbb{I} - 3\mathbf{b} \otimes \mathbf{b})\,. \tag{25c}$$

In connection with the heat flux, we shall also use the following tensors

$$\begin{aligned} \mathbb{H}_\wedge^h &:= \frac{1}{5} \cdot \frac{1}{2}\left\{(\mathbf{b} \times \mathbf{W}_h) \cdot (\mathbb{I} + 3\mathbf{b} \otimes \mathbf{b}) + [(\mathbf{b} \times \mathbf{W}_h) \cdot (\mathbb{I} + 3\mathbf{b} \otimes \mathbf{b})]^t\right\}\,, \\ \mathbb{H}_{||}^h &:= \frac{1}{5}(\mathbf{b} \cdot \mathbf{W}_h \cdot \mathbf{b})\,(\mathbb{I} - 3\mathbf{b} \otimes \mathbf{b})\,, \end{aligned} \tag{25d}$$

where $\mathbf{W}_h$ is given by (25a) with $\mathbf{u}$ replaced by the vector $\mathbf{h}$. Furthermore, we denote

$$\mathbf{h}_{||} := (\mathbf{b} \otimes \mathbf{b})\mathbf{h}\,, \qquad \mathbf{h}_\perp := (\mathbb{I} - \mathbf{b} \otimes \mathbf{b})\mathbf{h}\,, \qquad \mathbf{h}^\perp := \mathbf{h} \times \mathbf{b}\,. \tag{25e}$$

In the derivation of the closure relations the $\mathbf{B}$-field is assumed to be constant and along the z-direction, *i.e.* $\mathbf{B} = \mathbf{e}_z$, $|\mathbf{B}| = 1$. It is sometimes helpful to pass to cylindrical coordinates $(v_r, v_\theta, v_z) \in \mathbb{R}^+ \times [0, 2\pi) \times \mathbb{R}$ in velocity space, defined via

$$v_x = v_r \cos v_\theta\,, \qquad v_y = v_r \sin v_\theta\,, \qquad v_z = v_z\,,$$

$$|v|^2 = v_r^2 + v_z^2\,, \qquad dv_x dv_y dv_z = v_r\, dv_r dv_\theta dv_z\,.$$

We denote $\mathring{f}^\varepsilon(v_r, v_\theta, v_z) := f^\varepsilon(v_r \cos v_\theta, v_r \sin v_\theta, v_z)$ and omit the circle when no confusion is possible. One has

$$\frac{\partial f^\varepsilon}{\partial v_x} = \cos v_\theta \frac{\partial \mathring{f}^\varepsilon}{\partial v_r} - \frac{\sin v_\theta}{v_r}\frac{\partial \mathring{f}^\varepsilon}{\partial v_\theta}\,, \qquad \frac{\partial f^\varepsilon}{\partial v_y} = \sin v_\theta \frac{\partial \mathring{f}^\varepsilon}{\partial v_r} + \frac{\cos v_\theta}{v_r}\frac{\partial \mathring{f}^\varepsilon}{\partial v_\theta}\,. \tag{26}$$

In these coordinates the gyro operator has a particularly simple form:

$$-(\mathbf{v} \times \mathbf{e}_z) \cdot \nabla_v \mathring{f}^\varepsilon = \frac{\partial \mathring{f}^\varepsilon}{\partial v_\theta}\,. \tag{27}$$

We will also need the average $\overline{a}$ of a function $a(v_\theta)$ along the coordinate v_θ, called the gyro-average, and its fluctuation $a' := a - \overline{a}$, defined by

$$\overline{a} := \frac{1}{2\pi}\int_0^{2\pi} a(v_\theta)\, dv_\theta \qquad \Longrightarrow \qquad -(\mathbf{v}\times\mathbf{e}_z)\cdot\nabla_v \overline{a} = 0\,, \quad \frac{1}{2\pi}\int_0^{2\pi} a'\, dv_\theta = 0\,. \tag{28}$$

3.2. **Closure results.** Let us present here in compact form the closure relations for the stress tensor $\mathbb{P}^\varepsilon$ and for the heat flux $\mathbf{q}^\varepsilon$ obtained in this work. Theses relations may be inserted into the fluid system (9) in order to obtain approximate models for the kinetic equation (3), valid in the regime $\varepsilon \ll 1$. The detailed calculations leading to those results are written in the Sections 4-6. Results are presented for a constant magnetic field $\mathbf{B} = \mathbf{e}_z$. The three different collisional regimes introduced in Eq. (12) yield the following closures:

- *Results from Section 4: case (i) $\eta = const$.* This case signifies that the magnetic field-force is of the same order of magnitude as the collision term, stronger than the order of the transport operator:

$$\mathbf{q}^{\varepsilon,\eta} = \frac{\varepsilon}{\eta}\begin{pmatrix} \frac{\eta^2}{1+\eta^2} & -\frac{\eta}{1+\eta^2} & 0 \\ \frac{\eta}{1+\eta^2} & \frac{\eta^2}{1+\eta^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{h} + \mathcal{O}_\eta(\varepsilon^3)\,, \qquad \mathbf{h} = -\frac{5}{2}\, nT\,\nabla T\,, \tag{29a}$$

$$\mathbb{P}^{\varepsilon,\eta} = p\mathbb{I} + \frac{\varepsilon^2}{\eta^2}\mathbb{H}_{\|}^{h_\|} + \frac{\varepsilon^2}{\eta}\Big(\Pi_{\|}^{u_\|} - \mathbb{H}_{\wedge}^{h_\|}\Big) + \varepsilon^2\Big(- \Pi_{\wedge}^{u} + \mathbb{H}_{\|}^{h_\perp} + \mathbb{H}_{\wedge}^{h^\perp}\Big) + \varepsilon^2\mathcal{O}(\eta) + \mathcal{O}_\eta(\varepsilon^3)\,. \tag{29b}$$

Recalling that $\mathbf{b} = (0,0,1)^t$, the traceless matrices used in expression (29b) are given by the definitions (25):

$$\mathbb{H}_{||}^{h_{||}} = \frac{4}{15}\partial_z h_z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\,, \tag{30}$$

$$\Pi_{||}^{u_\|} = \frac{2}{3} p\,\partial_z u_z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\,, \tag{31}$$

$$\mathbb{H}_{\wedge}^{h_{||}} = \frac{2}{5}\begin{pmatrix} 0 & 0 & -\partial_y h_z \\ 0 & 0 & \partial_x h_z \\ -\partial_y h_z & \partial_x h_z & 0 \end{pmatrix}\,, \tag{32}$$

$$\Pi_\wedge^u = \frac{p}{2}\begin{pmatrix} -\partial_x u_y - \partial_y u_x & \partial_x u_x - \partial_y u_y & -2(\partial_y u_z + \partial_z u_y) \\ \partial_x u_x - \partial_y u_y & \partial_x u_y + \partial_y u_x & 2(\partial_x u_z + \partial_z u_x) \\ -2(\partial_y u_z + \partial_z u_y) & 2(\partial_x u_z + \partial_z u_x) & 0 \end{pmatrix}, \tag{33}$$

$$\mathbb{H}_{||}^{h_\perp} = -\frac{2}{15}(\partial_x h_x + \partial_y h_y)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{34}$$

$$\mathbb{H}_\wedge^{h^\perp} = \frac{1}{5}\begin{pmatrix} \partial_x h_x - \partial_y h_y & \partial_x h_y + \partial_y h_x & 2\partial_z h_x \\ \partial_x h_y + \partial_y h_x & \partial_y h_y - \partial_x h_x & 2\partial_z h_y \\ 2\partial_z h_x & 2\partial_z h_y & 0 \end{pmatrix}. \tag{35}$$

The terms of order $\varepsilon^2\mathcal{O}(\eta)$ have been omitted for clarity in Eq. (29b), i.e. we performed an expansion of $\mathbb{P}^{\varepsilon,\eta}$ in powers of η. The full result for $\mathbb{P}^{\varepsilon,\eta}$ can be found in Section 4. The notation $\mathcal{O}_\eta(\varepsilon^3)$ in (29) underlines the fact that these terms may depend on η; the explicit dependence on η is not known, and one has thus to be careful when choosing small η-values. In the higher orders ε^p, $p \geq 3$, we expect coefficients of the form $(\varepsilon/\eta)^p$. This is a first hint that the case $\eta = \varepsilon$ describes a certain threshold in which fluid closures may not be readily obtained. Insight into the η-dependence of the terms $\mathcal{O}_\eta(\varepsilon^3)$ is gained from the study of case (*ii*), $\eta = \sqrt{\varepsilon}$, which is discussed next.

- *Results from Section 5: case (ii) $\eta = \sqrt{\varepsilon}$.* This case signifies that the collision term is of an intermediate order of magnitude between the magnetic field-force and the transport operator $\mathbf{v}\cdot\nabla_x f^\varepsilon - \mathbf{E}\cdot\nabla_v f^\varepsilon$:

$$\mathbf{q}^\varepsilon = \sqrt{\varepsilon}\begin{pmatrix} \varepsilon & -\sqrt{\varepsilon} & 0 \\ \sqrt{\varepsilon} & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{h} + \mathcal{O}(\varepsilon^2), \tag{36a}$$

$$\mathbb{P}^\varepsilon = p\mathbb{I} + \varepsilon\mathbb{H}_{||}^{h_{||}} + \varepsilon\sqrt{\varepsilon}\Big(\Pi_{||}^{u_{||}} - \mathbb{H}_\wedge^{h_{||}}\Big) + \varepsilon^2\Big(-\Pi_\wedge^u + \mathbb{H}_{||}^{h_\perp} + \mathbb{H}_\wedge^{h^\perp}\Big) + \mathcal{O}(\varepsilon^{5/2}). \tag{36b}$$

The anisotropy in the heat flux as well as in the stress tensor is now embodied by the single parameter ε. We remark that the errors in the closure (36) are of order $\mathcal{O}(\varepsilon^2)$ for the heat flux and $\mathcal{O}(\varepsilon^{5/2})$ for the stress tensor, respectively, and are thus larger (in ε) than in (29). However, the errors in (29) may become important when $\eta \to 0$. The closure (36) tells us more about these errors as $\eta = \sqrt{\varepsilon} \to 0$; on the other hand, case (*i*) gives more physical insight because it shows explicitly the dependence of the different terms on the collision frequency via the parameter η.

- *Results from Section 6: case (iii) $\eta = \varepsilon$.* This case seems to be a threshold case; it corresponds to a situation where the magnetic field-force is stronger than the transport operator and the collision term, which are at the same order of magnitude. We were able to establish the Boltzmann relation for $\varepsilon \to 0$ and to identify the limit-model corresponding to the kinetic equation (3), which reads

$$
(L)\begin{cases}
f_0 = \mathcal{M}_0 = \mathcal{M}_{n_0,0,T_0}\,, \\[2mm]
n_0(t,\mathbf{x}_\perp,z) = c(t,\mathbf{x}_\perp)\,\exp\left(\dfrac{\phi(t,\mathbf{x}_\perp,z)}{T_0(t,\mathbf{x}_\perp)}\right)\,, \\[2mm]
\partial_t n_0 + \mathbf{u}_E\cdot\nabla_\perp n_0 + \partial_z (nu_z)^*_0 = 0\,, \qquad \underline{(nu_z)^*_0 = 0}\,, \\[2mm]
\partial_t(\underline{n_0 T_0}) + \underline{\mathbf{u}_E\cdot\nabla_\perp(n_0T_0)} - \dfrac{2}{3}\,\underline{(nu_z)^*_0\,\partial_z\phi} = 0\,,
\end{cases}
\tag{37}
$$

where $\mathbf{x}_\perp = (x,y)\in\mathbb{R}^2$ and we used the notation (44) for the mean and the fluctuation of a function along the magnetic field lines. The limit model (L) is a system for the unknowns $(c,(nu_z)^*_0,T_0)$ and is discussed in more more detail in section (3.4). The calculation of higher-order closures however, as obtained in the previous two cases, seems to be an interesting but complicated problem which will be studied in future works.

Let us make the following additional remarks concerning the obtained results:

- Setting in (29) formally $\eta = \sqrt{\varepsilon}$ and developing the heat flux in powers of $\sqrt{\varepsilon}$ permits to obtain the closure (36). This is true only when Remark 5 of subsection 5.1 holds, i.e. for boundary conditions that imply $h_\parallel = 0 \Rightarrow \nabla_\parallel T = 0$. Otherwise, one finds additional terms in the stress tensor resp. heat flux in case (ii), which appear at the orders $\mathcal{O}(\varepsilon\sqrt{\varepsilon})$ resp. $\mathcal{O}(\varepsilon^2)$. Hence, the result (36) justifies the use of the "conventional" closure (29) in the regime $\sqrt{\varepsilon}\le\eta<1$, also called the Pfirsch-Schlüter regime.

- With respect to the fluid closure derived by Braginskii, well-established in plasma physics [27], we observe that:
  - the heat flux $\mathbf{q}^{\varepsilon,\eta}$ in (29a) is identical with the Braginskii result up to the order $\mathcal{O}(\varepsilon\eta)$,
  - the parallel viscosity in the stress tensor $\mathbb{P}^{\varepsilon,\eta}$ given in (29b) is $\Pi^{u_\parallel}_\parallel$, whereas in Braginksii's closure, one finds $\Pi^u_\parallel$,
  - the gyro-viscosity $\Pi^u_\wedge$ is identical to the Braginskii result,
  - the stresses $\mathbb{H}$ related to gradients in the heat flux are not taken into account by the Braginskii closure. They are characteristic for the employed scaling (drift ordering) and have first been discovered by Tsypin et al. [26],
  - the friction terms, usually found in literature, do not occur in our closure relations due to the fact that we consider only electron-electron collisions,
  - the coefficients recovered in our closures are not identical to Braginskii's coefficients, as we are using a BGK collision operator.

- The contributions of parallel and perpendicular flow velocity and heat flux in the stress tensors $\mathbb{P}^{\varepsilon,\eta}$ resp. $\mathbb{P}^\varepsilon$ appear at different orders in $\varepsilon^{1/2}$. At first appear terms in relation with parallel components of $\mathbf{u}$ and $\mathbf{h}$, and only later (at higher orders) arise the perpendicular contributions, which are apparently also independent of the collisional regime, thus, not depending on the parameter η in (29b).

3.3. **A drift-fluid model.** The fluid models with closures (29) or (36) are reduced models compared to the full kinetic equation (3), but they are still quite involved and difficult to solve numerically. Further approximations are often desired. For example, in the study of large-scale plasma instabilities in a Tokamak it is not necessary to resolve the fastest phenomena, such as plasma waves or cyclotron waves. Plasma waves can be eliminated by the quasi-neutrality assumption; cyclotron waves can be filtered out via the drift approximation, which we shall present in the following.

Our simplified model originates from the fluid system (9) with the closures (36); it is derived by employing the traditional drift approximation, expressing the perpendicular flow velocity $\mathbf{u}_\perp$, up to an error, as the sum of several plasma drifts. To be more precise, taking the cross product of the momentum conservation law with $\mathbf{B}$ yields

$$n\,\mathbf{u}_\perp = \frac{n\,\mathbf{E}\times\mathbf{B}}{|\mathbf{B}|^2} + \frac{\nabla p\times\mathbf{B}}{|\mathbf{B}|^2} + \mathcal{O}(\varepsilon) =: n\,\mathbf{u}_E + n\,\mathbf{u}_D + \mathcal{O}(\varepsilon)\,, \tag{38}$$

where $\mathbf{u}_E$ denotes the electric-field drift velocity and $\mathbf{u}_D$ stands for the diamagnetic drift velocity. For $\mathbf{B} = \mathbf{e}_z$ and $\mathbf{E} = -\nabla_x\phi$ one obtains hence

$$\nabla_\perp\cdot(n\mathbf{u}_\perp) = \mathbf{u}_E\cdot\nabla_\perp n + \mathcal{O}(\varepsilon) = -\{n,\phi\}_{x,y} + \mathcal{O}(\varepsilon)\,. \tag{39}$$

Remark that due to the fact that the equation for the parallel particle flow $(n\mathbf{u}_\parallel)$ contains an error of order $\mathcal{O}(\varepsilon^{1/2})$, the errors we introduce using the perpendicular drift approximation are irrelevant. Therefore, in the drift approximation one obtains the following simplified model for the three scalar unknowns $(n,(nu_z),p)$:

$$\begin{cases} \partial_t n + \mathbf{u}_E\cdot\nabla_\perp n + \partial_z(nu_z) = \mathcal{O}(\varepsilon)\,, \\[2ex] \partial_t(nu_z) + \mathbf{u}_E\cdot\nabla_\perp(n\,u_z) + \partial_z(nu_z^2) + \dfrac{1}{\varepsilon^2}(\partial_z p + nE_z) - \dfrac{1}{\varepsilon}\dfrac{8}{15}\partial_{zz}h_z \\ \qquad - \dfrac{1}{\sqrt{\varepsilon}}\dfrac{4}{3}\partial_z(p\,\partial_z u_z) - \nabla^\perp\cdot(p\,\partial_z\mathbf{u}_\perp) - \dfrac{2}{3}\partial_z\nabla_\perp\cdot\mathbf{h}_\perp = \mathcal{O}(\sqrt{\varepsilon})\,, \\[2ex] \partial_t p + \mathbf{u}_E\cdot\nabla_\perp p + \partial_z(p\,u_z) + \dfrac{2}{3}\,p\,\partial_z u_z + \dfrac{1}{\sqrt{\varepsilon}}\dfrac{2}{3}\partial_z h_z = \mathcal{O}(\sqrt{\varepsilon})\,, \end{cases} \tag{40}$$

where we expressed the energy conservation in terms of the pressure $p = nT$. These manipulations of the moment equation significantly reduce the computational burden, reducing the number of unknowns, and keeping nevertheless still enough physics for a closed fluid plasma description in the adiabatic regime.

A future work will be concerned with the numerical discretization of this drift-model (40), focusing on ε-independent accuracy and stability, by making use of so-called Asymptotic-Preserving techniques. Such schemes can be of particular interest for plasma simulations where the parameter ε varies considerably in the simulation domain.

3.4. **The Boltzmann regime.** The adiabatic regime of electrons, defined via the relation (1), is readily obtained from the drift-fluid model (40) in the limit $\varepsilon \to 0$. Setting formally $\varepsilon = 0$ yields from the momentum and energy conservation laws, respectively, the relations

$$\partial_z p_0 + n_0 E_z = 0\,, \qquad \partial_z h_{z,0} = -\frac{5}{2}\partial_z\left(n_0 T_0\, \partial_z T_0\right) = 0\,. \tag{41}$$

Remark that we indicate unknowns by a zero in the asymptotic limit. On one hand, as $\varepsilon \to 0$, the pressure-gradient and electrostatic forces are in balance; on the other hand the heat-diffusivity along the magnetic field lines vanishes. It follows (c.f. remark 5 in Section 5.1) that

$$\partial_z T_0 = 0 \qquad \text{for periodic boundary conditions in } z\,, \tag{42}$$

and hence the Boltzmann relation

$$n_0(t,\mathbf{x}_\perp,z) = c(t,\mathbf{x}_\perp)\,\exp\left(\frac{\phi(t,\mathbf{x}_\perp,z)}{T_0(t,\mathbf{x}_\perp)}\right)\,, \quad \mathbf{x}_\perp = (x,y)\in\mathbb{R}^2\,,\ \ z\in\mathbb{R}\,,\ \ t\in\mathbb{R}\,, \tag{43}$$

with the functions $c(t,\mathbf{x}_\perp)$ and $T_0(t,\mathbf{x}_\perp)$ still to be determined. The physical meaning of this Boltzmann relation is that the electrons, being very light and hence mobile, accelerate to high energies very quickly, leaving behind them a region of large ion charges, which creates a retarding electric field. An equilibrium is hence achieved between the two antagonist forces.

It remains to identify the correct equations for the functions $c(t,\mathbf{x}_\perp)$ and $T_0(t,\mathbf{x}_\perp)$, which emerge in the asymptotic limit $\varepsilon \to 0$ from the drift-fluid model (40). For this, we denote by $\underline{a}$ and a^*, respectively, the mean and the fluctuation of a function a along the z-direction,

$$\underline{a}(x,y) := \frac{1}{L_z}\int_0^{L_z} a(x,y,z)\,dz\,, \qquad a^* := a - \underline{a}\,. \tag{44}$$

The evolution of the "constant" of integration $c(t,\mathbf{x}_\perp)$ is obtained by taking the mean in the particle conservation law,

$$\partial_t \underline{n_0} + \underline{\mathbf{u}_E\cdot\nabla_\perp n_0} = 0\,. \tag{45}$$

Clearly, for given T_0 this equation can be written as a transport equation for the function $c(t,\mathbf{x}_\perp)$. The evolution of the temperature $T_0(t,\mathbf{x}_\perp)$ is then obtained by averaging the energy equation, *i.e.*

$$\partial_t(\underline{n_0 T_0}) + \underline{\mathbf{u}_E\cdot\nabla_\perp(n_0 T_0)} + \frac{2}{3}\underline{n_0 T_0\,\partial_z u_{z,0}} = 0\,. \tag{46}$$

The term containing $u_{z,0}$ can be rewritten as

$$
\begin{aligned}
\underline{n_0 T_0\,\partial_z u_{z,0}} &= T_0\,\underline{n_0 \partial_z u_{z,0}} = -T_0\,\underline{u_{z,0}\,\partial_z n_0} = -T_0\,\underline{(nu_z)_0\,\partial_z(\ln n_0)} \\
&= -T_0\,\underline{[\underline{(nu_z)_0} + (nu_z)_0^*]\,\partial_z(\ln n_0)} = -T_0\,\underline{(nu_z)_0^*\,\partial_z(\ln n_0)} \\
&= -\underline{(nu_z)_0^*\,\partial_z \phi}\,,
\end{aligned}
\tag{47}
$$

where we used the periodicity in z and integration by parts. Therefore, supposing that n_0 and $(nu_z)_0^*$ are given, the temperature T_0 can be computed from (46). We now use the remaining information from the particle continuity equation to compute $(nu_z)_0^*$. Indeed, subtracting the average (45) from the particle conservation law yields

$$
\partial_z (nu_z)_0^* = -\partial_t n_0^* - (\mathbf{u}_E \cdot \nabla_\perp n_0)^*\,, \qquad \underline{(nu_z)_0^* = 0}\,. \tag{48}
$$

For given n_0 this last equation is readily solved by integrating with respect to z; the constant of integration is then determined via the integral constraint $\underline{(nu_z)_0^* = 0}$, which guarantees the uniqueness of the solution $(nu_z)_0^*$. Summarizing, we obtained the following limit system for the unknowns $(n_0, (nu_z)_0^*, T_0)$:

$$
(L)
\begin{cases}
\partial_t n_0 + \mathbf{u}_E \cdot \nabla_\perp n_0 + \partial_z (nu_z)_0^* = 0\,, \qquad \underline{(nu_z)_0^* = 0}\,, \\
\partial_z (n_0 T_0) + n_0 E_z = 0\,, \\
\partial_z \left(n_0 T_0\,\partial_z T_0\right) = 0\,, \\
\partial_t (\underline{n_0 T_0}) + \underline{\mathbf{u}_E \cdot \nabla_\perp (n_0 T_0)} - \dfrac{2}{3}\,\underline{(nu_z)_0^*\,\partial_z \phi} = 0\,.
\end{cases}
\tag{49}
$$

4. Closure of the fluid equations for $\eta = const.$

The first closure we study is derived from the kinetic equation (3) with $\eta = const.$, here repeated for clarity,

$$
\partial_t f^\varepsilon + \frac{1}{\varepsilon}\mathbf{v} \cdot \nabla_x f^\varepsilon - \frac{1}{\varepsilon}\left(\mathbf{E} + \frac{1}{\varepsilon}\mathbf{v} \times \mathbf{B}\right) \cdot \nabla_v f^\varepsilon = \frac{\eta}{\varepsilon^2} Q^\varepsilon_{BGK}(f^\varepsilon)\,. \tag{50}
$$

In this case, the magnetic field-force and the collision term are considered of the same order of magnitude, which seems not to be the right scaling for strongly magnetized fusion plasmas, as no particular difference in magnitude is observed for the dynamics in the parallel resp. perpendicular directions, as will be seen below. However a better description of such strongly-magnetized plasmas can be obtained by considering afterwards also small η-regimes.

In order to solve this equation approximately for $\varepsilon \ll 1$ and constant η, we seek for a solution f^ε under the form of a power series in ε, known as the Hilbert expansion, *i.e.*

$$
f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \mathcal{O}(\varepsilon^3)\,. \tag{51}
$$

Injecting this ansatz into (50) and equating the terms of equal power in ε leads to the following hierarchy of equations:

$$-(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_0 = \eta(\mathcal{M}_0 - f_0)\,, \tag{52a}$$

$$\mathbf{v}\cdot\nabla_x f_0 - \mathbf{E}\cdot\nabla_v f_0 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_1 = \eta(\mathcal{M}_1 - f_1)\,, \tag{52b}$$

$$\partial_t f_0 + \mathbf{v}\cdot\nabla_x f_1 - \mathbf{E}\cdot\nabla_v f_1 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_2 = \eta(\mathcal{M}_2 - f_2)\,, \tag{52c}$$

where $\mathcal{M}_i$ stands for the i-th order term in the expansion of the Maxwellian (4) in powers of ε, see Appendix A. For this development, the fluid variables $(n^\varepsilon, \mathbf{u}^\varepsilon, T^\varepsilon)$ have to be expanded in powers of ε as well,

$$n^\varepsilon = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \mathcal{O}(\varepsilon^3)\,, \tag{53a}$$

$$\mathbf{u}^\varepsilon = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \mathcal{O}(\varepsilon^3)\,, \tag{53b}$$

$$T^\varepsilon = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \mathcal{O}(\varepsilon^3)\,. \tag{53c}$$

The expansion coefficients of products $a^\varepsilon b^\varepsilon$ of fluid variables are defined as

$$(a^\varepsilon b^\varepsilon)_0 := a_0 b_0\,, \quad (a^\varepsilon b^\varepsilon)_1 := a_1 b_0 + a_0 b_1\,, \quad (a^\varepsilon b^\varepsilon)_2 := a_2 b_0 + a_1 b_1 + a_0 b_2\,, \quad \text{etc.} \tag{54}$$

Inserting the expansion of the macroscopic quantities (53) into Eqs. (9) permits to get the corresponding infinite fluid hierarchy, for $i \geq 0$,

$$\begin{cases} \partial_t n_i + \nabla\cdot(n\mathbf{u})_i = 0\,, \\ \partial_t (n\mathbf{u})_{i-2} + \nabla\cdot(n\mathbf{u}\otimes\mathbf{u})_{i-2} + \nabla\cdot\mathbb{P}_i = -n_i\mathbf{E} - (n\mathbf{u})_i \times \mathbf{B}\,, \\ \partial_t w_{i-1} + \nabla\cdot[(w\,\mathbf{u})_{i-1} + (\mathbb{P}\cdot\mathbf{u})_{i-1}] + \nabla\cdot\mathbf{q}_i = -(n\mathbf{u})_{i-1}\cdot\mathbf{E}\,, \end{cases} \tag{55}$$

where the energy at order i is given by

$$w_i = \frac{3}{2}p_i + (n|\mathbf{u}|^2/2)_{i-2} \quad\Longrightarrow\quad w_0 = \frac{3}{2}p_0\,, \quad w_1 = \frac{3}{2}p_1\,, \quad w_2 = \frac{3}{2}p_2 + (n|\mathbf{u}|^2/2)_0\,, \tag{56}$$

and where we employed the convention that quantities with a negative index are not taken into account.

Remark 1. *We point out that the fluid hierarchy* (55) *provides balance relations stemming from the momentum conservation law:*

$$\mathbf{E} = -\frac{1}{n_i}\nabla\cdot\mathbb{P}_i - \frac{(n\mathbf{u})_i}{n_i}\times\mathbf{B} \qquad\qquad \textit{for} \quad i \leq 1\,. \tag{57}$$

These force balance relations, or drift approximations for $\mathbf{u}_0$ and $\mathbf{u}_1$, respectively, will be used to simplify the expressions for the stress tensor and the heat flux. In particular, one will often go in the following forth and back between the kinetic and already established fluid equations.

The goal is now to solve the hierarchy (52) for the distribution functions f_i, compute the corresponding stress tensors $\mathbb{P}_i$ and heat fluxes $\mathbf{q}_i$, defined as the i-th order terms of (10)-(11), collect all the contributions and set down a truncated fluid model. It becomes clear that in order to obtain an evolution equation for the particle flux $(n\mathbf{u})_0$ (and hence a well-posed fluid system) one needs to go up to the order $i = 2$ in this fluid hierarchy and truncate the system there or at higher orders. The infinite hierarchy (55) can be truncated at any order $i \geq 2$, giving rise to more and more precise models.

Let us now start the truncation process, writing the power series of the stress tensor and the heat flux as

$$\mathbb{P}^\varepsilon = \mathbb{P}_0 + \varepsilon\mathbb{P}_1 + \varepsilon^2\mathbb{P}_2 + \mathcal{O}(\varepsilon^3)\,, \qquad \mathbf{q}^\varepsilon = \mathbf{q}_0 + \varepsilon\mathbf{q}_1 + \varepsilon^2\mathbf{q}_2 + \mathcal{O}(\varepsilon^3)\,, \tag{58}$$

where from (10) one obtains

$$\mathbb{P}_0 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v}\,, \tag{59a}$$

$$\mathbb{P}_1 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_0) f_0 d\mathbf{v}\,, \tag{59b}$$

$$\begin{aligned}\mathbb{P}_2 = &\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_2 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_0) f_1 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_1\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_1) f_0 d\mathbf{v} \\ &+ \mathbf{u}_0\otimes\mathbf{u}_0 \int_{\mathbb{R}^3} f_0 d\mathbf{v}\,,\end{aligned} \tag{59c}$$

and from (11)

$$\mathbf{q}_0 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_0 d\mathbf{v}\,, \tag{60a}$$

$$\mathbf{q}_1 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_1 d\mathbf{v} - \mathbf{u}_0\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0 d\mathbf{v} - \mathbf{u}_0\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v}\,, \tag{60b}$$

$$\begin{aligned}\mathbf{q}_2 = &\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_2 d\mathbf{v} - \mathbf{u}_0\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_1 d\mathbf{v} - \mathbf{u}_1\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0 d\mathbf{v} \\ &- \mathbf{u}_0\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v} - \mathbf{u}_1\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v} + |\mathbf{u}_0|^2\frac{1}{2}\int_{\mathbb{R}^3} \mathbf{v} f_0 d\mathbf{v} + \mathbf{u}_0\otimes\mathbf{u}_0\int_{\mathbb{R}^3} \mathbf{v} f_0 d\mathbf{v}\,.\end{aligned} \tag{60c}$$

To compute the distribution functions f_0, f_1 and f_2 from the hierarchy (52), we decompose each quantity f_i into its gyro-average (with respect to v_θ) and its fluctuation part, $f_i = \overline{f}_i + f_i'$, according to the definitions (28). Taking the gyro-average over Eqs. (52) yields

$$0 = \eta(\overline{\mathcal{M}}_0 - \overline{f}_0)\,, \tag{61a}$$

$$\overline{\mathbf{v}\cdot\nabla_x f_0} - \overline{\mathbf{E}\cdot\nabla_v f_0} = \eta(\overline{\mathcal{M}}_1 - \overline{f}_1)\,, \tag{61b}$$

$$\partial_t\overline{f}_0 + \overline{\mathbf{v}\cdot\nabla_x f_1} - \overline{\mathbf{E}\cdot\nabla_v f_1} = \eta(\overline{\mathcal{M}}_2 - \overline{f}_2)\,. \tag{61c}$$

This is a system of algebraic equations for the $\overline{f}_i$, which is readily solved:

$$\overline{f}_0 = \overline{\mathcal{M}}_0 = \mathcal{M}_0\,, \tag{62a}$$

$$\overline{f}_1 = \overline{\mathcal{M}}_1 - \frac{1}{\eta}\overline{\mathbf{v}\cdot\nabla_x f_0} + \frac{1}{\eta}\overline{\mathbf{E}\cdot\nabla_v f_0}\,, \tag{62b}$$

$$\overline{f}_2 = \overline{\mathcal{M}}_2 - \frac{1}{\eta}\,\partial_t\overline{f}_0 - \frac{1}{\eta}\overline{\mathbf{v}\cdot\nabla_x f_1} + \frac{1}{\eta}\overline{\mathbf{E}\cdot\nabla_v f_1}\,. \tag{62c}$$

By subtracting the Eqs. (61) from the Eqs. (52) we obtain

$$-(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_0' = -\eta f_0'\,, \tag{63a}$$

$$(\mathbf{v}\cdot\nabla_x f_0)' - (\mathbf{E}\cdot\nabla_v f_0)' - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_1' = \eta(\mathcal{M}_1' - f_1')\,, \tag{63b}$$

$$\partial_t f_0' + (\mathbf{v}\cdot\nabla_x f_1)' - (\mathbf{E}\cdot\nabla_v f_1)' - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_2' = \eta(\mathcal{M}_2' - f_2')\,. \tag{63c}$$

It is worth remarking that the system (61)-(63) can be solved sequentially, which means that for the computation of $f_{i\geq 1}$ one needs the distribution functions $f_{j<i}$. The fluctuations f_i' can be computed by passing to cylindrical coordinates in (63), which leads to ODEs of the type

$$\partial_t a(t) + \eta a(t) = r(t)\,, \qquad \eta = const.\,, \qquad t\in[0,2\pi)\,, \tag{64}$$

with r a function of zero average over $[0,2\pi)$. The solution of this equation is simply given by Duhamel's formula

$$a(t) = c_*\, e^{-\eta t} + \int_0^t r(s)e^{\eta(s-t)}\,ds\,, \quad c_* = \frac{1}{\eta}\,\frac{\int_0^{2\pi} r(s)e^{\eta(s-2\,\pi)}\,ds}{\int_0^{2\pi} e^{-\eta s}\,ds}\,. \tag{65}$$

The results for the corresponding stress tensor and heat flux, computed via the obtained distribution functions $f_{i\leq 2}$ and with the help of the software Maple [25], are stated in the following subsection.

4.1. **Expressions for $\mathbb{P}_{i\leq 2}$ and $\mathbf{q}_{i\leq 2}$ as functions of** $(n_{i\leq 2}, \mathbf{u}_{i\leq 2}, T_{i\leq 2})$**.** From (62a) and (63a) one obtains immediately

$$\overline{f}_0 = \mathcal{M}_0 = \frac{n_0}{(2\pi T_0)^{3/2}}\exp\left(-\frac{|\mathbf{v}|^2}{2T_0}\right)\,, \qquad f_0' = 0\,. \tag{66}$$

Therefore, the expressions (59) and (60) for the stress tensor and the heat flux take the following simpler form:

$$\mathbb{P}_0 = p_0\mathbb{I}\,, \tag{67a}$$

$$\mathbb{P}_1 = \int_{\mathbb{R}^3}\mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v}\,, \tag{67b}$$

$$\mathbb{P}_2 = \int_{\mathbb{R}^3}\mathbf{v}\otimes\mathbf{v} f_2 d\mathbf{v} - \int_{\mathbb{R}^3}(\mathbf{u}_0\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_0)f_1 d\mathbf{v} + n_0\,\mathbf{u}_0\otimes\mathbf{u}_0\,, \tag{67c}$$

and

$$\mathbf{q}_0 = 0\,, \tag{68a}$$

$$\mathbf{q}_1 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v} f_1 d\mathbf{v} - \frac{5}{2}\mathbf{u}_0\, p_0\,, \tag{68b}$$

$$\mathbf{q}_2 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v} f_2 d\mathbf{v} - \frac{5}{2}\mathbf{u}_1 p_0 - \mathbf{u}_0 \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_1 d\mathbf{v} - \mathbf{u}_0 \cdot \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v}\,. \tag{68c}$$

Remark 2. *The fact that $\mathbf{q}_0 = 0$ entails that the energy equation in (55) at order $i = 0$ contains no information for the fluid variables (it is identically zero). We thus shift the index in the energy equation, $i \to i+1$, such that a more meaningful fluid hierarchy reads*

$$\begin{cases} \partial_t n_i + \nabla\cdot(n\mathbf{u})_i = 0\,, \\ \partial_t (n\mathbf{u})_{i-2} + \nabla\cdot(n\mathbf{u}\otimes\mathbf{u})_{i-2} + \nabla\cdot\mathbb{P}_i = -n_i\mathbf{E} - (n\mathbf{u})_i \times \mathbf{B}\,, \\ \partial_t w_i + \nabla\cdot[(w\,\mathbf{u})_i + (\mathbb{P}\cdot\mathbf{u})_i] + \nabla\cdot\mathbf{q}_{i+1} = -(n\mathbf{u})_i\cdot\mathbf{E}\,. \end{cases} \tag{69}$$

For a given $i \geq 0$ we recall that terms with negative subscripts are not taken into account.

One might suspect that the new fluid hierarchy (69) leads to a closure problem because of the appearance of the heat flux $\mathbf{q}_{i+1}$ in the i-th order system. Indeed, we saw in the hierarchies (61) and (63) for the gyro-averages and the fluctuations that the distribution function f_i depends on the fluid variables $(n_{j\leq i}, \mathbf{u}_{j\leq i}, T_{j\leq i})$. Since the heat flux $\mathbf{q}_{i+1}$ depends on f_{i+1}, c.f. (68), a fluid system for given i issued from (69) might contain fluid variables of order $i+1$, and thus might not be closed. Nevertheless, the following Lemma shows that the closure of the fluid system (69) can be achieved at arbitrary order k.

Lemma 1. *For a given truncation order $k \geq 0$, the fluid equations (69) with $i \leq k$ and stress tensors $\mathbb{P}_i$ resp. heat fluxes $\mathbf{q}_{i+1}$ computed via (67) resp. (68), form a closed system for the fluid variables $(n_{i\leq k}, \mathbf{u}_{i\leq k}, T_{i\leq k})$.*

Proof. We have to show that the k-th order fluid system (69) contains only fluid variables $(n_{i\leq k}, \mathbf{u}_{i\leq k}, T_{i\leq k})$. From Eqs. (67) and (68) we learn that $\mathbb{P}_k = \mathbb{P}_k(f_{i\leq k})$ and $\mathbf{q}_k = \mathbf{q}_k(f_{i\leq k})$. It can be seen from (62) that the average $\overline{f}_k$ depends on the fluid variables of order $i \leq k$. Moreover, the fluctuations f'_k obtained from the hierarchy (63) depend on the fluid variables of order $i < k$. Therefore, for a fixed order k the only problem of closure can arise from the heat flux $\mathbf{q}_{k+1}$ in the energy equation. From the hierarchy (68) one observes that

$$\mathbf{q}_{k+1} = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v} f_{k+1} d\mathbf{v} + \mathcal{T}(f_{i\leq k}) = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}(\overline{f}_{k+1} + f'_{k+1}) d\mathbf{v} + \mathcal{T}(f_{i\leq k})\,, \tag{70}$$

where $\mathcal{T}(f_{i\leq k})$ stands for the rest terms dependent on $f_{i\leq k}$. We mentioned already that f'_{k+1} depends on $(n_i, \mathbf{u}_i, T_i)$, $i < k+1$, so that the fluctuating part in (70) does not pose a closure problem. Neither do the terms $\mathcal{T}(f_{i\leq k})$. In the average part $\overline{f}_{k+1}$ the highest-order terms $(n_{k+1}, \mathbf{u}_{k+1}, T_{k+1})$ stem from the averaged Maxwellian $\overline{\mathcal{M}}_{k+1}$, see (62). One can however observe from the form of the Maxwellian (4) and its expansion (see Appendix) that these terms always appear with even powers of the velocity $\mathbf{v}$, in particular with the terms $\mathcal{M}_0$ as well as $|\mathbf{v}|^2\,\mathcal{M}_0$ of the expansion of $\mathcal{M}_{k+1}$, such that they cancel out in the integration (70). □

We have now all the necessary ingredients to derive a closed fluid system, approximating the fully kinetic equations (50) in the regime $\varepsilon \ll 1$. Computing $\overline{f}_1$ from (62b) as well as f_1' from (63b), and inserting the result into the expression for $\mathbf{q}_1$, (68b), yields

$$\mathbf{q}_1 = \frac{1}{\eta}\begin{pmatrix} 0 \\ 0 \\ (h_z)_0 \end{pmatrix} + \frac{1}{1+\eta^2}\begin{pmatrix} -(h_y)_0 + \eta(h_x)_0 \\ (h_x)_0 + \eta(h_y)_0 \\ 0 \end{pmatrix} . \tag{71}$$

For the stress tensor from (67b) we compute

$$\mathbb{P}_1 = (nT)_1\mathbb{I} = p_1\mathbb{I} \,. \tag{72}$$

For $i = 2$, in order to treat the time derivative in (62c) separately, we define and compute first

$$\overline{g}_2 := \overline{\mathcal{M}}_2 - \frac{1}{\eta}\,\overline{\mathbf{v}\cdot\nabla_x f_1} + \frac{1}{\eta}\,\overline{\mathbf{E}\cdot\nabla_v f_1}\,. \tag{73}$$

Remarking that $f_0' \equiv 0$ and solving (63c) for f_2' permits to compute f_2 as follows

$$f_2 = \overline{f}_2 + f_2' = \overline{g}_2 - \frac{1}{\eta}\,\partial_t\mathcal{M}_0 + f_2'\,. \tag{74}$$

The integrals of the term with the Maxwellian are readily obtained,

$$\begin{aligned} &\frac{1}{\eta}\partial_t \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v}\mathcal{M}_0 d\mathbf{v} = \frac{1}{\eta}\,\partial_t p_0\mathbb{I}\,, \\ &\frac{1}{2\eta}\,\partial_t \int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v}\mathcal{M}_0 d\mathbf{v} = 0\,. \end{aligned} \tag{75}$$

Inserting then f_2 into (68c) leads to

$$\mathbf{q}_2 = \frac{1}{\eta}\begin{pmatrix} 0 \\ 0 \\ (h_z)_1 \end{pmatrix} + \frac{1}{1+\eta^2}\begin{pmatrix} -(h_y)_1 + \eta(h_x)_1 \\ (h_x)_1 + \eta(h_y)_1 \\ 0 \end{pmatrix} . \tag{76}$$

The components of the stress tensor $\mathbb{P}_2$ are computed from (67c):

$$
\begin{aligned}
\mathbb{P}_{2xx} = p_2+\frac{2}{3}\frac{1}{\eta^2(1+\eta^2)(4+\eta^2)} \Bigg\{ & \frac{8}{5}\partial_z(h_z)_0 \\
& +\eta\Bigg[-2n_0T_0(\partial_x u_{x0}+\partial_y u_{y0}-2\partial_z u_{z0})-2T_0\{n_0,T_0\}_{x,y}\Bigg] \\
& +\eta^2\left[\frac{2}{5}\partial_x(h_x)_0-2\partial_y(h_y)_0+2\partial_z(h_z)_0+3n_0T_0(\partial_x u_{y0}+\partial_y u_{x0})\right] \\
& +\eta^3\left[2\partial_x(h_y)_0+\frac{8}{5}\partial_y(h_x)_0-n_0T_0(4\partial_x u_{x0}+\partial_y u_{y0}-5\partial_z u_{z0})\right] \\
& +\eta^4\left[-\frac{4}{5}\partial_x(h_x)_0+\frac{2}{5}\partial_y(h_y)_0+\frac{2}{5}\partial_z(h_z)_0+3n_0T_0(\partial_x u_{y0}+\partial_y u_{x0})\right] \\
& +\eta^5\Bigg[n_0T_0(-2\partial_x u_{x0}+\partial_y u_{y0}+\partial_z u_{z0})\Bigg]\Bigg\} \ ,
\end{aligned}
\tag{77a}
$$

$$
\begin{aligned}
\mathbb{P}_{2yy} = p_2+\frac{2}{3}\frac{1}{\eta^2(1+\eta^2)(4+\eta^2)} \Bigg\{ & \frac{8}{5}\partial_z(h_z)_0 \\
& +\eta\Bigg[-2n_0T_0(\partial_x u_{x0}+\partial_y u_{y0}-2\partial_z u_{z0})-2T_0\{n_0,T_0\}_{x,y}\Bigg] \\
& +\eta^2\left[-2\partial_x(h_x)_0+\frac{2}{5}\partial_y(h_y)_0+2\partial_z(h_z)_0-3n_0T_0(\partial_x u_{y0}+\partial_y u_{x0})\right] \\
& +\eta^3\left[-\frac{8}{5}\partial_x(h_y)_0-2\partial_y(h_x)_0-n_0T_0(\partial_x u_{x0}+4\partial_y u_{y0}-5\partial_z u_{z0})\right] \\
& +\eta^4\left[\frac{2}{5}\partial_x(h_x)_0-\frac{4}{5}\partial_y(h_y)_0+\frac{2}{5}\partial_z(h_z)_0+3n_0T_0(\partial_x u_{y0}+\partial_y u_{x0})\right] \\
& +\eta^5\Bigg[n_0T_0(\partial_x u_{x0}-2\partial_y u_{y0}+\partial_z u_{z0})\Bigg]\Bigg\} \ ,
\end{aligned}
\tag{77b}
$$

$$
\begin{aligned}
\mathbb{P}_{2zz} = p_2+\frac{2}{3}\frac{1}{\eta^2(1+\eta^2)} \Bigg\{ & -\frac{4}{5}\partial_z(h_z)_0+\eta\Bigg[n_0T_0(\partial_x u_{x0}+\partial_y u_{y0}-2\partial_z u_{z0})+T_0\{n_0,T_0\}_{x,y}\Bigg] \\
& +\eta^2\left[\frac{2}{5}\partial_x(h_x)_0+\frac{2}{5}\partial_y(h_y)_0-\frac{4}{5}\partial_z(h_z)_0\right] \\
& +\eta^3\Bigg[n_0T_0(\partial_x u_{x0}+\partial_y u_{y0}-2\partial_z u_{z0})\Bigg]\Bigg\} \ ,
\end{aligned}
\tag{77c}
$$

$$\begin{aligned}\mathbb{P}_{2xy} = \mathbb{P}_{2yx} =& \frac{1}{(1+\eta^2)(4+\eta^2)} \Big\{ -2n_0T_0(\partial_x u_{x0} - \partial_y u_{y0}) + \frac{4}{5}\partial_x (h_y)_0 + \frac{4}{5}\partial_y (h_x)_0 \\ &+\eta\left[-\frac{6}{5}\partial_x (h_x)_0 + \frac{6}{5}\partial_y (h_y)_0 - n_0T_0(\partial_x u_{y0} + \partial_y u_{x0}) \right] \\ &+\eta^2\left[-\frac{2}{5}\partial_x (h_y)_0 - \frac{2}{5}\partial_y (h_x)_0 - 2n_0T_0(\partial_x u_{x0} - \partial_y u_{y0}) \right] \\ &+\eta^3\Big[- n_0T_0(\partial_x u_{y0} + \partial_y u_{x0}) \Big] \Big\} \,, \end{aligned} \tag{77d}$$

$$\begin{aligned}\mathbb{P}_{2xz} = \mathbb{P}_{2zx} =& \frac{1}{\eta(1+\eta^2)^2} \Big\{ \frac{2}{5}\partial_y (h_z)_0 \\ &+\eta\Big[n_0T_0(\partial_y u_{z0} + \partial_z u_{y0}) + T_0\{n_0, T_0\}_{x,z} \Big] \\ &+\eta^2\left[\frac{2}{5}\partial_y (h_z)_0 + \frac{4}{5}\partial_z (h_y)_0 - n_0T_0(\partial_x u_{z0} + \partial_z u_{x0}) \right] \\ &+\eta^3\left[-\frac{2}{5}\partial_x (h_z)_0 + \frac{2}{5}\partial_z (h_x)_0 + n_0T_0(\partial_y u_{z0} + \partial_z u_{y0}) \right] \\ &+\eta^4\Big[- n_0T_0(\partial_x u_{z0} + \partial_z u_{x0}) \Big] \Big\} \,, \end{aligned} \tag{77e}$$

$$\begin{aligned}\mathbb{P}_{2yz} = \mathbb{P}_{2zy} =& \frac{1}{\eta(1+\eta^2)^2} \Big\{ -\frac{2}{5}\partial_x (h_z)_0 \\ &+\eta\Big[- n_0T_0(\partial_x u_{z0} + \partial_z u_{x0}) + T_0\{n_0, T_0\}_{y,z} \Big] \\ &+\eta^2\left[-\frac{2}{5}\partial_x (h_z)_0 - \frac{4}{5}\partial_z (h_x)_0 - n_0T_0(\partial_y u_{z0} + \partial_z u_{y0}) \right] \\ &+\eta^3\left[-\frac{2}{5}\partial_y (h_z)_0 - \frac{2}{5}\partial_z (h_y)_0 - n_0T_0(\partial_x u_{z0} + \partial_z u_{x0}) \right] \\ &+\eta^4\Big[- n_0T_0(\partial_y u_{z0} + \partial_z u_{y0}) \Big] \Big\} \,. \end{aligned} \tag{77f}$$

4.2. **Truncation of the fluid equations.** The purpose is now to combine the obtained results $\mathbb{P}_{i\leq 2}$ and $q_{i\leq 2}$ in order to set up a closed fluid model approximating at a certain order in ε the kinetic equation (50) or equivalently the corresponding fluid model (9)-(11).

Let us define the fluid variables, truncated at order k (partial sums) as

$$\tilde{n}_k := \sum_{i=0}^{k} \varepsilon^i n_i \,, \qquad \tilde{\mathbf{u}}_k := \sum_{i=0}^{k} \varepsilon^i \mathbf{u}_i \,, \qquad \tilde{T}_k := \sum_{i=0}^{k} \varepsilon^i T_i \,, \qquad \tilde{w}_k := \sum_{i=0}^{k} \varepsilon^i w_i \,, \tag{78}$$

as well as the truncated stress tensor and heat flux

$$\tilde{\mathbb{P}}_k := \sum_{i=0}^{k} \varepsilon^i \mathbb{P}_i \,, \qquad \tilde{\mathbf{q}}_k := \sum_{i=1}^{k} \varepsilon^i \mathbf{q}_i \,, \qquad (\mathbf{q}_0 = 0)\,. \tag{79}$$

For products of partial sums we have the property

$$\tilde{a}_k \tilde{b}_k = (\widetilde{ab})_k + \mathcal{O}(\varepsilon^{k+1})\,. \tag{80}$$

Taking now the sum in the fluid hierarchy (69) up to order $k = 2$ and remarking that

$$\sum_{i=0}^{2} \varepsilon^i \partial_t (n\mathbf{u})_{i-2} = \varepsilon^2 \partial_t (\widetilde{n\mathbf{u}})_0 \,, \qquad \sum_{i=0}^{2} \varepsilon^i \nabla \cdot (n\mathbf{u} \otimes \mathbf{u})_{i-2} = \varepsilon^2 \nabla \cdot (\widetilde{n\mathbf{u} \otimes \mathbf{u}})_0 \tag{81}$$

$$\sum_{i=0}^{2} \varepsilon^i \nabla \cdot \mathbf{q}_{i+1} = \frac{1}{\varepsilon} \nabla \cdot \tilde{\mathbf{q}}_3 \,, \qquad \mathbf{q}_0 \equiv 0 \,, \tag{82}$$

one obtains

$$\begin{cases} \partial_t \tilde{n}_2 + \nabla \cdot \widetilde{(n\mathbf{u})}_2 = 0 \,, \\[2mm] \partial_t \widetilde{(n\mathbf{u})}_0 + \nabla \cdot (\widetilde{n\mathbf{u} \otimes \mathbf{u}})_0 + \dfrac{1}{\varepsilon^2} \nabla \cdot \tilde{\mathbb{P}}_2 = -\dfrac{1}{\varepsilon^2} \big(\tilde{n}_2 \, \mathbf{E} + \widetilde{(n\,\mathbf{u})}_2 \times \mathbf{B} \big) \,, \\[2mm] \partial_t \tilde{w}_2 + \nabla \cdot \big(\widetilde{(w\,\mathbf{u})}_2 + \widetilde{(\mathbb{P} \cdot \mathbf{u})}_2 \big) + \dfrac{1}{\varepsilon} \nabla \cdot \tilde{\mathbf{q}}_3 = -\widetilde{(n\mathbf{u})}_2 \cdot \mathbf{E} \,. \end{cases} \tag{83}$$

Replacing therein the terms $\tilde{a}_i$ of order $i \neq 2$ according to

$$\tilde{a}_0 = \tilde{a}_2 + \mathcal{O}(\varepsilon) \,, \qquad \tilde{a}_1 = \tilde{a}_2 + \mathcal{O}(\varepsilon^2) \,, \qquad \tilde{a}_3 = \tilde{a}_2 + \mathcal{O}(\varepsilon^3) \,, \tag{84}$$

then using (80) and setting simply

$$n := \tilde{n}_2 \,, \qquad \mathbf{u} := \tilde{\mathbf{u}}_2 \,, \qquad w := \tilde{w}_2 \,, \tag{85}$$

leads to the truncated fluid system

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0 + \mathcal{O}(\varepsilon^3) \,, \\[2mm] \partial_t (n\mathbf{u}) + \nabla \cdot (n\mathbf{u} \otimes \mathbf{u}) + \dfrac{1}{\varepsilon^2} \nabla \cdot \mathbb{P}^{\varepsilon,\eta} = -\dfrac{1}{\varepsilon^2} \, n \big(\mathbf{E} + \mathbf{u} \times \mathbf{B} \big) + \mathcal{O}(\varepsilon) \,, \\[2mm] \partial_t w + \nabla \cdot (w\,\mathbf{u} + \mathbb{P}^{\varepsilon,\eta} \cdot \mathbf{u}) + \dfrac{1}{\varepsilon} \nabla \cdot \mathbf{q}^{\varepsilon,\eta} = -n\mathbf{u} \cdot \mathbf{E} + \mathcal{O}(\varepsilon^2) \,. \end{cases} \tag{86}$$

The indicated errors are with respect to the non-truncated fluid system, i.e. the hierarchy (69) summed up to $k = \infty$. The stress tensor $\mathbb{P}^{\varepsilon,\eta}$ and the heat flux $\mathbf{q}^{\varepsilon,\eta}$ are computed with the results from subsection 4.1, which leads to the expressions given in (29). Remark that in order to get the simplified tensor formula (29b) we expanded $\mathbb{P}^{\varepsilon,\eta}$ in η and masked the terms of order $\mathcal{O}(\eta)$, due to the fact, that for magnetized plasmas we are interested in $\eta \ll 1$.

5. Closure of the fluid equations for $\eta = \sqrt{\varepsilon}$

The second closure to be studied stems from Eq. (3) with $\eta = \sqrt{\varepsilon}$, hence the rescaled kinetic equation is given by

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_x f^\varepsilon - \frac{1}{\varepsilon}\left(\mathbf{E} + \frac{1}{\varepsilon}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f^\varepsilon = \frac{1}{\varepsilon\sqrt{\varepsilon}}\, Q^\varepsilon_{BGK}(f^\varepsilon)\,. \tag{87}$$

In the same vein, to obtain closure relations for the associated fluid model, one takes the following asymptotic expansion of the distribution function f^ε in powers of $\varepsilon^{1/2}$:

$$f^\varepsilon = f_0 + \sqrt{\varepsilon} f_1 + \varepsilon f_2 + \varepsilon\sqrt{\varepsilon} f_3 + \varepsilon^2 f_4 + \mathcal{O}(\varepsilon^{5/2})\,. \tag{88}$$

Inserting the Hilbert expansion (88) into (87) and equating the terms of the same order in $\varepsilon^{1/2}$ yields the following hierarchy of equations:

$$-(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_0 = 0\,, \tag{89a}$$

$$-(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_1 = \mathcal{M}_0 - f_0\,, \tag{89b}$$

$$\mathbf{v}\cdot\nabla_x f_0 - \mathbf{E}\cdot\nabla_v f_0 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_2 = \mathcal{M}_1 - f_1\,, \tag{89c}$$

$$\mathbf{v}\cdot\nabla_x f_1 - \mathbf{E}\cdot\nabla_v f_1 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_3 = \mathcal{M}_2 - f_2\,, \tag{89d}$$

$$\partial_t f_0 + \mathbf{v}\cdot\nabla_x f_2 - \mathbf{E}\cdot\nabla_v f_2 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_4 = \mathcal{M}_3 - f_3\,, \tag{89e}$$

$$\partial_t f_1 + \mathbf{v}\cdot\nabla_x f_3 - \mathbf{E}\cdot\nabla_v f_3 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_5 = \mathcal{M}_4 - f_4\,. \tag{89f}$$

We point out that $\mathcal{M}_i$ in (89) stands for the expansion terms of order $\varepsilon^{i/2}$ of the Maxwellian (4), specified in Appendix A and obtained by developing the macroscopic quantities as follows:

$$n^\varepsilon = n_0 + \sqrt{\varepsilon} n_1 + \varepsilon n_2 + \varepsilon^{3/2} n_3 + \varepsilon^2 n_4 + \mathcal{O}(\varepsilon^{5/2})\,, \tag{90a}$$

$$\mathbf{u}^\varepsilon = \mathbf{u}_0 + \sqrt{\varepsilon}\mathbf{u}_1 + \varepsilon\mathbf{u}_2 + \varepsilon^{3/2}\mathbf{u}_3 + \varepsilon^2\mathbf{u}_4 + \mathcal{O}(\varepsilon^{5/2})\,, \tag{90b}$$

$$T^\varepsilon = T_0 + \sqrt{\varepsilon} T_1 + \varepsilon T_2 + \varepsilon^{3/2} T_3 + \varepsilon^2 T_4 + \mathcal{O}(\varepsilon^{5/2})\,. \tag{90c}$$

The corresponding fluid hierarchy is given by

$$\begin{cases} \partial_t n_i + \nabla\cdot(n\mathbf{u})_i = 0\,, \\ \partial_t (n\mathbf{u})_{i-4} + \nabla\cdot(n\mathbf{u}\otimes\mathbf{u})_{i-4} + \nabla\cdot\mathbb{P}_i = -n_i\mathbf{E} - (n\mathbf{u})_i\times\mathbf{B}\,, \\ \partial_t w_{i-2} + \nabla\cdot[(w\,\mathbf{u})_{i-2} + (\mathbb{P}\cdot\mathbf{u})_{i-2}] + \nabla\cdot\mathbf{q}_i = -(n\mathbf{u})_{i-2}\cdot\mathbf{E}\,, \end{cases} \tag{91}$$

where the energy at order $i \geq 0$ has the form

$$w_i = \frac{3}{2}p_i + (n|\mathbf{u}|^2/2)_{i-4}\,. \tag{92}$$

We remind the reader that the terms with negative subscripts are not taken into account.

Remark 3. *We point out that the fluid hierarchy* (91) *provides balance relations stemming from the momentum conservation law and from the energy conservation law:*

$$\mathbf{E} = -\frac{1}{n_i}\nabla \cdot \mathbb{P}_i - \frac{(n\mathbf{u})_i}{n_i} \times \mathbf{B} \qquad \textit{for} \quad i \le 3\,, \tag{93}$$

$$\nabla \cdot \mathbf{q}_i = 0 \qquad \textit{for} \quad i \le 1\,. \tag{94}$$

These relations (drift approximations for $\mathbf{u}_{i\le 3}$*) will be used to simplify the expressions for the stress tensor and the heat flux.*

Obtaining a well-posed closed fluid model from the infinite hierarchy (91) by truncation requires to go up to order $k = 4$ in order to get an evolution equation for the flux $(n\,\mathbf{u})_0$ and to compute the involved stress tensors $\mathbb{P}_i$ and heat fluxes $\mathbf{q}_i$, by solving the kinetic hierarchy (89) for f_i, $i \le 4$. To do this, let us expand the stress tensor and the heat flux in powers of $\varepsilon^{1/2}$,

$$\begin{aligned} \mathbb{P}^\varepsilon &= \mathbb{P}_0 + \sqrt{\varepsilon}\mathbb{P}_1 + \varepsilon\mathbb{P}_2 + \varepsilon^{3/2}\mathbb{P}_3 + \varepsilon^2\mathbb{P}_4 + \mathcal{O}(\varepsilon^{5/2})\,, \\ \mathbf{q}^\varepsilon &= \mathbf{q}_0 + \sqrt{\varepsilon}\mathbf{q}_1 + \varepsilon\mathbf{q}_2 + \varepsilon^{3/2}\mathbf{q}_3 + \varepsilon^2\mathbf{q}_4 + \mathcal{O}(\varepsilon^{5/2})\,, \end{aligned} \tag{95}$$

where the terms $\mathbb{P}_i$ and $\mathbf{q}_i$ are stated in Appendix C.

Let us now turn to the task of solving the hierarchy (89). For this, we decompose every function f_i into its gyro-average and its fluctuating part, $f_i = \overline{f}_i + f_i'$, $i \le 4$, according to the definitions (28). Taking now the gyro-average in (89) leads to

$$0 = \overline{\mathcal{M}}_0 - \overline{f}_0\,, \tag{96a}$$

$$\overline{\mathbf{v}\cdot\nabla_x f_0} - \overline{\mathbf{E}\cdot\nabla_v f_0} = \overline{\mathcal{M}}_1 - \overline{f}_1\,, \tag{96b}$$

$$\overline{\mathbf{v}\cdot\nabla_x f_1} - \overline{\mathbf{E}\cdot\nabla_v f_1} = \overline{\mathcal{M}}_2 - \overline{f}_2\,, \tag{96c}$$

$$\partial_t\overline{f_0} + \overline{\mathbf{v}\cdot\nabla_x f_2} - \overline{\mathbf{E}\cdot\nabla_v f_2} = \overline{\mathcal{M}}_3 - \overline{f}_3\,, \tag{96d}$$

$$\partial_t\overline{f_1} + \overline{\mathbf{v}\cdot\nabla_x f_3} - \overline{\mathbf{E}\cdot\nabla_v f_3} = \overline{\mathcal{M}}_4 - \overline{f}_4\,. \tag{96e}$$

These are algebraic equations for the gyro-averages $\overline{f}_i$. The equations for the fluctuations read

$$-(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_0' = 0\,, \tag{97a}$$

$$-(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_1' = \mathcal{M}_0' - f_0'\,, \tag{97b}$$

$$(\mathbf{v}\cdot\nabla_x f_0)' - (\mathbf{E}\cdot\nabla_v f_0)' - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_2' = \mathcal{M}_1' - f_1'\,, \tag{97c}$$

$$(\mathbf{v}\cdot\nabla_x f_1)' - (\mathbf{E}\cdot\nabla_v f_1)' - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_3' = \mathcal{M}_2' - f_2'\,, \tag{97d}$$

$$\partial_t f_0' + (\mathbf{v}\cdot\nabla_x f_2)' - (\mathbf{E}\cdot\nabla_v f_2)' - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_4' = \mathcal{M}_3' - f_3'\,. \tag{97e}$$

It is easily seen that the f_i can be computed recursively from the hierarchies (96) and (97). The fluctuations f_i' are computed by passing to cylindrical velocity coordinates in

(97) and solving then an ODE of the form

$$\partial_t a(t) = r(t)\,, \qquad t \in [0, 2\pi)\,, \tag{98}$$

with r a function of zero mean over $[0, 2\pi)$, i.e. $\overline{r} = 0$. The solution is given by

$$a(t) = \int_0^t r(s)\,ds + \frac{1}{2\pi}\int_0^{2\pi} r(s)s\,ds\,. \tag{99}$$

We computed the distribution functions $f_{i\leq 4}$ using the software Maple [25]. The results for the corresponding stress tensors and heat fluxes are stated in the following subsection.

5.1. **Expressions for $\mathbb{P}_{i\leq 4}$ and $\mathbf{q}_{i\leq 3}$ as functions of** $(n_{i\leq 4}, \mathbf{u}_{i\leq 4}, T_{i\leq 4})$**.** From (96a) and (97a) one obtains

$$\overline{f}_0 = \mathcal{M}_0 = \frac{n_0}{(2\pi T_0)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2T_0}\right)\,, \qquad f_0' \equiv 0\,. \tag{100}$$

Therefore, the expressions (159) and (160) for the stress tensor and the heat flux take the simpler form:

$$\mathbb{P}_0 = p_0\mathbb{I}\,, \tag{101a}$$

$$\mathbb{P}_1 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v}\,, \tag{101b}$$

$$\mathbb{P}_2 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_2 d\mathbf{v}\,, \tag{101c}$$

$$\mathbb{P}_3 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_3 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_0) f_1 d\mathbf{v}\,, \tag{101d}$$

$$\begin{aligned}\mathbb{P}_4 = &\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_4 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_0) f_2 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_1\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_1) f_1 d\mathbf{v} \\ &+ n_0\mathbf{u}_0\otimes\mathbf{u}_0\,,\end{aligned} \tag{101e}$$

and

$$\mathbf{q}_0 = 0\,, \tag{102a}$$

$$\mathbf{q}_1 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v} f_1 d\mathbf{v}\,, \tag{102b}$$

$$\mathbf{q}_2 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v} f_2 d\mathbf{v} - \frac{5}{2}\mathbf{u}_0 p_0\,, \tag{102c}$$

$$\mathbf{q}_3 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v} f_3 d\mathbf{v} - \frac{5}{2}\mathbf{u}_1 p_0 - \mathbf{u}_0 \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_1 d\mathbf{v} - \mathbf{u}_0 \cdot \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v}\,, \tag{102d}$$

$$\begin{aligned}\mathbf{q}_4 = &\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v} f_4 d\mathbf{v} - \frac{5}{2}\mathbf{u}_2 p_0 - \mathbf{u}_1 \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_1 d\mathbf{v} - \mathbf{u}_0 \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_2 d\mathbf{v} \\ &- \mathbf{u}_1 \cdot \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v} - \mathbf{u}_0 \cdot \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_2 d\mathbf{v}\,.\end{aligned} \tag{102e}$$

Remark 4. *The fact that $\mathbf{q}_0 = 0$ entails that the energy equation in* (91) *at order $i = 0$ contains no information for the fluid variables (it is identically zero). We thus shift the index in the energy equation, $i \to i+1$, such that a more meaningful fluid hierarchy reads*

$$\begin{cases} \partial_t n_i + \nabla\cdot(n\mathbf{u})_i = 0\,, \\ \partial_t (n\mathbf{u})_{i-4} + \nabla\cdot(n\mathbf{u}\otimes\mathbf{u})_{i-4} + \nabla\cdot\mathbb{P}_i = -n_i\mathbf{E} - (n\mathbf{u})_i \times \mathbf{B}\,, \\ \partial_t w_{i-1} + \nabla\cdot[(\mathbf{u}w)_{i-1} + (\mathbb{P}\cdot\mathbf{u})_{i-1}] + \nabla\cdot\mathbf{q}_{i+1} = -(n\mathbf{u})_{i-1}\cdot\mathbf{E}\,. \end{cases} \tag{103}$$

No closure problem arises due to the occurrence of the term $\mathbf{q}_{i+1}$ in the energy equation, which can be shown via a proof similar to the one of Lemma 1.

We may now derive a truncated fluid system approximating the exact equations (9)-(11) in the regime $\varepsilon \ll 1$. One obtains $\overline{f}_1$ from (96b) as well as $f_1' = 0$ from (97b). Inserting the results into the expressions $\mathbb{P}_1$ and $\mathbf{q}_1$ given in (101b) and (102b), respectively, yields

$$\mathbb{P}_1 = p_1\mathbb{I}\,, \qquad \mathbf{q}_1 = \begin{pmatrix} 0 \\ 0 \\ (h_z)_0 \end{pmatrix}\,. \tag{104}$$

In the same fashion at the second order $i = 2$ we obtain

$$\mathbb{P}_2 = p_2\mathbb{I}\,, \qquad \mathbf{q}_2 = \begin{pmatrix} -(h_y)_0 \\ (h_x)_0 \\ (h_z)_1 \end{pmatrix}\,. \tag{105}$$

For $i \geq 3$, in order to treat the time derivatives in (96) separately, we define

$$\overline{g}_3 := \overline{\mathcal{M}}_3 - \overline{\mathbf{v}\cdot\nabla_x f_2} + \overline{\mathbf{E}\cdot\nabla_v f_2}\,, \quad \Longrightarrow \quad f_3 = \overline{f}_3 + f_3' = \overline{g}_3 - \partial_t\mathcal{M}_0 + f_3'\,, \tag{106}$$

and

$$
\begin{aligned}
\overline{g}_4 :&= \overline{\mathcal{M}}_4 - \overline{\mathbf{v}\cdot\nabla_x(\overline{f}_3 + f_3')} + \overline{\mathbf{E}\cdot\nabla_v(\overline{f}_3 + f_3')}\,, \\
&= \overline{\mathcal{M}}_4 - \overline{\mathbf{v}\cdot\nabla_x(\overline{g}_3 + f_3')} + \overline{\mathbf{E}\cdot\nabla_v(\overline{g}_3 + f_3')} + \partial_t(v_z\partial_z\mathcal{M}_0) + E_z\partial_t\left(\frac{1}{T_0}v_z\mathcal{M}_0\right) \\
\Longrightarrow &\qquad f_4 = \overline{f}_4 + f_4' = \overline{g}_4 - \partial_t\overline{f}_1 + f_4'\,.
\end{aligned}
\tag{107}
$$

To evaluate the stress tensor and the heat flux for $i \geq 3$ we shall use the integrals (75) with $\eta = 1$ as well as the following formulae,

$$\partial_t \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v}\overline{f}_1 d\mathbf{v} = \partial_t \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v} = \partial_t p_1 \mathbb{I}\,, \tag{108}$$

$$\frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v}\overline{f}_1 d\mathbf{v} = \frac{1}{2}\partial_t \int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_1 d\mathbf{v} = \partial_t \mathbf{q}_1\,, \tag{109}$$

$$\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} v_z\mathcal{M}_0 d\mathbf{v} = 0\,, \tag{110}$$

$$\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} v_z\mathcal{M}_0 d\mathbf{v} = \frac{5}{2}n_0 T_0^2\,. \tag{111}$$

From (101d) we find the stress tensor

$$
\begin{aligned}
\mathbb{P}_3 = p_3\mathbb{I} &+ \left(\frac{4}{15}\partial_z(h_z)_1 + \frac{2}{3}p_0\partial_z u_{z0}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & \frac{2}{5}\partial_y(h_z)_0 \\ 0 & 0 & -\frac{2}{5}\partial_x(h_z)_0 \\ \frac{2}{5}\partial_y(h_z)_0 & -\frac{2}{5}\partial_x(h_z)_0 & 0 \end{pmatrix}\,,
\end{aligned}
\tag{112}
$$

and from (102d) the heat flux

$$\mathbf{q}_3 = \begin{pmatrix} -(h_y)_1 + (h_x)_0 \\ (h_x)_1 + (h_y)_0 \\ (h_z)_2 \end{pmatrix}\,. \tag{113}$$

Furthermore, from (101e) we find that the diagonal terms of the stress tensor $\mathbb{P}_4$ read

$$\begin{aligned}\mathbb{P}_{4xx} &= p_4 + \frac{4}{15}\partial_z(h_z)_2 + \frac{2}{3}(p\partial_z u_z)_1 \\ &+ \frac{1}{15}\partial_x(h_x)_0 - \frac{1}{3}\partial_y(h_y)_0 + \frac{1}{2}p_0(\partial_x u_{y0} + \partial_y u_{x0})\,,\end{aligned} \tag{114}$$

$$\begin{aligned}\mathbb{P}_{4yy} &= p_4 + \frac{4}{15}\partial_z(h_z)_2 + \frac{2}{3}(p\partial_z u_z)_1 \\ &- \frac{1}{3}\partial_x(h_x)_0 + \frac{1}{15}\partial_y(h_y)_0 - \frac{1}{2}p_0(\partial_x u_{y0} + \partial_y u_{x0})\,,\end{aligned} \tag{115}$$

$$\begin{aligned}\mathbb{P}_{4zz} &= p_4 - \frac{8}{15}\partial_z(h_z)_2 - \frac{4}{3}(p\partial_z u_z)_1 \\ &+ \frac{4}{15}\partial_x(h_x)_0 + \frac{4}{15}\partial_y(h_y)_0\,,\end{aligned} \tag{116}$$

For the off-diagonal terms one obtains

$$\mathbb{P}_{4xy} = \mathbb{P}_{4yx} = \frac{1}{5}\big[\partial_x(h_y)_0 + \partial_y(h_x)_0\big] + \frac{1}{2}p_0\big(\partial_y u_{y0} - \partial_x u_{x0}\big)\,, \tag{117}$$

$$\mathbb{P}_{4xz} = \mathbb{P}_{4zx} = \frac{2}{5}\partial_y(h_z)_1 + p_0\big(\partial_y u_{z0} + \partial_z u_{y0}\big) + \frac{2}{5}\partial_z(h_x)_0\,, \tag{118}$$

$$\mathbb{P}_{4yz} = \mathbb{P}_{4zy} = -\frac{2}{5}\partial_x(h_z)_1 - p_0\big(\partial_x u_{z0} + \partial_z u_{x0}\big) + \frac{2}{5}\partial_z(h_y)_0\,. \tag{119}$$

We refrain from giving here the rather tedious expression of $\mathbf{q}_4$ and remark that this term is not needed in the following.

Remark 5. *By inserting the result* (104) *for* $\mathbf{q}_1$ *into Eq.* (94) *one obtains* $\partial_z(p_0\partial_z T_0) = 0$*. Assuming that the pressure* $p_0 > 0$ *is positive and assuming periodic boundary conditions in* z *we deduce*

$$\partial_z T_0 = \frac{const.}{p_0} \qquad \Longrightarrow \qquad \partial_z T_0 = 0\,. \tag{120}$$

This assumption has been used to eliminate rather tedious terms in the expressions (113)-(116)*, which will appear at higher orders.*

5.2. **Truncation of the fluid equations.** Again, let us combine now all the obtained results $\mathbb{P}_{i\leq 4}$ and $\mathbf{q}_{i\leq 3}$, allowing for setting up a closed fluid system, approximating the fully kinetic one (87).

Let us define the truncated fluid variables (partial sums) as

$$\tilde n_k := \sum_{i=0}^k \varepsilon^{i/2} n_i\,, \qquad \tilde{\mathbf{u}}_k := \sum_{i=0}^k \varepsilon^{i/2}\mathbf{u}_i\,, \qquad \tilde T_k := \sum_{i=0}^k \varepsilon^{i/2} T_i\,, \qquad \tilde w_k := \sum_{i=0}^k \varepsilon^{i/2} w_i\,, \quad (121)$$

as well as the truncated stress tensor and heat flux

$$\tilde{\mathbb{P}}_k := \sum_{i=0}^k \varepsilon^{i/2}\mathbb{P}_i\,, \qquad \tilde{\mathbf{q}}_k := \sum_{i=1}^k \varepsilon^{i/2}\mathbf{q}_i\,, \qquad (\mathbf{q}_0 = 0)\,. \quad (122)$$

For products of partial sums we have the property

$$\tilde a_k \tilde b_k = (\widetilde{ab})_k + \mathcal{O}(\varepsilon^{(k+1)/2})\,. \quad (123)$$

Taking now the sum in the fluid hierarchy (103) up to order $k=4$, and observing that

$$\sum_{i=0}^4 \varepsilon^{i/2}\partial_t(n\mathbf{u})_{i-4} = \varepsilon^2\partial_t(\widetilde{n\mathbf{u}})_0\,, \qquad \sum_{i=0}^4 \varepsilon^{i/2}\nabla\cdot(n\mathbf{u}\otimes\mathbf{u})_{i-4} = \varepsilon^2\nabla\cdot(\widetilde{n\mathbf{u}\otimes\mathbf{u}})_0\,, \quad (124)$$

$$\sum_{i=0}^4 \varepsilon^{i/2} w_{i-1} = \sqrt{\varepsilon}\tilde w_3\,, \qquad \sum_{i=0}^4 \varepsilon^{i/2}\nabla\cdot\mathbf{q}_{i+1} = \frac{1}{\sqrt{\varepsilon}}\nabla\cdot\tilde{\mathbf{q}}_5\,, \qquad \mathbf{q}_0 \equiv 0\,, \quad (125)$$

one obtains

$$\begin{cases} \partial_t \tilde n_4 + \nabla\cdot\widetilde{(n\mathbf{u})}_4 = 0\,, \\[2mm] \partial_t\widetilde{(n\mathbf{u})}_0 + \nabla\cdot(\widetilde{n\mathbf{u}\otimes\mathbf{u}})_0 + \dfrac{1}{\varepsilon^2}\nabla\cdot\tilde{\mathbb{P}}_4 = -\dfrac{1}{\varepsilon^2}\left(\tilde n_4\,\mathbf{E} + \widetilde{(n\,\mathbf{u})}_4\times\mathbf{B}\right)\,, \\[2mm] \partial_t\tilde w_3 + \nabla\cdot[\widetilde{(w\,\mathbf{u})}_3 + \widetilde{(\mathbb{P}\cdot\mathbf{u})}_3] + \dfrac{1}{\varepsilon}\,\nabla\cdot\tilde{\mathbf{q}}_5 = -\widetilde{(n\mathbf{u})}_3\cdot\mathbf{E}\,. \end{cases} \quad (126)$$

Replacing here the partial sums $\tilde a_i$ of order $i\neq 4$ according to

$$\tilde a_0 = \tilde a_4 + \mathcal{O}(\varepsilon^{1/2})\,, \qquad \tilde a_3 = \tilde a_4 + \mathcal{O}(\varepsilon^2)\,, \qquad \tilde a_5 = \tilde a_3 + \mathcal{O}(\varepsilon^2)\,, \quad (127)$$

then using (123) and setting simply

$$n := \tilde n_4\,, \qquad \mathbf{u} := \tilde{\mathbf{u}}_4\,, \qquad w := \tilde w_4\,, \quad (128)$$

leads to the truncated fluid system

$$\begin{cases} \partial_t n + \nabla\cdot(n\mathbf{u}) = 0 + \mathcal{O}(\varepsilon^{5/2})\,, \\[2mm] \partial_t(n\mathbf{u}) + \nabla\cdot(n\mathbf{u}\otimes\mathbf{u}) + \dfrac{1}{\varepsilon^2}\nabla\cdot\mathbb{P}^\varepsilon = -\dfrac{1}{\varepsilon^2}\big(n\,\mathbf{E} + n\,\mathbf{u}\times\mathbf{B}\big) + \mathcal{O}(\varepsilon^{1/2})\,, \\[2mm] \partial_t w + \nabla\cdot(w\,\mathbf{u} + \mathbb{P}^\varepsilon\cdot\mathbf{u}) + \dfrac{1}{\varepsilon}\nabla\cdot\mathbf{q}^\varepsilon = -n\,\mathbf{u}\cdot\mathbf{E} + \mathcal{O}(\varepsilon)\,. \end{cases} \quad (129)$$

The indicated errors are with respect to the non-truncated fluid system, i.e. the hierarchy (103) summed up to $k=\infty$. The stress tensor $\mathbb{P}^{\varepsilon,\eta}$ and the heat flux $\mathbf{q}^{\varepsilon,\eta}$ are computed with the results from subsection 5.1, which leads to the expressions given in (36).

6. Study of the kinetic hierachy for $\eta = \varepsilon$

The third "closure" we study is derived from the kinetic equation (3) with $\eta = \varepsilon$, namely

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_x f^\varepsilon - \frac{1}{\varepsilon}\left(\mathbf{E} + \frac{1}{\varepsilon}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f^\varepsilon = \frac{1}{\varepsilon}Q^\varepsilon_{BGK}(f^\varepsilon)\,, \tag{130}$$

which describes a long-time scaling of the so-called guiding-center regime. Remark also that this situation corresponds once more to a situation where the magnetic force is stronger than the collision term. However, in contrast to the case $\eta = \sqrt{\varepsilon}$ this time the collision term appears at the same order as the transport terms $\mathbf{v}\cdot\nabla_x f^\varepsilon$ and $\mathbf{E}\cdot\nabla_v f^\varepsilon$, fact which will bring more difficulties in the mathematical study. Indeed, the hierarchy issued from a Hilbert ansatz like (51) reads

$$-(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_0 = 0\,, \tag{131a}$$

$$\mathbf{v}\cdot\nabla_x f_0 - \mathbf{E}\cdot\nabla_v f_0 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_1 = \mathcal{M}_0 - f_0\,, \tag{131b}$$

$$\partial_t f_0 + \mathbf{v}\cdot\nabla_x f_1 - \mathbf{E}\cdot\nabla_v f_1 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_2 = \mathcal{M}_1 - f_1 \tag{131c}$$

$$\partial_t f_1 + \mathbf{v}\cdot\nabla_x f_2 - \mathbf{E}\cdot\nabla_v f_2 - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_3 = \mathcal{M}_2 - f_2\,, \tag{131d}$$

where the Maxwellians $\mathcal{M}_i$ are given in Appendix A. The fluid variables are expanded according to (53) which yields the fluid hierarchy (55). The expansions of the stress tensor $\mathbb{P}^\varepsilon$ and of the heat flux $\mathbf{q}^\varepsilon$ are given in (59) and in (60), respectively. The resolution of the hierarchy (131) is the difficult question of the present section. Taking the gyro-average over the Eqs. (131) leads to

$$\overline{\mathbf{v}\cdot\nabla_x f_0} - \overline{\mathbf{E}\cdot\nabla_v f_0} = \overline{\mathcal{M}}_0 - \overline{f}_0\,, \tag{132a}$$

$$\partial_t\overline{f}_0 + \overline{\mathbf{v}\cdot\nabla_x f_1} - \overline{\mathbf{E}\cdot\nabla_v f_1} = \overline{\mathcal{M}}_1 - \overline{f}_1\,, \tag{132b}$$

$$\partial_t\overline{f}_1 + \overline{\mathbf{v}\cdot\nabla_x f_2} - \overline{\mathbf{E}\cdot\nabla_v f_2} = \overline{\mathcal{M}}_2 - \overline{f}_2\,. \tag{132c}$$

In order to determine the fluctuations f_i' we subtract the Eqs. (132) from the Eqs. (131) and obtain

$$-(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_0' = 0\,, \tag{133a}$$

$$(\mathbf{v}\cdot\nabla_x f_0)' - (\mathbf{E}\cdot\nabla_v f_0)' - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_1' = -f_0'\,, \tag{133b}$$

$$\partial_t f_0' + (\mathbf{v}\cdot\nabla_x f_1)' - (\mathbf{E}\cdot\nabla_v f_1)' - (\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_2' = \mathcal{M}_1' - f_1'\,. \tag{133c}$$

We immediately get $f_0' = 0$. The further fluctuations $f_{i\geq 1}'$ can be computed easily once f_{i-1}, $i \geq 1$, have been determined, again by passing to cylindrical coordinates in (133). Next, using relation (26), the mean Eqs. (132) can be written as

$$\overline{f}_0 + v_z\partial_z\overline{f}_0 - E_z\partial_{v_z}\overline{f}_0 = \overline{\mathcal{M}}_0 = \mathcal{M}_0\,, \tag{134a}$$

$$\overline{f}_1 + v_z\partial_z\overline{f}_1 - E_z\partial_{v_z}\overline{f}_1 = \overline{\mathcal{M}}_1 - \partial_t\overline{f}_0 - \overline{\mathbf{v}\cdot\nabla_x f_1'} + \overline{\mathbf{E}\cdot\nabla_v f_1'}\,, \tag{134b}$$

$$\overline{f}_2 + v_z\partial_z\overline{f}_2 - E_z\partial_{v_z}\overline{f}_2 = \overline{\mathcal{M}}_2 - \partial_t\overline{f}_1 - \overline{\mathbf{v}\cdot\nabla_x f_2'} + \overline{\mathbf{E}\cdot\nabla_v f_2'}\,. \tag{134c}$$

The resolution of this mean hierarchy (134) is the difficult part of this study. To compute $\overline{f}_0$ from (134a), let us first observe that in the asymptotic limit $\varepsilon \to 0$ the distribution function f_0 is a Maxwellian, fact which follows from the H-theorem. Strictly speaking, multiplying formally the kinetic equation (130) by $\log(f^\varepsilon)$ and integrating in the phase-space (x, v) yields

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} Q^\varepsilon_{BGK}(f^\varepsilon)\, \log(f^\varepsilon)\, dx\, dv = \varepsilon \int_{\mathbb{R}^3\times\mathbb{R}^3} \partial_t f^\varepsilon\, \log(f^\varepsilon)\, dx\, dv \to_{\varepsilon\to 0} 0\,, \tag{135}$$

permitting to establish that f_0 is a Maxwellian, in particular $f_0 = \mathcal{M}_0$, such that $\mathbb{P}_0 = n_0\, T_0\, \mathbb{I}$ and $\mathbf{q}_0 = 0$. Now, plugging this information into the stationary transport equation (134a) provides that this Maxwellian has to cancel the transport term, *i.e.*

$$v_z \partial_z f_0 - E_z \partial_{v_z} f_0 = 0\,, \quad \forall (v_r, v_z) \in \mathbb{R}^+ \times \mathbb{R}\,, \tag{136}$$

or equivalently

$$v_z \left[\frac{\partial_z n_0}{n_0} + \left(\frac{v_r^2 + v_z^2}{2T_0} - \frac{3}{2} \right) \frac{\partial_z T_0}{T_0} - \frac{\partial_z \phi}{T_0} \right] \mathcal{M}_0 = 0\,, \quad \forall (v_r, v_z) \in \mathbb{R}^+ \times \mathbb{R}\,. \tag{137}$$

This permits immediately to show that

$$\frac{\partial_z T_0}{T_0} \equiv 0 \quad \Rightarrow \quad \frac{\partial_z n_0}{n_0} - \frac{\partial_z \phi}{T_0} \equiv 0\,, \tag{138}$$

which yields the well-known Boltzmann relation

$$n_0(t, \mathbf{x}_\perp, z) = c(t, \mathbf{x}_\perp)\, \exp\left(\frac{\phi(t, \mathbf{x}_\perp, z)}{T_0(t, \mathbf{x}_\perp)} \right)\,, \quad \mathbf{x}_\perp = (x, y) \in \mathbb{R}^2\,,\ \ z \in \mathbb{R}\,,\ \ t \in \mathbb{R}\,. \tag{139}$$

After having computed f_0, one can solve immediately (133b) for f_1', giving

$$f_1' = \Big[\frac{\mathbf{v}_\perp \cdot \nabla^\perp T_0}{T_0} \Big(\frac{|\mathbf{v}|^2}{2T_0} - \frac{5}{2} \Big) + \frac{\mathbf{v}_\perp \cdot \mathbf{u}_{\perp,0}}{T_0} \Big] \mathcal{M}_0\,. \tag{140}$$

We now wish to resolve (134b) for $\overline{f}_1$. This is the hard part, due to the fact that the term $\partial_t \overline{f}_0$ contains $\nabla \cdot \mathbf{q}_1$, leading to an integro-differential equation for $\overline{f}_1$. However, if one is just interested in the limit model as $\varepsilon \to 0$, it is not necessary to compute the full distribution function f_1. The limit model can be obtained from the fluid system (69) with $i = 0$, where we write the energy conservation law in terms of the pressure $p_0 = n_0 T_0$, which leads to

$$\begin{cases} \partial_t n_0 + \nabla \cdot (n\mathbf{u})_0 = 0\,, \\[2mm] \nabla \cdot \mathbb{P}_0 + n_0 \mathbf{E} + (n\mathbf{u})_0 \times \mathbf{B} = 0\,, \\[2mm] \partial_t p_0 + \nabla \cdot (nT\mathbf{u})_0 + \dfrac{2}{3} p_0 \nabla \cdot \mathbf{u}_0 + \dfrac{2}{3} \nabla \cdot \mathbf{q}_1 = 0\,. \end{cases} \tag{141}$$

We will now show that this system permits to compute the functions $c(t, \mathbf{x}_\perp)$ and $T_0(t, \mathbf{x}_\perp)$ in the Boltzmann relation (139). For this we recall the notation (44) for the mean and the fluctuation of a function along the z-direction. The equation for the "constant" $c(t, \mathbf{x}_\perp)$ is obtained with the same arguments as in section 3.4, see in particular Eq. (45). Complications arise in the pressure equation, which should be used to compute $T_0(t, \mathbf{x}_\perp)$. Integrating this equation with respect to z yields (for periodic boundary conditions)

$$\partial_t(\underline{n_0 T_0}) + \nabla_\perp \cdot \underline{(nT\mathbf{u}_\perp)_0} + \frac{2}{3}\,T_0\,\underline{n_0 \nabla_\perp \cdot \mathbf{u}_{\perp,0}} + \frac{2}{3}\,T_0\,\underline{n_0 \partial_z u_{z,0}} + \frac{2}{3}\nabla_\perp \cdot \underline{\mathbf{q}_{\perp,1}} = 0\,. \tag{142}$$

The term containing $u_{z,0}$ does not present any difficulty as demonstrated in Eq. (47). The closure problem is due to the perpendicular heat flux $\mathbf{q}_{\perp,1}$. According to Eq. (68b) this heat flux is computed via

$$\mathbf{q}_{\perp,1} = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp \overline{f_1} d\mathbf{v} + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp f_1' d\mathbf{v} - \frac{5}{2}\mathbf{u}_{\perp,0}\, n_0 T_0\,. \tag{143}$$

The second term on the right-hand-side in this equation is computed via (140) and gives

$$\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp f_1' d\mathbf{v} = -\mathbf{h}^\perp + \frac{5}{2}\mathbf{u}_{\perp,0}\, n_0 T_0\,. \tag{144}$$

The trick is now to compute the first term on the right-hand-side of (143) without explicitly solving for $\overline{f_1}$. First we observe from (134b) that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp \overline{f_1} d\mathbf{v} &= \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp (-v_z \partial_z \overline{f}_1 + E_z \partial_{v_z} \overline{f}_1) d\mathbf{v} \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp (\overline{\mathcal{M}}_1 - \partial_t \overline{f}_0 - \overline{\mathbf{v}\cdot\nabla_x f_1'} + \overline{\mathbf{E}\cdot\nabla_v f_1'}) d\mathbf{v}\,.\end{aligned} \tag{145}$$

Since $\overline{f_1}$ does not depend on v_θ, it is easily seen that

$$\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp v_z \partial_z \overline{f}_1 d\mathbf{v} = \partial_z \int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp v_z \overline{f}_1 d\mathbf{v} = 0\,. \tag{146}$$

$$\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp E_z \partial_{v_z} \overline{f}_1 d\mathbf{v} = -2E_z \int_{\mathbb{R}^3} v_z \mathbf{v}_\perp \overline{f}_1 d\mathbf{v} = 0\,. \tag{147}$$

Moreover, a lengthy but straightforward calculation shows that

$$\begin{aligned}&\overline{\mathcal{M}}_1 - \partial_t \overline{f}_0 - \overline{\mathbf{v}\cdot\nabla_x f_1'} + \overline{\mathbf{E}\cdot\nabla_v f_1'} = \\ &= \frac{\mathcal{M}_0}{6 n_0 T_0^2}\Bigg[|\mathbf{v}|^2\Big(2n_0 T_0 \partial_z u_{z0} + 3n_0 T_1 + 2\nabla\cdot\mathbf{q}_1\Big) \\ &\qquad + v_r^2\Big(-n_0 T_0 \nabla_\perp\cdot\mathbf{u}_{\perp,0} + 6T_0\{T_0, n_0\}_{x,y}\Big) + v_z^2\Big(2n_0 T_0 \nabla_\perp\cdot\mathbf{u}_{\perp,0} + 3T_0\{T_0, n_0\}_{x,y}\Big) \\ &\qquad + v_z(6n_0 T_0 u_{z0}) + 6T_0^2 u_{z0}\partial_z n_0 - 15T_0^2\{T_0, n_0\}_{x,y} - 9n_0 T_0 T_1 + 6T_0^2 n_1 - 6T_0 \nabla\cdot\mathbf{q}_1\Bigg]\,.\end{aligned} \tag{148}$$

Thus, it is easily seen that

$$\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp (\overline{\mathcal{M}}_1 - \partial_t \overline{f}_0 - \overline{\mathbf{v}\cdot\nabla_x f_1'} + \overline{\mathbf{E}\cdot\nabla_v f_1'}) d\mathbf{v} = 0\,, \tag{149}$$

which leads us to the result

$$\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp \overline{f_1} d\mathbf{v} = 0\,. \tag{150}$$

Therefore, $\mathbf{q}_{\perp,1}$ has been determined entirely. We remark that

$$\frac{2}{3}\nabla_\perp \cdot \mathbf{q}_{\perp,1} = -\frac{2}{3}[\partial_x(h_y)_0 - \partial_y(h_x)_0] = \frac{5}{3}T_0\{n_0, T_0\}_{x,y} = -\frac{5}{3}n_0 T_0 \nabla_\perp \cdot \mathbf{u}_{\perp,0}\,, \tag{151}$$

where the last equality follows from the second equation in (141). We thus obtain, using the same arguments as in section 3.4, the asymptotic limit model corresponding to the kinetic equation (130), *i.e.*

$$(L)\begin{cases} f_0 = \mathcal{M}_0 = \mathcal{M}_{n_0,0,T_0}\,, \\ n_0 = c\,\exp\left(\dfrac{\phi}{T_0}\right)\,, \qquad c = c(t,\mathbf{x}_\perp)\,, \quad T_0 = T_0(t,\mathbf{x}_\perp)\,, \\ \partial_t n_0 + \mathbf{u}_E \cdot \nabla_\perp n_0 + \partial_z (nu_z)_0^* = 0\,, \qquad \underline{(nu_z)_0^* = 0}\,, \\ \partial_t(\underline{n_0 T_0}) + \underline{\mathbf{u}_E \cdot \nabla_\perp (n_0 T_0)} - \dfrac{2}{3}\,\underline{(nu_z)_0^*\,\partial_z \phi} = 0\,. \end{cases} \tag{152}$$

To summarize, in the case $\eta = \varepsilon$ we were able to identify the $\varepsilon \to 0$ limit model corresponding to the kinetic equation (130), higher order closures, similar to the other two cases, seem however very difficult to obtain and a detailed study is deferred for future works.

7. Conclusion

The principal purpose of the present work was to derive formally from the kinetic level closure relations for a plasma fluid model, describing the electron dynamics in the adiabatic regime. Braginskii closure terms have been found from our simple scaling. The main goal of this work was to obtain the macroscopic model (40), reducing the numerical burden related to the kinetic approach, being however more accurate than the asymptotic Boltzmann relation. Starting from the here obtained fluid models, our next concern affects the associated numerical simulations, relevant for thermonuclear fusion studies.

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Appendix A. Series expansions of the Maxwellian

A.1. **Expansion in powers of ε.** The series expansion of the Maxwellian (13b) in powers of ε reads

$$\mathcal{M}^{\varepsilon} = \mathcal{M}_0 + \varepsilon\mathcal{M}_1 + \varepsilon^2\mathcal{M}_2 + \mathcal{O}(\varepsilon^3)\,, \tag{153}$$

where

$$\mathcal{M}_0 = \frac{n_0}{(2\pi T_0)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2T_0}\right)\,, \tag{154a}$$

$$\mathcal{M}_1 = \mathcal{M}_0\left[\frac{n_1}{n_0} + \frac{\mathbf{v}\cdot\mathbf{u}_0}{T_0} - \frac{T_1}{T_0}\left(\frac{3}{2} - \frac{|\mathbf{v}|^2}{2T_0}\right)\right]\,, \tag{154b}$$

$$\begin{aligned}\mathcal{M}_2 = \mathcal{M}_0\Bigg[&\frac{n_2}{n_0} + \frac{\mathbf{v}\cdot\mathbf{u}_1}{T_0} + \frac{n_1}{n_0}\frac{\mathbf{v}\cdot\mathbf{u}_0}{T_0} - \frac{5}{2}\frac{T_1}{T_0}\frac{\mathbf{v}\cdot\mathbf{u}_0}{T_0} - \frac{3}{2}\frac{T_2}{T_0} - \frac{3}{2}\frac{n_1T_1}{n_0T_0} + \frac{15}{8}\frac{T_1^2}{T_0^2}\\ &-\frac{|\mathbf{u}_0|^2}{2T_0} + \frac{T_2}{2T_0^2}|\mathbf{v}|^2 + \frac{n_1}{n_0}\frac{T_1}{2T_0^2}|\mathbf{v}|^2 - \frac{5}{4}\frac{T_1^2}{T_0^3}|\mathbf{v}|^2 + \frac{(\mathbf{v}\cdot\mathbf{u}_0)^2}{2T_0} + \frac{T_1}{2T_0^3}|\mathbf{v}|^2(\mathbf{v}\cdot\mathbf{u}_0)\\ &+\frac{1}{8}\frac{T_1^2}{T_0^4}|\mathbf{v}|^4\Bigg]\,,\end{aligned} \tag{154c}$$

A.2. **Expansion in powers of $\sqrt{\varepsilon}$.** The series expansion of the Maxwellian (13b) in powers of $\varepsilon^{1/2}$ reads

$$\mathcal{M}_{n^\varepsilon,\varepsilon\mathbf{u}^\varepsilon,T^\varepsilon} = \mathcal{M}_0 + \sqrt{\varepsilon}\mathcal{M}_1 + \varepsilon\mathcal{M}_2 + \varepsilon^{3/2}\mathcal{M}_3 + \varepsilon^2\mathcal{M}_4 + \mathcal{O}(\varepsilon^{5/2})\,, \tag{155}$$

where

$$\mathcal{M}_0 = \frac{n_0}{(2\pi T_0)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2T_0}\right)\,, \tag{156a}$$

$$\mathcal{M}_1 = \mathcal{M}_0\left(\frac{n_1}{n_0} - \frac{3}{2}\frac{T_1}{T_0} + |\mathbf{v}|^2\frac{T_1}{2T_0^2}\right)\,. \tag{156b}$$

$$\begin{aligned}\mathcal{M}_2 = \mathcal{M}_0\Bigg(&\frac{n_2}{n_0} + \frac{\mathbf{v}\cdot\mathbf{u}_0}{T_0} - \frac{3}{2}\frac{n_1T_1}{n_0T_0} + \frac{15}{8}\frac{T_1^2}{T_0^2} - \frac{3}{2}\frac{T_2}{T_0} + \frac{T_2}{2T_0}|\mathbf{v}|^2 + \frac{n_1}{n_0}\frac{T_1}{2T_0^2}|\mathbf{v}|^2\\ &-\frac{5}{4}\frac{T_1^2}{T_0^3}|\mathbf{v}|^2 + \frac{1}{8}\frac{T_1^2}{T_0^4}|\mathbf{v}|^4\Bigg)\,.\end{aligned} \tag{156c}$$

The third order term reads

$$\begin{aligned}\mathcal{M}_3 =& \mathcal{M}_0 \left(\frac{90}{48} \frac{n_1 T_1^2}{n_0 T_0^2} - \frac{72}{48} \frac{n_1 T_2}{n_0 T_0} - \frac{72}{48} \frac{n_2 T_1}{n_0 T_0} + \frac{n_3}{n_0} - \frac{105}{48} \frac{T_1^3}{T_0^3} + \frac{180}{48} \frac{T_1 T_2}{T_0^2} - \frac{72}{48} \frac{T_3}{T_0} \right) \\ &+ \mathbf{v}\mathcal{M}_0 \cdot \left(\mathbf{u}_0 \frac{n_1}{n_0 T_0} + \frac{\mathbf{u}_1}{T_0} - \frac{120}{48} \mathbf{u}_0 \frac{T_1}{T_0^2} \right) \\ &+ |\mathbf{v}|^2 \mathcal{M}_0 \left(-\frac{60}{48} \frac{n_1 T_1^2}{n_0 T_0^3} + \frac{1}{2} \frac{n_1 T_2}{n_0 T_0^2} + \frac{1}{2} \frac{n_2 T_1}{n_0 T_0^2} - \frac{120}{48} \frac{T_1 T_2}{T_0^3} + \frac{1}{2} \frac{T_3}{T_0^2} + \frac{105}{48} \frac{T_1^3}{T_0^4} \right) \\ &+ |\mathbf{v}|^2 \mathbf{v}\mathcal{M}_0 \cdot \left(\frac{1}{2} \mathbf{u}_0 \frac{T_1}{T_0^3} \right) + |\mathbf{v}|^4 \mathcal{M}_0 \left(\frac{6}{48} \frac{n_1 T_1^2}{n_0 T_0^4} - \frac{21}{48} \frac{T_1^3}{T_0^5} + \frac{1}{4} \frac{T_1 T_2}{T_0^4} \right) \\ &+ |\mathbf{v}|^6 \mathcal{M}_0 \left(\frac{1}{48} \frac{T_1^3}{T_0^6} \right) .\end{aligned} \tag{157}$$

The fourth order term reads:

$$\begin{aligned}\mathcal{M}_4 =& \mathcal{M}_0 \Big[-\frac{840}{384} \frac{n_1 T_1^3}{n_0 T_0^3} + \frac{1440}{384} \frac{n_1 T_1 T_2}{n_0 T_0^2} + \frac{720}{384} \frac{n_2 T_1^2}{n_0 T_0^2} - \frac{576}{384} \frac{(n_1 T_3 + n_2 T_2 + n_3 T_1)}{n_0 T_0} + \frac{n_4}{n_0} \\ &\qquad + \frac{945}{384} \frac{T_1^4}{T_0^4} - \frac{2520}{384} \frac{T_1^2 T_2}{T_0^3} + \frac{720}{384} \frac{T_2^2}{T_0^2} + \frac{1440}{384} \frac{T_1 T_3}{T_0^2} - \frac{576}{384} \frac{T_4}{T_0} - \frac{192}{384} \frac{|\mathbf{u}_0|^2}{T_0} \Big] \\ &+ \mathbf{v}\mathcal{M}_0 \cdot \left(\mathbf{u}_1 \frac{n_1}{n_0 T_0} - \frac{960}{384} \mathbf{u}_0 \frac{n_1 T_1}{n_0 T_0^2} + \mathbf{u}_0 \frac{n_2}{n_0 T_0} + \frac{1680}{384} \mathbf{u}_0 \frac{T_1^2}{T_0^3} - \frac{960}{384} \mathbf{u}_0 \frac{T_2}{T_0^2} - \frac{960}{384} \mathbf{u}_1 \frac{T_1}{T_0^2} + \frac{\mathbf{u}_2}{T_0} \right) \\ &+ |\mathbf{v}|^2 \mathcal{M}_0 \Big[-\frac{960}{384} \frac{n_1 T_1 T_2}{n_0 T_0^3} + \frac{840}{384} \frac{n_1 T_1^3}{n_0 T_0^4} + \frac{192}{384} \frac{(n_1 T_3 + n_2 T_2 + n_3 T_1)}{n_0 T_0^2} - \frac{480}{384} \frac{n_2 T_1^2}{n_0 T_0^3} \\ &\qquad + \frac{2520}{384} \frac{T_1^2 T_2}{T_0^4} - \frac{480}{384} \frac{T_2^2}{T_0^3} - \frac{960}{384} \frac{T_1 T_3}{T_0^3} - \frac{1260}{384} \frac{T_1^4}{T_0^5} + \frac{192}{384} \frac{(T_4 + |\mathbf{u}_0|^2)}{T_0^2} \Big] \\ &+ |\mathbf{v}|^2 \mathbf{v}\mathcal{M}_0 \cdot \left(\frac{192}{384} \mathbf{u}_0 \frac{n_1 T_1}{n_0 T_0^3} - \frac{672}{384} \mathbf{u}_0 \frac{T_1^2}{T_0^4} + \frac{192}{384} \mathbf{u}_0 \frac{T_2}{T_0^3} + \frac{192}{384} \mathbf{u}_1 \frac{T_1}{T_0^3} \right) \\ &+ |\mathbf{v}|^4 \mathcal{M}_0 \Big(-\frac{168}{384} \frac{n_1 T_1^3}{n_0 T_0^5} + \frac{96}{384} \frac{n_1 T_1 T_2}{n_0 T_0^4} + \frac{48}{384} \frac{n_2 T_1^2}{n_0 T_0^4} + \frac{378}{384} \frac{T_1^4}{T_0^6} \\ &\qquad - \frac{504}{384} \frac{T_1^2 T_2}{T_0^5} + \frac{48}{384} \frac{T_2^2}{T_0^4} + \frac{96}{384} \frac{T_1 T_3}{T_0^4} \Big) \\ &+ |\mathbf{v}|^4 \mathbf{v}\mathcal{M}_0 \cdot \left(\frac{48}{384} \mathbf{u}_0 \frac{T_1^2}{T_0^5} \right) + |\mathbf{v}|^6 \mathcal{M}_0 \left(\frac{8}{384} \frac{n_1 T_1^3}{n_0 T_0^6} - \frac{36}{384} \frac{T_1^4}{T_0^7} + \frac{24}{384} \frac{T_1^2 T_2}{T_0^6} \right) \\ &+ |\mathbf{v}|^8 \mathcal{M}_0 \left(\frac{1}{384} \frac{T_1^4}{T_0^8} \right) .\end{aligned} \tag{158}$$

Appendix B. Gaussian integrals

The following integrals are important for the fluid expansion. Let $\mathcal{M}_0$ be the Maxwellian

$$\mathcal{M}_0 = \frac{n_0}{(2\pi T_0)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2T_0}\right) .$$

Then one has

$$\int_{\mathbb{R}^3} v_i^\alpha v_j^\beta v_k^\gamma \mathcal{M}_0 d\mathbf{v} = 0\,, \ \ \forall\, i,j,k\,, \quad \forall\, \alpha,\beta,\gamma \in \mathbb{N}\,, \ \text{ with odd } \ \alpha+\beta+\gamma\,,$$

$$\int_{\mathbb{R}^3} |\mathbf{v}|^{2\,l} \mathcal{M}_0 d\mathbf{v} = (2\,l+1)\,T_0 \int_{\mathbb{R}^3} |\mathbf{v}|^{2\,(l-1)} \mathcal{M}_0 d\mathbf{v}\,, \quad \forall l \geq 1\,,$$

$$\int_{\mathbb{R}^3} v_i v_j \mathcal{M}_0 d\mathbf{v} = \begin{cases} 0, & \text{if } i \neq j \\ n_0 T_0, & \text{if } i = j \end{cases} \implies \int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathcal{M}_0 d\mathbf{v} = 3 n_0 T_0\,,$$

$$\int_{\mathbb{R}^3} v_i^2 v_j v_k \mathcal{M}_0 d\mathbf{v} = \begin{cases} n_0 T_0^2, & \text{if } j = k \\ 3 n_0 T_0^2, & \text{if } i = j = k \\ 0, & \text{otherwise} \end{cases} \implies \int_{\mathbb{R}^3} |\mathbf{v}|^2 v_i^2 \mathcal{M}_0 d\mathbf{v} = 5 n_0 T_0^2\,,$$

$$\int_{\mathbb{R}^3} v_i^2 v_j^2 v_k^2 \mathcal{M}_0 d\mathbf{v} = n_0 T_0^3 \quad \forall\, i,j,k\,,$$

$$\int_{\mathbb{R}^3} v_i^3 v_j^2 v_k \mathcal{M}_0 d\mathbf{v} = 0 \quad \forall\, i,j,k\,,$$

$$\int_{\mathbb{R}^3} v_i^4 v_j v_k \mathcal{M}_0 d\mathbf{v} = \begin{cases} 3 n_0 T_0^3, & \text{if } j = k \\ 15 n_0 T_0^3, & \text{if } i = j = k \\ 0, & \text{otherwise} \end{cases} ,$$

$$\int_{\mathbb{R}^3} |\mathbf{v}|^4 v_i v_j \mathcal{M}_0 d\mathbf{v} = \begin{cases} 0, & \text{if } i \neq j \\ 35 n_0 T_0^3, & \text{if } i = j \end{cases} ,$$

$$\int_{\mathbb{R}^3} |\mathbf{v}|^6 v_i v_j \mathcal{M}_0 d\mathbf{v} = \begin{cases} 0, & \text{if } i \neq j \\ 315 n_0 T_0^4, & \text{if } i = j \end{cases} .$$

Appendix C. Coefficients of series expansions in powers of $\sqrt{\varepsilon}$

Corresponding to the series expansions (95), the coefficients for the stress tensor read

$$\mathbb{P}_0 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v}\,, \tag{159a}$$

$$\mathbb{P}_1 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v}\,, \tag{159b}$$

$$\mathbb{P}_2 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_2 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}_0) f_0 d\mathbf{v}\,, \tag{159c}$$

$$\mathbb{P}_3 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_3 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}_0) f_1 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_1\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}_1) f_0 d\mathbf{v}\,, \tag{159d}$$

$$\begin{aligned}\mathbb{P}_4 = &\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_4 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}_0) f_2 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_1\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}_1) f_1 d\mathbf{v} \\ &- \int_{\mathbb{R}^3} (\mathbf{u}_2\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{u}_2) f_0 d\mathbf{v} + \mathbf{u}_0\otimes\mathbf{u}_0 \int_{\mathbb{R}^3} f_0 d\mathbf{v}\,,\end{aligned} \tag{159e}$$

and for the heat flux one has

$$\mathbf{q}_0 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_0 d\mathbf{v}\,, \tag{160a}$$

$$\mathbf{q}_1 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_1 d\mathbf{v}\,, \tag{160b}$$

$$\mathbf{q}_2 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_2 d\mathbf{v} - \mathbf{u}_0\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0 d\mathbf{v} - \mathbf{u}_0\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v}\,, \tag{160c}$$

$$\begin{aligned}\mathbf{q}_3 = &\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_3 d\mathbf{v} - \mathbf{u}_1\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0 d\mathbf{v} - \mathbf{u}_0\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_1 d\mathbf{v} - \mathbf{u}_1\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v} \\ &- \mathbf{u}_0\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v}\,,\end{aligned} \tag{160d}$$

$$\begin{aligned}\mathbf{q}_4 = &\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_4 d\mathbf{v} - \mathbf{u}_2\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0 d\mathbf{v} - \mathbf{u}_1\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_1 d\mathbf{v} - \mathbf{u}_0\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_2 d\mathbf{v} \\ &- \mathbf{u}_2\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v} - \mathbf{u}_1\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v} - \mathbf{u}_0\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_2 d\mathbf{v} \\ &+ |\mathbf{u}_0|^2\frac{1}{2}\int_{\mathbb{R}^3} \mathbf{v} f_0 d\mathbf{v} + \mathbf{u}_0\otimes\mathbf{u}_0\int_{\mathbb{R}^3} \mathbf{v} f_0 d\mathbf{v}\,.\end{aligned} \tag{160e}$$

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