MATHEMATICAL STUDY OF A LAGRANGE-MULTIPLIER TECHNIQUE FOR SINGULARLY-PERTURBED PROBLEMS

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Abstract. The Lagrange-Multiplier technique is a multiscale numerical scheme designed to solve evolution problems containing some stiff terms. Such multiscale problems arise very often in kinetic models for the description of thermonuclear plasma dynamics. The particularity of this scheme is that it permits to capture (with no additional numerical costs) even the asymptotic limit, when the small parameter ε describing the stiffness of the problem, goes to zero. This property is called *asymptotic-preserving*, and has been validated numerically in previous works of the author. In the present work, the missing mathematical study of this Lagrange-Multiplier approach is performed, with special emphasize on the asymptotic-limits, when $\varepsilon \to 0$.

Keywords: Singularly-perturbed problems, stiff transport problems, anisotropic parabolic equations, asymptotic analysis, multiscale techniques, asymptotic-preserving approach.

1. Introduction/Motivation

Many problems in nature involve multiple scales, meaning that they contain features evolving at different time and/or space scales. To mention only some examples, think of chemical reactions, turbulent flows, plasma dynamics, biological systems, *etc.* The common feature of all these phenomena is that, after performing a physical scaling, their mathematical description renders visible the different scales via one or several small dimensionless parameters, denoted here simply by ε, representing for example the Mach number, the Reynolds number, the Knudsen number, the Peclet number *etc.*

When several scales occur in a mathematical problem, the theoretical as well as numerical investigation becomes very arduous. Questions like: *"What is the asymptotic behaviour of the solution, when one of the small parameters tends towards zero?", "What is the corresponding limit-model?", "Is it possible to design a numerical scheme able to follow this asymptotics on the discrete level, without too much numerical costs?"* A general, unified treatment of such, usually singularly-perturbed problems is not possible, so that a lot of theoretical as well as numerical techniques have been developed in literature, designed for each specific situation in particular.

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The present paper can be seen as the final work of a series of works of the author concerning a new multi-scale technique for the resolution of evolution problems containing some stiff terms (for ex. some stiff transport term), method called in these previous works *"Lagrange-Multiplier scheme"*. This method has been initially developed for thermonuclear fusion plasma simulations, in particular in the aim to solve the ion/electron kinetic equations in the strong magnetic field regime. However, due to its generality, it can be applied for a large variety of other singularly-perturbed problems. In the previous papers [13, 14], this (La)-method was tested and validated numerically firstly in the plasma kinetic framework, and secondly in a fluid mechanical framework, namely for the simulation of the incompressible 2D Navier-Stokes system (in the vorticity-streamfunction formulation) when the long-time (viscous-time) asymptotics is of interest. What was still missing in all these works was a detailed mathematical study of this (La)-approach in order to understand better its mathematical foundation. This will be the aim of the present work, concluding thus the validation of this new Asymptotic-Preserving approach.

In order to place this new approach in the context of existing methods and to point out in particular its differences with the existing techniques, we shall start by reviewing in Section 2 two other Asymptotic-Preserving approaches frequently used in literature. Section 3 explains then in detail the main features of the new (La)-scheme. And finally Section 4 is the main part of this work, focusing on the mathematical and asymptotic study of this scheme. All around the mathematical investigations of the (La)-scheme, Poincaré-type inequalities are required, such that the author decided to sum up in Section 5 some of the well-known Poincaré inequalities and point out the main "coercivity" problems of the transport operator, which is the dominant operator in this study.

1.1. **Multi-scale problems.** Let us start by saying some words about multiscale problems. The first question one has to answer when dealing with such problems, is what exactly is the aim of the investigation. If one is interested in the description of the microscopic scales (of the order ε), the best technique is to choose a single-scale microscopic approach, which is physically very accurate, however from a computational point of view very expensive, sometimes even unfeasible, when ε is too small. If one is interested only in the macroscopic details of the problem (of order one), a macroscopic approach is to be preferred. However, very often a closed macroscopic model is not available, taking also into account for the indispensable effects on the microscopic level. This macroscopic model has to be found via asymptotic studies, starting from the rescaled microscopic problem and performing rigorous or formal limits ($\varepsilon \to 0$), procedure which can turn out to be intractable in real physical situations. Hence, in practice macroscopic models often use empirical closure relations for the elimination/description of the inherent microscopic scales, closures that are not justified from a mathematical point of view, nor well understood, as for example the viscosity tensor terms in turbulent flows. As a consequence, such macroscopic models are questionable.

In such circumstances it is better to bring into play multiscale modeling, coupling for example different models which describe the phenomena on different scales, taking care to achieve a balance between accuracy of the numerical results and efficiency of the numerical method. Briefly, the main goal of multi-scale techniques is to design microscopic-macroscopic numerical schemes, which are more efficient than solving the full microscopic model and at the same time furnish the desired accuracy. Different multi-scale methods, based on various ideas, were introduced in literature, see for ex. the books of C. le Bris, M. H. Holmes and E. Weinan [33, 37, 45] as well as all the references therein.

Asymptotic-Preserving techniques are particular multiscale methods designed to cope with singularly perturbed problems P^ε. The solution of P^ε is supposed to converge, as the perturbation parameter tends to zero, towards the solution of a limit problem P^0, which is a well-posed problem. However, the fact that the singular limit $P^\varepsilon \to_{\varepsilon\to 0} P^0$ leads to a change in type of the equation, explains somehow the difficulties encountered when trying to solve numerically P^ε for too small ε-values. Two complications arise:

(a) restrictive stability issues in the case of explicit schemes;
(b) asymptotic accuracy issues in the case of implicit schemes.

The use of standard explicit numerical schemes for the resolution of singularly perturbed problems requires very restrictive time and/or space discretization step conditions, of the type $\Delta t, \Delta x \sim \mathcal{O}(\varepsilon^\gamma)$ with some $\gamma > 0$, and this due to stability reasons (a). These restrictive conditions become rapidly too costly from a numerical point of view and consequently a numerical asymptotic study and even numerical simulations for fixed but small ε-values, are out of reach. On the other hand, standard implicit schemes (even if computationally heavy) may be uniformly stable for $0 < \varepsilon < 1$, but yet provide a wrong solution in the limit $\varepsilon \to 0$, which means the scheme is not consistent with the limit problem P^0, in other words it captures not well the macroscopic behaviour of the solution for $\varepsilon \ll 1$ (b). Thus the design of robust numerical methods for singularly perturbed problems, whose stability and accuracy does not depend on the parameter ε (hence on the local scales of the singularity), allowing moreover to capture the limit $\varepsilon \to 0$, is a challenging and important point.

The main idea for the construction of AP-schemes is based on asymptotic arguments and consists in a mathematical reformulation of the singularly perturbed problem P^ε into an equivalent problem $(AP)^\varepsilon$, which is a regular perturbation of the limit problem P^0. The reformulation of P^ε into $(AP)^\varepsilon$ is a sort of "reorganization" of the problem into a form which is better suited for the limit $\varepsilon \to 0$. The same numerical scheme can then be used for the discretization of $(AP)^\varepsilon$ as well as for P^0, which means that AP-techniques allow for an automatic numerical transition from P^ε to P^0. Remark that the AP-reformulation is by no means unique, and several AP-schemes can be conceived for the same problem.

It is necessary to underline here that the asymptotic preserving techniques are not used to derive a simplified "macroscopic" model, which is then solved numerically. Rather the objective is to construct a numerical scheme, whose solution does not deteriorate as the singular limit is approached, and which can be used without additional numerical costs for all ε-regimes.

The concern of this paper is to study mathematically a new Asymptotic-Preserving scheme used to cope with stiff transport problems arising for example in plasma physics. This scheme was introduced in previous works [13, 14], however a detailed mathematical study was still lacking and shall be the aim of the present paper. To put this method in relation with other existing methods, we shall first briefly recall some of them, before passing to our main AP-scheme.

1.2. **Different regimes and multi-scale techniques in fusion plasma physics.** Thermonuclear fusion plasmas exhibit a large amount of temporal and spacial scales, which make the numerical treatment of its dynamics very challenging. Some of the main parameters characterizing such plasmas are the Debye length, the particle Larmor radius, the mean free path, the plasma frequency, the cyclotron frequency and so on. Depending on the physical phenomenon one wants to study, some of these parameters can be considered as small compared to others, and various asymptotic regimes can be considered. Some of these different scalings in the kinetic plasma description are briefly sketched here:

(a) Hydrodynamic regime [18, 22, 23, 28, 29]:

$$\partial_t f + \mathbf{v}\cdot\nabla_x f = \frac{1}{\varepsilon}\,Q(f)\,,$$

where $\varepsilon \ll 1$ stands here for the particle mean free path or Knudsen number. This kinetic equation is a diffusive (or collisional) equation and in the limit $\varepsilon \to 0$, one gets the compressible Euler equations.

(b) Drift-Diffusion scaling [19, 34–36, 38]:

$$\partial_t f + \frac{1}{\varepsilon}(\mathbf{v}\cdot\nabla_x f - \mathbf{E}\cdot\nabla_v f) = \frac{1}{\varepsilon^2}\,Q(f)\,,$$

where again $\varepsilon \ll 1$ stands for the Knudsen number as well as for long observation times. In the diffusive limit $\varepsilon \to 0$, one obtains the Drift-Diffusion model.

(c) High magnetic field scaling [6, 8, 9, 30, 31]:

$$\partial_t f + \mathbf{v}\cdot\nabla_x f + \mathbf{E}\cdot\nabla_v f + \frac{1}{\varepsilon}(\mathbf{v}\times\mathbf{B})\cdot\nabla_v f = 0\,,$$

where this time $\varepsilon \ll 1$ corresponds to the ion cyclotron period. This particular non-collisional kinetic equation is no more diffusive, and the asymptotic behaviour of the solutions f^ε is rather different from the previous ones (highly oscillating). In the limit $\varepsilon \to 0$, one gets the gyro-kinetic model.

(d) Adiabatic scaling [2, 20, 32, 41, 42]:

$$\partial_t f + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_x f - \frac{1}{\varepsilon}(\mathbf{E} + \frac{1}{\varepsilon}\,\mathbf{v}\times\mathbf{B})\cdot\nabla_v f = \frac{1}{\varepsilon}\,Q(f)\,,$$

where $\varepsilon \ll 1$ stands at the same time for the ion/electron mass ratio, the collisional period and the strong magnetic field. In the limit $\varepsilon \to 0$, one gets the electron Boltzmann relation.

In the just mentioned singularly-perturbed equations three different kinds of situations arise, namely relaxation procedures (a), diffusive processes (b) and highly oscillating situations (c,d). Each of these situations has to be treated in a specific, adequate way and different classes of methods have been used in literature.

(i) Penalization/IMEX methods [28, 46], especially for relaxation problems;
(ii) Micro-Macro methods [4, 19, 38], especially for diffusive problems;
(iii) Lagrange-Multiplier method [27], especially for highly oscillating problems.

We shall briefly recall in section 2 the first two AP-strategies (i) resp. (ii), by singling out two simple toy-models. Then Section 3 resp. 4 will focus on the Lagrange-Multiplier technique.

2. PENALIZATION, IMEX AND MICRO-MACRO TECHNIQUES

2.1. **Hydrodynamic regime via a penalization/IMEX method [28,46].** A first class of Asymptotic-Preserving methods we shall recall are the combined penalization/IMEX techniques. To present this technique, let us consider the kinetic equation describing the dynamics of a gas of particles in the hydrodynamic regime, namely

$$\partial_t f + \mathbf{v} \cdot \nabla_x f = \frac{1}{\varepsilon}\, Q(f)\,, \tag{1}$$

with $\varepsilon \ll 1$, meaning we are close to a local thermodynamic equilibrium, and the collision operator $Q(f)$ is in general the nonlinear Boltzmann operator [17]. Due to the nonlinearity as well as the non-locality (in v) of Q, important numerical difficulties arise when one wants to inverse this operator. An idea to overcome this problem is to penalize the Boltzmann collision operator with a simply invertible penalization operator $\mathcal{P}(f)$. Thus the ambition is to find an operator $\mathcal{P}(f)$, which is easy to invert but at the same time preserves the physical properties (conservations and equilibria for ex.) of the original collision operator Q. In this aim, a good choice could be a BGK operator, and the penalization reads then

$$(P)_\varepsilon \quad \partial_t f + \mathbf{v} \cdot \nabla_x f = \frac{Q(f) - \beta(\mathcal{M}_{n,\mathbf{u},T} - f)}{\varepsilon} + \frac{\beta\,(\mathcal{M}_{n,\mathbf{u},T} - f)}{\varepsilon}\,, \tag{2}$$

where $\mathcal{M}_{n,\mathbf{u},T}$ is the local thermodynamic equilibrium or Maxwellian, defined as

$$\mathcal{M}_{n,\mathbf{u},T} := \frac{n}{(2\,\pi\,T)^{3/2}}\, e^{-\frac{|v-\mathbf{u}|^2}{2\,T}}\,,$$

with the macroscopic quantities related to the distribution function f via

$$\begin{aligned}
n(t,\mathbf{x}) &:= \int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, && (3a)\\
n(t,\mathbf{x})\mathbf{u}(t,\mathbf{x}) &:= \int_{\mathbb{R}^3} \mathbf{v} f(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, && (3b)\\
w(t,\mathbf{x}) &:= \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f(t,\mathbf{x},\mathbf{v})d\mathbf{v} = \frac{3}{2}nT + \frac{1}{2}n\,|\mathbf{u}|^2\,, && (3c)\\
\frac{3}{2}n(t,\mathbf{x})T(t,\mathbf{x}) &:= \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}-\mathbf{u}|^2 f(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \qquad p := n(t,\mathbf{x})T(t,\mathbf{x}) && (3d)\\
\mathbb{P}(t,\mathbf{x}) &:= \int_{\mathbb{R}^3} (\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u})\, f d\mathbf{v}\,, \quad \mathbf{q}(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^3} (\mathbf{v}-\mathbf{u})|\mathbf{v}-\mathbf{u}|^2\, f d\mathbf{v}\,. && (3e)
\end{aligned}$$

Remark that $\beta \in \mathbb{R}$ has to be adequately tuned in order to approximate somehow the Fréchet derivative $dQ(\mathcal{M}_{n,\mathbf{u},T})$, which means one penalizes with the first order term in the Taylor development of $Q(f)$ around $\mathcal{M}_{n,\mathbf{u},T}$.

When ε is small, the solution f^ε of (1) being close to a local Maxwellian $\mathcal{M}_{n^\varepsilon,u^\varepsilon,T^\varepsilon}$, the first term on the right hand side of (2) is less or no more stiff, and consequently it can be explicitly discretized, whereas the second BGK-term has to be taken implicitly for stability reasons. However its inversion is no more a problem. Thus one gets the so-called IMEX semi-discretization in time of the reformulation (2)

$$\frac{f^{k+1}-f^k}{\Delta t} + \mathbf{v}\cdot\nabla_x f^k = \frac{Q(f^k) - \beta(\mathcal{M}_{n^k,\mathbf{u}^k,T^k} - f^k)}{\varepsilon} + \beta\,\frac{\mathcal{M}_{n^{k+1},\mathbf{u}^{k+1},T^{k+1}} - f^{k+1}}{\varepsilon}\,, \tag{4}$$

where the macroscopic unknowns $(n^{k+1},\mathbf{u}^{k+1},T^{k+1})$ are computed via the fluid system

$$\begin{cases}
\dfrac{n^{k+1}-n^k}{\Delta t} + \nabla_x\cdot(n^k\mathbf{u}^k) = 0\,,\\[2mm]
\dfrac{n^{k+1}\,\mathbf{u}^{k+1} - n^k\,\mathbf{u}^k}{\Delta t} + \nabla_x\cdot(n^k\mathbf{u}^k\otimes\mathbf{u}^k) + \nabla_x\cdot\mathbb{P}^k = 0\,,\\[2mm]
\dfrac{w^{k+1}-w^k}{\Delta t} + \nabla_x\cdot(w^k\,\mathbf{u}^k + \mathbb{P}^k\cdot\mathbf{u}^k + \mathbf{q}^k) = 0\,,
\end{cases} \tag{5}$$

starting from the known distribution f^k and the corresponding relations (3). An asymptotic study of (4)-(5) permits to show immediately that in the limit of small ε-values we approach a local thermodynamic equilibrium, namely a Maxwellian $f_0 = \mathcal{M}_{n_0,\mathbf{u}_0,T_0}$, with $(n_0,\mathbf{u}_0,T_0)$ solution of the compressible Euler system. Indeed, it was shown in [28] that regardless the initial condition f_{in}, for any $\varepsilon > 0$ and $\Delta t \gg \varepsilon$ there exists a time-index $K \in \mathbb{N}$ such that after a transition phase one has

$$f^k = \mathcal{M}_{n^k,\mathbf{u}^k,T^k} + \mathcal{O}(\varepsilon)\,, \quad \forall k \geq K\,,$$

which means that the scheme captures well the Euler limit, when $\varepsilon \to 0$. The asymptotic property of (1) is preserved hence even on the discrete level, which can be very beneficial.

The idea of using linear/simpler operators to penalize nonlinear/complex operators turns out to be a general approach, and the obtained scheme is rather simple to implement and has the desired AP-property when dealing with $\varepsilon \ll 1$. For each particular problem however, one needs to find an appropriate penalization operator which serves the required purposes of simplicity and adequacy with respect to the properties of the original collision operator. The design of such penalization operators can be a rather difficult task, requiring a detailed knowledge of the original collision operator. An additional difficulty comes from the choice of the parameter β which can be rather delicate.

2.2. **Drift-Diffusion regime via a micro-macro method [4, 19, 46].** A second class of AP-strategies is the so-called micro-macro approach. Unlike the penalization/IMEX idea, the MM-scheme separates the microscopic and macroscopic scales, by projecting the distribution function f onto the kernel of the dominant operator. A coupled system is obtained, composed of a kinetic equation for the microscopic scales and a fluid system for the macroscopic scales.

To introduce the main features of this approach, let us consider the kinetic equation describing the electron gas dynamics in a given electric field E and in the drift-diffusive regime, namely

$$\partial_t f + \frac{1}{\varepsilon}(\mathbf{v} \cdot \nabla_x f - \mathbf{E} \cdot \nabla_v f) = \frac{1}{\varepsilon^2}\, Q(f)\,. \tag{6}$$

The collision operator Q can be nonlinear, for simplicity reasons we shall however present the method only for the low-density, linear collision operator, defined as

$$Q(f)(\mathbf{v}) := \int_{\mathbb{R}^3} \sigma(\mathbf{v}, \mathbf{v}')\, [\mathcal{M}(\mathbf{v}) f(\mathbf{v}') - \mathcal{M}(\mathbf{v}') f(\mathbf{v})]\, d\mathbf{v}'\,, \tag{7}$$

where f belongs to a suitable functional space and $\mathcal{M}$ is the Maxwellian distribution function, given by

$$\mathcal{M}(\mathbf{v}) := \frac{1}{(2\,\pi)^{3/2}} e^{-|\mathbf{v}|^2/2}\,. \tag{8}$$

The cross section σ satisfies the following positivity, boundedness and symmetry (micro-reversibility principle) property

$$0 < \sigma_1 \le \sigma(\mathbf{v}, \mathbf{v}') = \sigma(\mathbf{v}', \mathbf{v}) \le \sigma_2\,. \tag{9}$$

Let us keep all over this section the variables $(t, \mathbf{x})$ as fixed (parameters) and consider the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^3; \mathcal{M}^{-1}(\mathbf{v})\, d\mathbf{v}) = \left\{ f \in L^2(\mathbb{R}^3) \ / \ \int_{\mathbb{R}^3} |f(\mathbf{v})|^2 \mathcal{M}^{-1}(\mathbf{v})\, d\mathbf{v} < \infty \right\},$$

associated with the following scalar product

$$(f, g)_{\mathcal{H}} := \int_{\mathbb{R}^3} f(\mathbf{v})\, g(\mathbf{v})\, \mathcal{M}^{-1}(\mathbf{v})\, d\mathbf{v}\,.$$

In order to identify the limit-model corresponding to (6) when $\varepsilon \to 0$, one has to investigate in more details the dominant collision operator Q.

Proposition 2.1. *[3, 44] Under the assumption (9), the collision operator Q defined in (7), satisfies the following properties :*
(i) The linear operator $Q : \mathcal{H} \to \mathcal{H}$ is bounded, symmetric and non-positive.
(ii) The kernel of Q is given by

$$\ker(Q) := \{\rho\mathcal{M}(\mathbf{v}) \ / \ \rho \in \mathbb{R}\} \,.$$

(iii) The $\mathcal{H}$-orthogonal to the kernel of Q is

$$(\ker(Q))^\perp := \left\{ f \in \mathcal{H} \ / \ \int_{\mathbb{R}^3} f(\mathbf{v})\, d\mathbf{v} = 0 \right\} .$$

(iv) $-Q$ is coercive on $(\ker(Q))^\perp$, i.e.

$$-\langle Q(f), f\rangle \geq C||f||^2_{\mathcal{H}} \,, \quad \forall f \in (\ker(Q))^\perp \,.$$

(v) The range $\Im m(Q)$ of Q is closed and coincides with $(\ker(Q))^\perp$. We have moreover the one-to-one mapping

$$Q : (\ker(Q))^\perp \to (\ker(Q))^\perp \,.$$

(vi) Let Π be the mapping defined by

$$\Pi : \mathcal{H} \to \ker(Q)\,, \quad \Pi(f)(\mathbf{v}) := \left(\int_{\mathbb{R}^3} f(\mathbf{v}')\, d\mathbf{v}' \right) \mathcal{M}(\mathbf{v}) = \langle f \rangle\, \mathcal{M}(\mathbf{v})\,, \quad \forall f \in \mathcal{H}\,. \tag{10}$$

Then, we have

$$(f - \Pi(f), g)_{\mathcal{H}} = 0\,, \quad \forall f \in \mathcal{H}\,, \ \forall g \in \ker(Q)\,,$$

which means that Π is an orthogonal projection on $\ker(Q)$.

All these properties permit now to identify the limit problem of the Boltzmann equation (6) when the perturbation parameter ε tends to zero. This limit leads necessary to a macroscopic description of the electron gas. Indeed, inserting the Hilbert-Ansatz

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

in the Boltzmann equation (6) and equating the terms of the same order in ε, yields first that $f_0(t, \mathbf{x}, \cdot) \in \ker(Q)$. This means that there exists a density function $\rho_0(t, \mathbf{x})$ such that $f_0 = \rho_0\, \mathcal{M}$. Moreover, the second equation permits to compute the unique $f_1(t, \mathbf{x}, \cdot) \in (\ker(Q))^\perp$ via

$$\mathbf{v} \cdot \nabla_x f_0 - \mathbf{E} \cdot \nabla_v f_0 = Q(f_1) \quad \Rightarrow f_1 = Q^{-1}(\mathbf{v}\,\mathcal{M}) \cdot (\nabla_x \rho_0 + \rho_0\, \mathbf{E})\,.$$

The third equation finally yields the limit model (L-model)

$$(L) \ \ \partial_t \rho_0 - \nabla_x \cdot [D(\nabla_x \rho_0 + \rho_0\, \mathbf{E})] = 0\,, \tag{11}$$

with the diffusion-matrix given by $D := -\langle \mathbf{v} \otimes Q^{-1}(\mathbf{v}\,\mathcal{M})\rangle$. This is the so-called Drift-Diffusion model, describing the evolution of the macroscopic density function ρ_0 in the limit of vanishing mean free path. Remark that the microscale information is contained in this equation in a homogenized way, via the diffusion matrix D.

The construction of a Micro-Macro method, which is a reformulation of (6) being better suited to pass numerically to the limit $\varepsilon \to 0$, will be based on all the information gathered up to now. The (MM)-scheme is essentially founded on the decomposition of the unknown f into a macroscopic part (equilibrium) belonging to the kernel of the dominant operator, and the microscopic, fluctuating part, namely

$$f = \rho\mathcal{M} + \varepsilon g\,, \quad \rho(t,\mathbf{x})\mathcal{M} = \Pi(f) \in \ker(Q)\,, \quad g := \frac{1}{\varepsilon}(Id - \Pi)\, f \in (\ker(Q))^{\perp}\,.$$

Inserting this Ansatz in the kinetic equation (6), yields

$$(\partial_t \rho)\, \mathcal{M} + \varepsilon\, \partial_t g + \frac{1}{\varepsilon}\left[\nabla_x \rho \cdot \mathbf{v}\, \mathcal{M} + \varepsilon \mathbf{v} \cdot \nabla_x g + \rho \mathbf{E} \cdot \mathbf{v}\, \mathcal{M} - \varepsilon \mathbf{E} \cdot \nabla_v g\right] = \frac{1}{\varepsilon} Q(g)\,.$$

Applying now the projection Π on this equation, and performing the subtraction $I - \Pi$ permits to get a micro-macro system for the unknowns (ρ, g)

$$(MM)_\varepsilon \quad \begin{cases} \partial_t \rho + \nabla_x \cdot \langle \mathbf{v} g \rangle = 0 \\ \varepsilon\, \partial_t g + (I - \Pi)(\mathbf{v} \cdot \nabla_x g) - \mathbf{E} \cdot \nabla_v g = \dfrac{1}{\varepsilon} Q(g) - \dfrac{1}{\varepsilon}(\nabla_x \rho) \cdot \mathbf{v}\mathcal{M} - \dfrac{1}{\varepsilon} \rho \mathbf{E} \cdot \mathbf{v}\mathcal{M}\,. \end{cases} \tag{12}$$

This formulation is by construction equivalent to the initial equation (6). Moreover, in the limit $\varepsilon \to 0$ it permits to get immediately the macroscopic diffusion model, allowing thus for a uniform, regular transition between the kinetic and the macroscopic models.

Even if preserving the asymptotics for $\varepsilon \ll 1$, a big disadvantage of this method is the obtention and delicate numerical implementation of the projection operator Π. It is probably for this reason that this method is today still not applied for real-life problems, but only for simple test-cases.

3. Lagrange-Multiplier technique

The goal of this section is now to present and investigate the new Lagrange-Multiplier method introduced in [27], by applying it to the resolution of the following linear, stiff transport problem

$$(V)^\varepsilon \quad \begin{cases} \partial_t f^\varepsilon + \dfrac{\mathbf{b}}{\varepsilon} \cdot \nabla f^\varepsilon = 0\,, \qquad \forall t \in (0,T)\,,\ \forall \mathbf{x} = (x,y) \in \Omega \subset \mathbb{R}^2\,, \\ f^\varepsilon(0,\mathbf{x}) = f^\varepsilon_{in}(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega\,, \end{cases} \tag{13}$$

associated with adequate boundary conditions. We use here a very simple model in order to illustrate the basic design principles of the Lagrange-Multiplier approach and also to perform the rigorous mathematical study. The great advantage of this method is however the fact that it can be easily applied to more general evolution problems, containing some stiff term, namely

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\, \mathbf{b} \cdot \nabla f^\varepsilon + \mathcal{L} f^\varepsilon = 0\,,$$

with $\mathcal{L}$ an arbitrary differential operator. It is thus a rather general strategy, simple to implement (no field-aligned mesh needed) and preserving the asymptotics when $\varepsilon \to 0$.

The following Hypothesis shall be assumed in the sequel:

Hypothesis A : *The time-independent vector-field* $\mathbf{b} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ *is supposed to be given, sufficiently smooth (for ex.* $\mathbf{b} \in W^{1,\infty}(\Omega)$*) and divergence-free, meaning* $\nabla \cdot \mathbf{b} = 0$*. The domain* $\Omega \subset \mathbb{R}^2$ *will be an infinite strip* $(L_1, L_2) \times \mathbb{R}$ *of the* (x, y)*-plane. We shall assume periodic boundary conditions in* x *and the field* $\mathbf{b}$ *is supposed to be periodic in* x*.*

For a normed functional space X we shall denote in the following by $X_\sharp$ the space of all functions belonging to X, which satisfy the boundary conditions given in Hypothesis A.

Remark now that letting formally $\varepsilon \to 0$ in (13), leads to the ill-posed problem $\mathbf{b} \cdot \nabla f^0 = 0$, which does not permit to compute in a unique manner the limit solution $f^0(t, x, y)$. The only information we get is that f^0 is constant along the field-lines of $\mathbf{b}$. To be able to construct an Asymptotic-Preserving scheme permitting to capture even the limit solution f^0, it is necessary to investigate in more details the asymptotic behaviour of the solution sequence $\{f^\varepsilon\}_{\varepsilon>0}$, as $\varepsilon \to 0$. A detailed study of the dominant operator, here $\mathcal{T} := \mathbf{b} \cdot \nabla$, is hence required, as in the case of the Drift-Diffusion regime (Section 2.2, proposition 2.1). However the properties of the transport operator are much subtler than it could seem at first sight.

3.1. The transport operator.

The dominant operator $\mathcal{T} : Q \subset L^2_\sharp(\Omega) \to L^2_\sharp(\Omega)$ is a linear, unbounded transport operator with definition domain $\mathcal{D}(\mathcal{T}) = Q$ and kernel given by

$$Q := \{u \in L^2_\sharp(\Omega) \ / \ \mathbf{b} \cdot \nabla u \in L^2_\sharp(\Omega)\} \,, \qquad \ker \mathcal{T} := \{u \in L^2_\sharp(\Omega) \ / \ \mathbf{b} \cdot \nabla u = 0\} \,.$$

Providing $L^2_\sharp(\Omega)$ with the standard scalar-product, then it can be immediately shown that $\mathcal{T}$ is conservative , *i.e.* $(\mathcal{T}u, u) = 0$, and maximal monotone. Let us define now the L^2-orthogonal projection on the kernel of $\mathcal{T}$. The kernel is composed of functions which are constant along the field lines of $\mathbf{b}$. Thus, the projection is nothing else than the average of a quantity q along the field lines and will be denoted by $\Pi(q)$. Briefly, if $Z(s; \mathbf{x})$ is the characteristic flow associated to the field $\mathbf{b}$, *i.e.*

$$\left\{ \begin{array}{l} \dfrac{d}{ds} Z(s; \mathbf{x}) = \mathbf{b}(Z(s; \mathbf{x})) \,, \\ Z(0; \mathbf{x}) = \mathbf{x} \,, \end{array} \right.$$

the average of a function $q \in L^2_\sharp(\Omega)$ over the field lines of $\mathbf{b}$ is defined as

$$\Pi(q)(\mathbf{x}) := \lim_{S \to \infty} \frac{1}{S} \int_0^S q(Z(s; \mathbf{x}))\, ds \quad \forall \mathbf{x} \in \Omega \,, \qquad \Pi : L^2_\sharp(\Omega) \to \ker \mathcal{T} \,. \tag{14}$$

One can show (after some hypothesis on the regularity of $\mathbf{b}$, see [7]) that Π is a well-defined, linear and continuous application, verifying

$$||\Pi||_{\mathcal{L}(L^2_\sharp(\Omega),L^2_\sharp(\Omega))} \leq 1\,.$$

Furthermore one can show that $\Pi(\mathcal{T}q) = 0$ for all $q \in Q$, $\Pi(q) = q$ for all $q \in \ker \mathcal{T}$ and

$$(q - \Pi(q), \vartheta)_{L^2} = 0\,, \quad \forall \vartheta \in \ker \mathcal{T} \text{ and } \forall q \in L^2_\sharp(\Omega)\,,$$

meaning that Π is an L^2-orthogonal projection on $\ker \mathcal{T}$ and furthermore $\ker \Pi = (\ker \mathcal{T})^\perp$.

With these definitions one can now decompose the space $L^2_\sharp(\Omega)$ into

$$L^2_\sharp(\Omega) = \ker \mathcal{T} \oplus^\perp (\ker \mathcal{T})^\perp\,, \qquad f^\varepsilon = p^\varepsilon + q^\varepsilon\,, \qquad p^\varepsilon := \Pi(f^\varepsilon)\,,$$

and $\mathcal{T} : Q \cap (\ker \mathcal{T})^\perp \to \Im m\, \mathcal{T}$ is a one-to-one mapping. Remark that $\Im m\, \mathcal{T} \subset (\ker \mathcal{T})^\perp$, however contrary to the Drift-Diffusion regime (see Prop. 2.1), one can not state any more that the mapping $\mathcal{T} : Q \cap (\ker \mathcal{T})^\perp \to (\ker \mathcal{T})^\perp$ is a bijection. For this, one would need that $\Im m\, \mathcal{T}$ is closed, and hence that $\Im m\, \mathcal{T} = (\ker \mathcal{T})^\perp$, or equivalently that a Poincaré type inequality holds, as for example

$$||u||_{L^2_\sharp(\Omega)} \leq C\, ||\mathbf{b} \cdot \nabla u||_{L^2_\sharp(\Omega)}\,, \quad \forall u \in Q \cap (\ker \mathcal{T})^\perp\,,$$

which is not always the case for the transport operator $\mathcal{T}$. More details about this delicate point are postponed to Section **??**.

3.2. **Identification of the limit model.** The first step to do before constructing an AP-scheme, is to try to identify the Limit model $(V)^0$ corresponding to (13). Let us suppose for this that f^ε admits the following Hilbert expansion

$$f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \dots\,. \tag{15}$$

Injecting this Ansatz in (13) leads to the infinite hierarchy of equations

$$\mathcal{T} f^0 = 0, \tag{16}$$

$$\partial_t f^0 + \mathcal{T} f^1 = 0, \tag{17}$$

$$\partial_t f^1 + \mathcal{T} f^2 = 0, \tag{18}$$

$$\vdots$$

Equation (16) reveals that $f^0 \in \ker \mathcal{T}$. However, this information is not enough to determine completely f^0. It is necessary to use the next equation (17), to get the missing information. To eliminate f^1 from this equation, one projects (17) on the kernel of $\mathcal{T}$. Altogether, the limit model $(V)^0$ writes thus

$$(V)^0 \quad \begin{cases} \partial_t f^0 = 0\,, & \forall (t, \mathbf{x}) \in (0,T) \times \Omega\,, \\ \mathbf{b} \cdot \nabla f^0 = 0\,, & \forall (t, \mathbf{x}) \in (0,T) \times \Omega\,, \\ f^0(0,\mathbf{x}) = \Pi(f^0_{in}(\mathbf{x}))\,, & \mathbf{x} \in \Omega\,, \end{cases} \tag{19}$$

where we assumed an expansion of the initial condition of the form $f^\varepsilon_{in} = f^0_{in} + \varepsilon f^1_{in} + \varepsilon^2 f^2_{in} + \ldots$. Let us underline here again that we decided to perform the mathematical study with the elementary transport equation (13) in order to render the theory not too complicated and in the aim to single out the main difficulties and to understand their treatment via the (La)-approach.

The following theorem proves rigorously the convergence of the solution f^ε of (13) towards the solution f^0 of the limit model (19), as $\varepsilon \to 0$.

Theorem 3.1. *[7] Consider $\mathbf{b} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ with Ω an infinite strip satisfying Hypothesis A. Assume $\mathbf{b} \in W^{1,\infty}_{loc}(\mathbb{R}^2)$, when extended periodically to the whole $\mathbb{R}^2$, and satisfying $\nabla \cdot \mathbf{b} = 0$ as well as the growth condition*

$$\exists C > 0 \ \ s.t. \ \ |\mathbf{b}(\mathbf{x})| \leq C\,(1 + |\mathbf{x}|)\,, \quad \forall \mathbf{x} \in \Omega\,.$$

Suppose furthermore that $f^\varepsilon_{in} \in L^2_\sharp(\Omega)$ satisfies $f^\varepsilon_{in} \rightharpoonup_{\varepsilon\to 0} f^0_{in}$ in $L^2_\sharp(\Omega)$. Then problems (13) resp. (19) admit unique weak solutions $f^\varepsilon, f^0 \in L^\infty(0,T; L^2_\sharp(\Omega))$ and one has $f^\varepsilon \underset{\epsilon\to 0}{\overset{\star}{\rightharpoonup}} f^0$, weakly-$\star$ in $L^\infty(0,T; L^2_\sharp(\Omega))$.
If the initial conditions are well prepared in the sense that f^ε_{in} is smooth enough and satisfies $f^\varepsilon_{in} \underset{\epsilon\to 0}{\to} f^0_{in}$ in $L^2_\sharp(\Omega)$ with $f^0_{in} \in \ker \mathcal{T}$, then one has even $f^\varepsilon \underset{\epsilon\to 0}{\to} f^0$ in $L^\infty(0,T; L^2_\sharp(\Omega))$.

3.3. **Micro-Macro reformulation.** The design of a multiscale numerical procedure for the resolution of problem (13) is now inspired by the asymptotic study performed in Section 3.2. To recover the missing microscopic information in the reduced model $\mathcal{T} f^0 = 0$, we shall decompose f^ε into a macroscopic and a microscopic part (see also the Drift-Diffusion decomposition in Section 2.1), as follows

$$f^\varepsilon = p^\varepsilon + \varepsilon\, q^\varepsilon\,, \quad \text{with } p^\varepsilon := \Pi(f^\varepsilon) \quad \Rightarrow \quad \mathbf{b} \cdot \nabla f^\varepsilon = \varepsilon\, \mathbf{b} \cdot \nabla q^\varepsilon\,. \tag{20}$$

This signifies that p^ε belongs to the kernel of the dominant operator $\mathcal{T} = \mathbf{b} \cdot \nabla$ and is considered as the macroscopic part, whereas q^ε is the so-called fluctuating or microscopic part of the distribution function f^ε.
Remark here that the decomposition (20) is slightly different from the Hilbert expansion (15). Indeed (20) is more closely related to a Chapmann-Enskog Ansatz, however with all higher order terms regrouped in q^ε.

Plugging (20) into (13) leads to the following extended system for the two unknowns $(f^\varepsilon, q^\varepsilon)$

$$(MM)_\varepsilon \quad \left\{ \begin{array}{l} \partial_t f^\varepsilon + \mathbf{b} \cdot \nabla q^\varepsilon = 0\,, \\[2mm] \mathbf{b} \cdot \nabla f^\varepsilon = \varepsilon\, \mathbf{b} \cdot \nabla q^\varepsilon\,, \\[2mm] \Pi(q^\varepsilon) = 0\,, \end{array} \right. \tag{21}$$

associated with the initial condition $f^\varepsilon(0,\cdot) = f^\varepsilon_{in}$ and adequate boundary conditions (*Hypothesis A*). Make here also the link between (21) and the Hilbert hierarchy (16)-(17).

System (21) is for all $\varepsilon > 0$ completely equivalent to the stiff kinetic equation (13), however it behaves better in the limit $\varepsilon \to 0$. Indeed, one gets formally

$$(MM)_0 \quad \begin{cases} \partial_t f^0 + \mathbf{b} \cdot \nabla q^0 = 0\,, \\ \mathbf{b} \cdot \nabla f^0 = 0\,, \\ \Pi(q^0) = 0\,, \end{cases} \tag{22}$$

which yields $f^0 \in \ker \mathcal{T}$, $q^0 \in (\ker \mathcal{T})^\perp$, both being uniquely determined via the first equation. The function q^0 can be seen as a Lagrange multiplier, corresponding to the constraint $\mathbf{b} \cdot \nabla f^0 = 0$. To see that (22) is equivalent to the limit problem (19), let us take the projection Π of the first equation, which leads to $\partial_t \Pi(f^0) = 0$ such that, with $f^0 \in \ker \mathcal{T}$, one has $f^0 = \Pi(f^0) = \Pi(f^0_{in})$. With this, the first equation reduces to $\mathbf{b} \cdot \nabla q^0 = 0$, thus $q^0 \in \ker \mathcal{T}$, which together with $q^0 \in (\ker \mathcal{T})^\perp$ yields $q^0 \equiv 0$. We recognize thus the limit model (19).
Attracting again the attention of the reader to the similarity between (22) and the Hilbert hierarchy (16)-(17), one can understand somehow the third equation in (22) as a manner to close the infinite Hilbert hierarchy in the limit $\varepsilon \to 0$.

The constraint $\Pi(q^\varepsilon) = 0$ in (21) is very important to fix the values of q^ε on the field-lines of $\mathbf{b}$, fact which is nothing else than rendering the decomposition (20) unique. Other ways are possible to settle the values of q^ε on the field-lines of $\mathbf{b}$, leading necessary to different micro-macro decompositions of f^ε, keeping however always $p^\varepsilon \in \ker \mathcal{T}$. For example in the case of a flow $\mathbf{b}$ with field lines which enter and escape from the domain (no closed field lines), one can impose $q^\varepsilon_{|\Gamma_{in}} \equiv 0$, meaning $p^\varepsilon = f^\varepsilon_{|\Gamma_{in}}$. The corresponding decomposition $f^\varepsilon = p^\varepsilon + \varepsilon q^\varepsilon$ is no more orthogonal in $L^2_\sharp(\Omega)$. This choice works very well in such flows, however is not adequate for closed field line configurations. Important is to observe that the value of q^ε along the field lines of $\mathbf{b}$ is of no importance for the computation of our physical unknown f^ε, as only $\mathbf{b} \cdot \nabla q^\varepsilon$ is occurring in the system (21). Thus any arbitrary choice could do the work.

Let us finally remark that it is more efficient to implement instead of (21), the slightly changed, but equivalent system

$$(MM)_\varepsilon \quad \begin{cases} \partial_t f^\varepsilon + \mathbf{b} \cdot \nabla q^\varepsilon = 0\,, & \forall (t, \mathbf{x}) \in (0,T) \times \Omega\,, \\ \mathbf{b} \cdot \nabla f^\varepsilon = \varepsilon\, \mathbf{b} \cdot \nabla q^\varepsilon - \sigma \Pi(q^\varepsilon)\,, & \forall (t, \mathbf{x}) \in (0,T) \times \Omega\,, \end{cases} \tag{23}$$

with some parameter $\sigma > 0$. Indeed, one can remark immediately that taking the average of the second equation over the field-lines yields automatically the constraint $\Pi(q^\varepsilon) = 0$.

The Micro-Macro idea (21) or (23) is very nice from a mathematical point of view, however, if one is thinking at the numerical implementation, one has to average over the field lines of $\mathbf{b}$, in order to discretize the new term $\Pi(q^\varepsilon)$ in the second equation of (23). This procedure is rather hard (we are working on Cartesian grids, not-aligned with the field $\mathbf{b}$) and it can introduce moreover ε-dependent error terms in the results. Thus we shall leave this idea behind, and search for a more practical one.

3.4. **The regularized AP-reformulation.** An alternative idea to render q^ϵ unique in (23), is to use a regularization technique. Regularization is a very broad field in mathematics, and is devoted to the design and analysis of methods for obtaining stable solutions of ill-posed problems. In particular, the usual regularization technique consists in replacing the ill-posed problem by a nearby (slightly-perturbed) well-posed problem, whose resolution poses no difficulties (uniqueness, stability of the solution). The original solution is recovered only in the limit of vanishing regularization/perturbation parameter. The choice of the perturbation term as well as the strength of the perturbation parameter is essential and constitutes the key point of the method. There is a rich literature on regularization techniques, we refer the interested reader to the references [5, 10, 16, 24].

Our Asymptotic-Preserving scheme for an efficient resolution of the anisotropic transport equation (13) is based on the resolution of the following reformulated system

$$(La)_\varepsilon^\sigma \quad \begin{cases} \partial_t f^{\varepsilon,\sigma} + \mathbf{b}\cdot\nabla q^{\varepsilon,\sigma} = 0\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,, \\ \mathbf{b}\cdot\nabla f^{\varepsilon,\sigma} = \varepsilon\,\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma} - \sigma\, q^{\varepsilon,\sigma}\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,, \end{cases} \tag{24}$$

with $\sigma > 0$ a small parameter to be fixed later on. This system is completed by an initial condition $f^{\varepsilon,\sigma}(0,\cdot) = f^\varepsilon_{in}(\cdot)$ and adequate boundary conditions (*Hypothesis A*). Let us underline here the difference between (23) and (24). Both procedures are fixing the value of the auxiliary variable q on the field lines of $\mathbf{b}$ by imposing zero mean $\Pi(q^{\varepsilon,\sigma}) = 0$. To see this in (24), it is enough to take the average/projection Π of the second equation. However, while (23) is completely equivalent to the starting model (13), the system (24) introduces an error, as the supplementary term we introduced, $\sigma q^{\varepsilon,\sigma}$, is no more zero but contains also the non-zero fluctuation part of $q^{\varepsilon,\sigma}$. The big advantage of (24) with respect to (23) is that this time we have no more to discretize the average operator Π.

The ε-regularity of the system (24) allows now to pass directly to the $\varepsilon \to 0$ limit in (24) to get the corresponding limit model, *i.e.*

$$(La)_0^\sigma \quad \begin{cases} \partial_t f^{0,\sigma} + \mathbf{b}\cdot\nabla q^{0,\sigma} = 0\,, \\ \mathbf{b}\cdot\nabla f^{0,\sigma} + \sigma\, q^{0,\sigma} = 0\,. \end{cases} \tag{25}$$

Eliminating $q^{0,\sigma}$ from this system, yields the degenerate diffusion equation

$$\partial_t f^{0,\sigma} - \frac{1}{\sigma}\nabla\cdot\left[(\mathbf{b}\otimes\mathbf{b})\,\nabla f^{0,\sigma}\right] = 0\,, \tag{26}$$

which shows clearly what the regularization term is doing in the limit $\varepsilon \to 0$, in particular it forces the solution $f^{0,\sigma}$ to diffuse rapidly along the field lines of $\mathbf{b}$ and to approach thus a function belonging to the kernel of $\mathcal{T}$.

4. Mathematical study of the Lagrange-Multiplier reformulation

The rigorous mathematical study of the existence and uniqueness of a solution $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma})$ to the following AP-reformulation (with fixed $\varepsilon \geq 0$ and $\sigma \geq 0$)

$$(La)_\varepsilon^\sigma \quad \begin{cases} \partial_t f^{\varepsilon,\sigma} + \mathbf{b} \cdot \nabla q^{\varepsilon,\sigma} = 0\,, & \forall (t, \mathbf{x}) \in (0,T) \times \Omega\,, \\ \mathbf{b} \cdot \nabla f^{\varepsilon,\sigma} = \varepsilon\, \mathbf{b} \cdot \nabla q^{\varepsilon,\sigma} - \sigma\, q^{\varepsilon,\sigma}\,, & \forall (t, \mathbf{x}) \in (0,T) \times \Omega\,, \end{cases} \tag{27}$$

along with the rigorous study of the various limits $\varepsilon, \sigma \to 0$, especially its $\varepsilon \to 0$ limit towards (25), is an important point for our AP-scheme design and shall be treated in this section. Boundary conditions are specified in Hypothesis A. Figure 1 briefly sketches the different problems we want to investigate and their corresponding asymptotic relations.

$$\begin{tikzcd}
(La)_\varepsilon^\sigma \arrow[r, "\varepsilon \to 0"] \arrow[d, "\sigma \to 0"] & (La)_0^\sigma \arrow[d, "\sigma \to 0"] \\
(La)_\varepsilon^0 \arrow[r, "\varepsilon \to 0"] & (La)_0^0
\end{tikzcd}$$

Figure 1. Relations between the different models

4.1. **Existence and uniqueness results.** In the following we shall suppose Hypothesis A to be verified and we shall treat the cases $[\varepsilon > 0,\ \sigma > 0]$, $[\varepsilon = 0,\ \sigma > 0]$, $[\varepsilon > 0,\ \sigma = 0]$ and $[\varepsilon = 0,\ \sigma = 0]$ separately. Our aim is to prove for fixed (ε, σ) the existence and uniqueness of a solution to $(La)_\varepsilon^\sigma$.

Let us start the study by introducing the mathematical framework, especially some useful functional spaces and operators. Thus, let V, Q, W be three Hilbert-spaces, defined as

$$V := L^2_\sharp(\Omega)\,, \quad Q := \{ g \in V \ / \ \mathbf{b} \cdot \nabla g \in L^2_\sharp(\Omega) \}\,, \quad W := \{ g \in Q \ / \ \nabla \cdot (\mathbf{b} \otimes \mathbf{b}\, \nabla g) \in L^2_\sharp(\Omega) \}\,,$$

where V is associated with the standard L^2 scalar-product and the other two spaces with

$$(u,v)_Q := (u,v)_{L^2} + (\mathbf{b} \cdot \nabla u, \mathbf{b} \cdot \nabla v)_{L^2}\,, \quad (u,v)_W := (u,v)_Q + (\nabla \cdot (\mathbf{b} \otimes \mathbf{b}\, \nabla u), \nabla \cdot (\mathbf{b} \otimes \mathbf{b}\, \nabla v))_{L^2}\,.$$

Remark that Q is densely embedded in V such that one has the evolution triple $Q \subset V \equiv V^* \subset Q^*$. The embedding $Q \subset V$ is not necessarily compact, as Ω is not a bounded domain.

Case I: Let us first consider the case of fixed $\varepsilon > 0$ *and* $\sigma > 0$. Using the above introduced definitions we denote now the transport operator by $\mathcal{T} \in \mathcal{L}(Q, V)$ where $\mathcal{T} u := \mathbf{b} \cdot \nabla u$ and its "regularization" by $\mathcal{A}_{\varepsilon,\sigma}$, defined as

$$\mathcal{A}_{\varepsilon,\sigma} : Q \to L^2_\sharp(\Omega)\,, \qquad \mathcal{A}_{\varepsilon,\sigma} q := \sigma q - \varepsilon\, \mathbf{b}\cdot\nabla q\,.$$

The operator $\mathcal{A}_{\varepsilon,\sigma}$ is linear and continuous, thus belongs to $\mathcal{L}(Q,V)$, and it is furthermore a bijective mapping (for fixed $\varepsilon > 0$ and $\sigma > 0$), such that one can define its inverse

$$\mathcal{M}_{\varepsilon,\sigma} : L^2_\sharp(\Omega) \to Q\,, \qquad \mathcal{M}_{\varepsilon,\sigma} := \mathcal{A}^{-1}_{\varepsilon,\sigma} \in \mathcal{L}(V,Q)\,.$$

The bijectivity of $\mathcal{A}_{\varepsilon,\sigma}$ can be immediately proven by rewriting this operator as $\mathcal{A}_{\varepsilon,\sigma} = \sigma\, Id - \varepsilon\,\mathcal{T}$, where we recall that the transport operator $-\mathcal{T} : Q \subset L^2_\sharp(\Omega) \to L^2_\sharp(\Omega)$ is a maximal monotone operator.

As ε and σ are fixed in this subsection, we shall skip for simplicity reasons in the following the indices for the two operators $\mathcal{A}_{\varepsilon,\sigma}$ and $\mathcal{M}_{\varepsilon,\sigma}$. Let us mention furthermore that $\mathcal{A}$ as well as $\mathcal{M}$ can be also seen as linear, continuous and bijective operators within the following spaces

$$\mathcal{A} : L^2_\sharp(\Omega) \to Q^*\,, \qquad \mathcal{M} : Q^* \to L^2_\sharp(\Omega)\,, \qquad \mathcal{A} : W \to Q\,, \qquad \mathcal{M} : Q \to W\,.$$

Moreover, let us precise some continuity and coercivity estimates, needed in the sequel

$$\min\{\varepsilon,\sigma\}\, ||v||_Q \le ||\mathcal{A}\, v||_V \le \max\{\varepsilon,\sigma\}\, ||v||_Q\,, \quad \forall v \in Q\,,$$

$$(\max\{\varepsilon,\sigma\})^{-1}\, ||u||_V \le ||\mathcal{M}\, u||_Q \le (\min\{\varepsilon,\sigma\})^{-1}\, ||u||_V\,, \quad \forall u \in V\,.$$

With all this, we can reformulate system (27) for $\varepsilon > 0$ and $\sigma > 0$ in terms of the only unknown $f^{\varepsilon,\sigma}$. The second equation reads indeed

$$\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma} = -\mathcal{A}\, q^{\varepsilon,\sigma} \;\Leftrightarrow\; -\mathcal{M}(\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma}) = q^{\varepsilon,\sigma} \;\Leftrightarrow\; -\mathbf{b}\cdot\nabla(\mathcal{M} f^{\varepsilon,\sigma}) = q^{\varepsilon,\sigma}\,. \tag{28}$$

Inserting (28) in the first equation of (27) yields finally

$$\partial_t f^{\varepsilon,\sigma} - \nabla\cdot(\mathbf{b}\otimes\mathbf{b}\,\nabla(\mathcal{M} f^{\varepsilon,\sigma})) = 0 \quad \text{or} \quad \partial_t(\mathcal{A} g^{\varepsilon,\sigma}) - \nabla\cdot(\mathbf{b}\otimes\mathbf{b}\,\nabla g^{\varepsilon,\sigma}) = 0\,, \tag{29}$$

where we have introduced a new unknown via the relation $\mathcal{A}\, g^{\varepsilon,\sigma} = f^{\varepsilon,\sigma}$. We shall study the existence and uniqueness of a solution of this last equation (29) via the Hille-Yosida theorem, thus a last linear operator will be introduced

$$\mathcal{C} : Q \subset L^2_\sharp(\Omega) \to L^2_\sharp(\Omega)\,, \quad \mathcal{C} u := -\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\,\nabla(\mathcal{M} u))\,, \quad \mathcal{C} \in \mathcal{L}(Q,V)\,.$$

The aim of the next theorem is to investigate for fixed $\varepsilon > 0$ and $\sigma > 0$ the well-posedness of the Cauchy-problem

$$\left\{ \begin{array}{l} \partial_t f^{\varepsilon,\sigma} + \mathcal{C} f^{\varepsilon,\sigma} = 0 \\ f^{\varepsilon,\sigma}(0,\cdot) = f^{\varepsilon}_{in}\,. \end{array} \right. \tag{30}$$

Theorem 4.1. *Let Hypothesis A be satisfied and $\varepsilon > 0$, $\sigma > 0$ be fixed. Then for every initial condition $f^{\varepsilon}_{in} \in Q$ there exists a unique solution $f^{\varepsilon,\sigma} \in C^1([0,\infty), L^2_\sharp(\Omega)) \cap C^0([0,\infty), Q)$ of the evolution problem (30).*

Proof. The proof is simply based on Hille-Yosida's theorem [43]. We have only to show that the operator $\mathcal{C}$ is maximal monotone. The monotonicity of $\mathcal{C}$ is due to

$$(\mathcal{C}u, u)_{L^2} = -(\nabla \cdot (\mathbf{b} \otimes \mathbf{b}\, \nabla(\mathcal{M}u)), u)_{L^2} = -(\nabla \cdot (\mathbf{b} \otimes \mathbf{b}\, \nabla v), \mathcal{A}v)_{L^2}\,, \quad \forall u \in Q\,,$$

where we introduced $v := \mathcal{M}u \in W$ such that $u = \mathcal{A}v$. Now one has

$$-(\nabla \cdot (\mathbf{b} \otimes \mathbf{b}\, \nabla v), \mathcal{A}v)_{L^2} = \sigma ||\mathbf{b} \cdot \nabla v||^2_{L^2} \geq 0\,, \quad \forall v \in W\,.$$

To prove that $\mathcal{C}$ is indeed maximal monotone, it remains to show that there exists a constant $\lambda > 0$ for which $\lambda\, Id + \mathcal{C} : Q \to L^2_\sharp(\Omega)$ is surjective. To show this, let us consider for a given $\theta \in L^2_\sharp(\Omega)$ the equation

$$\lambda\, u + \mathcal{C}u = \theta\,,$$

and show the existence of some solution $u \in Q$. Let us define again $v := \mathcal{M}u \in W$ such that $u = \mathcal{A}v = \sigma v - \varepsilon\, \mathbf{b} \cdot \nabla v$. Making this chance of variables in the previous equation leads to

$$\sigma\, \lambda\, v - \varepsilon\, \lambda\, \mathbf{b} \cdot \nabla v - \nabla \cdot (\mathbf{b} \otimes \mathbf{b}\, \nabla v) = \theta\,, \tag{31}$$

associated with the boundary conditions explicited in Hypothesis A. This equation has indeed, for each $\theta \in L^2_\sharp(\Omega)$ and fixed $\lambda > 0, \varepsilon > 0, \sigma > 0$ a unique solution $v \in W$, implying that $\mathcal{C}$ is in fact maximal monotone, concluding thus the proof. The well-posedness of (31) is based on Lax-Milgram's theorem, with the bilinear, continuous and coercive form $\mathfrak{M} : Q \times Q \to \mathbb{R}$ defined as

$$\mathfrak{M}(v, w) := \sigma\, \lambda\, (v, w)_{L^2} - \varepsilon\, \lambda\, (\mathbf{b} \cdot \nabla v, w)_{L^2} + (\mathbf{b} \cdot \nabla v, \mathbf{b} \cdot \nabla w)_{L^2}\,, \quad \forall (v, w) \in Q \times Q\,.$$

□

Knowing now that there exists a unique solution $f^{\varepsilon,\sigma} \in C^1([0,\infty), L^2_\sharp(\Omega)) \cap C^0([0,\infty), Q)$ to (30), we can immediately say something about the auxiliary Lagrange multiplier $q^{\varepsilon,\sigma}$ arising in (27). Indeed, the relation

$$-\mathcal{M}(\mathbf{b} \cdot \nabla f^{\varepsilon,\sigma}) = q^{\varepsilon,\sigma}\,, \tag{32}$$

permits to show that $q^{\varepsilon,\sigma} \in C^1([0,\infty), L^2_\sharp(\Omega)) \cap C^0([0,\infty), Q)$. This is only possible due to the regularization term $\sigma\, q^{\varepsilon,\sigma}$, introduced expressly to be able to solve the second equation in (27) for $q^{\varepsilon,\sigma}$ via (32).

Thus, altogether, we proved the following existence and uniqueness theorem for our reformulated AP-problem (27).

Theorem 4.2. ***(Existence/uniqueness for*** $\varepsilon > 0$ ***and*** $\sigma > 0$***)*** *Let Hypothesis A be satisfied and* $\varepsilon > 0$, $\sigma > 0$ *be fixed. Then for every initial condition* $f^\varepsilon_{in} \in Q$ *there exists a unique solution* $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma}) \in (C^1([0,\infty), L^2_\sharp(\Omega)))^2 \cap (C^0([0,\infty), Q))^2$ *of the AP-problem* (27)*. Moreover one has even* $q^{\varepsilon,\sigma} \in C^0([0,\infty), Q \cap (\ker \mathcal{T})^\perp)$*.*

Case II: Let us treat now the special two cases $[\varepsilon = 0,\ \sigma > 0]$ *and* $[\varepsilon > 0,\ \sigma = 0]$. In the case $\varepsilon = 0$ the problem (27) reduces, for the unknown $f^{0,\sigma}$, to the degenerate parabolic equation

$$\partial_t f^{0,\sigma} - \frac{1}{\sigma}\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\nabla f^{0,\sigma}) = 0\,. \tag{33}$$

Lions' theory for evolution problems [12] as well as the relation $q^{0,\sigma} = -\frac{1}{\sigma}\,\mathbf{b}\cdot\nabla f^{0,\sigma}$ permits in this case to prove the following existence/uniqueness result.

Theorem 4.3. ***(Existence/uniqueness for*** $\varepsilon = 0$ ***and*** $\sigma > 0$***)*** *Let Hypothesis A be satisfied and* $\varepsilon = 0$, $\sigma > 0$ *be fixed. Then for any fixed* $T > 0$ *and every initial condition* $f^0_{in} \in Q$ *there exists a unique solution of the corresponding AP-problem* (27)*, satisfying* $f^{0,\sigma} \in L^2(0,T;W)$ *and* $\partial_t f^{0,\sigma} \in L^2(0,T;L^2_\sharp(\Omega))$*, in particular* $f^{0,\sigma} \in C([0,T],Q)$*. Moreover the auxiliary variable satisfies* $q^{0,\sigma} \in L^2(0,T;Q\cap(\ker\mathcal{T})^\perp)$*.*

Remark 4.4. *Let us remark here that we could use the Hille-Yosida theorem for the existence and uniqueness proof of a solution to* (33)*. In the more regular case* $f^0_{in} \in W$*, we would have the existence of a unique solution* $f^{0,\sigma} \in C^1([0,\infty);V)\cap C^0([0,\infty);W)$ *and thus* $q^{0,\sigma} \in C^0([0,\infty);Q\cap(\ker\mathcal{T})^\perp)$*. However, for* $f^0_{in} \in Q$*, one should make use of the generalized Hille-Yosida theorem [11], which states then that the unique solution is less regular, namely* $f^{0,\sigma} \in C^0([0,\infty);Q)\cap C^1((0,\infty);V)\cap C^0((0,\infty);W)$*. Observe that* $\partial_t f^{0,\sigma}$ *could explode for* $t\to 0^+$*, in particular*

$$||\partial_t f^{0,\sigma}(t,\cdot)||_V = \frac{1}{\sigma}\,||\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\nabla f^{0,\sigma})||_V \le \frac{1}{t}\,||f^0_{in}||_V\,,\quad \forall t>0\,. \tag{34}$$

Now if $\sigma = 0$, the problem (27) reduces, for the unknown $f^{\varepsilon,0}$, to the following transport equation, which is nothing else than our original problem (13), namely

$$\partial_t f^{\varepsilon,0} + \frac{1}{\varepsilon}\,\mathbf{b}\cdot\nabla f^{\varepsilon,0} = 0\,. \tag{35}$$

Thus, the standard transport theory (Hille-Yosida theorem) yields

Theorem 4.5. ***(Existence/uniqueness for*** $\varepsilon > 0$ ***and*** $\sigma = 0$***)*** *Let Hypothesis A be satisfied and* $\varepsilon > 0$, $\sigma = 0$ *be fixed. Then for every initial condition* $f^\varepsilon_{in} \in Q$ *the corresponding problem* (27) *admits a unique solution* $f^{\varepsilon,0} \in C^1([0,\infty),L^2_\sharp(\Omega))\cap C^0([0,\infty),Q)$ *and the auxiliary variable* $q^{\varepsilon,0}$ *is unique in* $C^0([0,\infty),Q\cap(\ker\mathcal{T})^\perp)$*.*

Proof. The existence/uniqueness of $q^{\varepsilon,0} \in C^0([0,\infty),Q\cap(\ker\mathcal{T})^\perp)$ comes from the relation $\mathbf{b}\cdot\nabla q^{\varepsilon,0} = \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla f^{\varepsilon,0}$ and the fact that $\mathcal{T}$ is a bijective mapping between the following spaces

$$\mathcal{T} : Q\cap(\ker\mathcal{T})^\perp \to \Im m\,\mathcal{T}\,.$$

Remark that $q^{\varepsilon,0}$ is not unique in $C^0([0,\infty),Q)$. ☐

Case III: Finally, let us treat the last case $[\varepsilon = 0,\ \sigma = 0]$.

Theorem 4.6. ***(Existence/uniqueness for*** $\varepsilon = 0$ ***and*** $\sigma = 0$***)*** *Let Hypothesis A be satisfied and* $\varepsilon = 0$, $\sigma = 0$ *be fixed. Then for every initial condition* $f^0_{in} \in \ker\mathcal{T}$ *there exists a unique solution* $f^{0,0} \in C^\infty([0,\infty),\ker\mathcal{T})$ *of the corresponding problem* (27)*, namely*

$f^{0,0} \equiv f^0_{in}$. *The auxiliary variable* $q^{0,0}$ *is unique in* $C^\infty([0,\infty), Q \cap (\ker \mathcal{T})^\perp)$ *and equal to zero.*

Proof. For $\varepsilon = 0$ and $\sigma = 0$ problem (27) writes

$$(La)^0_0 \quad \begin{cases} \partial_t f^{0,0} + \mathbf{b} \cdot \nabla q^{0,0} = 0\,, & \forall (t,\mathbf{x}) \in (0,T) \times \Omega\,, \\ \mathbf{b} \cdot \nabla f^{0,0} = 0\,, & \forall (t,\mathbf{x}) \in (0,T) \times \Omega\,. \end{cases} \tag{36}$$

The last equation permits to show that $f^{0,0}(t,\cdot) \in \ker \mathcal{T}$, such that $\Pi(f^{0,0}) = f^{0,0}$. Taking now the projection Π of the first equation yields

$$\partial_t \Pi(f^{0,0}) = \partial_t f^{0,0} = 0\,,$$

such that one immediately has the solution $f^{0,0} = f^0_{in}$. Furthermore, with this information the first equation becomes $\mathbf{b} \cdot \nabla q^{0,0} = 0$, meaning $q^{0,0}(t,\cdot) \in \ker \mathcal{T}$. Thus $q^{0,0} \equiv 0$ in $L^2(0,T; Q \cap (\ker \mathcal{T})^\perp)$, which finishes the proof. ☐

Remark 4.7. *The difference between the original, singularly-perturbed problem* (13) *and the "reformulation"* (30) *comes from the regularization term* $\sigma q^{\varepsilon,\sigma}$, *which yields* $\mathcal{A}_{\varepsilon,\sigma}$ *invertible for all* $\varepsilon \geq 0$, *thus leading to a regular problem* (30) *when considering the limit* $\varepsilon \to 0$. *Observe also that equation* (13) *is a local transport problem, whereas* (30) *is a non-local "parabolic" equation, fact which arises from the operator* $\mathcal{M}_{\varepsilon,\sigma}$.

Remark 4.8. *Again, due to the regularization term* $\sigma q^{\varepsilon,\sigma}$ *one has* $q^{\varepsilon,\sigma}(t,\cdot) \in \Im m \mathcal{T} \subset (\ker \mathcal{T})^\perp$ *for all* $\varepsilon \geq 0$ *and* $\sigma > 0$. *In the case* $\sigma = 0$, *the auxiliary variable* $q^{\varepsilon,\sigma}$ *is no more unique in* $C^0([0,\infty), Q)$, *but in* $C^0([0,\infty), Q \cap (\ker \mathcal{T})^\perp)$. *To have uniqueness, one has thus to enforce the property* $q^{\varepsilon,\sigma}(t,\cdot) \in (\ker \mathcal{T})^\perp$ *by asking for example* $\Pi(q^{\varepsilon,\sigma}(t,\cdot)) = 0$, *which is nothing else than system* (21).

4.2. **A priori estimates and weak asymptotic limits $\varepsilon \to 0$ and $\sigma \to 0$.** The aim of this section is to understand what happens when both parameters $\varepsilon \to 0$ and $\sigma \to 0$ vanish independently (see Figure 1). Remark that σ is a regularization parameter, so that its limit towards zero will be more problematic. In this subsection, we shall not care about the initial conditions, the only hypothesis shall be that $\{f^\varepsilon_{in}\}$ is independent on σ and bounded in Q with respect to ε.

Theorem 4.9. ***(A priori estimates)*** *Let us consider problem* (27) *for* $\sigma > 0$ *and* $\varepsilon \geq 0$ *with its corresponding unique solutions and* σ*-independent initial conditions* $\{f^\varepsilon_{in}\}_\varepsilon \subset Q$. *Then one has the following estimates, with* $C > 0$ *some constants independent on* ε *and* σ, *but possibly dependent on the final time* $T > 0$:
(i) For $\sigma > 0$ *and* $\varepsilon > 0$:

$$||f^{\varepsilon,\sigma}||_{L^\infty(\mathbb{R}^+,Q)} \leq ||f^\varepsilon_{in}||_Q\,, \quad ||\partial_t f^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \leq C\, \min\{1/\varepsilon^2,\, 1/\sigma\}\, ||\mathbf{b} \cdot \nabla f^\varepsilon_{in}||^2_V\,,$$

as well as

$$||q^{\varepsilon,\sigma}||_{L^\infty(\mathbb{R}^+,V)} \leq ||\mathbf{b} \cdot \nabla f^\varepsilon_{in}||_V/\sigma\,, \quad ||\mathbf{b} \cdot \nabla q^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \leq C\, \min\{1/\varepsilon^2,\, 1/\sigma\}\, ||\mathbf{b} \cdot \nabla f^\varepsilon_{in}||^2_V\,.$$

(ii) For $\sigma > 0$ *and* $\varepsilon = 0$:

$$||f^{0,\sigma}||_{L^\infty(\mathbb{R}^+,V)} \leq ||f^0_{in}||_V\,, \quad ||\mathbf{b} \cdot \nabla f^{0,\sigma}||^2_{L^2(0,T;V)} \leq C\, \sigma\, ||f^0_{in}||^2_V\,,$$

and

$$\sigma\,||q^{0,\sigma}||^2_{L^2(0,T;V)} \le C\,||f^0_{in}||^2_V\,.$$

One can also show that slightly detached from $t=0$, namely for each fixed $\delta>0$, we have

$$||\partial_t f^{0,\sigma}||_{L^\infty(\delta,\infty;V)} \le \frac{1}{\delta}\,||f^0_{in}||_V\,, \quad ||\mathbf{b}\cdot\nabla q^{0,\sigma}||_{L^\infty(\delta,\infty;V)} \le \frac{1}{\delta}\,||f^0_{in}||_V\,.$$

Proof. (i) Case $[\varepsilon>0,\ \sigma>0]$. Let $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma}) \in \left[C^1([0,\infty), L^2_\sharp(\Omega)) \cap C^0([0,\infty), Q)\right]^2$ be the unique solution of the AP-problem (27). Considering the equivalent form (30), namely

$$\partial_t f^{\varepsilon,\sigma} - \nabla\cdot(\mathbf{b}\otimes\mathbf{b}\,\nabla(\mathcal{M} f^{\varepsilon,\sigma})) = 0\,, \tag{37}$$

multiplying this equation with $f^{\varepsilon,\sigma}$ and integrating in space only, permits to get

$$\frac{1}{2}\,\frac{d}{dt}||f^{\varepsilon,\sigma}||^2_V + \sigma\,||\mathbf{b}\cdot\nabla(\mathcal{M} f^{\varepsilon,\sigma})||^2_V = 0\,,$$

thus

$$||f^{\varepsilon,\sigma}||_{L^\infty(\mathbb{R}^+,V)} \le ||f^\varepsilon_{in}||_V\,. \tag{38}$$

Applying now the operator $\mathbf{b}\cdot\nabla$ on (37) (in a distributional sense) and multiplying this time with $\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma} \in C^0([0,\infty), L^2_\sharp(\Omega))$, leads to

$$\frac{1}{2}\,\frac{d}{dt}||\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma}||^2_V + \sigma\,||\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\nabla(\mathcal{M} f^{\varepsilon,\sigma}))||^2_V = 0\,, \tag{39}$$

thus

$$||\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma}||_{L^\infty(\mathbb{R}^+,V)} \le ||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V\,. \tag{40}$$

These two estimates permit to have, independently on $\varepsilon>0$ and $\sigma>0$, the bound

$$||f^{\varepsilon,\sigma}||_{L^\infty(\mathbb{R}^+,Q)} \le ||f^\varepsilon_{in}||_Q\,. \tag{41}$$

Let us try now to find some estimates for $\partial_t f^{\varepsilon,\sigma}$ as well as $q^{\varepsilon,\sigma}$. Firstly, integrating (39) over $(0,T)$ yields

$$\frac{1}{2}\,||\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma}(T)||^2_V + \sigma\,||\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\nabla(\mathcal{M} f^{\varepsilon,\sigma}))||^2_{L^2(0,T;V)} = \frac{1}{2}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,,$$

which leads to

$$||\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\nabla(\mathcal{M} f^{\varepsilon,\sigma}))||^2_{L^2(0,T;V)} \le \frac{1}{2\,\sigma}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,.$$

Recalling equations (37) as well as (27), permits to get the estimates

$$||\partial_t f^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \le \frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V}{2\,\sigma}\,, \qquad ||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \le \frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V}{2\,\sigma}\,.$$

To go one step further, let us multiply the second equation of (27) by $q^{\varepsilon,\sigma}$ and integrate in $\mathbf{x}$, permitting to show (with the help of (40)) that

$$\sigma\,||q^{\varepsilon,\sigma}||_V \le ||\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma}||_V \quad \Rightarrow \quad ||q^{\varepsilon,\sigma}||_{L^\infty(\mathbb{R}^+,V)} \le ||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V/\sigma\,. \tag{42}$$

Finally, the same last equation of (27) yields also

$$\varepsilon\,||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}||_V \le ||\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma}||_V + \sigma\,||q^{\varepsilon,\sigma}||_V \le 2\,||\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma}||_V\,,$$

and hence

$$||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}||_{L^\infty(\mathbb{R}^+,V)} \le 2\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V/\varepsilon \quad \Rightarrow \quad ||\partial_t f^{\varepsilon,\sigma}||_{L^\infty(\mathbb{R}^+,V)} \le 2\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V/\varepsilon\,.$$

Altogether one has

$$||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \le C\,\min\{\frac{1}{\varepsilon^2},\,\frac{1}{\sigma}\}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,, \quad ||\partial_t f^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \le C\,\min\{\frac{1}{\varepsilon^2},\,\frac{1}{\sigma}\}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,, \tag{43}$$

with a constant $C > 0$ independent on ε and σ, but dependent on T.

Remark that the only estimate which shall create problems in our $\sigma \to 0$ asymptotic study, will be (42). In the case $\Im m\,\mathcal{T}$ is closed however, we have a Poincaré inequality of the type

$$||q||_V \le C\,||\mathbf{b}\cdot\nabla q||_V\,, \quad \forall q \in Q \cap (\ker\mathcal{T})^\perp\,,$$

with a constant $C > 0$ independent on q. Then inequality (43) would automatically lead to

$$||q^{\varepsilon,\sigma}||^2_{L^2(0,T;Q)} \le C\,\min\{1/\varepsilon^2,\,1/\sigma\}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V \qquad \forall q^{\varepsilon,\sigma} \in L^2(0,T;Q\cap(\ker\mathcal{T})^\perp)\,. \tag{44}$$

The problem is (as explained in detail in Section **??**) that such a Poincaré inequality is not always available, such that we cannot count on an estimate like (44).

(ii) Case $[\varepsilon = 0,\ \sigma > 0]$. Let $(f^{0,\sigma}, q^{0,\sigma})$ be the unique solution of the AP-problem (27).

Starting from (33) and multiplying this equation with $f^{0,\sigma}$, yields after integration in space and time

$$\frac{1}{2}\,||f^{0,\sigma}||^2_V(t) + \frac{1}{\sigma}\,||\mathbf{b}\cdot\nabla f^{0,\sigma}||^2_{L^2(0,t;V)} = \frac{1}{2}\,||f^0_{in}||^2_V\,, \quad \forall t \in [0,T]\,.$$

Now, this together with the fact that $\mathbf{b}\cdot\nabla f^{0,\sigma} = -\sigma q^{0,\sigma}$ gives the desired estimates.

Finally, the generalized Hille-Yosida theorem, especially estimate (34), permits to show for each fixed $\delta > 0$ that

$$||\partial_t f^{0,\sigma}||_{L^\infty(\delta,\infty;V)} \le \frac{1}{\delta}\,||f^0_{in}||_V\,, \quad ||\mathbf{b}\cdot\nabla q^{0,\sigma}||_{L^\infty(\delta,\infty;V)} \le \frac{1}{\delta}\,||f^0_{in}||_V\,.$$

□

Theorem 4.10. ***(Weak convergences)*** *Let $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma})$ be the unique solutions of the corresponding AP-problem (27) for arbitrary $\varepsilon \ge 0$ and $\sigma > 0$. Furthermore, let us assume that the σ-independent initial condition $f^\varepsilon_{in} \in Q$ satisfies $||f^\varepsilon_{in}||_Q \le C$, with a constant $C > 0$ independent on ε. Then one can prove for each fixed $T > 0$ the following weak convergences:*

(i) For fixed $\sigma > 0$ and $\varepsilon \to 0$ one has

$$f^{\varepsilon,\sigma} \rightharpoonup_{\varepsilon\to 0} f^{0,\sigma} \quad in \quad L^2(0,T;Q)\,, \qquad \partial_t f^{\varepsilon,\sigma} \rightharpoonup_{\varepsilon\to 0} \partial_t f^{0,\sigma} \quad in \quad L^2(0,T;V)\,,$$

with $f^{0,\sigma}$ *solution of the limit problem*

$$\partial_t f^{0,\sigma} - \frac{1}{\sigma}\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\nabla f^{0,\sigma}) = 0\,. \tag{45}$$

Furthermore, one has for the corresponding auxiliary variable

$$q^{\varepsilon,\sigma} \rightharpoonup_{\varepsilon\to 0} q^{0,\sigma} \;\; in \;\; L^2(0,T;Q)\,,$$

and $(f^{0,\sigma}, q^{0,\sigma})$ *is the unique solution to* $(La)_0^\sigma$.

(ii) For fixed $\varepsilon > 0$ *and* $\sigma \to 0$ *one has*

$$f^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} f^{\varepsilon,0} \quad in \quad L^2(0,T;Q)\,, \qquad \partial_t f^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} \partial_t f^{\varepsilon,0} \quad in \quad L^2(0,T;V)\,,$$

with $f^{\varepsilon,0}$ *solution of the limit problem*

$$\partial_t f^{\varepsilon,0} + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla f^{\varepsilon,0} = 0\,. \tag{46}$$

What can be said about the convergence of the auxiliary variable is that

$$\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} \mathbf{b}\cdot\nabla q^{\varepsilon,0} \quad in \quad L^2(0,T;V)\,, \qquad \sigma\, q^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} 0 \quad in \quad L^2(0,T;V)\,,$$

with $(f^{\varepsilon,0}, q^{\varepsilon,0})$ *the unique solution of* $(La)_\varepsilon^0$. *If* $\Im m\,\mathcal{T}$ *is closed however, one has even*

$$q^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} q^{\varepsilon,0} \;\; in \;\; L^2(0,T;Q\cap(\ker\mathcal{T})^\perp)\,.$$

(iii) For fixed $\varepsilon = 0$ *and* $\sigma \to 0$ *one has*

$$f^{0,\sigma} \rightharpoonup_{\sigma\to 0} f^{0,0} \quad in \quad L^2(0,T;Q)\,, \qquad \mathbf{b}\cdot\nabla f^{0,\sigma} \to_{\sigma\to 0} 0 \quad in \quad L^2(0,T;V)\,,$$

with $f^{0,0} \in L^2(0,T;\ker\mathcal{T})$. *Furthermore*

$$\partial_t f^{0,\sigma} \rightharpoonup_{\sigma\to 0} 0\,, \quad \sigma\, q^{0,\sigma} \to_{\sigma\to 0} 0\,, \quad \mathbf{b}\cdot\nabla q^{0,\sigma} \rightharpoonup_{\sigma\to 0} 0 \quad in \quad L^2(0,T;V)\,,$$

with $f^{0,0}$ *solution of the limit problem*

$$\begin{cases} \partial_t f^{0,0} = 0\,, \\ \mathbf{b}\cdot\nabla f^{0,0} = 0\,. \end{cases} \tag{47}$$

If $\Im m\,\mathcal{T}$ *is closed however, one has even*

$$q^{0,\sigma} \rightharpoonup_{\sigma\to 0} 0 \;\; in \;\; L^2(0,T;Q\cap(\ker\mathcal{T})^\perp)\,.$$

Remark 4.11. *The case* $\sigma = 0$ *and* $\varepsilon \to 0$ *has been treated in [7] (see theorem 3.1 of this paper).*

Proof. Let $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma})$ be the unique solution of the AP-problem (27) with the regularity given in theorems 4.2-4.6. The proof relies on *a priori* estimates and compactness arguments.

1*st*. Step. *Study of the convergence* $\varepsilon \to 0$ *with fixed* $\sigma > 0$.
Fixing now $\sigma > 0$, recalling the estimates obtained above and letting $\varepsilon \to 0$ allows to

show the existence of some limit functions $f^{0,\sigma} \in L^2(0,T;Q) \cap H^1(0,T;L^2_\sharp(\Omega))$ resp. $q^{0,\sigma} \in L^2(0,T;Q)$, such that up to some subsequences

$$f^{\varepsilon,\sigma} \rightharpoonup_{\varepsilon\to 0} f^{0,\sigma} \quad \text{in} \quad L^2(0,T;Q)\,; \quad q^{\varepsilon,\sigma} \rightharpoonup_{\varepsilon\to 0} q^{0,\sigma} \quad \text{in} \quad L^2(0,T;Q)\,,$$

$$\partial_t f^{\varepsilon,\sigma} \rightharpoonup_{\varepsilon\to 0} \partial_t f^{0,\sigma} \quad \text{in} \quad L^2(0,T;V) \qquad \text{s.t.} \qquad f^{\varepsilon,\sigma} \to_{\varepsilon\to 0} f^{0,\sigma} \quad \text{in} \quad C_w([0,T],V)\,.$$

Passing to the weak limit in (27) yields the limit model

$$(La)_0^\sigma \quad \begin{cases} \partial_t f^{0,\sigma} + \mathbf{b}\cdot\nabla q^{0,\sigma} = 0\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,, \\ \mathbf{b}\cdot\nabla f^{0,\sigma} = -\sigma\, q^{0,\sigma}\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,. \end{cases} \tag{48}$$

Eliminating the auxiliary variable $q^{0,\sigma}$ permits to rewrite this limit model as

$$\partial_t f^{0,\sigma} - \frac{1}{\sigma}\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\nabla f^{0,\sigma}) = 0\,.$$

***2nd.* Step.** *Study of the convergence $\sigma \to 0$ with fixed $\varepsilon > 0$.*
On the contrary, let us now fix $\varepsilon > 0$, recall the estimates obtained in Theorem 4.9 and let $\sigma \to 0$. This proves the existence of some limit functions $f^{\varepsilon,0} \in L^2(0,T;Q) \cap H^1(0,T;L^2_\sharp(\Omega))$, $(\xi,\chi) \in (L^2(0,T;V))^2$ such that

$$\left.\begin{array}{ll} f^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} f^{\varepsilon,0} & \text{in} \quad L^2(0,T;Q) \\ \partial_t f^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} \partial_t f^{\varepsilon,0} & \text{in} \quad L^2(0,T;V) \end{array}\right\} \Rightarrow \quad f^{\varepsilon,\sigma} \to_{\sigma\to 0} f^{\varepsilon,0} \quad \text{in} \quad C_w([0,T],V)\,,$$

$$\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} \xi \quad \text{in} \quad L^2(0,T;V)\,,$$

$$\sigma\, q^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} \chi \quad \text{in} \quad L^2(0,T;V)\,.$$

Passing now to the weak-limit in (27) yields the limit model satisfied by these limit quantities

$$(La)_\varepsilon^0 \quad \begin{cases} \partial_t f^{\varepsilon,0} + \xi = 0\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,, \\ \mathbf{b}\cdot\nabla f^{\varepsilon,0} = \varepsilon\,\xi - \chi\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,, \end{cases} \tag{49}$$

which can be rewritten, eliminating ξ, as follows

$$\partial_t f^{\varepsilon,0} + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla f^{\varepsilon,0} = -\frac{1}{\varepsilon}\chi\,.$$

Now, the fact that $\Pi(q^{\varepsilon,\sigma}) = 0$ for all $\sigma > 0$ and all $\varepsilon \geq 0$ (with Π the projection operator on $\ker\mathcal{T}$ defined in (14)), means that $q^{\varepsilon,\sigma} \in L^2(0,T;Q\cap(ker\mathcal{T})^\perp)$, which automatically leads to $\chi \in L^2(0,T;(ker\mathcal{T})^\perp)$. But we have also that $\mathbf{b}\cdot\nabla\chi = 0$ meaning $\chi \in L^2(0,T;ker\mathcal{T})$. Indeed, this is a simple consequence of the fact that $\sigma\,\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma} \to_{\sigma\to 0} 0$. Thus altogether $\chi \in L^2(0,T;ker\mathcal{T}) \cap L^2(0,T;(ker\mathcal{T})^\perp)$ and hence $\chi \equiv 0$ such that the limit model reduces to

$$\partial_t f^{\varepsilon,0} + \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla f^{\varepsilon,0} = 0\,.$$

Now, the second equation of (49) yields that

$$\xi = \frac{1}{\varepsilon}\mathbf{b}\cdot\nabla f^{\varepsilon,0} \in \Im m\,\mathcal{T}\,,$$

such that there exists a unique $\theta \in L^2(0,T;Q\cap (ker\mathcal{T})^\perp)$ satisfying $\mathbf{b}\cdot\nabla\theta = \xi$ and hence, denoting $q^{\varepsilon,0} := \theta$, we have

$$\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} \mathbf{b}\cdot\nabla q^{\varepsilon,0} \quad \text{in} \quad L^2(0,T;V)\,.$$

If one knows additionally that $\Im m\,\mathcal{T}$ is closed, then we have even

$$q^{\varepsilon,\sigma} \rightharpoonup_{\sigma\to 0} q^{\varepsilon,0} \ \text{ in } \ L^2(0,T;Q\cap(\ker\mathcal{T})^\perp)$$

due to (44).

***3rd.* Step.** Study of the convergence $\sigma\to 0$ with fixed $\varepsilon = 0$.
Finally, fixing $\varepsilon = 0$, recalling the estimates obtained in Theorem 4.9 and letting $\sigma\to 0$ implies immediately

$$f^{0,\sigma} \rightharpoonup_{\sigma\to 0} f^{0,0} \quad \text{in} \quad L^2(0,T;Q)\,, \qquad \mathbf{b}\cdot\nabla f^{0,\sigma} \to_{\sigma\to 0} 0 \quad \text{in} \quad L^2(0,T;V)\,,$$

with $f^{0,0}\in L^2(0,T;\ker\mathcal{T})$. In a distributional sens, on has then

$$\partial_t f^{0,\sigma} \rightharpoonup_{\sigma\to 0} \partial_t f^{0,0}\,, \quad \mathbf{b}\cdot\nabla q^{0,\sigma} \rightharpoonup_{\sigma\to 0} \xi\,.$$

Now, using the fact that $L^2_\sharp(\Omega) = \ker\mathcal{T} \oplus^\perp (\ker\mathcal{T})^\perp$ one can decompose

$$f^{0,\sigma} = \Pi(f^{0,\sigma}) + (Id-\Pi)f^{0,\sigma} =: p^{0,\sigma} + h^{0,\sigma}\,.$$

Taking then the projection Π of the first equation in $(La)_0^\sigma$, leads to the property $\partial_t\Pi(f^{0,\sigma}) = \partial_t p^{0,\sigma} = 0$ for all $\sigma>0$, which means we can rewrite our system as

$$(La)_0^\sigma \quad \begin{cases} \partial_t h^{0,\sigma} + \mathbf{b}\cdot\nabla q^{0,\sigma} = 0\,, & \forall (t,\mathbf{x})\in(0,T)\times\Omega\,,\\ \mathbf{b}\cdot\nabla h^{0,\sigma} = -\sigma\, q^{0,\sigma}\,, & \forall (t,\mathbf{x})\in(0,T)\times\Omega\,,\end{cases}$$

with $h^{0,\sigma}(t,\cdot)\in(ker\mathcal{T})^\perp$ for all $\sigma>0$. This system can be reformulated as

$$\partial_t h^{0,\sigma} - \frac{1}{\sigma}\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\nabla h^{0,\sigma}) = 0\,,$$

yielding immediately that $h^{0,\sigma}\rightharpoonup_{\sigma\to 0} h^{0,0}$ and $\mathbf{b}\cdot\nabla h^{0,\sigma}\to_{\sigma\to 0} 0$ in $L^2(0,T;V)$, such that $h^{0,0}(t,\cdot)\in ker\mathcal{T}\cap(ker\mathcal{T})^\perp$, meaning $h^{0,0}\equiv 0$. This leads to the estimates

$$\partial_t f^{0,\sigma} \rightharpoonup_{\sigma\to 0} 0\,, \quad \sigma\, q^{0,\sigma} \to_{\sigma\to 0} 0\,, \quad \mathbf{b}\cdot\nabla q^{0,\sigma} \rightharpoonup_{\sigma\to 0} 0 \quad \text{in} \quad L^2(0,T;V)\,,$$

with $f^{0,0}$ solution of the limit problem

$$\begin{cases} \partial_t f^{0,0} = 0\,,\\ \mathbf{b}\cdot\nabla f^{0,0} = 0\,.\end{cases} \tag{50}$$

If $\Im m\,\mathcal{T}$ is closed however, one has even

$$q^{0,\sigma} \rightharpoonup_{\sigma\to 0} 0 \ \text{ in } \ L^2(0,T;Q\cap(\ker\mathcal{T})^\perp)\,.$$

□

4.3. **Strong asymptotic limits $\varepsilon \to 0$ and $\sigma \to 0$.** Let us now study in more details the $\varepsilon \to 0$ and $\sigma \to 0$ convergences, with particular focus on the rate of convergence as well as on the initial conditions. Remark that initially our aim was to solve numerically the singularly-perturbed problem (13). Instead, due to several numerical difficulties, we shall solve numerically, via an efficient AP-scheme, the problem (27). The question one naturally poses is then: *How far is $f^{\varepsilon,\sigma}$ from f^ε, and how does this distance depends on the parameter ε?*

Theorem 4.12. ***(Rate of convergence in ε for fixed $\sigma > 0$)*** *Let $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma})$ resp. $(f^{0,\sigma}, q^{0,\sigma})$ be the unique solutions of the corresponding AP-problem (27). Furthermore, let us assume that $f^\varepsilon_{in} \in Q$ satisfies $||f^\varepsilon_{in} - f^0_{in}||_Q \le C\varepsilon$ as well as $f^0_{in} \in \ker \mathcal{T}$ (well-prepared initial conditions). Then one can show for each fixed $T > 0$ the following estimates*

$$||f^{\varepsilon,\sigma} - f^{0,\sigma}||_{L^2(0,T;Q)} \le C\,\varepsilon\,\theta_{\varepsilon,\sigma}\,, \quad ||\partial_t f^{\varepsilon,\sigma} - \partial_t f^{0,\sigma}||_{L^2(0,T;Q^*)} \le C\,\frac{\varepsilon}{\sqrt{\sigma}}\,\theta_{\varepsilon,\sigma}\,,$$

as well as

$$||q^{\varepsilon,\sigma} - q^{0,\sigma}||_{L^2(0,T;V)} \le C\,\frac{\varepsilon}{\sqrt{\sigma}}\,\theta_{\varepsilon,\sigma}\,, \quad ||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma} - \mathbf{b}\cdot\nabla q^{0,\sigma}||_{L^2(0,T;Q^*)} \le C\,\frac{\varepsilon}{\sqrt{\sigma}}\,\theta_{\varepsilon,\sigma}\,,$$

where we used the notation $\theta_{\varepsilon,\sigma} := 1 + \frac{1}{\sqrt{\sigma}}\min\{1, \varepsilon/\sqrt{\sigma}\}$, remarking that $\theta_{\varepsilon,\sigma} \to_{\varepsilon\to 0} 1$.

Proof. Let us introduce the differences $\tilde{f}^\varepsilon := f^{\varepsilon,\sigma} - f^{0,\sigma}$ and $\tilde{q}^\varepsilon := q^{\varepsilon,\sigma} - q^{0,\sigma}$, where $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma})$ resp. $(f^{0,\sigma}, q^{0,\sigma})$ are the solutions of the corresponding problem (27). Subtracting now these two systems yields

$$\begin{cases} \partial_t \tilde{f}^\varepsilon + \mathbf{b}\cdot\nabla\tilde{q}^\varepsilon = 0\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,, \\ \mathbf{b}\cdot\nabla\tilde{f}^\varepsilon + \sigma\,\tilde{q}^\varepsilon = \varepsilon\,\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,, \end{cases} \tag{51}$$

associated with an initial condition $\tilde{f}^\varepsilon_{in} \in Q$, which satisfies $||\tilde{f}^\varepsilon_{in}||_Q \le C\,\varepsilon$. This system can be rewritten under the form

$$\partial_t \tilde{f}^\varepsilon - \frac{1}{\sigma}\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\,\nabla\tilde{f}^\varepsilon) = -\frac{\varepsilon}{\sigma}\,\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\,\nabla q^{\varepsilon,\sigma})\,,$$

and we recall that we have the estimate $||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \le C\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,\min\{1/\varepsilon^2,\, 1/\sigma\}$.

Now, usual arguments permit to show that

$$\begin{aligned} ||\tilde{f}^\varepsilon||^2_V(t) + \frac{1}{\sigma}||\mathbf{b}\cdot\nabla\tilde{f}^\varepsilon||^2_{L^2(0,t;V)} &\le \frac{\varepsilon^2}{\sigma}\,||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}||^2_{L^2(0,t;V)} + ||\tilde{f}^\varepsilon_{in}||^2_V \\ &\le C\,\frac{1}{\sigma}\,\min\{1,\,\varepsilon^2/\sigma\}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V + C\varepsilon^2 \\ &\le C\,\frac{\varepsilon^2}{\sigma}\,\min\{1,\,\varepsilon^2/\sigma\} + C\varepsilon^2 \le C\varepsilon^2\left[1 + \frac{1}{\sigma}\,\min\{1,\,\varepsilon^2/\sigma\}\right]\,, \end{aligned}$$

yielding

$$||\tilde{f}^\varepsilon||_{L^2(0,T;V)} \le C\varepsilon\left[1 + \frac{1}{\sqrt{\sigma}}\min\{1,\,\varepsilon/\sqrt{\sigma}\}\right]\,, \quad ||\mathbf{b}\cdot\nabla\tilde{f}^\varepsilon||_{L^2(0,T;V)} \le C\sqrt{\sigma}\varepsilon\left[1 + \frac{1}{\sqrt{\sigma}}\,\min\{1,\,\varepsilon/\sqrt{\sigma}\}\right]\,,$$

and

$$||\partial_t \tilde{f}^\varepsilon||_{L^2(0,T;Q^*)} \leq C\,\frac{\varepsilon}{\sqrt{\sigma}}\left[1+\frac{1}{\sqrt{\sigma}}\min\{1,\,\varepsilon/\sqrt{\sigma}\}\right].$$

This gives immediately the estimates for $\tilde{q}^\varepsilon$, namely

$$||\tilde{q}^\varepsilon||_{L^2(0,T;V)} + ||\mathbf{b}\cdot\nabla\tilde{q}^\varepsilon||_{L^2(0,T;Q^*)} \leq C\,\frac{\varepsilon}{\sqrt{\sigma}}\left[1+\frac{1}{\sqrt{\sigma}}\min\{1,\,\varepsilon/\sqrt{\sigma}\}\right].$$

$\square$

Remark 4.13. *Theorem 4.12 shows clearly that for well-prepared initial conditions we have strong $\varepsilon \to 0$ convergences of the corresponding solutions of* (27)*, and in particular $||f^{\varepsilon,\sigma} - f^{0,\sigma}||_{L^2(0,T;Q)} \leq C\,\varepsilon$ with a constant $C > 0$ independent on $\sigma > 0$ if ε is small enough, fact which underlines the AP-property of the reformulation.*

Let us come now to the last case, fixing $\varepsilon \geq 0$ and investigating the $\sigma \to 0$ convergence. This is also an important part, as we want to measure the error one introduces when introducing the regularization term $\sigma\, q^{\varepsilon,\sigma}$ in the problem. For some parts of the proof, we shall need an additional hypothesis on Ω, namely that $\Omega \subset \mathbb{R}^2$ is bounded. Let us thus introduce the following hypothesis, replacing Hypothesis A when needed:

Hypothesis B : *The time-independent vector-field $\mathbf{b} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is supposed to be given, sufficiently smooth (for ex. $\mathbf{b} \in W^{1,\infty}(\Omega)$) and divergence-free, meaning $\nabla\cdot\mathbf{b} = 0$. The bounded domain $\Omega \subset \mathbb{R}^2$ will be a rectangle $(L_1, L_2) \times (H_1, H_2)$ of the (x, y)-plane. We shall assume periodic boundary conditions in x and homogeneous Dirichlet boundary conditions in y, and the field $\mathbf{b}$ is supposed to be periodic in x.*

Theorem 4.14. ***(Rate of convergence in σ for fixed $\varepsilon \geq 0$)*** *Let $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma})$ resp. $(f^{\varepsilon,0}, q^{\varepsilon,0})$ be the unique solution of the corresponding AP-problem* (27)*. Let us assume that the initial condition $f^\varepsilon_{in} \in Q$ does not depend on σ for $\varepsilon > 0$, however for $\varepsilon = 0$ one shall keep in mind that $f^{0,\sigma}_{in} = f^0_{in} \in Q$ whereas $f^{0,0}_{in} = \Pi(f^0_{in}) \in \ker\mathcal{T}$. Then one can show for each $T > 0$ that:*
(i) For fixed $\varepsilon > 0$, one has

$$||f^{\varepsilon,\sigma} - f^{\varepsilon,0}||_{L^2(0,T;V)} \leq C\,\frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V}{\varepsilon},$$

$$||\mathbf{b}\cdot\nabla f^{\varepsilon,\sigma} - \mathbf{b}\cdot\nabla f^{\varepsilon,0}||_{L^2(0,T;V)} \leq C\sqrt{\sigma}\min\{1,\,\frac{\sqrt{\sigma}}{\varepsilon}\}\,\frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V}{\varepsilon},$$

as well as

$$||\partial_t f^{\varepsilon,\sigma} - \partial_t f^{\varepsilon,0}||_{L^2(0,T;V)} \leq C\,\chi_{\varepsilon,\sigma}\,\frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V}{\varepsilon}, \quad ||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma} - \mathbf{b}\cdot\nabla q^{\varepsilon,0}||_{L^2(0,T;V)} \leq C\,\chi_{\varepsilon,\sigma}\,\frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V}{\varepsilon}.$$

If $\Im m\,\mathcal{T}$ is closed, we have even

$$||f^{\varepsilon,\sigma} - f^{\varepsilon,0}||_{L^2(0,T;Q)} \leq C\sqrt{\sigma}\min\{1,\,\frac{\sqrt{\sigma}}{\varepsilon}\}\,\frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||_V}{\varepsilon},$$

and

$$||\partial_t f^{\varepsilon,\sigma} - \partial_t f^{\varepsilon,0}||_{L^2(0,T;V)} + ||q^{\varepsilon,\sigma} - q^{\varepsilon,0}||_{L^2(0,T;Q\cap(\ker\mathcal{T})^\perp)} \le C\frac{\sqrt{\sigma}}{\varepsilon}\,\min\{1,\,\frac{\sqrt{\sigma}}{\varepsilon}\}\,\frac{||\mathbf{b}\cdot\nabla f^{\varepsilon}_{in}||_V}{\varepsilon}\,,$$

where we used the notation $\chi_{\varepsilon,\sigma} := 1 + \frac{\sqrt{\sigma}}{\varepsilon}\min\{1,\,\sqrt{\sigma}/\varepsilon\}$, *remarking that* $\chi_{\varepsilon,\sigma} \to_{\sigma\to 0} 1$.
(ii) For fixed $\varepsilon = 0$, *let us suppose that* $\Im m\,\mathcal{T}$ *is closed and that Hypothesis B is satisfied. Then one has*

$$||f^{0,\sigma} - f^{0,0}||^2_{L^2(0,T;V)} \le C\,\sigma\,||f^0_{in}||^2_V \quad ||\mathbf{b}\cdot\nabla f^{0,\sigma}||^2_{L^2(0,T;V)} \le C\,\sigma\,||f^0_{in}||^2_V\,,$$

as well as for an arbitrary but fixed $\delta > 0$

$$||\partial_t f^{0,\sigma} - \partial_t f^{0,0}||^2_{L^2(\delta,T;\,V)} \le \frac{C}{\sigma}\,e^{-C\delta/\sigma}\,||f^0_{in}||^2_V\,, \quad ||q^{0,\sigma}||_{L^2(\delta,T;\,Q\cap(\ker\mathcal{T})^\perp)} \le \frac{C}{\sigma}\,e^{-C\delta/\sigma}\,||f^0_{in}||^2_V.$$

Proof. Let us fix $\varepsilon \ge 0$ and introduce the differences $\tilde{f}^\sigma := f^{\varepsilon,\sigma} - f^{\varepsilon,0}$ and $\tilde{q}^\sigma := q^{\varepsilon,\sigma} - q^{\varepsilon,0}$, where $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma})$ resp. $(f^{\varepsilon,0}, q^{\varepsilon,0})$ are the unique solutions of (27) for $\sigma > 0$ resp. $\sigma = 0$. Subtracting these two systems yields

$$\begin{cases} \partial_t \tilde{f}^\sigma + \mathbf{b}\cdot\nabla\tilde{q}^\sigma = 0\,, & \forall(t,\mathbf{x}) \in (0,T)\times\Omega\,,\\ \mathbf{b}\cdot\nabla\tilde{f}^\sigma = \varepsilon\,\mathbf{b}\cdot\nabla\tilde{q}^\sigma - \sigma\,q^{\varepsilon,\sigma}\,, & \forall(t,\mathbf{x}) \in (0,T)\times\Omega\,, \end{cases} \tag{52}$$

associated with an initial condition $\tilde{f}^\sigma_{in} \equiv 0$ for $\varepsilon > 0$ and $\tilde{f}^\sigma_{in} = f^0_{in} - \Pi(f^0_{in})$ for $\varepsilon = 0$.
(i) In the case $\varepsilon > 0$ one can rewrite the system (52) as

$$\partial_t \tilde{f}^\sigma + \frac{1}{\varepsilon}\,\mathbf{b}\cdot\nabla\tilde{f}^\sigma = -\frac{\sigma}{\varepsilon}\,q^{\varepsilon,\sigma}\,.$$

Recall now that one has

$$||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \le C\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,\min\{1/\varepsilon^2,\,1/\sigma\}\,, \quad ||q^{\varepsilon,\sigma}||^2_{L^2(0,T;V)} \le C\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V/\sigma^2\,.$$

In the case $\Im m\,\mathcal{T}$ is closed, one has even

$$||q^{\varepsilon,\sigma}||^2_{L^2(0,T;Q)} \le C\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,\min\{1/\varepsilon^2,\,1/\sigma\} \qquad \forall q^{\varepsilon,\sigma} \in L^2(0,T;Q\cap(\ker\mathcal{T})^\perp)\,.$$

Standard arguments permit now to show that

$$||\tilde{f}^\sigma||^2_{L^2(0,T;V)} \le C\,\frac{\sigma^2}{\varepsilon^2}\,||q^{\varepsilon,\sigma}||^2_{L^2(0,T;V)}\,, \quad ||\mathbf{b}\cdot\nabla\tilde{f}^\sigma||^2_{L^2(0,T;V)} \le C\,\frac{\sigma^2}{\varepsilon^2}\,||\mathbf{b}\cdot\nabla q^{\varepsilon,\sigma}||^2_{L^2(0,T;V)}\,,$$

yielding immediately the estimates

$$||\tilde{f}^\sigma||^2_{L^2(0,T;V)} \le C\,\frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V}{\varepsilon^2}\,, \quad ||\mathbf{b}\cdot\nabla\tilde{f}^\sigma||^2_{L^2(0,T;V)} \le C\,\frac{\sigma}{\varepsilon^2}\,\min\{1,\,\frac{\sigma}{\varepsilon^2}\}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,,$$

and

$$||\partial_t\tilde{f}^\sigma||^2_{L^2(0,T;V)} \le C\,\left[1 + \frac{\sigma}{\varepsilon^2}\,\min\{1,\,\sigma/\varepsilon^2\}\right]\,\frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V}{\varepsilon^2}\,,$$

$$||\mathbf{b}\cdot\nabla\tilde{q}^\sigma||^2_{L^2(0,T;V)} \le C\,\left[1 + \frac{\sigma}{\varepsilon^2}\,\min\{1,\,\sigma/\varepsilon^2\}\right]\,\frac{||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V}{\varepsilon^2}\,.$$

In the case $\Im m\,\mathcal{T}$ is closed, one has even

$$||\tilde{f}^\sigma||^2_{L^2(0,T;Q)} \le C\,\frac{\sigma}{\varepsilon^2}\,\min\{1,\,\frac{\sigma}{\varepsilon^2}\}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,,$$

and

$$||\partial_t \tilde{f}^\sigma||^2_{L^2(0,T;V)} \leq C\,\frac{\sigma}{\varepsilon^4}\,\min\{1,\frac{\sigma}{\varepsilon^2}\}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,, \quad ||\mathbf{b}\cdot\nabla \tilde{q}^\sigma||^2_{L^2(0,T;V)} \leq C\,\frac{\sigma}{\varepsilon^4}\,\min\{1,\frac{\sigma}{\varepsilon^2}\}\,||\mathbf{b}\cdot\nabla f^\varepsilon_{in}||^2_V\,.$$

(ii) In the case $\varepsilon = 0$ one has

$$\left\{ \begin{array}{ll} \partial_t \tilde{f}^\sigma + \mathbf{b}\cdot\nabla\tilde{q}^\sigma = 0\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,,\\ \mathbf{b}\cdot\nabla\tilde{f}^\sigma = -\sigma\, q^{0,\sigma}\,, & \forall (t,\mathbf{x}) \in (0,T)\times\Omega\,. \end{array}\right. \tag{53}$$

Remark that $\tilde{f}^\sigma_{in} = f^0_{in} - \Pi(f^0_{in})$ as well as the fact that $q^{0,0} \equiv 0$ such that the previous system can be rewritten as

$$\left\{ \begin{array}{l} \partial_t \tilde{f}^\sigma + \mathbf{b}\cdot\nabla\tilde{q}^\sigma = 0\\ \mathbf{b}\cdot\nabla\tilde{f}^\sigma = -\sigma\,\tilde{q}^\sigma \end{array}\right. \quad \text{or} \quad \left\{ \begin{array}{l} \partial_t \tilde{f}^\sigma - \dfrac{1}{\sigma}\nabla\cdot(\mathbf{b}\otimes\mathbf{b}\,\nabla\tilde{f}^\sigma) = 0\\ \tilde{f}^\sigma_{in} = f^0_{in} - \Pi(f^0_{in}) \in (\ker\mathcal{T})^\perp\,. \end{array}\right.$$

Thus one has

$$\frac{1}{2}\,||\tilde{f}^{0,\sigma}||^2_V(t) + \frac{1}{\sigma}\,||\mathbf{b}\cdot\nabla\tilde{f}^{0,\sigma}||^2_{L^2(0,t;V)} = \frac{1}{2}\,||\tilde{f}^\sigma_{in}||^2_V\,,$$

hence, recalling that $q^{0,0} \equiv 0$ and $f^{0,0}(t,\cdot) \in \ker\mathcal{T}$, we have firstly that $||f^{0,\sigma} - f^{0,0}||^2_{L^\infty(\mathbb{R}^+;V)} \leq ||\tilde{f}^\sigma_{in}||^2_V$ and

$$||\mathbf{b}\cdot\nabla f^{0,\sigma}||^2_{L^2(0,T;V)} \leq C\,\sigma\,||\tilde{f}^\sigma_{in}||^2_V \quad \Rightarrow \quad \sigma\,||q^{0,\sigma}||^2_{L^2(0,T;V)} \leq C\,||\tilde{f}^\sigma_{in}||^2_V\,.$$

To be more precise in the σ-convergence, let us make use of spectral theory and suppose for this that $\Omega \subset \mathbb{R}^2$ is bounded (Hypothesis B) and that $\Im m\,\mathcal{T}$ is closed, meaning there is a Poincaré inequality of the type

$$||u||_{L^2(\Omega)} \leq C_\star\,||\mathbf{b}\cdot\nabla u||_{L^2(\Omega)}\,, \quad \forall u \in Q \cap (\ker\mathcal{T})^\perp\,. \tag{54}$$

Denoting by $\mathcal{A} := Q \cap (\ker\mathcal{T})^\perp$, one has:

- $Q \hookrightarrow V$ is dense and compact, hence $\mathcal{A} \hookrightarrow (\ker\mathcal{T})^\perp$ is dense and compact;
- The bilinear form

  $$d : \mathcal{A}\times\mathcal{A} \to \mathbb{R}\,, \quad d(u,v) := \frac{1}{\sigma}\,(\mathcal{T}u, \mathcal{T}v)_{L^2}\,,$$

  is continuous, symmetric, positive and coercive, due to the Poincaré inequality (54).

Then the spectral theory for compact, selfadjoint operators states that:

- $\exists\,\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ a sequence of eigenvalues satisfying

  $$0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \quad \text{with} \quad \lim_{k\to\infty}\lambda_k = \infty\,,$$

- $\exists\,\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{A}$ a sequence of eigenvectors, forming an orthonormal basis for $\mathcal{A}$ wrt. the scalar product $d(\cdot,\cdot)$, and $\{w_k\}_{k\in\mathbb{N}} \subset \mathcal{A} \subset (\ker\mathcal{T})^\perp$ defined as $w_k := \lambda_k^{1/2} u_k$, forming an orthonormal basis in $(\ker\mathcal{T})^\perp$ wrt. the L^2-scalar product, such that

$$d(w_k, v) = \lambda_k (w_k, v)_{L^2} \quad \forall v \in \mathcal{A}\,, \quad \forall k \in \mathbb{N}\,.$$

Now, decomposing the initial condition $\tilde{f^\sigma_{in}} \in \mathcal{A}$ as well as the unknown $\tilde{f^\sigma}(t,\cdot) \in \mathcal{A}$ in the basis $\{w_k\}_{k\in\mathbb{N}}$ leads to

$$\tilde{f^\sigma_{in}} = \sum_{k=1}^{\infty} \alpha_k^0\, w_k\,, \quad \tilde{f^\sigma}(t,\cdot) = \sum_{k=1}^{\infty} \alpha_k(t)\, w_k\,.$$

Inserting these formulae in the variational form

$$\langle \partial_t \tilde{f^\sigma}, v\rangle_{Q^*,Q} + d(\tilde{f^\sigma}, v) = 0 \quad \forall v \in \mathcal{A}\,,$$

and taking as test function $v = w_l$ for $l \in \mathbb{N}$, yields the system of ODEs

$$\frac{d}{dt}\alpha_k(t) + \lambda_k\, \alpha_k(t) = 0 \quad \forall k \in \mathbb{N}\,,$$

with initial conditions $\alpha_k(0) = \alpha_k^0$. The solution is hence given by

$$\alpha_k(t) = \alpha_k^0\, e^{-\lambda_k\, t} \quad \Rightarrow \quad f^{0,\sigma}(t,\cdot) = \sum_{k=1}^{\infty} \alpha_k^0\, e^{-\lambda_k\, t}\, w_k\,, \quad \forall t \geq 0\,. \tag{55}$$

To estimate $\tilde{f^\sigma}$ one needs now some estimates for $\alpha_k(t)$. The Rayleigh quotient

$$\lambda_1 = \min_{v\in\mathcal{A},\, v\neq 0} \frac{d(v,v)}{||v||^2_{L^2}} \geq \frac{C_\star}{\sigma}\,,$$

where $C_\star > 0$ is the Poincaré constant of the transport operator, permits to show that

$$|\alpha_k(t)| \leq |\alpha_k^0|\, e^{-C_\star t/\sigma}\,, \quad \forall t \geq 0\,, \quad \forall k \in \mathbb{N}\,.$$

Thus, one has immediately

$$||\tilde{f^\sigma}(t,\cdot)||^2_V \leq \sum_{k=1}^{\infty} |\alpha_k^0|^2\, e^{-2\lambda_k\, t} \leq e^{-2C_\star\, t/\sigma} \sum_{k=1}^{\infty} |\alpha_k^0|^2 = e^{-2C_\star\, t/\sigma}\, ||\tilde{f^\sigma_{in}}||^2_V\,.$$

Integrating in time yields finally that

$$||\tilde{f^\sigma}||^2_{L^2(0,T;V)} \leq C\,\sigma\,(1 - e^{-2C_\star\, T/\sigma})\, ||\tilde{f^\sigma_{in}}||^2_V \leq C\,\sigma\, ||\tilde{f^\sigma_{in}}||^2_V\,.$$

To estimate the time-derivative we observe that

$$\partial_t \tilde{f^\sigma}(t,\cdot) = -\sum_{k=1}^{\infty} \alpha_k^0\, \lambda_k\, e^{-\lambda_k\, t}\, w_k \quad \Rightarrow \quad ||\partial_t \tilde{f^\sigma}(t,\cdot)||^2_V \leq \sum_{k=1}^{\infty} |\alpha_k^0|^2\, \lambda_k^2\, e^{-2\lambda_k\, t}\,.$$

For a more detailed investigation of this last estimate, let us study the function $h(\xi) := e^{-a\,\xi}\,\xi^2$, where we denote for simplicity $\xi := \lambda_k$ and $a = 2t$. We have that $h'(\xi) = (2 - a\,\xi)\;\xi e^{-a\,\xi}$ which is equal to zero for $\xi = 2/a$ positive on $(0, 2/a)$ and negative on $(2/a, \infty)$. Thus $h(\xi)$ is decreasing for $\xi \geq 2/a$.

Fixing now an arbitrary, but small $\delta > 0$, there exists a $\sigma_\delta > 0$ such that $\lambda_1 > \frac{C_\star}{\sigma} > \frac{C_\star}{\sigma_\delta} > \frac{1}{\delta} \geq \frac{1}{t}$ for all $0 < \sigma < \sigma_\delta$ and $t \in [\delta, \infty)$. Hence, under these conditions we have

$$\lambda_k^2 \, e^{-2\lambda_k \, t} \leq \lambda_1^2 \, e^{-2\lambda_1 \, t} \leq \left(\frac{C_\star}{\sigma} \right)^2 \, e^{-2\, C_\star t/\sigma} \, ,$$

yielding

$$||\partial_t \tilde{f^\sigma}(t, \cdot)||_V^2 \leq \left(\frac{C_\star}{\sigma} \right)^2 \, e^{-2\, C_\star t/\sigma} \, ||\tilde{f^\sigma_{in}}||_V^2 \, .$$

Integrating in time over (δ, T) yields finally

$$||\partial_t \tilde{f^\sigma}||^2_{L^2(\delta,T;V)} \leq \frac{C_\star}{2\,\sigma} \, \left(e^{-2\, C_\star \delta/\sigma} - e^{-2\, C_\star T/\sigma} \right) \, ||\tilde{f^\sigma_{in}}||_V^2 \leq \frac{C_\star}{2\,\sigma} \, e^{-2\, C_\star \delta/\sigma} \, ||\tilde{f^\sigma_{in}}||_V^2 \, .$$

which completes the proof. □

Remark 4.15. *The constants in Theorem 4.12 as well as Theorem 4.14 depend on* σ *and* ε*, such that one has to be careful now with the limits. Remark also that there seems to be a certain relation to be satisfied between* σ *and* ε^2*.*
However, let us investigate better what is needed in practice. One is interested usually to have a result with a given, acceptable error denoted here $\tau > 0$*. Furthermore, let us remark that in practice the parameter* $\varepsilon > 0$ *is given, so probably small, but fixed. Thus, the only parameter we have access to is* $\sigma > 0$*, to be chosen, such that*

$$||f^{\varepsilon,\sigma} - f^{\varepsilon,0}||_{L^2(0,T;Q)} \leq \tau \, .$$

For closed $\Im m \, \mathcal{T}$ *and* $||\mathbf{b} \cdot \nabla f^\varepsilon_{in}||_V \leq C\varepsilon$*, this bound seems to be obtained for sufficiently small* σ*, as stated in Theorem 4.14. Improving these estimates is a work in preparation.*

5. Some observations on the Poincaré inequality

Let us state here some well-known Poincaré inequalities, in order to facilitate the understanding of the mathematical part of this paper. The Poincaré inequality is an essential tool in functional analysis, in particular for the study of the existence and regularity of PDEs, for coercivity arguments as well as for the obtention of energy estimates.

Let us recall the singularly-perturbed problem (13) with dominant transport operator $\mathcal{T} := \mathbf{b} \cdot \nabla$, which is a linear, un-bounded L^2-operator defined as

$$\mathcal{T} : Q \subset L^2_\sharp(\Omega) \to L^2_\sharp(\Omega) \, ,$$

with definition domain and kernel given by

$$Q := \{ u \in L^2_\sharp(\Omega) \; / \; \mathbf{b} \cdot \nabla u \in L^2_\sharp(\Omega) \} \, , \qquad \ker \mathcal{T} := \{ u \in L^2_\sharp(\Omega) \; / \; \mathbf{b} \cdot \nabla u = 0 \} \, .$$

While studying this singularly-perturbed transport problem, one queries very often if there is a sort of Poincaré inequality of the form

$$||u||_{L^2(\Omega)} \leq C \, ||\mathbf{b} \cdot \nabla u||_{L^2(\Omega)} \, , \quad \forall u \in Q \cap (\ker \mathcal{T})^\perp \, , \tag{56}$$

with a constant $C > 0$ independent on u. This would permit to state that $\mathcal{T}$ is a bijective linear mapping

$$\mathcal{T} : Q \cap (\ker \mathcal{T})^{\perp} \to (\ker \mathcal{T})^{\perp}.$$

Unfortunately (56) is not satisfied generally, such that one gets problems for example when trying to solve equations of the form

$$\mathcal{T} u = g\,, \quad \text{for } g \in (\ker \mathcal{T})^{\perp}\,.$$

Let us thus review some Poincaré inequalities here and try to understand in which cases (56) is valid.

Theorem 5.1. *(Generalized Poincaré inequality) [21]*
Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary. Furthermore, let us consider a continuous semi-norm

$$\mathcal{N} : W^{1,p}(\Omega) \to \mathbb{R}\,, \qquad p \in [1, \infty).$$

Then, there exists a constant $C > 0$, depending only on Ω, n, p such that

$$||u||_{W^{1,p}(\Omega)} \le C \left[||\nabla u||_{L^p(\Omega)} + \mathcal{N}(u) \right]\,.$$

Remark 5.2. *Some examples of continuous semi-norms are:*

- *$\mathcal{N}(u) := \int_\Gamma |u(x)|\, d\sigma$ with Ω of classe C^1 and $\Gamma \subset \partial\Omega$ with $|\Gamma| > 0$;*
- *$\mathcal{N}(u) := \langle u \rangle$ with $\langle u \rangle := \frac{1}{|\Omega|} \int_\Omega u\, dx$.*

Theorem 5.3. *(Poincaré-Wirtinger inequality) [11, 26]*
Let $\Omega \subset \mathbb{R}^d$ be a connected open and bounded set of regularity C^1 and let $p \in [1, \infty]$. Then there exists a constant $C > 0$ depending only on Ω, n, p such that

$$||u - \langle u \rangle||_{L^p(\Omega)} \le C\, ||\nabla u||_{L^p(\Omega)}\,, \qquad \forall u \in W^{1,p}(\Omega)\,.$$

Theorem 5.4. *(Inflow-Poincaré inequality) [11, 26]*
Let $\Omega \subset \mathbb{R}^d$ be an open bounded set and let $p \in [1, \infty)$. Then there exists a constant $C > 0$ depending only on Ω, n, p such that

$$||u||_{L^p(\Omega)} \le C\, ||\nabla u||_{L^p(\Omega)}\,, \qquad \forall u \in W_0^{1,p}(\Omega)\,.$$

Remark 5.5. *This last Poincaré inequality remains valid for domains which are bounded only in one direction (strip-like domains) or for functions which vanish only on part of the boundary $\Gamma \subset \partial\Omega$ with non-zero measure.*

Let us come now to some "directional" Poincaré inequalities. For this we have to introduce a vector field along which one wants to have the Poincaré inequality.

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain of class C^1 and U a neighbourhood of this domain, namely $\overline{\Omega} \subset U$. Furthermore, let $\mathbf{b} : U \to \mathbb{R}^d$ be a $W^{1,\infty}$ vector-field satisfying $\nabla \cdot \mathbf{b} = 0$. Denoting by $\nu(x)$ the outer, unit normal to $\partial\Omega$, one can define the borders

$$\Gamma^{\pm} := \{x \in \partial\Omega \ / \ \mathbf{b}(x) \cdot \nu(x) \gtrless 0\}\,, \quad \Gamma_0 := \{x \in \partial\Omega \ / \ \mathbf{b}(x) \cdot \nu(x) = 0\}\,,$$

as well as the Hilbert-spaces

$$H(\mathcal{T},\Omega) := \{u \in L^2(\Omega) \ / \ \mathbf{b}\cdot\nabla u \in L^2(\Omega)\}\,, \quad H_0^{\pm}(\mathcal{T},\Omega) := \{u \in H(\mathcal{T},\Omega) \ / \ u|_{\Gamma_0^{\pm}} = 0\}\,.$$

Remark 5.6. *Let us observe here that for* $u \in H(\mathcal{T},\Omega)$ *the normal trace, denoted here for simplicity reasons by* $u_{|\partial\Omega}$, *is well-defined in* $H^{-1/2}(\partial\Omega)$. *Indeed, for* $u \in H(\mathcal{T},\Omega)$ *one has* $\mathbf{b}\,u \in H(div,\Omega)$, *such that* $u\,\mathbf{b}\cdot\nu \in H^{-1/2}(\partial\Omega)$. *We shall thus denote this normal trace by* $u_{|\partial\Omega} := u\,\mathbf{b}\cdot\nu$, *and* $u_{|\Gamma_0^{\pm}} = 0$ *signifies that* $\langle u\,\mathbf{b}\cdot\nu,\varphi\rangle_{H^{-1/2},H^{1/2}} = 0$ *for all* $\varphi \in H^{1/2}(\partial\Omega)$ *with supp*$(\varphi) \subset \Gamma_0^{\pm}$.

One can now associate to the vector field $\mathbf{b}$ the characteristic flow $Z(s;0,x)$ defined as

$$\begin{cases} \dfrac{d}{ds}Z(s) = \mathbf{b}(Z(s))\,, \\ Z(0) = x\,. \end{cases}$$

Definition 5.7. *A vector-field* $\mathbf{b}$ *is called* Ω*-filling, if there exists a finite time* $T > 0$ *and a negligible set* $\mathcal{K} \subset \Omega$ *with* $|\mathcal{K}| = 0$, *such that for all* $x \in \overline{\Omega}\backslash\mathcal{K}$ *there exists* $x_0 \in \Gamma^-$ *and* $t \in [0,T]$ *such that*

$$x = Z(t;0,x_0)\,,$$

which means that the trajectories starting from the inflow border Γ^-, *do fill the domain* $\overline{\Omega}$ *in a finite time, except for a negligible set.*

Theorem 5.8. *(Curved Poincaré inequality) [1]*
Let $\Omega \subset \mathbb{R}^d$ *be an open bounded domain, governed by the divergence-free vector-field* $\mathbf{b}$, *which is* Ω*-filling. Then there exists a constant* $C > 0$ *depending only on* Ω, n, p, *such that*

$$||u||_{L^2(\Omega)} \le C\,||\mathbf{b}\cdot\nabla u||_{L^2(\Omega)}\,, \quad \forall u \in H_0^-(\mathcal{T},\Omega)\,.$$

Remark 5.9. *The proof of this theorem is based on a coordinate transformation (in order to redress the advection field) and a subsequent use of the standard Poincaré inequality (with inflow boundary).*

The situation gets now more dramatic if the vector-field $\mathbf{b}$ is not Ω-filling. Let us thus consider the case of this paper (see Hypothesis A).

Theorem 5.10. *(Poincaré for uniformly-closed trajectories) [7]*
Let $\mathbf{b} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ *be a vector-field satisfying Hypothesis A. Extending* $\mathbf{b}$ *by periodicity to the whole* $\mathbb{R}^2$, *we shall assume that* $\mathbf{b} \in W^{1,\infty}_{loc}(\mathbb{R}^d)$, *with* $d \ge 2$, $\nabla\cdot\mathbf{b} = 0$ *and that the following growth condition (for some constant* $C > 0$*) is verified*

$$|\mathbf{b}(x)| \le C(1 + |x|)\,, \quad \forall x \in \mathbb{R}^d\,.$$

Assume now that all trajectories corresponding to the vector-field $\mathbf{b}$ *are uniformly closed in time, meaning there exists a finite time* $T > 0$ *such that for all* $y \in \Omega$ *there is a* $T_y \in [0,T]$ *such that* $y = Z(T_y;0,y)$. *Then* $\Im m\,\mathcal{T}$ *is closed and one has* $\Im m\,\mathcal{T} = \ker\Pi$, *the operator* $\Pi(\cdot)$ *being the projection operator on the kernel, defined in* (14). *Furthermore, we have the Poincaré inequalities*

$$||u||_{L^2(\Omega)} \le 2T\,||\mathbf{b}\cdot\nabla u||_{L^2(\Omega)}\,, \quad \forall u \in Q \cap (\ker\mathcal{T})^{\perp}\,.$$

and

$$||u - \Pi(u)||_{L^2(\Omega)} \le 2T\,||\mathbf{b}\cdot\nabla u||_{L^2(\Omega)}\,, \quad \forall u \in Q\,.$$

The assumptions of this last theorem are however not always satisfied, and one has generally only $\overline{\Im m\,\mathcal{T}} = \ker\Pi$. For example, in the interesting case for plasma physics, namely $\mathbf{b} := (v, -E(x))$, the field lines are not uniformly closed. Even in the more simple case $\mathbf{b} := (v, 0)$ no Poincaré inequality can be obtained. To see this, one has only to consider the sequence of functions $\{u_n\}_{n\in\mathbb{R}}$ defined as $u_n(x,v) := e^{-v^2/n}\sin(x)$, thus concentrated around $v \approx 0$, and to show that there is no constant $C_P > 0$ independent on n such that

$$||u_n||^2_{L^2} = \int_{\mathbb{R}}\int_{-\pi}^{\pi} |u_n(x,v)|^2\,dxdv \le C_P \int_{\mathbb{R}}\int_{-\pi}^{\pi} |v|^2\,|\partial_x u_n(x,v)|^2\,dxdv = C_P\,||\mathbf{b}\cdot\nabla u_n||^2_{L^2}\,.$$

Indeed, after some computations, one gets for the left side integral I_L as well as the right side integral I_R the following values

$$I_L = \pi\sqrt{\frac{\pi\,n}{2}}\,, \qquad I_R = \frac{n\,I_L}{4} = \frac{n\,\pi}{4}\sqrt{\frac{\pi n}{2}}\,,$$

fact which shows the impossibility to have a Poincaré inequality, when $n \to 0$.

At this point let us mention that from a numerical point of view the spaces are no more of dimension infinity, such that in the discretized framework (for a fixed grid) one has always $\overline{\Im m\,\mathcal{T}} = \Im m\,\mathcal{T} = \ker\Pi$, such that a Poincaré type inequality always holds. But one has to keep in mind that the Poincaré constant will depend in this case on the discretization parameters.

Let us try now to change a little bit the mathematical framework in order to enable a Poincaré inequality for the transport operator, meaning choosing other norms and enlarging the spaces. For example, let us consider the subspace $L^2_\sharp(\Omega)\backslash\ker\mathcal{T} = (\ker\mathcal{T})^\perp$ and define on it a different norm than the L^2-norm, namely

$$||u||_\star := \sup_{q\in Q,\ q\neq 0} \frac{(u, \mathbf{b}\cdot\nabla q)_{L^2}}{||q||_Q}\,, \quad \forall u \in L^2_\sharp(\Omega)\backslash\ker\mathcal{T}\,. \tag{57}$$

To be sure that this is a norm on $L^2_\sharp(\Omega)\backslash\ker\mathcal{T}$, one can remark that for each $u \in L^2_\sharp(\Omega)$ there exists a unique $u^\star \in Q$, solution to the problem

$$(u^\star, q)_Q = (u, \mathbf{b}\cdot\nabla q)_{L^2} \qquad \forall q \in Q\,. \tag{58}$$

Remark that for $u \in \ker\mathcal{T}$ the solution is $u^\star \equiv 0$. For $u \notin \ker\mathcal{T}$, we have $u^\star \neq 0$ and we shall decompose each $q \in Q$ as follows $q = \alpha\,u^\star + \eta$ with $(u^\star, \eta)_Q = 0$, meaning $\alpha = \frac{(q,u^\star)_Q}{||u^\star||^2_Q}$ to obtain

$$\frac{(u, \mathbf{b}\cdot\nabla q)_{L^2}}{||q||_Q} = \frac{(u^\star, q)_Q}{||q||_Q} = \frac{\alpha||u^\star||^2_Q}{\sqrt{\alpha^2\,||u^\star||^2_Q + ||\eta||^2_Q}} \le ||u^\star||_Q = \frac{(u, \mathbf{b}\cdot\nabla u^\star)_{L^2}}{||u^\star||_Q}\,.$$

This permits to show that the supremum in (57) for $u \notin \ker \mathcal{T}$ is taken in $u^\star$, *i.e.*

$$||u||_\star = \sup_{q \in Q,\, q \neq 0} \frac{(u, \mathbf{b} \cdot \nabla q)_{L^2}}{||q||_Q} = \frac{(u, \mathbf{b} \cdot \nabla u^\star)_{L^2}}{||u^\star||_Q} = ||u^\star||_Q\,, \quad \forall u \in (\ker \mathcal{T})^\perp \backslash \{0\}\,.$$

For $u \equiv 0$ the equality is obvious. So, if $||u||_\star = 0$ for some $u \in (\ker \mathcal{T})^\perp$ then automatically $u^\star = u = 0$ and we showed hence that $(L^2_\sharp(\Omega) \backslash \ker \mathcal{T}, ||\cdot||_\star)$ is a normed space. Remark that the two norms $||\cdot||_{L^2}$ and $||\cdot||_\star$ are not equivalent. Furthermore, with the new norm (57) on $L^2_\sharp(\Omega) \backslash \ker \mathcal{T}$, one has now

$$||u||_\star \leq ||\mathcal{T} u||_{L^2}\,, \quad \forall u \in Q \cap (\ker \mathcal{T})^\perp\,.$$

To simplify the notation in the following, let us denote by $\mathcal{A} := Q \cap (\ker \mathcal{T})^\perp$ the space of function with zero *Average*. Then, we shall introduce the closure of this space with respect to the new norm $||\cdot||_\star$

$$\tilde{\mathcal{A}} := \overline{\mathcal{A}}^{||\cdot||_\star} \cap L^2_\sharp(\Omega)\,, \tag{59}$$

such that $(\tilde{\mathcal{A}}, ||\cdot||_\star)$ is now a Hilbert space, with the scalar product $(u, v)_\star := (u^\star, v^\star)_Q$, where $u^\star$ resp. $v^\star$ are the solutions of the problem (58) corresponding to u resp. v.

With this definition, we shall now extend the transport operator as follows

$$\mathcal{T}_\star : \tilde{\mathcal{A}} \to \Im m\, \mathcal{T}_\star\,,$$

and show that this time $\Im m\, \mathcal{T}_\star$ is closed in $(L^2_\sharp(\Omega), ||\cdot||_2)$, meaning $\mathcal{T}_\star$ satisfies a Poincaré inequality of the type

$$||u||_\star \leq C_\star ||\mathcal{T}_\star u||_{L^2}\,, \qquad \forall u \in \tilde{\mathcal{A}}\,,$$

with $C_\star > 0$ a Poincaré constant independend on u.

The extension is defined as follows. For $u \in \mathcal{A}$ we simply put $\mathcal{T}_\star u := \mathcal{T} u = \mathbf{b} \cdot \nabla u$. Now, let us choose an arbitrary $u \in \tilde{\mathcal{A}} \backslash \mathcal{A}$. Due to the dense embedding $\mathcal{A} \subset \tilde{\mathcal{A}}$, there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that $u_n \to_{n \to \infty} u$ in $\tilde{\mathcal{A}}$. This permits to show that $u_n^\star \to_{n \to \infty} \xi$ in Q, where $u_n^\star$ are the solutions of (58) corresponding to u_n, and thus

$$(\mathcal{T} u_n, q)_{L^2} \to_{n \to \infty} -(\xi, q)_Q\,, \quad \forall q \in Q\,.$$

This means $\mathcal{T} u_n$ converges weakly$-\star$ in $Q^\star$ towards an element we shall denote $\mathcal{T}_\star u \in Q^\star$. Density arguments permit to extend uniquely this $\mathcal{T}_\star u$ to an element $\mathcal{T}_\star u \in (L^2_\sharp(\Omega))^\star \equiv L^2_\sharp(\Omega)$, such that $||\mathcal{T}_\star u||_{Q^\star} = ||\mathcal{T}_\star u||_{L^2}$. With this, we have defined the extension of $\mathcal{T}$ and one can show now that

$$||u||_\star \leq C_\star ||\mathcal{T}_\star u||_{L^2}\,, \quad \forall u \in \tilde{\mathcal{A}}\,,$$

which means that $\mathcal{T}_\star$ is of closed range. To show this, let $u \in \tilde{\mathcal{A}} \backslash \mathcal{A}$. Then, there exists a unique $\xi \in Q$ such that $(\xi, q)_Q = \lim_{n \to \infty} (u_n, \mathbf{b} \cdot \nabla q)_{L^2}$ for all $q \in Q$ and $||\xi||_Q = ||u||_\star$. Thus one has

$$||\xi||_Q^2 = -(\mathcal{T}_\star u, \xi)_{L^2} \leq ||\mathcal{T}_\star u||_{L^2} ||\xi||_{L^2} \leq ||\mathcal{T}_\star u||_{L^2} ||\xi||_Q\,,$$

which yields the desired Poincaré inequality.

Altogether we have thus proved:

Theorem 5.11. *(Poincaré inequality for the transport operator)* *Let* $\mathbf{b} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ *be a vector-field satisfying Hypothesis A or B and* $(\tilde{\mathcal{A}}, (\cdot,\cdot)_\star)$ *the Hilbert space defined in* (57), (59)*. The extended transport operator*

$$\mathcal{T}_\star : \tilde{\mathcal{A}} \to \Im m\, \mathcal{T}_\star\,, \quad \textit{where} \quad \mathcal{T}_\star u := \mathbf{b} \cdot \nabla u \quad \textit{for } u \in Q \cap (\ker \mathcal{T})^\perp\,,$$

satisfies a Poincaré inequality of the type

$$||u||_\star \leq C_\star\, ||\mathcal{T}_\star u||_{L^2}\,, \qquad \forall u \in \tilde{\mathcal{A}}\,, \tag{60}$$

with $C_\star > 0$ *a Poincaré constant independent on* u*.*

6. Conclusion

The aim of the present paper was to conclude a series of works consacrated to the introduction and study of a new multiscale asymptotic-preserving scheme, called *Lagrange-Multiplier scheme.* In previous works the author validated numerically this new numerical procedure in the context of thermonuclear fusion plasmas, with particular emphasize on its asymptotic-preserving properties. In this last work, the missing mathematical study was carried out, permitting to understand deeply the particular features of this new scheme. To summarize, the advantages of the new Lagrange-Multiplier scheme are the following:

- standard discretization methods can be employed to solve the reformulated (La)-problem, fact which permits the use of very performant, existing methods;
- there is no need for field aligned coordinates, a Cartesian mesh is perfectly adequate;
- the Lagrange-Multiplier scheme is asymptotic-preserving, meaning the choice of the discretization mesh has only to be adapted to the desired physics, and not to stability requirements, fact which permits large computational savings.

There are however also some disadvantages, namely:

- the (La)-technique requires the determination of two unknowns, namely $(f^{\varepsilon,\sigma}, q^{\varepsilon,\sigma})$, fact which can lead to supplementary numerical costs;
- the choice of the regularization parameter $\sigma > 0$ is rather delicate.

Thus, the reader has to make a careful investigation before using this (La)-technique, in order to identify whether this scheme is adapted or not for his particular situation. In the author's opinion, three situations can be singled out:

- $\varepsilon \sim 1$: in this case, the best thing to do is to solve directly the singularly-perturbed problem $(P)_\varepsilon$ via explicit schemes;
- $\varepsilon \sim 0$: in this case, the most rapid procedure could be to solve directly the limit model $(P)_0$, if it does not require the discretization of some difficult terms, as for example averages along the field-lines of $\mathbf{b}$;
- $\varepsilon \in [0, 1]$ variable: this case is the most interesting case for the use of the $(La)_\varepsilon^\sigma$ scheme, which switches automatically between the different ε-regimes, without having to adapt the mesh.

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