Geometric optics for quasilinear hyperbolic boundary value problems

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Hyp2022
June 21, 2022

Introduction

We consider the following problem

$$\begin{cases} L(u^\varepsilon, \partial_z)\, u^\varepsilon := \partial_t u^\varepsilon + \displaystyle\sum_{j=1}^{d} A_j(u^\varepsilon)\, \partial_{x_j} u^\varepsilon = 0 & \text{in } \Omega_T, \\ B\, u^\varepsilon_{|x_d=0} = \varepsilon\, g^\varepsilon & \text{on } \omega_T, \\ u^\varepsilon_{|t\leqslant 0} = 0, \end{cases} \tag{1}$$

where

- $\Omega_T := (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $\omega_T := (-\infty, T] \times \mathbb{R}^{d-1}$, with $T > 0$,
- we denote $z = (t, x) = (t, y, x_d) \in \Omega_T = (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$, and $z' := (t, y) \in \omega_T = (-\infty, T] \times \mathbb{R}^{d-1}$,
- the unknown u^ε is a (regular) function from Ω_T to $\mathbb{R}^N$, $N \geqslant 2$,
- for all $j = 1, \ldots, d-1$, A_j is a regular map from $\mathbb{R}^N$ into $\mathcal{M}_N(\mathbb{R})$,
- B belongs to $\mathcal{M}_{M,N}(\mathbb{R})$ for some $1 \leqslant M \leqslant N$ and is of maximal rank.

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The dependency in ε of the system comes from the boundary term $\varepsilon\, g^\varepsilon$, where g^ε is given by, for $z' \in \omega_T$,

$$g^\varepsilon(z') = G\left(z', \frac{z' \cdot \zeta_1}{\varepsilon}, \frac{z' \cdot \zeta_2}{\varepsilon}\right),$$

where G belongs to $H^\infty(\omega_T \times \mathbb{T}^2)$, zero for negative times t, and ζ_1, ζ_2 are in $\mathbb{R}^d \setminus \{0\}$.

The aim is to construct an approximate solution of (1) in the high frequency asymptotic, namely for $\varepsilon \to 0$, in the form of a WKB expansion. More precisely we expect that

$$u^\varepsilon(z) \sim \varepsilon\, U_1\Big(z, \frac{\Phi_1(z)}{\varepsilon}\Big) + \varepsilon^2\, U_2\Big(z, \frac{\Phi_2(z)}{\varepsilon}\Big) + \cdots .$$

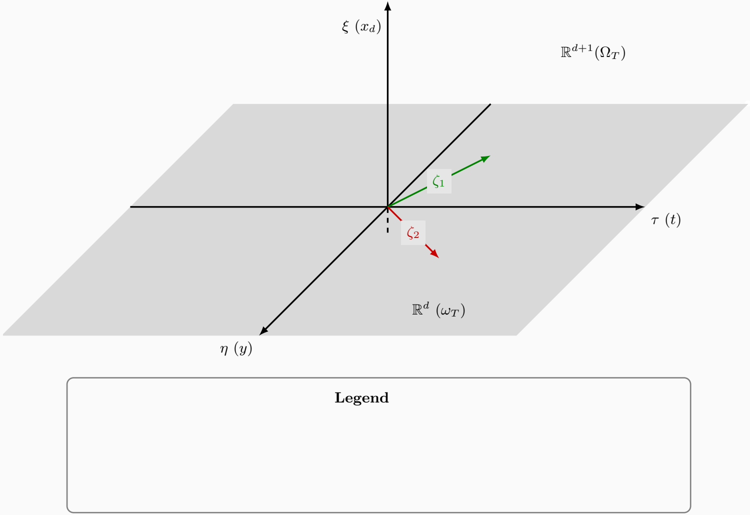
In this work we construct the first term of the asymptotic expansion, adapting

> Jean-Luc Joly, Guy Métivier et Jeffrey Rauch. "Coherent and focusing multidimensional nonlinear geometric optics". In : *Ann. Sci. École Norm. Sup. (4)* 28.1 (1995), p. 51-113

which deals with Cauchy problems, to the framework of boundary value problems.

Functional framework

Set of frequencies inside the domain



ζ_1, ζ_2 on the boundary

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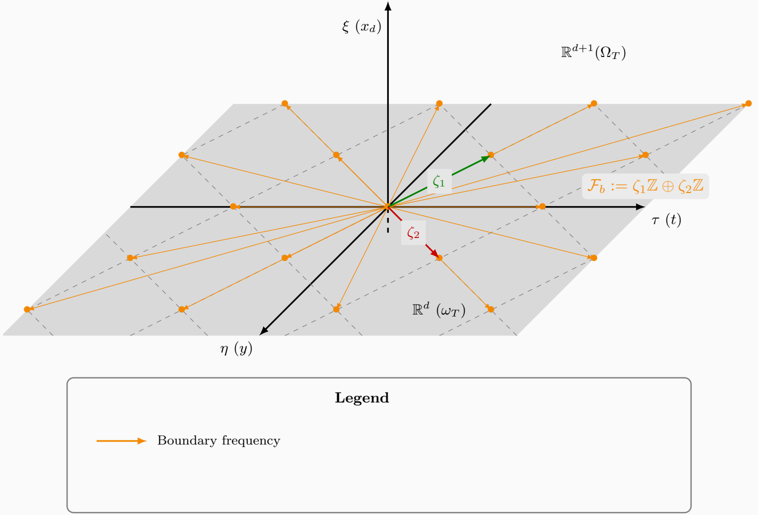

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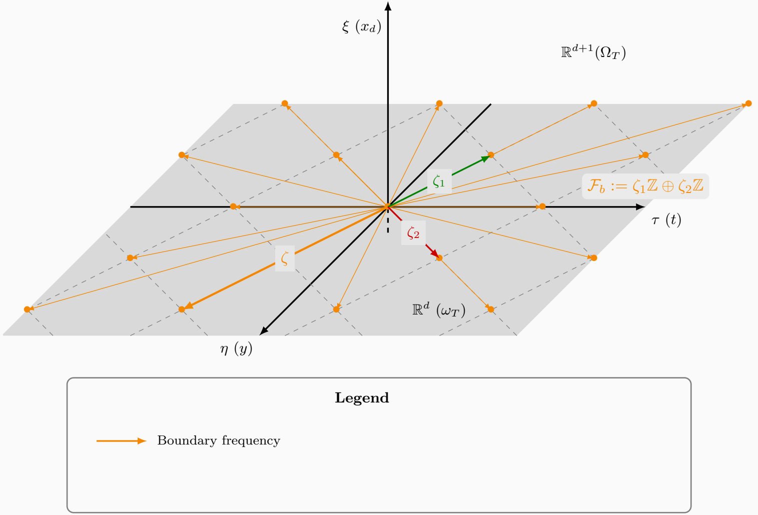

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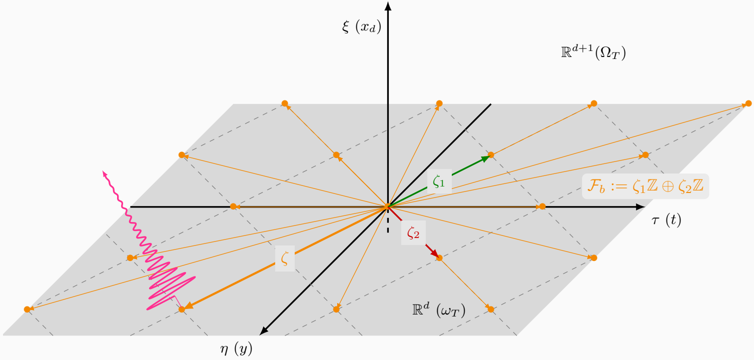

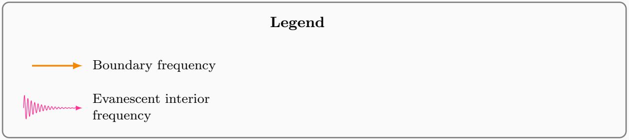

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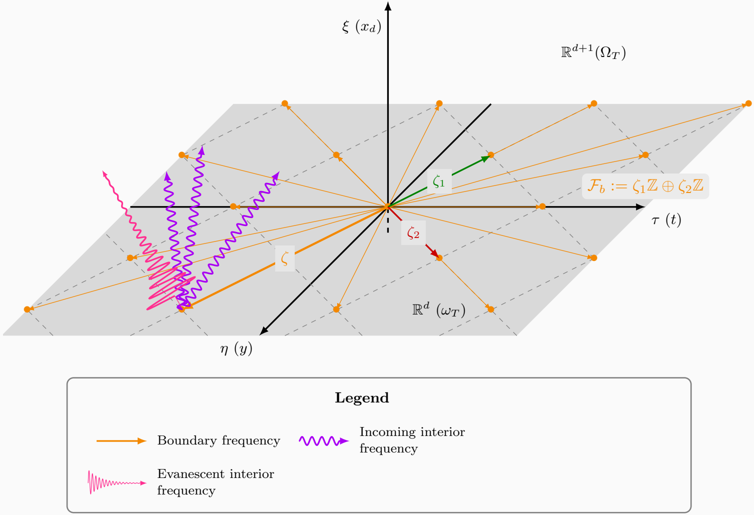

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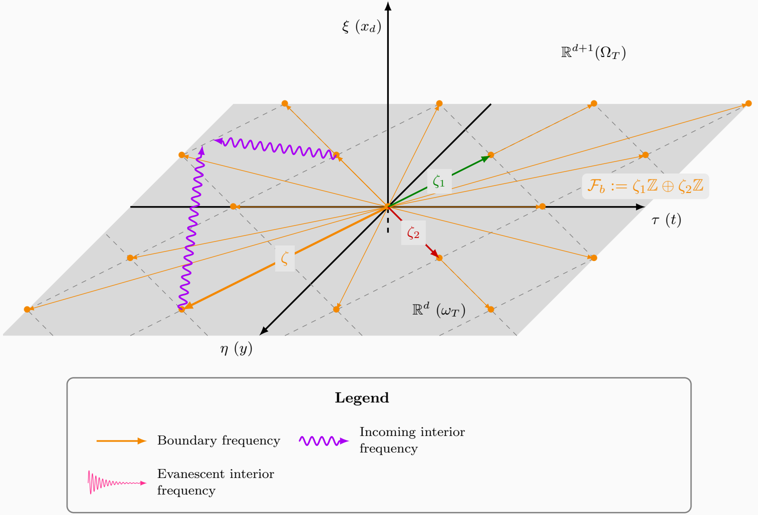

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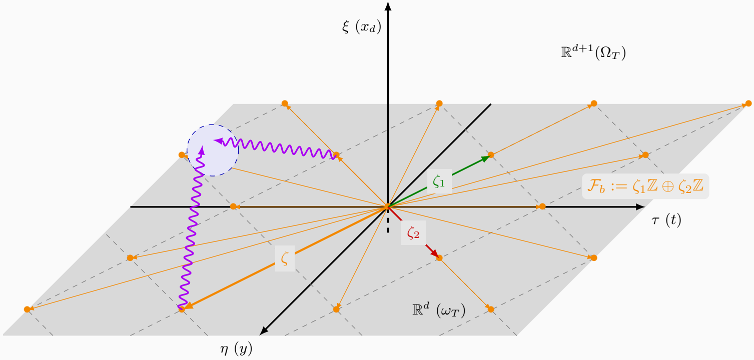

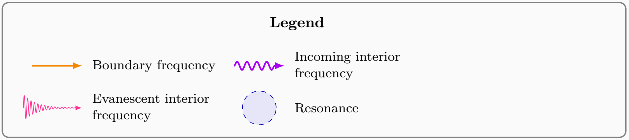

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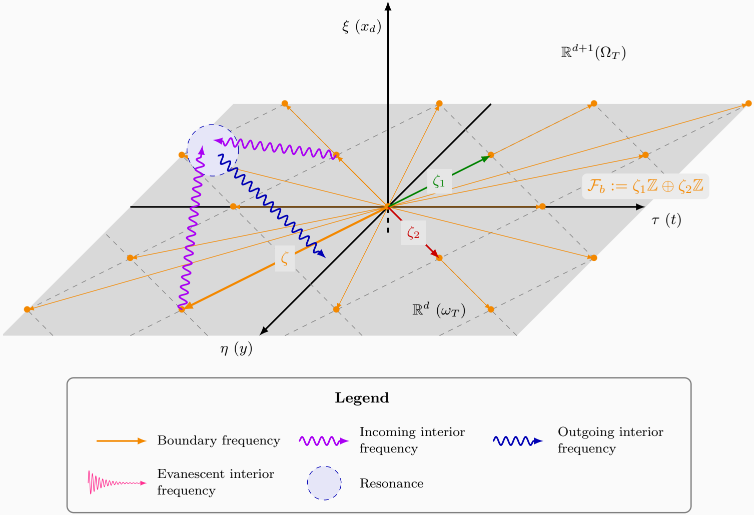

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Functional framework : almost-periodic functions

We need a functional framework allowing to consider functions of the form

$$\sum_{\alpha \in \mathcal{F}} U_\alpha(z)\, e^{iz\cdot\alpha/\varepsilon}.$$

If $\alpha = (\zeta, \xi)$ with $\zeta = n_1\zeta_1 + n_2\zeta_2 \in \zeta_1\mathbb{Z} \oplus \zeta_2\mathbb{Z}$, and $z = (z', x_d) \in \omega_T \times \mathbb{R}_+$,

$$U_\alpha(z)\, e^{iz\cdot\alpha/\varepsilon} = U_\alpha(z)\, e^{in_1 z'\cdot\zeta_1/\varepsilon}\, e^{in_2 z'\cdot\zeta_2/\varepsilon}\, e^{i\xi\, x_d/\varepsilon} = U_\alpha(z)\, e^{in_1\,\theta_1}\, e^{in_2\,\theta_2}\, e^{i\xi\,\psi_d},$$

with $\theta = (\theta_1, \theta_2) = (z'\cdot\zeta_1/\varepsilon, z'\cdot\zeta_2/\varepsilon) \in \mathbb{T}^2$ and $\psi_d := x_d/\varepsilon \in \mathbb{R}_+$ the new fast variables.

We use a framework of almost-periodic functions in the sense of Bohr. A function U of $\mathcal{C}_b(\mathbb{R}^+_{x_d} \times \mathbb{R}^+_{\psi_d}, H^s(\omega_T \times \mathbb{T}^2))$ is called a *trigonometric polynomial* (in ψ_d) if it writes as a finite sum

$$U(z, \theta, \psi_d) = \sum_{\xi \in \mathbb{R}} U_\xi(z, \theta)\, e^{i\,\psi_d\,\xi}.$$

Definition

The space of profiles $\mathcal{P}_{s,T}$ is defined as the closure in $\mathcal{C}_b(\mathbb{R}^+_{x_d} \times \mathbb{R}^+_{\psi_d}, H^s(\omega_T \times \mathbb{T}^2))$ of the set of trigonometric polynomials.

System satisfied by the main profile

Ansatz and main result

We look for an approximate solution of (1) under the form of a formal series $u^{\varepsilon,\mathrm{app}}(z) = v(z, z' \cdot \zeta_1/\varepsilon, z' \cdot \zeta_2/\varepsilon, x_d/\varepsilon)$, where v is given by

$$v(z, \theta, \psi_d) := \sum_{k \geqslant 1} \varepsilon^k \, U_k(z, \theta, \psi_d),$$

with U_1 in $\mathcal{P}_{s,T}$ and U_k a formal trigonometric series, for $k \geqslant 2$.

Theorem (K. 2021)

Under the uniform Kreiss-Lopatinskii condition and with assumptions on the set of resonances, for $s \geqslant 0$ large enough, there exists a time $T > 0$ and a leading profile U_1 solution in $\mathcal{P}_{s,T}$ of the problem (2) given below, that governs the evolution of the leading profile.

Remark. This theorem applies to the compressible isentropic Euler equations in dimension two.

WKB cascade

For $u^{\varepsilon,\mathrm{app}}$ to formally satisfy the system (1), a WKB study and a decoupling of the cascade obtained shows that the leading profile U_1 has to satisfy the following system

$$\mathbf{E}\, U_1 = U_1 \tag{2a}$$

$$\mathbf{E}\Big[L(0,\partial_z)\, U_1 + \mathcal{M}(U_1, U_1)\Big] = 0 \tag{2b}$$

$$B\, U_1|_{x_d=0,\psi_d=0} = G \tag{2c}$$

$$U_1|_{t\leqslant 0} = 0. \tag{2d}$$

with $\mathbf{E}$ a projector on $\mathcal{P}_{s,T}$.

Existence of a solution to (2) is obtained using energy estimates without loss of derivative. Two terms have to be treated.

Transport part

If U_1 reads as

$$U_1(z, \theta, \psi_d) = \sum_\alpha U^1_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d},$$

then the transport part $\mathbf{E}\Big[L(0, \partial_z)\, U_1\Big]$ reads as a sum of transport terms

$$\mathbf{E}\Big[L(0, \partial_z)\, U_1\Big] = \sum_\alpha \big(\partial_t + v_\alpha \cdot \nabla_x\big)\, U^1_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d},$$

which are easy to treat in energy estimates.

Remark. The sign of the x_d-component of v_α determines if the frequency α is incoming or outgoing.

Quadratic part

As for the quadratic term $\mathbf{E}\left[\mathcal{M}(U_1, U_1)\right]$, we have

$$\mathbf{E}\left[\mathcal{M}(U_1, U_1)\right] = \sum_{\alpha,\alpha'} \pi_{\alpha+\alpha'}\, L_1\left(U^1_\alpha, n'_1\,\zeta_1 + n'_2\,\zeta_2\right) U^1_{\alpha'}\, e^{i(n_1+n'_1)\theta_1}\, e^{i(n_2+n'_2)\theta_2} e^{i(\xi+\xi')\psi_d}.$$

Remarks.

- The coefficient $\left(n'_1\,\zeta_1 + n'_2\,\zeta_2\right)$ acts as a derivative in θ.
- Only the terms for which the frequency $\alpha + \alpha'$ is characteristic remain, this is resonance.
  - → If α and α' are collinear, there is always resonance, this is called self-interaction, and generates terms of Burgers type.
  - → If not, this is a real resonance, and generates terms of convolution type, that are more difficult to handle.
- The main additional difficulty compared to [Joly-Métivier-Rauch 1995] is the lack of symmetry in the resonance terms.

Perspectives

Justification of the expansion

To prove that the approximate solution actually approaches the exact solution to (1), two methods are possible.

- Having an exact solution on a time interval independent of ε. Proving that the time of existence for (1) does not depend on ε seems out of reach for the moment.
- Using a large number of correctors of the asymptotic expansion. A lot more frequencies may occur in the correctors, making more complex the functional framework needed.

Outgoing frequencies

One assumption of the theorem ensures that all outgoing frequencies are zero, allowing to determine beforehand the trace on the boundary of profiles associated with incoming frequencies.

Allowing outgoing frequencies to occur in the solution would also make more complex the functional framework, since

Incoming frequencies $\longrightarrow$ propagation in x_d

Outgoing frequencies $\longrightarrow$ evolution in t.

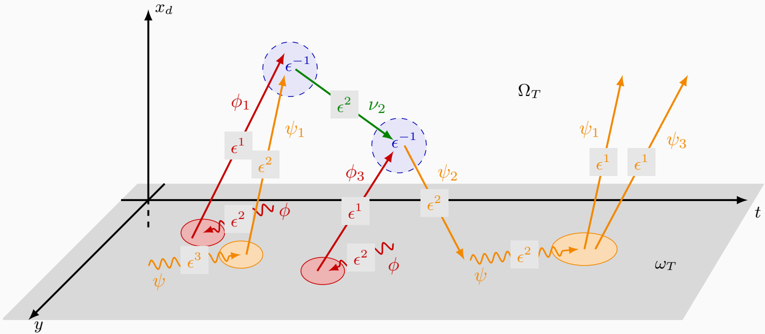
Weakly stable problems

Weakening the uniform Kreiss-Lopatinskii condition on the boundary allows amplification to happen on the boundary.

Considering a perturbation H of small amplitude $O(\varepsilon^M)$ ($M \geqslant 3$) of a periodic forcing boundary term G of amplitude $O(\varepsilon^2)$,

$$\varepsilon\, g^\varepsilon(z') = \varepsilon^2\, G\left(z', \frac{z' \cdot \varphi}{\varepsilon}\right) + \varepsilon^M\, H\left(z', \frac{z' \cdot \psi}{\varepsilon}\right),$$

with a particular configuration of boundary frequencies φ and ψ, we prove (K. 2022), on a study model, that an instability may be created.



Thank you for your attention !