Geometric optics for quasilinear hyperbolic boundary value problems

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Introduction

We consider the following problem

$$\begin{cases} L(u^\varepsilon, \partial_z)\, u^\varepsilon := \partial_t u^\varepsilon + \displaystyle\sum_{j=1}^{d-1} A_j(u^\varepsilon)\, \partial_{x_j} u^\varepsilon + \partial_{x_d} u^\varepsilon = 0 & \text{in } \Omega_T, \\ B\, u^\varepsilon_{|x_d=0} = \varepsilon\, g^\varepsilon & \text{on } \omega_T, \\ u^\varepsilon_{|t\leqslant 0} = 0, & \end{cases} \tag{1}$$

where

- $\Omega_T := (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $\omega_T := (-\infty, T] \times \mathbb{R}^{d-1}$, with $T > 0$,
- we denote $z = (t, x) = (t, y, x_d) \in \Omega_T = (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$, and $z' := (t, y) \in \omega_T = (-\infty, T] \times \mathbb{R}^{d-1}$,
- the unknown u^ε is a (regular) function from Ω_T to $\mathbb{R}^N$, $N \geqslant 2$,
- for all $j = 1, \ldots, d-1$, A_j is a regular map from $\mathbb{R}^N$ into $\mathcal{M}_N(\mathbb{R})$,
- B belongs to $\mathcal{M}_{M,N}(\mathbb{R})$ for some $1 \leqslant M \leqslant N$ and is of maximal rank.

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The dependency in ε of the system comes from the boundary term $\varepsilon\, g^\varepsilon$, where g^ε is given by, for $z' \in \omega_T$,

$$g^\varepsilon(z') = G\left(z', \frac{z' \cdot \zeta_1}{\varepsilon}, \frac{z' \cdot \zeta_2}{\varepsilon}\right),$$

where G belongs to $H^\infty(\omega_T \times \mathbb{T}^2)$, zero for negative times t, and ζ_1, ζ_2 are in $\mathbb{R}^d \setminus \{0\}$.

The aim is to construct an approximate solution of (1) in the high frequency asymptotic, namely for $\varepsilon \to 0$, in the form of a WKB expansion. More precisely we expect that

$$u^\varepsilon(z) \sim \varepsilon\, U_1\Big(z, \frac{\Phi_1(z)}{\varepsilon}\Big) + \varepsilon^2\, U_2\Big(z, \frac{\Phi_2(z)}{\varepsilon}\Big) + \cdots .$$

Outline of the presentation

$\longrightarrow$ State of the art.

$\longrightarrow$ Functional framework.

$\longrightarrow$ Ansatz and main result.

$\longrightarrow$ WKB cascade and resolution of some fast problem.

$\longrightarrow$ Polarization and evolution equation.

$\longrightarrow$ Boundary conditions.

$\longrightarrow$ Conclusion and perspectives.

State of the art I

- Same boundary value problem, but with only one phase on the boundary, in a quasi-periodic functional framework :
  - → Mark Williams. "Singular pseudodifferential operators, symmetrizers, and oscillatory multidimensional shocks". In : *J. Funct. Anal.* 191.1 (2002), p. 132-209,
  - → Jean-Francois Coulombel, Olivier Gues et Mark Williams. "Resonant leading order geometric optics expansions for quasilinear hyperbolic fixed and free boundary problems". In : *Comm. Partial Differential Equations* 36.10 (2011), p. 1797-1859,
  - → Matthew Hernandez. "Resonant leading term geometric optics expansions with boundary layers for quasilinear hyperbolic boundary problems". In : *Comm. Partial Differential Equations* 40.3 (2015), p. 387-437.

State of the art II

- Multiple phases for a semilinear problem, in an almost-periodic (Wiener algebras) functional framework :
  - → Jean-Luc Joly, Guy Métivier et Jeffrey Rauch. "Coherent nonlinear waves and the Wiener algebra". In : *Ann. Inst. Fourier (Grenoble)* 44.1 (1994), p. 167-196,
  - → Mark Williams. "Nonlinear geometric optics for hyperbolic boundary problems". In : *Comm. Partial Differential Equations* 21.11-12 (1996), p. 1829-1895.
- Multiple phases for the quasilinear Cauchy problem, in a framework of almost periodic functions (in the sense of Bohr) :
  - → Jean-Luc Joly, Guy Métivier et Jeffrey Rauch. "Coherent and focusing multidimensional nonlinear geometric optics". In : *Ann. Sci. École Norm. Sup. (4)* 28.1 (1995), p. 51-113.

Characteristic Frequencies

Note that with $\alpha = (\tau, \eta, \xi) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$, we have

$$L(0, \partial_z)\, e^{iz\cdot\alpha/\varepsilon} = \left[\partial_t + \sum_{j=1}^{d-1} A_j(0)\,\partial_{x_j} + \partial_{x_d}\right] e^{iz\cdot\alpha/\varepsilon} = \frac{i}{\varepsilon}\left[\tau I + \sum_{j=1}^{d-1} \eta_j A_j(0) + \xi I\right] e^{iz\cdot\alpha/\varepsilon},$$

so if we denote $L(0, \alpha) := \tau I + \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi I$ the symbol associated to α, we have

$$L(0, \partial_z)\left\{U(z)\, e^{iz\cdot\alpha/\varepsilon}\right\} = \left[L(0, \partial_z)\, U(z) + \frac{i}{\varepsilon}\, L(0, \alpha)\, U(z)\right] e^{iz\cdot\alpha/\varepsilon}.$$

The frequency α is said to be *characteristic* if $L(0, \alpha)$ is not invertible.

Assumption

For all α in $\mathbb{R}^{d+1}$, we have

$$\mathbb{C}^N = \ker L(0, \alpha) \oplus \operatorname{Im} L(0, \alpha).$$

For all α in $\mathbb{R}^{d+1}$, we define π_α the projector onto $\ker L(0, \alpha)$ along $\operatorname{Im} L(0, \alpha)$.

Characteristic frequencies for Euler equations

For the isentropic compressible Euler system in dimension two, if $\alpha = (\tau, \eta, \xi) \in \mathbb{R}^3$, we have

$$\det L(0, \alpha) = (\tau + \xi u_0) \times \Big((\tau + \xi u_0)^2 - c_0^2 (\eta^2 + \xi^2) \Big),$$

with $u_0 > 0$ an incoming subsonic velocity and c_0 the sound velocity in the fluid at a fixed volume $v_0 > 0$.

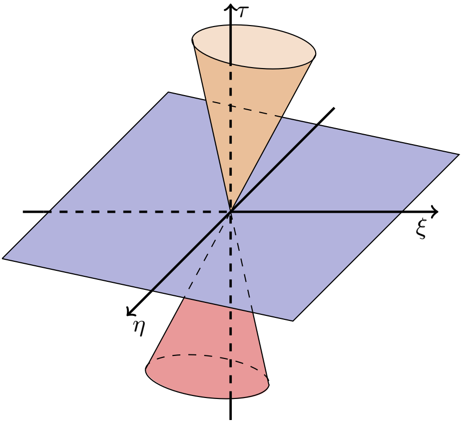

FIGURE – Characteristic frequencies for isentropic compressible Euler equations in dimension two

Set of frequencies inside the domain

Which frequencies may occur in the solution ?

- By nonlinearity, the two frequencies on the boundary ζ_1 and ζ_2 generate the group of frequencies

$$\mathcal{F}_b := \zeta_1 \mathbb{Z} \oplus \zeta_2 \mathbb{Z}$$

  on the boundary.

- Then, each frequency $\zeta \in \mathcal{F}_b \setminus \{0\}$ is lifted into real characteristic frequencies $\alpha = (\zeta, \xi)$, with $\xi \in \mathbb{R}$ and frequencies $\alpha = (\zeta, \xi)$ with $\operatorname{Im} \xi > 0$ (boundary layers).

The infinite countable set of frequencies inside the domain is therefore given by

$$\mathcal{F} := \{0\} \cup \left\{ \alpha = (\zeta, \xi) \,\middle|\, \zeta \in \mathcal{F}_b \setminus \{0\} \right\}.$$

In our framework there are an infinite countable number of resonances allowed, that is triplets of non-collinear characteristic frequencies $(\alpha_1, \alpha_2, \alpha_3)$ such that

$$\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}^*, \quad \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3 = 0.$$

Functional framework : almost-periodic functions

We need a functional framework allowing to consider functions of the form

$$\sum_{\alpha\in\mathcal{F}} U_\alpha(z)\, e^{iz\cdot\alpha/\varepsilon}.$$

If $\alpha = (\zeta,\xi)$ with $\zeta = n_1\zeta_1 + n_2\zeta_2 \in \zeta_1\mathbb{Z}\oplus\zeta_2\mathbb{Z}$, and $z = (z', x_d) \in \omega_T \times \mathbb{R}_+$,

$$U_\alpha(z)\, e^{iz\cdot\alpha/\varepsilon} = U_\alpha(z)\, e^{in_1 z'\cdot\zeta_1/\varepsilon}\, e^{in_2 z'\cdot\zeta_2/\varepsilon}\, e^{i\xi\, x_d/\varepsilon} = U_\alpha(z)\, e^{in_1\,\theta_1}\, e^{in_2\,\theta_2}\, e^{i\xi\,\psi_d},$$

with $\theta = (\theta_1,\theta_2) = (z'\cdot\zeta_1/\varepsilon, z'\cdot\zeta_2/\varepsilon) \in \mathbb{T}^2$ and $\psi_d := x_d/\varepsilon \in \mathbb{R}_+$ the new fast variables.

We use a framework of almost-periodic functions in the sense of Bohr. A function U of $\mathcal{C}_b(\mathbb{R}^+_{x_d} \times \mathbb{R}^+_{\psi_d}, H^s(\omega_T \times \mathbb{T}^2))$ is called a *trigonometric polynomial* (in ψ_d) if it writes as a finite sum

$$U(z,\theta,\psi_d) = \sum_{\xi\in\mathbb{R}} U_\xi(z,\theta)\, e^{i\,\psi_d\,\xi}.$$

Definition

The space of profiles $\mathcal{P}_{s,T}$ is defined as the closure in $\mathcal{C}_b(\mathbb{R}^+_{x_d} \times \mathbb{R}^+_{\psi_d}, H^s(\omega_T \times \mathbb{T}^2))$ of the set of trigonometric polynomials.

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$$U_\alpha(z)\, e^{iz\cdot\alpha/\varepsilon} = U_\alpha(z)\, e^{in_1 z'\cdot\zeta_1/\varepsilon}\, e^{in_2 z'\cdot\zeta_2/\varepsilon}\, e^{i\xi\, x_d/\varepsilon} = U_\alpha(z)\, e^{in_1\,\theta_1}\, e^{in_2\,\theta_2}\, e^{i\xi\,\psi_d},$$

with $\theta = (\theta_1, \theta_2) = (z'\cdot\zeta_1/\varepsilon, z'\cdot\zeta_2/\varepsilon) \in \mathbb{T}^2$ and $\psi_d := x_d/\varepsilon \in \mathbb{R}_+$ the new fast variables.

We can think of an element U of $\mathcal{P}_{s,T}$ as a series

$$U(z, \theta, \psi_d) = \sum_{\substack{\alpha = (n_1\zeta_1 + n_2\zeta_2, \xi) \\ \in \mathcal{F}_b \times \mathbb{R}}} U_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d}.$$

Ansatz and main result

We look for an approximate solution of (1) under the form of a formal series $u^{\varepsilon,\text{app}}(z) = v(z, z' \cdot \zeta_1/\varepsilon, z' \cdot \zeta_2/\varepsilon, x_d/\varepsilon)$, where v is given by

$$v(z,\theta,\psi_d) := \sum_{k\geqslant 1} \varepsilon^k \, U_k(z,\theta,\psi_d),$$

with U_1 in $\mathcal{P}_{s,T}$ and U_k a formal trigonometric series, for $k \geqslant 2$.

Theorem

Under the uniform Kreiss-Lopatinskii condition and with assumptions on the set of resonances, for $s \geqslant 0$ large enough, there exists a time $T > 0$ and a leading profile U_1 solution in $\mathcal{P}_{s,T}$ of the problem (3) given below, that governs the evolution of the leading profile.

Remark. This theorem applies to the compressible isentropic Euler equations in dimension two.

WKB cascade

We want $u^{\varepsilon,\text{app}}$ to formally satisfy the system

$$\begin{cases} L(u^{\varepsilon,app}, \partial_z)\, u^{\varepsilon,app} = 0 & \text{in } \Omega_T, \\ B\, u^{\varepsilon,app}_{|x_d=0} = \varepsilon\, g^{\varepsilon} & \text{on } \omega_T, \\ u^{\varepsilon,app}_{|t\leqslant 0} = 0. \end{cases} \tag{1}$$

Thus, using a formal study, we obtain the following necessary conditions

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U_1 = 0, \tag{2a}$$

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U_2 + L(0, \partial_z)\, U_1 + \mathcal{M}(U_1, U_1) = 0, \tag{2b}$$

where the fast operator $\mathcal{L}(\partial_\theta, \partial_{\psi_d})$ is defined as

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d}) := L(0, \zeta_1)\, \partial_{\theta_1} + L(0, \zeta_2)\, \partial_{\theta_2} + \partial_{\psi_d},$$

and the quadratic operator $\mathcal{M}$ as

$$\mathcal{M}(U, V) := \sum_{j=1}^{d-1} dA_j(0) \cdot U \left(\zeta_1^j\, \partial_{\theta_1} + \zeta_2^j\, \partial_{\theta_2} \right) V.$$

Thus we wish to determine the kernel and the range of the fast operator $\mathcal{L}(\partial_\theta, \partial_{\psi_d})$.

Resolution of the fast problem

We want to solve

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U = H,$$

with U and H given in $\mathcal{P}_{s,T}$ by

$$U(z,\theta,\psi_d) = \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)\in\mathcal{F}_b\times\mathbb{R}} U_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d},$$

$$H(z,\theta,\psi_d) = \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)\in\mathcal{F}_b\times\mathbb{R}} H_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d}.$$

We have

$$\begin{aligned}
\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U &= \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)} \Big\{ L(0,\zeta_1)\,\partial_{\theta_1} + L(0,\zeta_2)\,\partial_{\theta_2} + \partial_{\psi_d} \Big\}\, U_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d} \\
&= \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)} i\Big\{ L(0,n_1\,\zeta_1) + L(0,n_2\,\zeta_2) + \xi\, I \Big\}\, U_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d} \\
&= \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)} iL(0,\alpha)\, U_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d}.
\end{aligned}$$

Kernel and range of the fast operator

Thus if $H = 0$,

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U = 0 \quad \Leftrightarrow \quad \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)} L(0,\alpha)\, U_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d} = 0$$

$$\Leftrightarrow \quad L(0,\alpha)\, U_\alpha = 0,\ \forall\alpha \quad \Leftrightarrow \quad \pi_\alpha\, U_\alpha = U_\alpha,\ \forall\alpha.$$

On an other hand,

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U = H \quad \Leftrightarrow \quad iL(0,\alpha)\, U_\alpha = H_\alpha,\ \forall\alpha \quad \Rightarrow \quad \pi_\alpha\, H_\alpha = 0,\ \forall\alpha.$$

Thus if $\mathbf{E}$ is the projector defined on $\mathcal{P}_{s,T}$ as

$$\mathbf{E}\, U = \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)} \pi_\alpha\, U_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d},$$

then

$$\ker \mathcal{L}(\partial_\theta, \partial_{\psi_d}) = \operatorname{Im} \mathbf{E}, \qquad \operatorname{Im} \mathcal{L}(\partial_\theta, \partial_{\psi_d}) = \ker \mathbf{E}.$$

System satisfied by the leading profile

Therefore

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U_1 = 0 \qquad \text{and} \qquad \mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U_2 + L(0, \partial_z)\, U_1 + \mathcal{M}(U_1, U_1) = 0$$

imply that the leading profile U_1 has to satisfy the following system

$$\mathbf{E}\, U_1 = U_1 \tag{3a}$$

$$\mathbf{E}\Big[L(0, \partial_z)\, U_1 + \mathcal{M}(U_1, U_1)\Big] = 0 \tag{3b}$$

together with the boundary and initial conditions

$$B\, U_1|_{x_d=0, \psi_d=0} = G \tag{3c}$$

$$U_1|_{t \leqslant 0} = 0. \tag{3d}$$

Remark. The main additional difficulty compared to [Joly-Métivier-Rauch 1995] is the lack of symmetry in the resonance terms.

Polarization of the leading profile

Recall that the projector π_α onto $\ker L(0,\alpha)$ is zero for α noncharacteristic. Thus, if U_1 writes in $\mathcal{P}_{s,T}$ as

$$U_1(z,\theta,\psi_d) = \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)} U^1_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d},$$

then the polarization equation $\mathbf{E}\, U_1 = U_1$ writes

$$\sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)} U^1_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d} = \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)} \pi_\alpha\, U^1_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d}$$

and therefore for all α noncharacteristic,

$$U^1_\alpha = \pi_\alpha\, U^1_\alpha = 0,$$

and for all α characteristic,

$$U^1_\alpha = \pi_\alpha\, U^1_\alpha \quad \Leftrightarrow \quad U^1_\alpha \in \ker L(0,\alpha).$$

Thus

$$U_1(z,\theta,\psi_d) = \sum_{\substack{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)\\ \alpha \text{ charact}}} U^1_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\psi_d},$$

Evolution equation : transport term

In equation (3b), there is a transport term $\mathbf{E}\left[L(0,\partial_z)\,U_1\right]$ that writes

$$\mathbf{E}\Bigg[\sum_{\substack{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)\\ \alpha \text{ charact}}} L(0,\partial_z)\,U^1_\alpha(z)\,e^{in_1\theta_1}\,e^{in_2\theta_2}e^{i\xi\psi_d}\Bigg]$$

$$= \sum_{\substack{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)\\ \alpha \text{ charact}}} \pi_\alpha\,L(0,\partial_z)\,\pi_\alpha\,U^1_\alpha(z)\,e^{in_1\theta_1}\,e^{in_2\theta_2}e^{i\xi\psi_d}.$$

Lemma (Lax Lemma)

If $\alpha \in \mathbb{R}^{d+1}\setminus\{0\}$ is a characteristic frequency, then there exists $v_\alpha \in \mathbb{R}^d$ such that

$$\pi_\alpha\,L(0,\partial_z)\,\pi_\alpha = \left(\partial_t + v_\alpha\cdot\nabla_x\right)\pi_\alpha.$$

Therefore,

$$\mathbf{E}\left[L(0,\partial_z)\,U_1\right] = \sum_{\substack{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)\\ \alpha \text{ charact}}} \left(\partial_t + v_\alpha\cdot\nabla_x\right)U^1_\alpha(z)\,e^{in_1\theta_1}\,e^{in_2\theta_2}e^{i\xi\psi_d}.$$

Remark. The sign of the x_d-component of v_α determines if the frequency α is incoming or outgoing.

Evolution equation : quadratic term

As for the quadratic term $\mathbf{E}\Big[\mathcal{M}(U_1, U_1)\Big]$, recalling that

$$\mathcal{M}(U, V) = \sum_{j=1}^{d-1} dA_j(0) \cdot U \left(\zeta_1^j\,\partial_{\theta_1} + \zeta_2^j\,\partial_{\theta_2}\right) V,$$

we have

$$\mathbf{E}\Big[\mathcal{M}(U_1, U_1)\Big] = \sum_{\substack{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)\\ \alpha \text{ charact}}} \sum_{\substack{\alpha'=(n_1'\zeta_1+n_2'\zeta_2,\xi')\\ \alpha \text{ charact}}} \pi_{\alpha+\alpha'}$$

$$\left\{\sum_{j=1}^{d-1} dA_j(0) \cdot U^1_\alpha \, i\left(n_1'\,\zeta_1^j + n_2'\,\zeta_2^j\right) U^1_{\alpha'}\right\} e^{i(n_1+n_1')\theta_1}\, e^{i(n_2+n_2')\theta_2}\, e^{i(\xi+\xi')\psi_d}.$$

Remarks.

- The coefficient $\left(n_1'\,\zeta_1^j + n_2'\,\zeta_2^j\right)$ acts as a derivative in θ.
- Only the terms for which the frequency $\alpha + \alpha'$ is characteristic remain, this is resonance.
  - $\rightarrow$ If α and α' are collinear, there is always resonance, this is called self-interaction, and generates terms of Burgers type.
  - $\rightarrow$ If not, this is a real resonance, and generates terms of convolution type, that are more difficult to handle.

Boundary conditions

The boundary condition $B\,U_1|_{x_d=0,\psi_d=0} = G$ writes

$$B \sum_{\substack{\alpha=(n_1\zeta_1+n_2\zeta_2,\xi)\\ \alpha \text{ charact}}} U^1_\alpha|_{x_d=0}(z')\, e^{in_1\theta_1}\, e^{in_2\theta_2} = \sum_{n_1,n_2\in\mathbb{Z}} G_{n_1,n_2}(z')\, e^{in_1\theta_1}\, e^{in_2\theta_2},$$

if G decomposes as the second sum. Thus,

$$B \sum_{\substack{\xi\in\mathbb{R} \text{ s.t.}\\ \alpha=(n_1\zeta_1+n_2\zeta_2,\xi) \text{ charact}}} U^1_\alpha|_{x_d=0}(z') = G_{n_1,n_2}(z'), \qquad \forall n_1, n_2 \in \mathbb{Z}.$$

Therefore we would like the sum to belong to a space on which the matrix B is invertible. But

$$\sum_{\substack{\xi\in\mathbb{R} \text{ s.t.}\\ \alpha=(n_1\zeta_1+n_2\zeta_2,\xi) \text{ charact}}} U^1_\alpha \in \bigoplus_{\xi} \ker L(0,\alpha),$$

and, according to the uniform Kreiss-Lopatinskii condition, B is invertible on the later space if if no frequency α is outgoing.

Conclusion

End of the proof. Show a priori estimates for the evolution equation.

- Prove that there is no outgoing term.
- Use the assumption on the resonance set to make up for the lack of symmetry in the resonance terms.

Conclusion. Doing that achieves to prove that there exists a unique leading profile U_1 solution in $\mathcal{P}_{s,T}$ of the problem

$$\begin{aligned} \mathbf{E}\, U_1 &= U_1 \\ \mathbf{E}\Big[L(0,\partial_z)\, U_1 + \mathcal{M}(U_1, U_1)\Big] &= 0 \\ B\, U_1|_{x_d=0,\psi_d=0} &= G \\ U_1|_{t\leqslant 0} &= 0. \end{aligned}$$

Perspectives

- To justify that the asymptotic expansion is actually an approximation of the exact solution, there are two possibilities :
  - $\rightarrow$ to have an exact solution on a time interval independent of ε,
  - $\rightarrow$ to construct a large number of correctors U_k, $k \geqslant 2$ of the asymptotic expansion.

  These two points are open problems at this stage.

- Weakening the uniform Kreiss-Lopatinskii condition gives rise to amplification which could creates instability. This is the subject of a current work.

- The assumptions on the set of resonances could be changed to allow outgoing frequencies to appear inside the domain.

Thank you for your attention !