

Geometric optics for quasilinear hyperbolic boundary value problems

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Introduction

We consider the following problem

$$\left\{ \begin{array}{ll} L(u^\varepsilon, \partial_z) u^\varepsilon := \partial_t u^\varepsilon + \sum_{j=1}^{d-1} A_j(u^\varepsilon) \partial_{x_j} u^\varepsilon + \partial_{x_d} u^\varepsilon = 0 & \text{in } \Omega_T, \\ B u^\varepsilon|_{x_d=0} = \varepsilon g^\varepsilon & \text{on } \omega_T, \\ u^\varepsilon|_{t \leq 0} = 0, & \end{array} \right. \quad (1)$$

where

- $\Omega_T := (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $\omega_T := (-\infty, T] \times \mathbb{R}^{d-1}$, with $T > 0$,
- we denote $z = (t, x) = (t, y, x_d) \in \Omega_T = (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$, and $z' := (t, y) \in \omega_T = (-\infty, T] \times \mathbb{R}^{d-1}$,
- the **unknown** u^ε is a (regular) function from Ω_T to \mathbb{R}^N , $N \geq 2$,
- for all $j = 1, \dots, d-1$, A_j is a **regular** map from \mathbb{R}^N into $\mathcal{M}_N(\mathbb{R})$,
- B belongs to $\mathcal{M}_{M,N}(\mathbb{R})$ for some $1 \leq M \leq N$ and is of **maximal rank**.

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The **dependency in ε** of the system comes from the **boundary term** $\varepsilon g^\varepsilon$, where g^ε is given by, for $z' \in \omega_T$,

$$g^\varepsilon(z') = G\left(z', \frac{z' \cdot \zeta_1}{\varepsilon}, \frac{z' \cdot \zeta_2}{\varepsilon}\right),$$

where G belongs to $H^\infty(\omega_T \times \mathbb{T}^2)$, zero for negative times t , and ζ_1, ζ_2 are in $\mathbb{R}^d \setminus \{0\}$.

The aim is to construct an approximate solution of (1) in the **high frequency** asymptotic, namely for $\varepsilon \rightarrow 0$, in the form of a **WKB expansion**. More precisely we expect that

$$u^\varepsilon(z) \sim \varepsilon U_1\left(z, \frac{\Phi_1(z)}{\varepsilon}\right) + \varepsilon^2 U_2\left(z, \frac{\Phi_2(z)}{\varepsilon}\right) + \dots$$

Outline of the presentation

- State of the art.
- Functional framework.
- Ansatz and main result.
- System satisfied by the leading profile.
- Polarization and evolution equation.
- Conclusion.

- Same boundary value problem, but with only **one phase on the boundary**, in a **quasi-periodic** functional framework :
 - Mark WILLIAMS. “Singular pseudodifferential operators, symmetrizers, and oscillatory multidimensional shocks”. In : *J. Funct. Anal.* 191.1 (2002), p. 132-209,
 - Jean-Francois COULOMBEL, Olivier GUES et Mark WILLIAMS. “Resonant leading order geometric optics expansions for quasilinear hyperbolic fixed and free boundary problems”. In : *Comm. Partial Differential Equations* 36.10 (2011), p. 1797-1859,
 - Matthew HERNANDEZ. “Resonant leading term geometric optics expansions with boundary layers for quasilinear hyperbolic boundary problems”. In : *Comm. Partial Differential Equations* 40.3 (2015), p. 387-437.

State of the art II

- Multiple phases for a **semilinear** problem, in an **almost-periodic (Wiener algebras)** functional framework :
 - Jean-Luc JOLY, Guy MÉTIVIER et Jeffrey RAUCH. "Coherent nonlinear waves and the Wiener algebra". In : *Ann. Inst. Fourier (Grenoble)* 44.1 (1994), p. 167-196,
 - Mark WILLIAMS. "Nonlinear geometric optics for hyperbolic boundary problems". In : *Comm. Partial Differential Equations* 21.11-12 (1996), p. 1829-1895.
- Multiple phases for the quasilinear **Cauchy problem**, in a framework of **almost periodic functions (in the sense of Bohr)** :
 - Jean-Luc JOLY, Guy MÉTIVIER et Jeffrey RAUCH. "Coherent and focusing multidimensional nonlinear geometric optics". In : *Ann. Sci. École Norm. Sup. (4)* 28.1 (1995), p. 51-113.

Characteristic Frequencies

Note that with $\alpha = (\tau, \eta, \xi) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$, we have

$$L(0, \partial_z) e^{iz \cdot \alpha / \varepsilon} = \left[\partial_t + \sum_{j=1}^{d-1} A_j(0) \partial_{x_j} + \partial_{x_d} \right] e^{iz \cdot \alpha / \varepsilon} = \frac{i}{\varepsilon} \left[\tau I + \sum_{j=1}^{d-1} \eta_j A_j(0) + \xi I \right] e^{iz \cdot \alpha / \varepsilon},$$

so if we denote $L(0, \alpha) := \tau I + \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi I$ the **symbol associated to α** , we have

$$L(0, \partial_z) \left\{ U(z) e^{iz \cdot \alpha / \varepsilon} \right\} = \left[L(0, \partial_z) U(z) + \frac{i}{\varepsilon} L(0, \alpha) U(z) \right] e^{iz \cdot \alpha / \varepsilon}.$$

The frequency α is said to be **characteristic** if $L(0, \alpha)$ is **not invertible**.

Assumption

For all α in \mathbb{R}^{d+1} , we have

$$\mathbb{C}^N = \ker L(0, \alpha) \oplus \operatorname{Im} L(0, \alpha).$$

For all α in \mathbb{R}^{d+1} , we define π_α the **projector** onto $\ker L(0, \alpha)$ along $\operatorname{Im} L(0, \alpha)$.

Set of frequencies inside the domain

Which frequencies may occur in the solution ?

- By nonlinearity, the two frequencies on the boundary ζ_1 and ζ_2 generate the group of frequencies

$$\mathcal{F}_b := \zeta_1 \mathbb{Z} \oplus \zeta_2 \mathbb{Z}$$

on the boundary.

- Then, each frequency $\zeta \in \mathcal{F}_b \setminus \{0\}$ is lifted into real characteristic frequencies $\alpha = (\zeta, \xi)$, with $\xi \in \mathbb{R}$ and frequencies $\alpha = (\zeta, \xi)$ with $\text{Im } \xi > 0$ (boundary layers).

The infinite countable set of frequencies inside the domain is therefore given by

$$\mathcal{F} := \{0\} \cup \{ \alpha = (\zeta, \xi) \mid \zeta \in \mathcal{F}_b \setminus \{0\} \}.$$

In our framework there are an infinite countable number of resonances allowed, that is triplets of non-collinear characteristic frequencies $(\alpha_1, \alpha_2, \alpha_3)$ such that

$$\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}^*, \quad \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 \alpha_3 = 0.$$

Functional framework : almost-periodic functions

We need a **functional framework** allowing to consider functions of the form

$$\sum_{\alpha \in \mathcal{F}} U_{\alpha}(z) e^{iz \cdot \alpha / \varepsilon}.$$

If $\alpha = (\zeta, \xi)$ with $\zeta = n_1 \zeta_1 + n_2 \zeta_2 \in \zeta_1 \mathbb{Z} \oplus \zeta_2 \mathbb{Z}$, and $z = (z', x_d) \in \omega_T \times \mathbb{R}_+$,

$$U_{\alpha}(z) e^{iz \cdot \alpha / \varepsilon} = U_{\alpha}(z) e^{im_1 z' \cdot \zeta_1 / \varepsilon} e^{im_2 z' \cdot \zeta_2 / \varepsilon} e^{i\xi x_d / \varepsilon} = U_{\alpha}(z) e^{im_1 \theta_1} e^{im_2 \theta_2} e^{i\xi \psi_d},$$

with $\theta = (\theta_1, \theta_2) = (z' \cdot \zeta_1 / \varepsilon, z' \cdot \zeta_2 / \varepsilon) \in \mathbb{T}^2$ and $\psi_d := x_d / \varepsilon \in \mathbb{R}_+$ the new **fast variables**.

We use a framework of **almost-periodic functions in the sense of Bohr**. A function U of $\mathcal{C}_b(\mathbb{R}_{x_d}^+ \times \mathbb{R}_{\psi_d}^+, H^s(\omega_T \times \mathbb{T}^2))$ is called a **trigonometric polynomial** (in ψ_d) if it writes as a **finite sum**

$$U(z, \theta, \psi_d) = \sum_{\xi \in \mathbb{R}} U_{\xi}(z, \theta) e^{i\psi_d \xi}.$$

Definition

The **space of profiles** $\mathcal{P}_{s, T}$ is defined as the **closure** in $\mathcal{C}_b(\mathbb{R}_{x_d}^+ \times \mathbb{R}_{\psi_d}^+, H^s(\omega_T \times \mathbb{T}^2))$ of the set of **trigonometric polynomials**.

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$$U_{\alpha}(z) e^{iz \cdot \alpha / \varepsilon} = U_{\alpha}(z) e^{in_1 z' \cdot \zeta_1 / \varepsilon} e^{in_2 z' \cdot \zeta_2 / \varepsilon} e^{i\xi x_d / \varepsilon} = U_{\alpha}(z) e^{in_1 \theta_1} e^{in_2 \theta_2} e^{i\xi \psi_d},$$

with $\theta = (\theta_1, \theta_2) = (z' \cdot \zeta_1 / \varepsilon, z' \cdot \zeta_2 / \varepsilon) \in \mathbb{T}^2$ and $\psi_d := x_d / \varepsilon \in \mathbb{R}_+$ the new **fast variables**.

We can think of an element U of $\mathcal{P}_{s,T}$ as a **series**

$$U(z, \theta, \psi_d) = \sum_{\substack{\alpha = (n_1 \zeta_1 + n_2 \zeta_2, \xi) \\ \in \mathcal{F}_b \times \mathbb{R}}} U_{\alpha}(z) e^{in_1 \theta_1} e^{in_2 \theta_2} e^{i\xi \psi_d}.$$

Ansatz and main result

We look for an **approximate solution** of (1) under the form of a **formal series**

$u^{\varepsilon, \text{app}}(z) = v(z, z' \cdot \zeta_1/\varepsilon, z' \cdot \zeta_2/\varepsilon, x_d/\varepsilon)$, where v is given by

$$v(z, \theta, \psi_d) := \sum_{k \geq 1} \varepsilon^k U_k(z, \theta, \psi_d),$$

with U_1 in $\mathcal{P}_{s, T}$ and U_k a formal trigonometric series, for $k \geq 2$.

Theorem

Under the **uniform Kreiss-Lopatinskii condition** and with **assumptions on the set of resonances**, for $s \geq 0$ large enough, **there exists a time $T > 0$ and a leading profile U_1 solution in $\mathcal{P}_{s, T}$ of the problem (2) given below, that governs the evolution of the leading profile.**

Remark. This theorem applies to the compressible isentropic **Euler equations** in dimension two.

WKB cascade

For $u^{\varepsilon, \text{app}}$ to **formally satisfy** the system (1), a **BKW study** and a **decoupling of the cascade** obtained shows that the **leading profile U_1** has to **satisfy the following system**

$$\mathbf{E} U_1 = U_1 \quad (2a)$$

$$\mathbf{E} \left[L(0, \partial_z) U_1 + \mathcal{M}(U_1, U_1) \right] = 0 \quad (2b)$$

$$B U_1|_{x_d=0, \psi_d=0} = G \quad (2c)$$

$$U_1|_{t \leq 0} = 0. \quad (2d)$$

where the **projector \mathbf{E}** is defined on $\mathcal{P}_{s, T}$ as

$$\mathbf{E} \left[\sum_{\alpha=(n_1 \zeta_1 + n_2 \zeta_2, \xi)} U_\alpha(z) e^{in_1 \theta_1} e^{in_2 \theta_2} e^{i\xi \psi_d} \right] = \sum_{\alpha=(n_1 \zeta_1 + n_2 \zeta_2, \xi)} \pi_\alpha U_\alpha(z) e^{in_1 \theta_1} e^{in_2 \theta_2} e^{i\xi \psi_d},$$

and the **quadratic operator \mathcal{M}** as

$$\mathcal{M}(U, V) := \sum_{j=1}^2 L_1(U, \zeta_j) \partial_{\theta_j} V, \quad L_1(U, \zeta_j) = \sum_{k=1}^{d-1} \zeta_j^k dA_k(0) \cdot U.$$

Remark. The **main additional difficulty** compared to [Joly-Métivier-Rauch 1995] is the **lack of symmetry** in the **resonance terms**.

Polarization of the leading profile

Recall that the projector π_α onto $\ker L(0, \alpha)$ is zero for α noncharacteristic. Thus, if U_1 writes in $\mathcal{P}_{s, T}$ as

$$U_1(z, \theta, \psi_d) = \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2, \xi)} U_\alpha^1(z) e^{im_1\theta_1} e^{in_2\theta_2} e^{i\xi\psi_d},$$

then the polarization equation $\mathbf{E} U_1 = U_1$ writes

$$\sum_{\alpha=(n_1\zeta_1+n_2\zeta_2, \xi)} U_\alpha^1(z) e^{im_1\theta_1} e^{in_2\theta_2} e^{i\xi\psi_d} = \sum_{\alpha=(n_1\zeta_1+n_2\zeta_2, \xi)} \pi_\alpha U_\alpha^1(z) e^{im_1\theta_1} e^{in_2\theta_2} e^{i\xi\psi_d}$$

and therefore for all α noncharacteristic,

$$U_\alpha^1 = \pi_\alpha U_\alpha^1 = 0,$$

and for all α characteristic,

$$U_\alpha^1 = \pi_\alpha U_\alpha^1 \Leftrightarrow U_\alpha^1 \in \ker L(0, \alpha).$$

Thus

$$U_1(z, \theta, \psi_d) = \sum_{\substack{\alpha=(n_1\zeta_1+n_2\zeta_2, \xi) \\ \alpha \text{ charact}}} U_\alpha^1(z) e^{im_1\theta_1} e^{in_2\theta_2} e^{i\xi\psi_d},$$

Evolution equation : transport term

In equation (2b), there is a **transport term** $\mathbf{E}[L(0, \partial_z) U_1]$ that writes

$$\begin{aligned} \mathbf{E} \left[\sum_{\substack{\alpha=(n_1 \zeta_1 + n_2 \zeta_2, \xi) \\ \alpha \text{ charact}}} L(0, \partial_z) U_\alpha^1(z) e^{in_1 \theta_1} e^{in_2 \theta_2} e^{i\xi \psi_d} \right] \\ = \sum_{\substack{\alpha=(n_1 \zeta_1 + n_2 \zeta_2, \xi) \\ \alpha \text{ charact}}} \pi_\alpha L(0, \partial_z) \pi_\alpha U_\alpha^1(z) e^{in_1 \theta_1} e^{in_2 \theta_2} e^{i\xi \psi_d}. \end{aligned}$$

Lemma (Lax Lemma)

If $\alpha \in \mathbb{R}^{d+1} \setminus \{0\}$ is a **characteristic frequency**, then there exists $v_\alpha \in \mathbb{R}^d$ such that

$$\pi_\alpha L(0, \partial_z) \pi_\alpha = (\partial_t + v_\alpha \cdot \nabla_x) \pi_\alpha.$$

Therefore,

$$\mathbf{E}[L(0, \partial_z) U_1] = \sum_{\substack{\alpha=(n_1 \zeta_1 + n_2 \zeta_2, \xi) \\ \alpha \text{ charact}}} (\partial_t + v_\alpha \cdot \nabla_x) U_\alpha^1(z) e^{in_1 \theta_1} e^{in_2 \theta_2} e^{i\xi \psi_d}.$$

Remark. The **sign** of the x_d -component of v_α determines if the **frequency** α is **incoming** or **outgoing**.

Evolution equation : quadratic term

As for the quadratic term $\mathbf{E}[\mathcal{M}(U_1, U_1)]$, recalling that

$$\mathcal{M}(U, V) := \sum_{j=1}^2 L_1(U, \zeta_j) \partial_{\theta_j} V, \quad L_1(U, \zeta_j) = \sum_{k=1}^{d-1} \zeta_j^k dA_k(0) \cdot U.$$

we have

$$\mathbf{E}[\mathcal{M}(U_1, U_1)] = \sum_{\substack{\alpha=(n_1 \zeta_1 + n_2 \zeta_2, \xi) \\ \alpha \text{ charact}}} \sum_{\substack{\alpha'=(n'_1 \zeta_1 + n'_2 \zeta_2, \xi') \\ \alpha' \text{ charact}}} \pi_{\alpha+\alpha'} L_1(U_\alpha^1, n'_1 \zeta_1 + n'_2 \zeta_2) U_{\alpha'}^1 e^{i(n_1+n'_1)\theta_1} e^{i(n_2+n'_2)\theta_2} e^{i(\xi+\xi')\psi_d}.$$

Remarks.

- The coefficient $(n'_1 \zeta_1 + n'_2 \zeta_2)$ acts as a derivative in θ .
- Only the terms for which the frequency $\alpha + \alpha'$ is characteristic remain, this is resonance.
 - If α and α' are collinear, there is always resonance, this is called self-interaction, and generates terms of Burgers type.
 - If not, this is a real resonance, and generates terms of convolution type, that are more difficult to handle.

Conclusion

End of the proof. Show *a priori estimates* for the evolution equation.

- Prove that there is *no outgoing term*.
- Use the *assumption on the resonance set* to make up for the *lack of symmetry* in the resonance terms.

Conclusion. Doing that achieves to prove that *there exists a unique leading profile* U_1 solution in $\mathcal{P}_{s,\tau}$ of the problem

$$\begin{aligned} \mathbf{E} U_1 &= U_1 \\ \mathbf{E} \left[L(0, \partial_z) U_1 + \mathcal{M}(U_1, U_1) \right] &= 0 \\ B U_1|_{x_d=0, \psi_d=0} &= G \\ U_1|_{t \leq 0} &= 0. \end{aligned}$$

Thank you for your attention !