Darboux theory of integrability in the sparse case

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Abstract

Darboux's theorem and Jouanolou's theorem deal with the existence of first integrals and rational first integrals of a polynomial vector field. These results are given in terms of the degree of the polynomial vector field. Here we show that we can get the same kind of results if we consider the size of a Newton polytope associated to the vector field. Furthermore, we show that in this context the bound is optimal.

Keywords: Darboux integrability, Rational first integral, Newton polytope, optimal bound
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Introduction

In this paper we study the following polynomial differential system in $\mathbb{C}^n$:

$$\frac{dX_1}{dt} = A_1(X_1, \dots, X_n), \; \dots, \; \frac{dX_n}{dt} = A_n(X_1, \dots, X_n),$$

where $A_i \in \mathbb{C}[X_1, \dots, X_n]$ and $\deg A_i \leq d$. We associate to this polynomial differential system the polynomial derivation $D = \sum_{i=1}^{n} A_i(X_1, \dots, X_n)\partial_{X_i}$.

The computation of first integrals of such polynomial differential systems is an old and classical problem. The situation is the following: we want to

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compute a function $\mathcal{F}$ such that the hypersurfaces $\mathcal{F}(X_1, \ldots, X_n) = c$, where c are constants, give orbits of the differential system. Thus we want to find a function $\mathcal{F}$ such that $D(\mathcal{F}) = 0$.

In 1878, G. Darboux [5] has given a strategy to find first integrals. One of the tools developed by G. Darboux is now called *Darboux polynomials*.
A polynomial f is said to be a *Darboux polynomial*, if $D(f) = g.f$, where g is a polynomial. The polynomial g is called the cofactor. A lot of properties of a polynomial differential system are related to Darboux polynomials of the corresponding derivation D, see e.g.[8, 6].
There exists a lot of different names in the literature for Darboux polynomials, for example we can find: special integrals, eigenpolynomials, algebraic invariant hypersurfaces, special polynomials or second integrals.

G. Darboux shows in [4] that *if the derivation D has at least $\binom{n+d-1}{n} + 1$ irreducible Darboux polynomials then D has a first integral which can be expressed by means of these polynomials.* More precisely the first integral has the following form: $\prod_i f_i^{\lambda_i}$ where f_i are Darboux polynomials and λ_i are complex numbers. This kind of integral is called today a Darboux first integral.

In 1979, J.-P. Jouanolou shows in his book [9], that *if a derivation has at least $\binom{n+d-1}{n} + n$ irreducible Darboux polynomials then the derivation has a rational first integral.* We recall that a rational first integral is a first integral which belongs to $\mathbb{C}(X_1, \ldots, X_n)$. As in Darboux's theorem, the rational first integral $\mathcal{F}$ can be expressed by means of these irreducible Darboux polynomials. We have $\mathcal{F} = \prod_i f_i^{e_i}$, where now $e_i \in \mathbb{Z}$.
Several authors have given simplified proof for this result. M. Singer proves this result in $\mathbb{C}^2$, see [19]. This approach is based on a work of Rosenlicht [17]. J.-A. Weil generalizes this strategy and gives a proof for derivations in $\mathbb{C}^n$, see [21]. J. Llibre and X. Zhang gives a direct proof of Jouanolou's result in [14].

Darboux and Jouanolou's theorem are improved in [12, 13]. The authors show that we get the same kind of result if we take into account the multiplicity of Darboux polynomials. The multiplicity of a Darboux polynomial is defined and studied in [3].

The Darboux theory of integrability has been successfully used in the study of some physical problems, see e.g. [20, 11], and in the study of limit cycles and centers, see e.g. [1, 18, 10]. Unfortunately, to the author knowledge, there do not exist example showing if these bounds are optimal. In this note we are going to study the situation in the sparse case. This means that we are going to consider polynomials A_i with some coefficients equals to zero. In this situation, the size of the polynomials A_i is not measured by the degree but by the size of its Newton polytope. We recall that the Newton polytope of a Laurent polynomial $f(\underline{X}) = \sum_{\alpha} c_\alpha X^\alpha$, where $\underline{X} = X_1, \dots, X_n$ and α is a multi-index $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, is the convex hull in $\mathbb{R}^n$ of the exponent α of all nonzero terms of f. We denote this polytope by $\mathcal{N}(f)$.

In this note we prove a result improving Darboux and Jouanolou theorem. Our result depends on the size of a Newton polytope associated to the derivation and not on the degree d. Furthermore, in this context we can give example showing that the bound is optimal. This is our result:

Theorem 1. *Let $D = \sum_{i=1}^n A_i(X_1, \dots, X_n)\partial_{X_i}$ be a derivation. Consider generic values $(x_1, \dots, x_n)$ in $\mathbb{C}^n$ and the polytope $N_D = \mathcal{N}\Big(\sum_{i=1}^n x_i \dfrac{A_i}{X_i}\Big)$. Let B be the number of integer points in $N_D \cap \mathbb{N}^n$, then*

1. *if D has at least $B+1$ irreducible Darboux polynomials then D has a Darboux first integral,*
2. *if D has at least $B+n$ irreducible Darboux polynomials then D has a rational first integral.*

Furthermore, these bounds are optimal.

We can remark that if we consider dense polynomials A_i with degree d, that is to say each coefficient of A_i is nonzero, then $B = \binom{n+d-1}{n}$. Thus Theorem 1 gives the classical bounds in the dense case.

Now, we illustrate why these bounds can be better than the classical ones. We give an example with $n = 2$ in order to give a picture. If A_i has the following form: $A_i(X_1, X_2) = c_{e,e}X_1^eX_2^e + c_{e-1,e}X_1^{e-1}X_2^e + c_{e,e-1}X_1^eX_2^{e-1} + c_{0,0}$, then $B = 3e + 2$, and $d = \deg(A_i) = 2e$. In this situation we have $\binom{n+d-1}{2} = 2e(2e + 1)/2$. Thus for such examples Theorem 1 gives a linear bound in e instead of a quadratic bound.

Figure 1 shows the Newton polygon of $A_i(X_1, X_2)$, when $e = 3$. The triangle corresponds to the Newton polygon of dense polynomials with total degree 6. In this situation, Jouanolou's theorem says that if we have 23 Darboux polynomials then we have a rational first integral. Here, our bound improves this result and means that 13 Darboux polynomials are sufficient to construct a rational first integral.

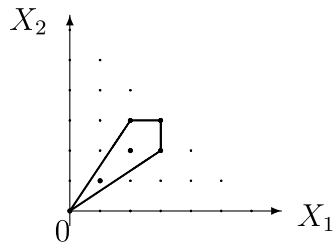

Figure 1: Newton polygon $\mathcal{N}\big(A_i(X_1, X_2)\big)$.

The Newton polygon of $\dfrac{A_i(X_1, X_2)}{X_1}$ corresponds to a translation of the Newton polygon of $A_i(X_1, X_2)$. In Figure 2 we show $\mathcal{N}\Big(\dfrac{A_i(X_1, X_2)}{X_1}\Big)$.

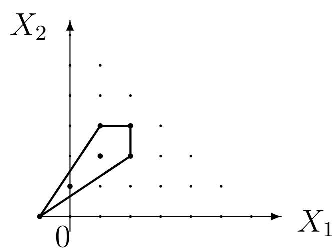

Figure 2: Newton polygon $\mathcal{N}\Big(\dfrac{A_i(X_1, X_2)}{X_1}\Big)$.

Figure 3 shows the part of the Newton polygon of $x_1\dfrac{A_1(X_1, X_2)}{X_1} + x_2\dfrac{A_2(X_1, X_2)}{X_2}$ in $\mathbb{N}^2$, when x_1, x_2 are generic.

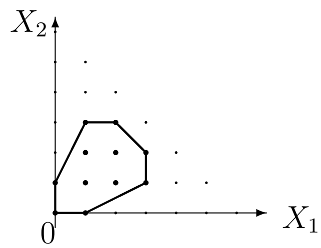

Figure 3: Newton polygon $\mathcal{N}\Big(x_1\dfrac{A_1(X_1,X_2)}{X_1}+x_2\dfrac{A_2(X_1,X_2)}{X_2}\Big)\cap\mathbb{N}^2$.

Structure of the paper

In Section 1 we give some results about Newton polytopes and weighted degree. In Section 2 we prove Theorem 1 and we show with examples that our bounds are optimal.

1. ToolBox

In this section we introduce some notations and results that will be useful in Section 2. These kinds of tools are already present in the work of Ostrowski, see [16]. For more results about sparse polynomials, see e.g. [7, 2].

Definition 2. Let P be a polytope, then H is a supporting hyperplane of P if

1. $H \cap P \neq \emptyset$,
2. P is fully contained in one of the two halfspaces defined by H.

In our situation, as we consider Newton polytopes, the equation of H has integer coefficients. More precisely, the equation of H is $\nu.m = a$, where "." denotes the usual scalar product, ν is a vector with integer coefficients, and a is an integer.

We can represent a Newton polytope with the equations of its supporting hyperplanes:

$$\mathcal{N}(f) = \{m \in \mathbb{Z}^n \mid \nu_j.m \leq a_j, \text{ for } j = 1, \dots, k\},$$

where k is the number of supporting hyperplanes.

Now we define a degree related to a vector $\nu \in \mathbb{Z}^n$.

Definition 3. Let $\nu \in \mathbb{Z}^n$, we set $\deg_\nu(f) = \max_{m \in \mathcal{N}(f)} \nu.m$.

Now, we explain why we can call $\deg_\nu(f)$ a degree.

Proposition 4. *Let f and g be two polynomials in $\mathbb{C}[X_1, \ldots, X_n]$, $\nu = (\nu_1, \ldots, \nu_n)$ in $\mathbb{Z}^n$ and x_1, x_2 two generic elements in $\mathbb{C}^2$.*

1. $\deg_\nu(f+g) \leq \max\big(\deg_\nu(f), \deg_\nu(g)\big)$,
2. $\deg_\nu(x_1 f + x_2 g) = \max\big(\deg_\nu(f), \deg_\nu(g)\big)$,
3. $\deg_\nu(f.g) = \deg_\nu(f) + \deg_\nu(g)$,
4. $\deg_\nu(\partial_{X_i} f) \leq \deg_\nu(f) - \nu_i = \deg_\nu(f/X_i)$.

Proof.
1. As $\mathcal{N}(f+g)$ is included in the convex hull of $\mathcal{N}(f) \cup \mathcal{N}(g)$, we have
$$\deg_\nu(f+g) = \max_{m \in \mathcal{N}(f+g)} \nu.m \leq \max_{m \in Conv(\mathcal{N}(f) \cup \mathcal{N}(g))} \nu.m,$$
where $Conv(.)$ denotes the convex hull.
Furthemore, as we consider a convex set, we deduce that this maximum is reached on a point in $\mathcal{N}(f) \cup \mathcal{N}(g)$. Thus
$$\begin{aligned} \deg_\nu(f+g) &\leq \max_{m \in \mathcal{N}(f) \cup \mathcal{N}(g)} \nu.m \\ &\leq \max\big(\max_{m \in \mathcal{N}(f)} \nu.m, \max_{m \in \mathcal{N}(g)} \nu.m\big) \\ &\leq \max\big(\deg_\nu(f), \deg_\nu(g)\big). \end{aligned}$$
2. As x_1, x_2 are generic then $\mathcal{N}(x_1 f + x_2 g)$ is equal to the convex hull of $\mathcal{N}(f) \cup \mathcal{N}(g)$. Indeed, with generic x_1 and x_2 we avoid simplifications in the sum $x_1 f + x_2 g$. Then the proof in this case proceeds as before.
3. This result comes from the well-known result by Ostrowski, [16], which gives: $\mathcal{N}(f.g) = \mathcal{N}(f) + \mathcal{N}(g)$, where $+$ in this situation is the Minkowski sum.
4. Let e_i be the i-th vector of the canonical basis of $\mathbb{R}^n$, then we have
$$\max_{m \in \mathcal{N}(\partial_{X_i} f)} \nu.m \leq \max_{m \in \mathcal{N}(f)} \nu.(m - e_i) = \max_{m \in \mathcal{N}(f)} \nu.m - \nu_i = \deg_\nu(f) - \nu_i.$$
We also have
$$\max_{m \in \mathcal{N}(f)} \nu.(m - e_i) = \max_{m \in \mathcal{N}(f/X_i)} \nu.m$$

this completes the proof.

$\square$

Newton polytopes and degree $\deg_\nu$ are related by the following proposition.

Proposition 5. *Let $f \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be a Laurent polynomial with corresponding Newton polytope:*

$$\mathcal{N}(f) = \{m \in \mathbb{Z}^n \mid \nu_j.m \le a_j, \text{ for } j = 1, \dots, k\},$$

where $\nu_j.m = a_j$, whith $a_j \in \mathbb{Z}$ and $\nu_j \in \mathbb{Z}^n$, are the equations of the k supporting hyperplanes of $\mathcal{N}(f)$.
Let $g \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ such that for $j = 1, \dots, k$, $\deg_{\nu_j}(g) \le \deg_{\nu_j}(f)$ then $\mathcal{N}(g)$ is included in $\mathcal{N}(f)$.

Proof. We just have to remark that

$$\max_{n \in \mathcal{N}(g)} \nu_j.n = \deg_{\nu_j}(g) \le \deg_{\nu_j}(f) = \max_{m \in \mathcal{N}(f)} \nu_j.m = a_j.$$

Thus each element in $\mathcal{N}(g)$ satisfies the equations of $\mathcal{N}(f)$. $\square$

2. Proof of Theorem 1

2.1. Newton polytope and cofactors

We are going to show that if A_i are sparse then the cofactors are sparse. This property will be the main tool of the proof of Theorem 1.

Proposition 6. *Let $D = \sum_{i=1}^n A_i(X_1, \dots, X_n)\partial_{X_i}$ be a derivation. Let f be a Darboux polynomial with corresponding cofactor g.*
Consider generic values $(x_1, \dots, x_n)$ in $\mathbb{C}^n$ and let N_D be the convex set $\mathcal{N}\Big(\sum_{i=1}^n x_i \dfrac{A_i}{X_i}\Big)$ then

$$\mathcal{N}(g) \subset N_D \cap \mathbb{N}^n.$$

Proof. First, obviously $\mathcal{N}(g) \in \mathbb{N}^n$ since g is a polynomial.
Second, we have $g.f = \sum_{i=1}^n A_i \partial_{X_i} f$, thus for all $\nu \in \mathbb{Z}^n$ we have $\deg_\nu(g.f) = \deg_\nu(\sum_{i=1}^n A_i \partial_{X_i} f)$. Thanks to Proposition 4, we deduce these inequalities

$$\begin{aligned}
\deg_\nu(g) + \deg_\nu(f) &\le \max_i(\deg_\nu(A_i)\partial_{X_i} f) \\
&\le \max_i(\deg_\nu(A_i) + \deg_\nu \partial_{X_i} f) \\
&\le \max_i(\deg_\nu(A_i) + \deg_\nu(f) - \nu_i).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\deg_\nu(g) &\leq \max_i(\deg_\nu(A_i) - \nu_i) \\
\deg_\nu(g) &\leq \max_i \Big(\deg_\nu(\frac{A_i}{X_i}) \Big) \\
\deg_\nu(g) &\leq \deg_\nu \Big(\sum_{i=1}^{n} x_i . \frac{A_i}{X_i} \Big).
\end{aligned}$$

Now, we apply Proposition 5 with $\nu \in \mathbb{Z}^n$ corresponding to supporting hyperplanes of $\mathcal{N}\Big(\sum_{i=1}^{n} x_i . \frac{A_i}{X_i} \Big)$ and we get the desired result. □

We can now prove easily Theorem 1. Indeed, we use the classical strategy to prove Darboux Theorem in our situation.
As for all cofactors g_{f_i} associated to a Darboux polynomial f_i, we have, by Proposition 6,

$$\mathcal{N}(g_{f_i}) \subset N_D \cap \mathbb{N}^n,$$

then all cofactors belong to a $\mathbb{C}$-vector space of dimension B, where B is the number of integer points in $N_D \cap \mathbb{N}^n$. Thus if we have $B+1$ cofactors, then there exists a relation between them:

$$(\star) \ \sum_{i \in I} \lambda_i g_{f_i} = 0,$$

where λ_i are complex numbers. Now, we recall a fundamental and straightforward result on Darboux polynomials: $g_{f_1.f_2} = g_{f_1} + g_{f_2}$. Thus relation $(\star)$ gives the Darboux first integral $\prod_{i \in I} f_i^{\lambda_i}$. This proves the first part of Theorem 1.

Now, in order to prove the second part of our theorem, we can use the strategy proposed in [14]. In [14] the authors show that if we have n relations of the type $(\star)$ then we can deduce a relation with integer coefficients, i.e. $\lambda_i \in \mathbb{Z}$. This gives a first integral of this kind: $\prod_{i \in I} f_i^{\lambda_i}$ with $\lambda_i \in \mathbb{Z}$, thus this first integral belongs to $\mathbb{C}(X_1, \dots, X_n)$ and we have a rational first integral. In the sparse case as the cofactors belong to a $\mathbb{C}$-vector space of dimension B, if we have $B+n$ irreducible Darboux polynomials then we have $B+n$ cofactors and then we deduce n relations between the cofactors. With the strategy used in [14] we obtain that the derivation has a rational first integral.

2.2. Optimality

2.2.1. Darboux first integrals

We are going to prove that the bound $B+1$ is optimal if we are looking for a Darboux first integral:

Consider the X_iX_{i+1}-derivation, i.e.

$$D = \Big(\sum_{i=1}^{n-1} X_iX_{i+1}\partial_{X_i}\Big) + X_nX_1\partial_{X_n}.$$

It is proved in [15] that D has no Darboux first integrals. Indeed, it is proved that the only irreducible Darboux polynomials are: $X_1, X_2, \dots, X_n$. Thus, if we have a Darboux first integral then it must be of the following form: $\mathcal{F} = c\prod_{i=1}^n X_i^{\lambda_i}$, where $c, \lambda_i \in \mathbb{C}$. Then $D(\mathcal{F}) = 0$ implies $\lambda_n X_1 + \sum_{i=1}^{n-1}\lambda_i X_{i+1} = 0$, and it follows: D has no non-trivial Darboux first integrals.
For this derivation, we get $A_i/X_i = X_{i+1}$, for $i = 1, \dots, n-1$, and $A_n/X_n = X_1$. Thus $B = n$. This implies: D has B irreducible Darboux polynomials but no Darboux first integrals.

From this example we deduce that we cannot improve the sparse bound $B+1$ given in Theorem 1, since there exists a derivation without Darboux first integrals which has B irreducible Darboux polynomials.

2.2.2. Rational first integrals

Now, we are going to show that the bound $B+n$ is optimal if we are looking for a rational first integral.

Consider a squarefree polynomial $p(X_1) \in \mathbb{C}[X_1]$ with degree d. Let α be a root of p and $\xi_2, \dots, \xi_n$ be distinct complex numbers such that $p'(\alpha), \xi_2, \dots, \xi_n$ are $\mathbb{Z}$-independent. We denote by D the following derivation:

$$D = p(X_1)\partial_{X_1} + \xi_2X_2\partial_{X_2} + \dots + \xi_nX_n\partial_{X_n}.$$

This derivation has no non-trivial rational first integrals, by Corollary 5.3 in [8]. Indeed, $(\alpha, 0, \dots, 0)$ is a fixed point of the polynomial vector field, and the corresponding eigenvalues are distinct and $\mathbb{Z}$ independent.

By Proposition 6, if g is a cofactor then $\mathcal{N}(g) \subset N_D \cap \mathbb{N}^n$. Here, $N_D \cap \mathbb{N}^n$ is the set of univariate polynomials in X_1 with degree smaller than $d-1$. Thus $B = d$.

Furthermore, $(X_1 - \alpha)$, $(X_1 - \alpha_2), \dots, (X_1 - \alpha_d)$ where α_i are roots of p, and $X_2, \dots, X_n$ are Darboux polynomials. Thus we have $d + n - 1$ Darboux polynomials.

In conclusion, we cannot improve the sparse bound $B + n$ given in Theorem 1, since there exists a derivation without rational first integrals which has $B + n - 1$ irreducible Darboux polynomials.

Final remarks

The bound B is dependent of the coordinates. Indeed, after a generic linear change of coordinates a sparse derivation becomes a dense one.

In the family of examples used to show the optimality, the success comes from the fact that for these derivations the set of exponents of the cofactors is exactly the set of integer points in $N_D \cap \mathbb{N}^n$. Thus in these cases, the dimension of the vector space generated by the cofactors is equal to B.

It is not possible to get the same kind of results for a lacunary derivation. That is to say, it is not possible to have a bound in term of the number of non-zero coefficients in the derivations.
For example: $D = (X_1^d - 1)\partial_{X_1} - (dX_1^{d-1}X_2 + 1)\partial_{X_2}$ has 4 non-zero coefficients and D has the following rational first integral $f(X_1, X_2) = X_2(X_1^d - 1) + X_1$. However, if we just consider the following d irreducible Darboux polynomials $X_1 - \omega$, where $\omega^d = 1$, then we cannot reconstruct f. Thus we cannot get a result like Jouanolou's theorem if we just use the number of non-zero coefficients in the derivation.

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