SEMI-CONTINUITY OF SIEGEL DISKS UNDER PARABOLIC IMPLOSION

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Abstract. We transfer the lower semi-continuity of Siegel disks with fixed Brjuno type rotation numbers to geometric limits. Here, we restrict to Lavaurs maps associated to quadratic polynomials with a parabolic fixed point.

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1. Introduction

We will prove here what was called hypothesis 4 in [C]. There is no breakthrough here. The essential difficulty resides in getting a correct definition of perturbed horn maps, which we need to tend to the extended horn map of the limit parabolic point.

2. Statement

Let

$$P_\theta(z) = e^{i2\pi\theta} z + z^2.$$

The map $P_{p/q}$ with p/q irreducible and $q > 0$ has a parabolic point at $z = 0$. The theory of parabolic implosion associates to this two families of (related) maps: the horn maps h, living in the cylinder and the Lavaurs maps g, living in the dynamical plane of $P_{p/q}$. Both depend on a parameter $\sigma \in \mathbb{C}$ called the *phase*. Let us choose the phase such that the virtual multiplier of h at one end of the cylinder is equal to $e^{i2\pi\theta}$, with θ a Brjuno number:

$$\theta \in \mathcal{B}.$$

To determine which end of the cylinder we will choose, we need to introduce a few definitions. Let us expand the p/q in continued fraction:

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}$$

with $a_i \in \mathbb{N}^*$, which we more concisely write $p/q = [a_0, a_1, \ldots, a_m]$. Let $\theta \in \mathcal{B}$ and

$$\theta_n = a_0 + \cfrac{1}{\ddots + \cfrac{1}{a_m + \cfrac{1}{n+\theta}}}$$

Let us recall that p/q has two such expansions, one of the form $[a_0, \ldots, a_k]$ with $a_k \geq 2$ and one of the form $[a_0, \ldots, a_k - 1, 1]$. In one case, $m = k$, in the other case $m = k+1$. We have

$$\theta_n \longrightarrow \frac{p}{q},$$

$$\operatorname{sign}\left(\theta_n - \frac{p}{q}\right) = (-1)^m.$$

The end of the cylinder we consider in the upper end if $\theta_n > p/q$ and the lower end of $\theta_n < p/q$.

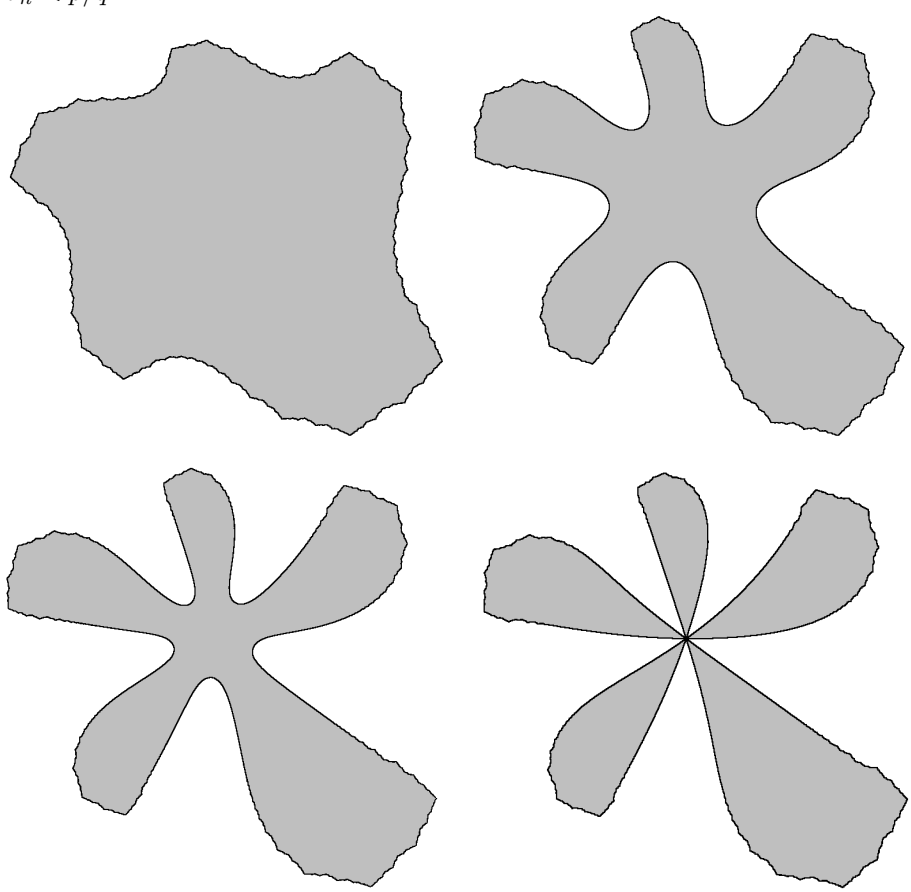

Figure 1: Let $\theta_n = [0, 2, 2, n+\theta]$ with $\theta = \frac{\sqrt{5}-1}{2}$, the first three images show the Siegel disk of P_{θ_n} for $n = 10, 500, 10000$, and the last is the virtual Siegel disk of $P_{2/5}$ they tend to. Here $\theta_n > p/q$.

Theorem. *For all compact subset C of the virtual Siegel disk, $\exists N$ such that $\forall n \geq N$, C is contained in the Siegel disk of P_{θ_n}.*

Remark. The theorem extends to a more general setting than quadratic polynomials.

3. Tools of the proof

3.1. **Semi-continuity of Siegel disks.** The following theorem is due to E. Risler, it is a corollary of [R]. I wrote a specific proof in [C]. Let S_θ be the set of analytic maps $f : \mathbb{D} \to \mathbb{C}$ fixing 0 with multiplier $e^{i2\pi\theta}$ (we do not assume injectivity). The Siegel disk $\Delta(f)$ is the maximal subset of $\mathbb{D}$ containing 0 and on which f is conjugate to a rotation. Let

$$R_\theta(z) = e^{i2\pi\theta} z.$$

Theorem 1. *$\forall\theta \in \mathcal{B}$, $\forall\varepsilon > 0$, $\exists\eta > 0$ such that if $f \in S_\theta$ and $\|f - R_\theta\|_{sup} < \eta$ then $\Delta(f) \supset D(0, 1-\varepsilon)$. (For all Brjuno number, all maps in S_α sufficiently close to the rotation have a Siegel disk containing any given compact subset of $\mathbb{D}$.)*

3.2. **Parabolic implosion.** [1]

For all n big enough, there are q fixed points $z_1, \dots, z_q$ of $P^q_{\theta_n}$ that tend to 0 as $n \longrightarrow +\infty$. We have

$$z_i^q \underset{n\to+\infty}{\sim} \Big(\theta_n - \frac{p}{q}\Big)c$$

for some constant $c \in \mathbb{C}^*$, and moreover there is exactly one point z_i close to each of the q-th roots of $(\theta_n - \frac{p}{q})c$. Thus they tend to 0 asymptotically to q equally spaced axes. We can label the axes with the symbols 1 to q and label the q fixed points according to the axis they are close to. We will focus our attention on one of them, let us say z_1. Let

$$p(z) = z/|z| = e^{i\arg(z)}.$$

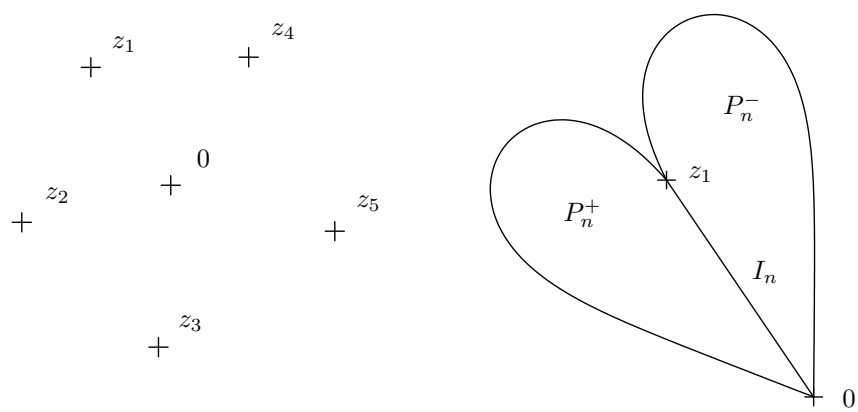

Figure 2: The fixed points and the petal P_n.

We will work on a small ball $B(0, r)$, where r will be fixed later. Let us consider the map $\mathrm{pow}_n : z \mapsto (z/p(z_1))^q$ defined on the sector "$\arg(z/z_i) \in]-\frac{\pi}{q}, \frac{\pi}{q}[$" (so that pow_n is injective). Let us consider the two biggest open disks whose boundary passes by $0, 1$ and that are contained in the image of $B(0, r) \cap S$ by pow_n, i.e. in $B(0, r^q)\backslash]-\infty, 0]$. Let P_n be the preimage by pow_n of the union of these two disks (see the picture). This is the petal associated to z_1. Let us cut the union of two disks by the real line. It corresponds to a cut $I_n = [0, z_1]$ of P_n. If $\theta_n > p/q$, we

[1] The theory of parabolic implosion (perturbation of parabolic points leading to geometric limits) has been developed by several authors, among which (alphabetical order) A. Douady, A. Epstein, R. Oudkerk, P. Lavaurs, M. Shishikura, and is still under development. Here we simply state the results that will be useful for our proof, after introducing the necessary definitions. We do not pretend to give an introduction to the topic.

will call P_n^+ the preimage of the part above the real line, and P_n^- the other one. If $\theta_n < p/q$ this will be the other way round.

Let $\zeta_n : P_n \to \mathbb{C}$ be defined by

$$\zeta_n(z) = \frac{-1}{i2\pi q^2(\theta_n - p/q)} \log\Big(1 - (z_1/z)^q\Big).$$

The map $1 - z \mapsto (z_1/z)^q$ sends P_n to a sector centered on 0, on which a branch of log is chosen. The map ζ_n is univalent and sends P_n to a vertical band, mapping 0 to the upper end and z_1 to the lower end if $\theta_n > p/q$ and the other way round otherwise.

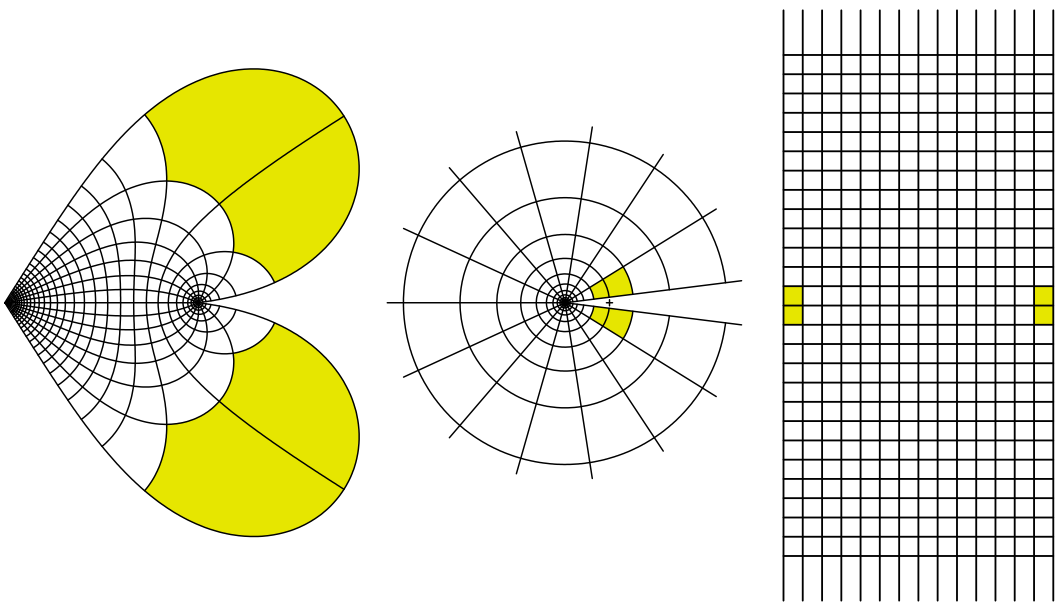

Figure 3: The petal (rotated), its image under the map $z \mapsto 1 - \frac{z_1}{z^q}$, and its image under ζ_n. The grid and the yellow parts correspond to each other. In the middle image, the yellow boxes are close to the point $z = 1$ which is marked by a cross.

Let α_1 be the argument of the axis z_1 is close to. Let pow : $z \mapsto (z/e^{i\alpha_1})^q$ be defined on the sector "$\arg(z/e^{i\alpha_1}) \in]-\frac{\pi}{q}, \frac{\pi}{q}[$". Let P be the preimage by pow of the union of the two biggest disks tangent to $\mathbb{R}$ at 0 and contained in $B(0, r^q)$ (these are the limits of the previous disks). Then every compact subset of P is eventually contained in P_n. Let P_+ and P_- be the analogs of P_n^+ and P_n^-. Then, provided r is small enough, P_+ is a repelling petal and P_- is an attracting petal for $P_{p/q}$.

Lemma 1. *There exist $k_+, k_- \in \mathbb{Z}$ (with $k_- - k_+ = (-1)^m$) such that*

$$\zeta_n(z) + \frac{k_+}{q^2(\theta_n - p/q)} \longrightarrow \frac{c}{i2\pi q^2 z^q}$$

on every compact subset of P^+ and

$$\zeta_n(z) + \frac{k_-}{q^2(\theta_n - p/q)} \longrightarrow \frac{c}{i2\pi q^2 z^q}$$

on every compact subset of P^-.

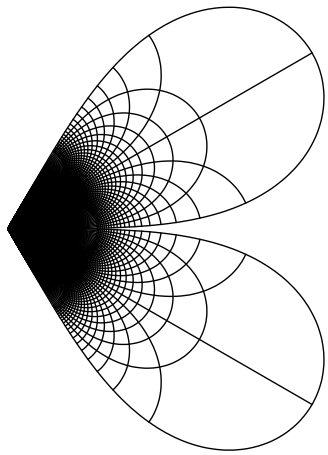

Figure 4: The limit petal $P_+ \cup P_-$ (rotated), decorated by the preimage of the orthogonal grid by the limit map $z \mapsto \frac{c}{i2\pi q^2 z^q}$.

Let $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ be the quotient map. Then $\exists r_0 > 0$, $\forall r < r_0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, the following two theorems hold.

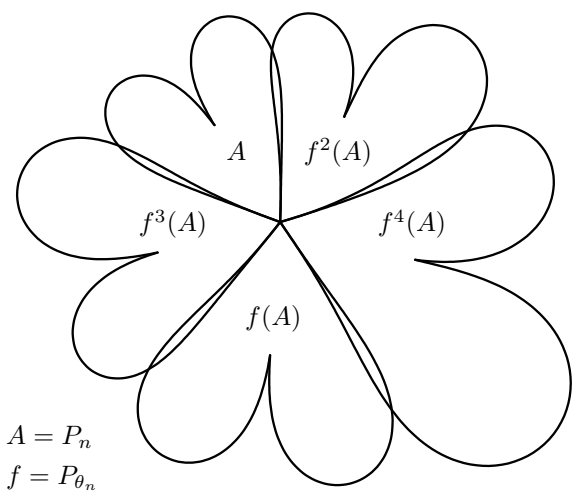

Figure 5: This picture illustrates the domain P_n and its first 4 iterates by P_{θ_n}. Notice the overlaps.

Theorem 2. [2]

(1) *The following sets are pairwise disjoint:* P_n^-, $P_{\theta_n}(P_n^-)$, ..., $P_{\theta_n}^{q-1}(P_n^-)$.
(2) *The closures of two of them intersect only at* 0.
(3) *The sames hold for* P_-, ..., $P_{p/q}^{q-1}(P_-)$.
(4) *If* $z \in P_n^+$ *and* $0 < k < q$ *then* $P_{\theta_n}^k(z) \notin I_n$.

[2]Points (1) through (5) are elementary consequence of P_{θ_n} being close to R_{θ_n} near $z = 0$. Points (6) through (8) are consequences of the following estimate: $\left|\zeta_n \circ P_{\theta_n}^q(z) \circ \zeta_n^{-1} - (z+1)\right| \leq \varepsilon$ for r small enough (and N big enough, depending on r). Point (9) is a consequence of the same estimate for an other branch of ζ_n.

(5) *$\exists r' < r$ (r' depends on n) such that every point in $B(0,r')$ eventually falls in P_n^- under iteration of P_{θ_n}.*
(6) *For all $z \in \partial P_n^- \setminus I_n$, $P_{\theta_n}^q(z) \in P_n^-$.*
(7) *Every point of P_n^- is eventually mapped to P_n^+ under iteration of $P_{\theta_n}^q$.*
(8) *If $z \in P_n^- \cap P_{\theta_n}^k(P_n^+)$ with $0 \le k < q$ then $P_{\theta_n}^q(z) \in P_n^-$.*
(9) *Note $]a,b[\times\mathbb{R} = \zeta_n(P_n)$. There exists $M > 0$ (which depends on r but not on n) such that every point $z \in P_n^+$ with $\mathrm{Im}(\zeta_n(z)) > M$ and $\mathrm{Re}(\zeta_n(z)) > b-2$ is eventually mapped to P_n^- under iteration of P_{θ_n}.*

Theorem 3 (perturbed Fatou coordinates)**.**

(10) *There exists a univalent function $\Phi_n : P_n \to \mathbb{C}$ such that $\Phi_n \circ P_{\theta_n}^q = T_1 \circ \Phi_n$ holds on $P_n \cap (P_{\theta_n}^q)^{-1}(P_n)$.*
(11) *The derivative of $\Phi_n \circ \zeta_n^{-1}$ stays in $B(1,1/4)$.*
(12) *The image of P_n^+ under Φ_n is surjectively mapped to $\mathbb{C}/\mathbb{Z}$ by π. The same holds for P_n^-.*
(13) *If both z and $z+n$ belong to $\Phi_n(P_n^+)$, then $z+k \in \Phi_n(P_n^+)$ for $0<k<n$. The same holds for P_n^-.*
(14) $\mathrm{sign}(\theta_n - p/q).\mathrm{Im}(\Phi_n(z)) \longrightarrow +\infty \iff z \longrightarrow 0$
(15) $\Phi_n(z) \underset{z\longrightarrow 0}{=} \dfrac{\log z}{i2\pi(q\theta_n - p)} + \mathrm{cst} + o(1)$.
(16) *For any given $a \in P_+$ (resp. P_-), $\Phi_n(z) - \Phi_n(a)$ tends uniformly on every compact subset of P_+ (resp. P_-) to a Fatou coordinate for $P_{p/q}$.*

Points (12), (13) and (14) follow from (11).

Point (16) motivates the following definition: we fix any $a_+ \in P_+$ and $a_- \in P_-$, and define $\Phi_+ : P_+ \to \mathbb{C}$ to be the limit of $\Phi_n(z) - \Phi_n(a_+)$ for $z \in P_+$ and analogously for $\Phi_- : P_- \to \mathbb{C}$.

4. Proof

4.1. **Defining perturbed horn maps.** This definition requires some care.

We first define a function k on $\mathbb{C}$:

$$k(z) = \inf\left\{k \in \mathbb{N} \,\middle|\, P_{\theta_n}^k(z) \in P_n^-\right\} \in \mathbb{N} \cup \{+\infty\}.$$

Since P_n^- is open, k is upper semi-continuous.

Then we let

$$U = \left\{z \in P_n^+ \,\middle|\, k(z) < +\infty \text{ and } P_{\theta_n}^k(z) \notin I_n \text{ for } 0 \le k \le k(z)\right\}.$$

The set U is open and $U \subset P_+$.

From (4) and (8), we get:

Lemma 2. *If $z \in P_n^+$ and $P_{\theta_n}^q(z) \in P_n^+$ then $z \in U \iff P_{\theta_n}^q(z) \in U$.*

This, together with (10) and (13) implies that $\Phi_n(U) = \Phi_n(P_n^+) \cap \pi^{-1}(\widetilde{U})$ for some subset $\widetilde{U}$ of $\mathbb{C}/\mathbb{Z}$.

We set

$$\mathrm{def}(h_n) = \widetilde{U}$$

which is an open subset of $\mathbb{C}/\mathbb{Z}$. For $w \in \widetilde{U}$ we let $w' \in \Phi_n(P_n^+)$ be any representative of w. Then $w' = \Phi_n(z)$ for a unique $z \in U$ and we let

$$h_n(w) = \Phi_n(P_{\theta_n}^{k(z)}(z)) \bmod \mathbb{Z} \in \mathbb{C}/\mathbb{Z}.$$

Claim: $h_n(w)$ is independent of the representative w' of w we chose.

Proof. Indeed, if $w_1 = \Phi_n(z_1)$ and $w_2 = w_1 + m = \Phi_n(z_2)$ are two representatives, then (13) and (10) imply $P_{\theta_n}^{jq}(z_1) \in P_n^+$ for $0 \le j \le m$ and $z_2 = P_{\theta_n}^{mq}(z_1)$. Let $u_k = P_{\theta_n}^k(z_1)$.
Case 1: $k(z_1) \ge mq$. Then $k(z_2) = k(z_1) - mq$ and thus $P_{\theta_n}^{k(z_2)}(z_2) = P_{\theta_n}^{k(z_1)}(z_1)$.
Case 2: $k(z_1) \le mq$. Then point (8) implies $u_{k(z_1)+jq} \in P_n^-$ for all j such that $k(z_1) \le k(z_1) + jq < mq + q$. By (1), these are the only values of k such that $k(z_1) \le k < mq + q$ and $u_k \in P_n^-$. In particular, $k(z_2) = b$ where $b \in [0, q[$ is the remainder of the Euclidean division $k(z_1) = aq + b$. By point (10), we have $\Phi_n(u_{k(z_1)+jq}) = j + \Phi_n(u_{k(z_1)})$. Whence $\Phi_n(P_{\theta_n}^{k(z_2)}(z_2)) = \Phi_n(u_{mq+b}) = m - a + \Phi_n(u_{k(z_1)}) = m - a + \Phi_n(P_{\theta_n}^{k(z_1)}(z_1))$. □

Claim. The map h_n is analytic.

Proof. It is obvious for points $w \in \widetilde{U}$ such that k is locally constant at z. If $w_0 = \pi(\Phi_n(z_0))$ with $z_0 \in U$ is such that k is not locally constant at z_0, it means that there is $k' < k(z_0)$ such that $P_{\theta_n}^{k'}(z_0) \in \partial P_n^-$. Since I_n must be avoided, point (6) says $P_{\theta_n}^{k'+q}(z_0) \in P_n^-$. Now by (1), $k(z_0) = k' + q$. In particular, k' is unique. By (2), one concludes that in a neighborhood V of z_0, $k(z) \in \{k', k' + q\}$. We can choose V small enough so that $P_{\theta_n}^{k'+q}(V) \subset P_n^-$. For $z \in V$ so that $k(z) = k'$, (10) implies $\Phi_n(P_{\theta_n}^{k'+q}(z)) = \Phi_n(P_{\theta_n}^{k'}(z)) \bmod \mathbb{Z}$. Thus $h_n(w) = \Phi_n(P_{\theta_n}^{k'+q}(z)) \bmod \mathbb{Z}$ holds on a neighborhood of w_0. □

By (7), we get:

Proposition 1. *$\forall z \in P_n$, if $\Phi_n(z) \in \mathrm{def}(h_n) \bmod \mathbb{Z}$ then $\exists k \in \mathbb{N}$ such that[3] $k > 0$, $z' = P_{\theta_n}^k(z) \in P_n^+$ and $\Phi_n(z') = h_n(\Phi_n(z)) \bmod \mathbb{Z}$.*

Corollary 1. *$\forall z \in P_n$, if $\Phi_n(z) \bmod \mathbb{Z}$ has an infinite orbit under $\mathrm{def}(h_n)$, then the orbit of z under P_{θ_n} passes an infinite number of times in P_n^+.*

By (12), (14) and (5), the set $\mathrm{def}(h_n)$ is a neighborhood of ν, and h_n has a continuous (and thus analytic) extension $\widehat{h}_n$ fixing it.[4]

Let ν be the upper end if $\theta_n > p/q$ and the lower end if $\theta_n < p/q$.

Proposition 2 (folk.)**.** *The multiplier of $\widehat{h}_n$ at the end ν is equal to*

$$\exp\left((-1)^{m+1} i 2\pi\theta\right).$$

This is a well known corollary of point (15), of a local combinatorial study, and of the theory of continued fractions.

The (unperturbed) horn map h can be defined as follows: given $w \in \mathbb{C}/\mathbb{Z}$ let $w' \in \Phi_+(P_+)$ be any representative of w in $\mathbb{C}$ and $z = \Phi_+^{-1}(w')$. If there is a $k \in \mathbb{N}$ such that $z' = P_{\theta_n}^k(z) \in P_-$, then we define

$$h(w) = \Phi_-(z') \bmod \mathbb{Z}.$$

Otherwise we decide that $z \notin \mathrm{def}(h)$. Analogously, the map h is well defined, the set $\mathrm{def}(h)$ is open, h is analytic, $\mathrm{def}(h)$ is a neighborhood of ν, and h has an analytic continuation $\widehat{h}$ fixing it (cf. [DH], [E], [S], etc...).

Lemma 3.

$$T_{-\Phi_n(a_-)} \circ \widehat{h}_n \circ T_{\Phi_n(a_+)} \underset{n\to+\infty}{\longrightarrow} \widehat{h}$$

in the sense described above.

[3] $k > 0$ is an important point

[4] We consider the cylinder completed by its two ends as a Riemann surface isomorphic to the Riemann sphere (via the map $(z \bmod \mathbb{Z}) \mapsto e^{i2\pi z}$).

Proof. Let $w \in \operatorname{def}(\widehat{h})$.

Case 1: $w \neq \nu$. By definition and (16), $T_{-\Phi_n(a_+)} \circ \Phi_n \longrightarrow \Phi_+$ uniformly on every compact subsets of P_+. Thus $\Phi_n^{-1} \circ T_{\Phi_n(a_+)} \longrightarrow \Phi_+^{-1}$ uniformly on every compact subsets of $\Phi_+(P_+)$. If $z \in P_+$ is eventually mapped to P_- under iteration of $P_{p/q}$, say $P_{p/q}^k(z) \in P_-$, then there is a neighborhood V of z and N such that $\forall n \geq N$, $P_{\theta_n}^k(V) \subset P_n^-$. By taking a slightly smaller V, we may assume also that $0 \notin \overline{P_{p/q}^k(V)}$. Since $I_n \longrightarrow \{0\}$ as $n \longrightarrow +\infty$, we know that for all n big enough, V avoids I_n before getting in P_n^- under P_{θ_n} (otherwise, $\overline{P_{p/q}^k(V)}$ would contain 0). Therefore, there is a neighborhood of w that is contained in $\operatorname{def}(h_n)$ for n big enough. Now, either z does not fall in ∂P_- before getting in P_-, in which case there is a neighborhood $V' \subset V$ of z and $N' \geq N$ such that $\forall n \geq N'$, $\forall z' \in V'$, $k_n(z') = k(z)$. Or $\exists k' < k(z)$ such that $P_{p/q}^{k'}(z) \in \partial P_-$. Since P_- is a petal, we know by (3) that $k(z) = k' + q$, whence k' is unique. This implies, by (3) and (10) that we have $\widehat{h}_n(w) = \Phi_n(P_{\theta_n}^{k'+q}(z)) \bmod \mathbb{Z}$ which holds for all n big enough on a neighborhood of w independent of n. In both cases, the convergence then follows from the definitions and from $T_{-\Phi_n(a_-)} \circ \Phi_n \longrightarrow \Phi_-$.

Case 2: $w = \nu$. We will prove that there is a neighborhood V of ν that is eventually contained in $\operatorname{def}(\widehat{h}_n)$. Then, using case 1 on a circle $A \subset V$ around ν, we deduce that the limit $h_n \longrightarrow h$ must be uniform on A. But since $\widehat{h}_n(\nu) = \nu$, the limit $\widehat{h}_n \longrightarrow \widehat{h}$ must be uniform above A for the spherical metrics, which ends the proof of case 2. For the existence of V, consider the set U of points in the band $\zeta_n(P_n) =]a, b[\times \mathbb{R}$ whose real part are $\geq b - 2$ and imaginary part are $\geq M$ with M given by (9). Since ζ_n converges on compact subsets of P_+ (cf. lemma 1), it converges at a_+, and this implies by (11) that the sets $\pi \circ T_{-\Phi_n(a_+)} \circ \Phi_n \circ \zeta_n^{-1}(U)$ contain a common[5] neighborhood of ν in $\mathbb{C}/\mathbb{Z}$. Let us call it V. By (9), $V \subset \operatorname{def}(\widehat{h}_n)$.

□

Since $\widehat{h}$ has a non zero multiplier at the end ν, we have $\widehat{h}(z) - z \longrightarrow \tau$ as $z \longrightarrow \nu$. Together with proposition 2 this gives:

$$\Phi_n(a_+) - \theta - \Phi_n(a_-) \underset{n \to +\infty}{\longrightarrow} \tau$$

Thus we can rephrase lemma 3 in a more convenient way:

$$T_{-\Phi_n(a_+)} \circ \widehat{h}_n \circ T_{\Phi_n(a_+)} \underset{n \to +\infty}{\longrightarrow} T_{-\theta-\tau} \circ \widehat{h}.$$

In other words

$$\widehat{f}_n \underset{n \to +\infty}{\longrightarrow} \widehat{f}$$

with

$$\widehat{f}_n = T_{-\Phi_n(a_+)} \circ \widehat{h}_n \circ T_{\Phi_n(a_+)}$$

and

$$\widehat{f} = T_{-\theta-\tau} \circ \widehat{h},$$

and $\widehat{f}$ and $\widehat{f}_n$ have the same multiplier at ∞ (the one given by proposition 2). Since this number is a Brjuno number, we can apply the semi-continuity result (theorem 1) and deduce from this:

Lemma 4. *Every compact subset of $\Delta(\widehat{f}_n)$ is eventually contained in $\Delta(\widehat{f})$. Note that we include the end ν in these Siegel disks, so in particular, there is a neighborhood of ν that is eventually contained in $\Delta(\widehat{f}_n)$.*

[5] i.e. independent of n

4.2. **Transferring the semi-continuity.** Let us define the shorthands $\Delta(f) = \pi^{-1}(\Delta(\widehat{f}))$, and $\Delta(f_n) = \pi^{-1}(\Delta(\widehat{f}_n))$. The *virtual Siegel disk* is defined[6] as

$$\Delta_\infty = \bigcup_{k\in\mathbb{N}} P^k_{p/q}(\Phi_+^{-1}(\Delta(f))).$$

Let Δ_n be the (usual) Siegel disk of P_{θ_n}.[7] By lemma 4, $\exists y_0 > 0$, $\exists N \in \mathbb{N}$, such that $\forall n \geq N$, $\Delta(f_n)$ contains "$\mathrm{Im}(z) > y_0$".

Claim: $\forall n \geq N$ and $\forall z \in P_n$, if $\mathrm{Im}(\Phi_n(z) - \Phi_n(a_+)) > y_0$ then $z \in \Delta_n$.

Proof. By (11), $\Phi_n(P_n) \cap$ "$\mathrm{Im}(z) >$ something" is connected. Thus Φ_n^{-1}("$\mathrm{Im}(z - \Phi_n(a_+)) > y_0$") is a connected open set whose points have by corollary 1 an orbit under P_{θ_n} which passes an infinite number of times in the bounded set P_n^+. Therefore, it is contained in the Fatou set of the polynomial P_{θ_n}. Using (11) again, we see that this open set adheres to 0. Since P_{θ_n} is linearizable at 0, the connected open set is contained in the Siegel disk. □

Let z_0 be in the virtual Siegel disk. Let us write $z_0 = P^{k_0}_{p/q}(\Phi_+^{-1}(w_0))$ with $w_0 \bmod \mathbb{Z} \in \Delta(f)$. Let $\gamma : [0,1] \to \Delta(f)$ a path from w_0 to any point w_1 in "$\mathrm{Im}(z) > y_0$". By lemma 4, the curve γ is contained in $\Delta(f_n)$ for n big enough. Since this curve is compact, there exists $m_0 > 0$ such that $T_{-m_0} \circ \gamma([0,1]) \subset \Phi_+(P_+)$. By definition and (16), there exists a neighborhood U of $\gamma([0,1])$ such that $T_{-m_0}(U) \subset T_{-\Phi_n(a_+)} \circ \Phi_n(P_n)$ for n big enough. Up to taking a smaller neighborhood, we will choose U open, connected and contained in $\Delta(f)$. Then, by corollary 1, $\Phi_n^{-1} \circ T_{\Phi_n(a_+)} \circ T_{-m_0}(U)$ is contained in the Fatou set, and by the claim above, it contains a point in Δ_n (the image of $\gamma(1)$). Being moreover connected, it is contained in Δ_n. Therefore

$$P^{k_0}_{\theta_n} \circ \Phi_n^{-1} \circ T_{\Phi_n(a_+)} \circ T_{-m_0}(U) \subset \Delta_n.$$

Since the limit $P^{k_0}_{p/q} \circ \Phi_+^{-1} \circ T_{-m_0}$ of the functions $P^{k_0}_{\theta_n} \circ \Phi_n^{-1} \circ T_{\Phi_n(a_+)} \circ T_{-m_0}$ is non constant, the image of U and contains some neighborhood of z_0 for n big enough. This ends the proof of the main theorem.

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[6] There are several equivalent definitions of the virtual Siegel disk. For instance we may have used the attracting Fatou coordinate Φ_- instead of Φ_+. We may also have used the maximal extension of Φ_+^{-1} instead of taking forward images by P_n. Or the maximal extension of Φ_- together with covering properties of Φ_-.

[7] Linearizability of P_{θ_n} can be justified in several ways. Invariance of $\mathcal{B}$ under $\mathrm{PSL}(\mathbb{Z}^2)$ and $\theta \in \mathcal{B}$ imply that $\theta_n \in \mathcal{B}$. Or one may argue that $\widehat{h}_n$ is a (Yoccoz) renormalization (high enough in the cylinder) of P_{θ_n} and that linearizability of $\widehat{h}_n$ thus implies that of P_n.