UNIFORMITY OF THE ŚWIĄTEK DISTORTION FOR COMPACT FAMILIES OF BLASCHKE PRODUCTS

M.R. HERMAN

Translator's note: this document is a sequel to *Quasisymmetric conjugacy of analytic circle homeomorphisms to rotations* [H]. Herman wrote it shorlty after [H] and as the latter, it was a preliminary version. Thus, I found useful to add a few notes at the end of this document. Here is a short abstract of the article: in [H], Herman proved that an analytic circle homeomorphism with bounded type rotation number is conjugated to a rotation by a quasisymmetric map. Here, he shows that for all *compact* families of Blaschke products with bounded degree and inducing homeomorphisms of the circle with fixed rotation number, the quasisymmetry constant is bounded.

0. Introduction

We propose to show that, if $(g_i)_{i\in\mathbb{N}}$ is a sequence of rational fractions that induce on $\mathbb{S}^1$ orientation preserving homeomorphisms, and that converge on a neighborhood of $\mathbb{S}^1$, then (theorem §2)

$$\sup_i SD(\widetilde{g}_i) < +\infty$$

($\widetilde{g}_i$ refers to a lift to $\mathbb{R}$ and $SD(\widetilde{g}_i)$ to the Świątek distortion of $\widetilde{g}_i$ defined in §9). Świątek proved [S] that for all i

$$SD(\widetilde{g}_i) < +\infty.$$

One has to show the uniformity with respect to i. The main difficulty is near the critical points: several critical points in the complex plane may tend to a point.

One must first show that the crossratio distorition stays bounded even if several critcal points converge in $\widehat{\mathbb{C}}$ to a common limit on $\mathbb{S}^1$. It is the object of §4 and 5 (the idea of the proof of 4.2 has been communicated to me by J.C. Yoccoz in 1987). The §3 reduces this to a local problem.

Then, one has to show that the Świątek distortion stays bounded, locally near critical points, even if several ones merge. It is the object of §6. The intervals where the Schwarzian derivative is < 0 are not a problem. What allows everything to work, is that the total variation of $\log D\widetilde{g}_i$ stays bounded on the intervals where $S(\widetilde{g}_i) \geqslant 0$. If theorem §2 is true then one shall be able to prove it. It's easier said than done if one wants to avoid inextricable[1] combinatorial problems.

We use the notations of [H] and at paragraphs 5 and 10 we indicate only the changes that are needed to come to the conclusion.

1. Let $(g_i)_{i\in\mathbb{N}}$ be a sequence of rational Blaschke fractions satisfying

- $g_i\big|_{\mathbb{S}^1}$ is an orientation preserving homeomorphism;
- $\text{degree}(g_i) = d_i \leqslant d < +\infty$;

[1] tn: I am not sure of the translation

- (g_i) converges uniformly on $\{1-\varepsilon < |z| < 1+\varepsilon\}$ to g, where $\varepsilon > 0$ is given. $g\big|_{\mathbb{S}^1}$ is necessarily a homeomorphism.

2. Theorem. *One has*

$$\sup_{i\in\mathbb{N}} SD(\widetilde{g}_i) < +\infty$$

where SD refers to the Świątek distortion of the lift $\widetilde{g}_i$ of g_i to the universal cover $\mathbb{R}$, $\mathbb{R} \xrightarrow{e^{2\pi i}} \mathbb{S}^1$, of $\mathbb{S}^1$. (For the definition of $SD(\widetilde{g}_i)$, see §9).

The proof can be found in §10.

2.1. To prove the theorem, it is enough to draw the conclusion of the theorem for an extracted subsequence (g_{i_k}), $0 < i_k < i_{k+1}$ of the sequence g_i.

Świątek proved that for all i, $SD(\widetilde{g}_i) < +\infty$. We must prove a uniformity, we may assume $i > k_0$.

2.2. Let us remark that.

1. By Cauchy's inequalities, $g_i \longrightarrow g$ for the C^∞ topology.
2. g_j has at most $2d-2$ critical points $\widehat{c}_1^{(j)}, \ldots, \widehat{c}_p^{(j)}$.

3.1. Up to extracting a subsequence (g_{i_k}) of the sequence (g_i) we may assume that the critical points $(\widehat{c}_j^{(l)})_{1\leqslant j\leqslant q}$ of $(g_{i_k})_k$ can be split into $n+1$ disjoint sets $I_i \subset \{1,\ldots,q\}$

$$I_1 \amalg \cdots \amalg I_{n+1} = \{1,\ldots,q\}$$

where $\mathrm{Card}(I_j)$ is independent of j and satisfying:

3.2. If $j \in I_{q_1}$ and $q_1 \neq n+1$ then if $k \longrightarrow +\infty$,

$$\widehat{c}_j^{(k)} \longrightarrow z_{q_1} \in \mathbb{S}^1$$

with $z_{q_1} \neq z_{q_2}$ for $q_1 \neq q_2$.

3.3. If $j \in I_{n+1}$,

$$\mathrm{distance}(\widehat{c}_j^{(k)}, \mathbb{S}^1) \geqslant \delta_1 > 0$$

for a $\delta_1 > 0$ independent of i_k.

3.4. For all[2] $\delta_2 > 0$, we can find $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$ then[3] the disks $\{|z - z_{q_1}| \leqslant 2\delta_2\}$, $q_1 = 1,\ldots,n$ are disjoint and $\widehat{c}_j^{(k)} \in \{|z - z_{q_1}| < \delta_2/2\}$ $\forall j \in I_{q_1}$.

3.3 and 3.4 imply, using Rouché's theorem

$$\min_{k\geqslant k_0} \min_{z\in\mathbb{S}^1 - \bigcup_1^n \{|z-z_{q_1}|<\delta_2\}} \left|Dg_{i_k}(z)\right| \geqslant \delta_3 > 0 \tag{3.5}$$

where δ_3 depends on δ_2 and $\delta_3 \longrightarrow 0$ if $\delta_2 \longrightarrow 0$.

3.6. Using 3.4 and assuming $0 < \delta_2 \leqslant \delta_2'$ for $j = 1,\ldots,n$ we can find Möbius transformations $h_1, \ldots, h_n$ independent of δ_2' satisfying:

$$h_j(z_j) = 0, \qquad h_j : \mathbb{S}^1 - \{1 \text{ point}\} \to \mathbb{R}$$

[2]tn: sufficiently small

[3]tn: See the illustration at the end of the article

and for all $\delta_2 \leqslant \delta_2'$, h_j is a diffeomorphism from $\big\{|z-z_j| \leqslant 2\delta_2,\ |z|=1\big\}$ to $[-3\delta, 3\delta]$ where δ depends on δ_2,

$$\delta \longrightarrow 0 \quad \text{si} \quad \delta_2 \longrightarrow 0. \tag{3.6.1}$$

Moreover, we may assume that

$$h_j\big(\big\{|z-z_j| \leqslant \delta_2,\ |z|=1\big\}\big) \subset [-\delta, \delta].$$

3.7. For $1 \leqslant r \leqslant n$, $r \in \mathbb{N}$, let $\lambda_{r,k} \in \mathbb{S}^1$ such that $\lambda_{r,k} g_{i_k}(z_r) = z_r$.

Assuming δ_2' is small enough so that $k \geqslant k_1$ since $\widehat{c}_j^{(k)} \longrightarrow z_r$ we have

$$\lambda_{r,k} \cdot g_{i_k}\big(\{|z-z_r| \leqslant 2\delta_2,\ |z|=1\}\big) \subset \{|z-z_r| \leqslant 2\delta_2,\ |z|=1\}. \tag{3.8}$$

We define[4][5] $f_k = h_r \circ (\lambda_{r,k} \cdot g_{i_k}) \circ h_r^{-1}$, that satisfies, for $k \geqslant \sup(k_0, k_1)$,

$$f_k([-3\delta, 3\delta]) \subset \mathbb{R}; \tag{3.9}$$

(3.10) f_k is a C^∞ homeomorphism to its image that converges for the C^∞ topology to a homeomorphism $f : [-3\delta, 3\delta] \to$ its image.

Moreover, the sequence (f_k) satisfies the following conditions:

3.11.

$$Df_k(x) = \phi_k(x) \prod_{j=1}^{l_1} (x - c_j^{(k)})^{2q_j} \prod_{j=1}^{l_2} \big((x - b_j^{(k)})^2 + (\varepsilon_j^{(k)})^2\big)^{p_j}$$

where q_j, p_j are non negative integers, independent of k,
l_1, l_2 are non negative integers, independent of k, and satisfy $l_1 + l_2 \geqslant 1$,
$c_j^{(k)} \in [-\delta, \delta]$,
$b_j^{(k)} \in [-\delta, \delta]$,
$\varepsilon_j^{(k)} > 0$,
$c_j^{(k)} \longrightarrow 0$, $b_j^{(k)} \longrightarrow 0$, $\varepsilon_j^{(k)} \longrightarrow 0$, if $k \longrightarrow +\infty$,
and the sequence

$$(\log \phi_k)_k : x \in [-3\delta, 3\delta] \mapsto \log \phi_k(x)$$

is bounded for the C^∞ topology (C^2 will be enough in the sequel) if $k \longrightarrow +\infty$ (this follows from 2.2.1, 3.3, 3.4 using Rouché's theorem).

The sequences (f_k), (ϕ_k), $c_j^{(k)}$, ..., depend on $r = 1, \ldots, n$; we will remove this dependence and consider abstractly a sequence $(f_k)_{k \geqslant k_3}$ satisfying the conditions (3.9) to 3.11 that we stated above.

4.1. If $\hat{l} \in \mathcal{L}_1 = \big\{(a,b,c) \in \mathbb{R},\ a < b < c\big\}$ we set

$$b(\hat{l}) = \frac{b-a}{c-a}$$

[4] tn: Note that this is not any more a dynamical system. Indeed, the §3 4 and 5 study the crossratio distortion under *one* iterate of g_k.

[5] tn: A reason why Herman preferred this change of variables to $z \mapsto e^{2i\pi z}$ is that he bounded the Świątek distortion of f_k in terms of, among others, the number of intervals where the Schwarzian derivative Sf_k is non-negative, number he bounded in terms of the number of zeroes of the Schwarzian derivative. With his definition, f_k remains a rational fraction, with the same degree as f, whence a bound on the number of zeroes of Sf_k. But this is not the essential point.

and if $f : K \to \mathbb{R}$ is a homeomorphism on its image, where K is a compact interval, we set

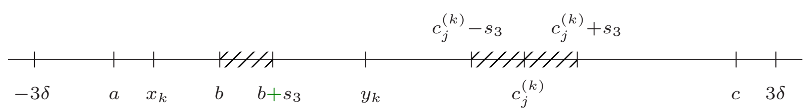
$$|f|_{D_1,K} = \sup_{\substack{\hat{l} \in \mathcal{L}_1 \\ [a,c] \subset K}} \frac{b(f(\hat{l}))}{b(\hat{l})}.$$

4.2. Proposition. *One has*

$$\sup_k |f_k|_{D_1,[-3\delta,3\delta]} < +\infty$$

$$= \sup_k \sup_{-3\delta<a<b<c<3\delta} \frac{f_k(a)-f_k(b)}{a-b} \, \frac{a-c}{f_k(a)-f_k(c)} < +\infty.$$

4.3. Proof. (The idea of the proof has be communicated to me by J.C. Yoccoz in 1987).



We set

$$c - b = s, \qquad b - a = s_1.$$

We may (and will) assume that

$$s_1 < \frac{s}{2}$$

since

$$s_1 \geqslant \frac{s}{2} \qquad \text{implies}$$

$$\frac{f_k(a)-f_k(b)}{a-b} \, \frac{a-c}{f_k(a)-f_k(c)} \leqslant \frac{s_1+s}{s_1} \leqslant 3.$$

We set $s_3 = \dfrac{s}{4(l_1+l_2+1)}$, where l_1, l_2 are the integers defined in 3.11. We set

$$W^k = [b,c] - \Big[ ]b, b+s_3[\, \cup \Big(\bigcup_{j=1}^{l_1}]c_j^k - s_3, c_j^k + s_3[\Big) \cup \Big(\bigcup_{j=1}^{l_2}]b_j^k - s_3, b_j^k + s_3[\Big) \Big].$$

We have

$$\frac{f_k(a)-f_k(b)}{a-b} \leqslant Df_k(x_k)$$

where

$$Df_k(x_k) = \sup_{x\in[a,b]} Df_k(x).$$

$$\begin{aligned}
\frac{f_k(c)-f_k(a)}{c-a} \;&\geqslant\; \frac{1}{c-a}\int_a^c Df_k(y)dy \;\geqslant\; \frac{1}{c-a}\int_{W^{(k)}} Df_k(y)dy \\
&\geqslant\; \frac{1}{c-a} Df_k(y_k) \int_{W^{(k)}} 1\, dy \\
&\geqslant\; Df_k(y_k)\frac{s/2}{s+s_1} \;\geqslant\; Df_k(y_k)/3
\end{aligned}$$

where $Df_k(y_k) = \min_{y\in W^{(k)}} Df_k(y)$.

It is enough to show that

$$\sup_k \frac{Df_k(x_k)}{Df_k(y_k)} < +\infty. \tag{4.4}$$

We have by 3.11

$$\sup_k \log\left(\frac{\phi_k(x_k)}{\phi_k(y_k)}\right) < +\infty. \tag{4.5}$$

We set $p_{k,j}(x) = |x - c_j^{(k)}|$ et $Q_{k,j}(x) = (x - b_j^{(k)})^2 + (\varepsilon_j^{(k)})^2$. To get 4.4 since

$$\frac{Df_k(x_k)}{Df_k(y_k)} = \frac{\phi_k(x_k)}{\phi_k(y_k)} \prod_{j=1}^{l_1} \left(\frac{p_{k,j}(x_k)}{p_{k,j}(y_k)}\right)^{2q_j} \prod_{j=1}^{l_2} \left(\frac{Q_{k,j}(x_k)}{Q_{k,j}(y_k)}\right)^{p_j}$$

it is enough, by 4.5, to show that

$$\sup_{j,k} \frac{p_{k,j}(x_k)}{p_{k,j}(y_k)} < +\infty; \tag{4.6}$$

$$\sup_{j,k} \frac{Q_{k,j}(x_k)}{Q_{k,j}(y_k)} < +\infty. \tag{4.7}$$

We assume

$$c_j^{(k)} \leqslant x_k \quad \text{et} \quad b_{j'}^{(k)} \leqslant x_k. \tag{4.8}$$

Since

$$\begin{aligned} |x_k - c_j^{(k)}| &\leqslant |y_k - c_j^{(k)}|; \\ |x_k - b_{j'}^{(k)}| &\leqslant |y_k - b_{j'}^{(k)}|; \end{aligned}$$

we have

$$\begin{aligned} \frac{p_{k,j}(x_k)}{p_{k,j}(y_k)} &\leqslant 1; \\ \frac{Q_{k,j'}(x_k)}{Q_{k,j'}(y_k)} &\leqslant 1. \end{aligned}$$

If j satisfies

$$x_k \leqslant c_j^{(k)} \leqslant b, \qquad x_k \leqslant b_{j'}^{(k)} \leqslant b. \tag{4.9}$$

We have

$$\begin{cases} |x_k - c_j^{(k)}| \leqslant s/2; \\ |y_k - c_j^{(k)}| \geqslant \frac{s}{4(l_1+l_2+1)} = s_3 \qquad \text{(cf. the definition of } W^{(k)}\text{)}; \end{cases} \tag{4.10}$$

and thus

$$\frac{p_{k,j}(x_k)}{p_{k,j}(y_k)} \leqslant 2(l_1 + l_2 + 1).$$

We have the same inequalities as 4.10, replacing $c_j^{(k)}$ by $b_{j'}^{(k)}$ whence

$$\frac{Q_{k,j}(x_k)}{Q_{k,j}(y_k)} \leqslant \frac{\frac{s^2}{4} + \left(\varepsilon_{j'}^{(k)}\right)^2}{\frac{s^2}{16(l_1+l_2+1)^2} + \left(\varepsilon_{j'}^{(k)}\right)^2} = u_{k,j}$$

and we conclude by using

$$u_{k,j} - 1 \leqslant 4(l_1 + l_2 + 1)^2.$$

Finally, we consider the j's such that

$$b \leqslant c_j^{(k)}, \qquad b \leqslant b_{j'}^{(k)}. \tag{4.11}$$

Then

$$\begin{cases} |x_k - c_j^{(k)}| \leqslant c - a \leqslant \frac{3}{2}s \\ |y_k - c_j^{(k)}| \geqslant \frac{s}{4(l_1+l_2+1)} = s_3 \qquad \text{cf. the definition of } W^{(k)} \end{cases} \tag{4.12}$$

and we have the same inequalities as 4.12, remplacing $c_j^{(k)}$ by $b_{j'}^{(k)}$. We conclude

$$\begin{aligned}\frac{p_{k,j}(x_k)}{p_{k,j}(y_k)} &\leqslant 6(l_1+l_2+1);\\ \frac{Q_{k,j'}(x_k)}{Q_{k,j'}(y_k)} &\leqslant 36(l_1+l_2+1)^2+1.\end{aligned}$$

4.8, 4.9 and 4.11 exhaust all the cases and we have ideed proved 4.6 and 4.7. ■

4.13. By changing x to $-x$ we obtain:

Proposition. *One has*

$$\sup_k \sup_{-3\delta<b<c<d<3\delta} \frac{f_k(d)-f_k(c)}{d-c}\,\frac{d-b}{f_k(d)-f_k(b)} < +\infty.$$

5.1. Let us assume that we are under hypotheses of 4.1. We assume that $h : K_1 \to K$ and $g : K \to K_2$ are bi-lipschitz homeomorphisms. Then

$$|g\circ f\circ h|_{D_1,K_1} \leqslant \operatorname{Lip}(g)\operatorname{Lip}(g^{-1})\operatorname{Lip}(h)\operatorname{Lip}(h^{-1})|f|_{D_1,K}. \tag{5.2}$$

5.3. Let $\widetilde{g}_{i_k}$ be the lifted sequence.
Proposition. *One has*[6]

$$\sup_k \sup_{l\in\mathcal{L}_1} D(l,\widetilde{g}_{i_k}) < +\infty.$$

5.4. Proof. By 2.2

$$\sup_k \|D\widetilde{g}_{i_k}\|_{C^0} < +\infty. \tag{5.5}$$

$(\widetilde{g}_i)$ converges uniformly to $\widetilde{g}\in\mathcal{D}^0(\mathbb{T}^1)$ and thus

$$\widetilde{g}_i^{-1} \text{ converges uniformly to } \widetilde{g}^{-1}. \tag{5.6}$$

Let $\varepsilon>0$, $l=(a,b,c)\in\mathcal{L}_1$, $c-a\geqslant\varepsilon$, then

$$D(l,\widetilde{g}_i) \leqslant \|D\widetilde{g}_i\|_{C^0}\frac{\varepsilon}{\widetilde{g}_i(c)-\widetilde{g}_i(a)}$$

and thus using 5.5 and 5.6, for all fixed $\varepsilon>0$

$$\sup_i \sup_{\substack{l\in\mathcal{L}_1\\ c-a\geqslant\varepsilon}} D(l,\widetilde{g}_i) < +\infty. \tag{5.5bis}$$

Near critical points, we use 5.1[7], 4.2, 3.6 and 3.8. The proof of proposition 2, §6 of [H] applies without modification using 2.2 and 3.5, and allows to obtain a uniformy and to conclude. ■

We obtain by the same arguments and using §6 of [H]:

5.6bis. Proposition. One has

$$\sup_k \sup_{\mathcal{L}_2} D(l,\widetilde{g}_{i_k}) < +\infty;$$

which implies with 5.3

$$\sup_k \sup_{\mathcal{L}} D(l,\widetilde{g}_{i_k}) < +\infty.$$

[6]tn: See the notes at the end of the article for the definition of $D(l,\widetilde{g}_{i_k})$.
[7]tn: indeed we pass from $\widetilde{g}_{i_k}$ to f_k by a change of variable (exponential then Möbius map)

6.1. Let $f : [-3\delta, 3\delta] \to \mathbb{R}$ be a homeomorphism[8] to its image of class C^ω (f may have critical points). Let $C(f) = \big\{x \in [-3\delta, 3\delta],\ Df(x) = 0\big\}$. We define on $[-3\delta, 3\delta] - C(f)$ the Schwarzian derivative of f

$$S(f) = D^2 \log Df - \frac{1}{2}\Big(D \log Df\Big)^2.$$

The set $\Delta = \big\{x,\ S(f)(x) \geqslant 0\big\}$ is compact in $[-3\delta, 3\delta]$ and included in $[-3\delta, 3\delta] - C(f)$.

6.2. We assume that $\Delta \neq \varnothing$. We can write

$$\Delta = \bigcup_{j=1}^{p} \Delta_j$$

where Δ_j is a compact interval, and $\Delta_{j_1} \cap \Delta_{j_2} = \varnothing$ if $j_1 \neq j_2$.

6.3. If f is a rational fraction of degree $\leqslant d$ then $S(f)$ is a rational fraction of degree $\leqslant 2(3d-2)$ and thus $p \leqslant 2(3d-2)+1$.[9]

6.4. On each Δ_j we have $D^2 \log Df(x) > 0$ if $x \in \mathrm{Int}(\Delta_j)$ and thus $\log Df$ is *convex* on Δ_j.

6.5. If $\eta : \Delta_j \to \mathbb{R}$ is a function, we set

$$\underset{\Delta_j}{\mathrm{Osc}}(\eta) = \max_{\Delta_j} \eta(x) - \min_{\Delta_j} \eta(x).$$

If $(\eta_i)_{1\leqslant i\leqslant l}$ is a finite sequence of functions on Δ_j we have

$$\underset{\Delta_j}{\mathrm{Osc}}\Big(\sum_1^l \eta_i\Big) \leqslant \sum_1^l \underset{\Delta_j}{\mathrm{Osc}}(\eta_i).$$

6.6. Since $\log Df$ is convex on Δ_j we have

$$\mathrm{Var}_{\Delta_j}(\log Df) \leqslant 2 \underset{\Delta_j}{\mathrm{Osc}}(\log Df)$$

and

$$\mathrm{Var}_{\Delta}(\log Df) \leqslant 2 \sum_{j=1}^{p} \underset{\Delta_j}{\mathrm{Osc}}(\log Df)$$

where Var_Δ denotes the total variation on Δ.

6.7. We assume that

$$\begin{aligned} &Df(x) = \phi(x) \prod_1^{l_1} P_j(x)^{2q_j} \prod_1^{l_2} Q_j(x)^{p_j} && (6.8)\\ &P_j(x) = |x - c_j|, \qquad c_j \in [-\delta, \delta];\\ &Q_j(x) = (x - b_j)^2 + \varepsilon_j^2, \quad b_j \in [-\delta, \delta];\\ &p_j \in \mathbb{N} \text{ and the } \varepsilon_j \text{ are small, } \varepsilon_j > 0, \text{ and } l_1 + l_2 \geqslant 1. \end{aligned}$$

We have

$$D^2 \log Df(x) = D^2 \log \phi(x) + \sum_1^{l_1} \frac{-2q_j}{(x-c_j)^2} + \sum_1^{l_2} 2p_j \frac{\varepsilon_j^2 - (x-b_j)^2}{((x-b_j)^2 + \varepsilon_j^2)^2}. \tag{6.9}$$

[8] tn: increasing

[9] tn: these bounds can be enhanced

Let

$$\Phi_j(x) = \frac{\varepsilon_j^2 - (x-b_j)^2}{((x-b_j)^2+\varepsilon_j^2)^2} \tag{6.10}$$

that satisfies

(6.11) $\Phi_j(x) \geqslant 0 \implies |x-b_j| \leqslant \varepsilon_j$.

(6.12) $\sup_x \Phi_j(x) = \frac{1}{\varepsilon_j^2}$.

([10])

(6.14) $\Phi(x) \leqslant -\frac{2}{25}\,\frac{1}{(x-b_j)^2}$, if $|x-b_j| \geqslant 2\varepsilon_j$.

Let $q>0$. Then there exists $C(q)>0$ such that

$$\operatorname*{Osc}_{|x\text{-}b_j|\leqslant q\varepsilon_j} (\log Q_j) \leqslant C(q) = \log(1+q^2). \tag{6.15}$$

(Indeed

$$\frac{1}{1+q^2} \leqslant \frac{(x-b_j)^2+\varepsilon_j^2}{(y-b_j)^2+\varepsilon_j^2} \leqslant 1+q^2$$

if $|x-b_j| \leqslant q\varepsilon_j$ and $|y-b_j| \leqslant q\varepsilon_j$.)

6.16. We assume that δ_1 is small and $D^2\log\phi$ is bounded independently of $\delta<\delta_1$. Since $l_1+l_2 \geqslant 1$ we have (using (6.14))

$$\bigcup_1^p \Delta_j \subset \bigcup_{j=1}^{l_2} \big\{|x-b_j| \leqslant 2\varepsilon_j\big\}. \tag{6.17}$$

6.18. We consider the points $b_j \pm 10\varepsilon_j$ contained in Δ_k. Together with the tips of Δ_k, these points cut Δ_k in at most $2l_2+1$ intervals

$$\bigcup_{i=1}^{p_2} I_i^k = \Delta_k$$

$\mathrm{Int}(I_{i_1}^k) \cap \mathrm{Int}(I_{i_2}^k) = \varnothing$, if $i_1 \neq i_2$. By (6.17)

$$\mathrm{Int}(I_i^k) \cap \big\{|x-b_j| \leqslant 10\varepsilon_j\big\} \neq \varnothing \tag{6.19}$$

for at least one $j \geqslant 1$.

6.20. Lemma. *If we assume (6.19) then*

$$I_i^k \subset \big\{|x-b_j| \leqslant 10\varepsilon_j\big\}.$$

Proof. $\mathrm{Int}(I_i^k) \cap \big\{|x-b_j| \leqslant 10\varepsilon_j\big\}$ is closed in $\mathrm{Int}(I_i^k)$. It is open because, by the choice of the tips of I_i^k,

$$\mathrm{Int}(I_i^k) \cap \big\{|x-b_j| \leqslant 10\varepsilon_j\big\} = \mathrm{Int}(I_i^k) \cap \big\{|x-b_j| < 10\varepsilon_j\big\}$$

and we conclude by using (6.19). ■

The essential proposition in the sequel is the following.

6.21. Proposition. *We assume*[11] *that* $l_2 \geqslant 1$, $\varepsilon'>0$ *and*

$$\sup_{-3\delta\leqslant x\leqslant 3\delta} \Big(\big|\log\phi(x)\big|, \big|D\log\phi(x)\big|, \big|D^2\log\phi(x)\big|\Big) = w < \varepsilon'^{-1}. \tag{6.22}$$

[10] tn: entry 6.13 has been removed.

[11] tn: We are under the hypotheses of 6.7.

There exist $\delta_1 > 0$*,* $0 < \varepsilon_0 < \varepsilon'$ *such that if* $0 < \delta < \delta_1$*,* $\sup_j \varepsilon_j \leqslant \varepsilon_0$ *then*

$$\operatorname*{Var}_{I_i^k}(\log Df) \leqslant 2 \operatorname*{Osc}_{I_i^k}(\log \phi) + C$$

where C *is a constant independent of* c_j*,* $b_{j'}$*,* I_i^k*,* $\varepsilon_{j'}$ *and depending only on* l_2 *and* p_j*,* $1 \leqslant j \leqslant l_2$*.*

6.23. Proof. According to 6.5 and 6.6 it is enough to bound $\operatorname*{Osc}_{I_i^k}(\log Q_j)$ and $\operatorname*{Osc}_{I_i^k}(\log P_{j'})$ from above, with $1 \leqslant j \leqslant l_2$, $1 \leqslant j' \leqslant l_1$. We may assume, if δ_1 is small enough, that 6.16 is satisfied if $\delta < \delta_1$.

6.24. Using 6.20 one can find an integer $\nu \geqslant 1$ such that

$$I_i^k \subset \big\{|x - b_\nu| \leqslant 10\varepsilon_\nu\big\}$$

and

$$\varepsilon_j \geqslant \varepsilon_\nu \text{ if } I_i^k \subset \big\{|x - b_j| \leqslant 10\varepsilon_j\big\}.$$

We consider the

$$j\text{'s such that } \varepsilon_j \geqslant \varepsilon_\nu. \tag{6.25}$$

We want to bound $\operatorname*{Osc}_{I_i^k}(\log Q_j)$ from above. We consider 2 cases:

1. $|b_j - b_\nu| \leqslant 20\varepsilon_\nu$ (we allow $j = \nu$).
We have

$$I_i^k \subset [b_j - c_2\varepsilon_j, b_j + c_2\varepsilon_j]$$

where $c_2 \leqslant \dfrac{30\varepsilon_\nu}{\varepsilon_j} \leqslant 30$ and we apply 6.15 to obtain

$$\operatorname*{Osc}_{I_i^k}(\log Q_j) \leqslant \log(1 + c_2^2). \tag{6.26}$$

2. $|b_j - b_\nu| \geqslant 20\varepsilon_\nu$.
We have, if $x, y \in [u_1, u_2] = I_i^k$,

$$\begin{cases} |b_j - x| \leqslant |b_j - b_\nu| + 10\varepsilon_\nu \leqslant \frac{3}{2}|b_j - b_\nu|; \\ |b_j - y| \geqslant |b_j - b_\nu| - 10\varepsilon_\nu \geqslant \frac{1}{2}|b_j - b_\nu|. \end{cases} \tag{6.26bis}$$

6.27. Lemma. *Let* $c_3 > 0$*,* $c_4 > 0$*,* $x \neq 0$ *and* $\varepsilon > 0$*. Then*

$$\inf(1, c_3/c_4) \leqslant \psi(x) = \frac{c_3x^2 + \varepsilon^2}{c_4x^2 + \varepsilon^2} \leqslant \sup(1, c_3/c_4).$$

Proof.
Let $0 < k \leqslant 1$ with $kc_3 \leqslant c_4$, then $\frac{k}{k}\psi(x) \leqslant \frac{1}{k}$.
Let $0 < k \leqslant 1$ with $kc_4 \leqslant c_3$, then $\frac{k}{k}\psi(x) \geqslant k$. ∎

From this, it follows from that if $x, y \in I_i^k$, using 6.26bis,

$$\frac{1}{9} \leqslant \frac{Q_j(x)}{Q_j(y)} \leqslant 9 \iff \operatorname*{Osc}_{I_i^k}(\log Q_j) \leqslant \log 9. \tag{6.28}$$

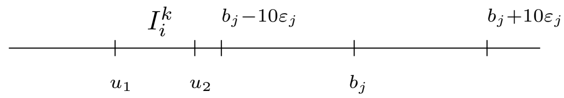
We consider the

$$j\text{'s such that } \varepsilon_j < \varepsilon_\nu. \tag{6.29}$$

We have by definition of ν in 6.24 and 6.20

$$\operatorname{Int}(I_i^k) \cap [b_j - 10\varepsilon_j, b_j + 10\varepsilon_j] = \varnothing.$$

We may assume that we have the following figure, up to a change of the orientation (we a priori allow $u_2 = b_j - 10\varepsilon_j$)



By 6.4, 6.9, 6.12 and 6.14, if $x \in I_i^k$,[12]

$$\begin{aligned} 0 \leqslant D^2 \log Df(x) &\leqslant \sup_x D^2 \log \phi(x) + \frac{2\sum_1^{l_2} p_{j'}}{\varepsilon_\nu^2} - \frac{2}{25}\frac{1}{(x-b_j)^2} \\ &\underset{6.22}{\leqslant} (1 + 2\sum_1^{l_2} p_{j'})/\varepsilon_\nu^2 - \frac{2}{25}\frac{1}{(x-b_j)^2} \end{aligned}$$

and thus

$$|x - b_j| \geqslant c_5 \varepsilon_\nu$$

where c_5 is a constant.

From this we deduce using $|u_1 - u_2| \leqslant 20\varepsilon_\nu$ that

$$\operatorname*{Osc}_{x \in I_i^l} \log |x - b_j| \leqslant c_6$$

and conclude using 6.27

$$\operatorname*{Osc}_{I_i^k}(\log Q_j) \leqslant c_7 \tag{6.30}$$

where c_6 and c_7 are constants depending only on $(p_l)_{1 \leqslant l \leqslant l_2}$.

The same proof as above shows that, if $x \in I_i^k$,

$$|x - c_j| \geqslant c_{10} \varepsilon_\nu$$

whence

$$\operatorname*{Osc}_{x \in I_i^k} \big(\log |x - c_j| \big) \leqslant c_9;$$

which implies that for all j, $\quad 1 \leqslant j \leqslant l_1$,

$$\operatorname*{Osc}_{I_i^k}(\log P_j) \leqslant c_8 \tag{6.31}$$

where c_7 and c_8 depend only on $(p_i)_{1 \leqslant i \leqslant l_2}$. The proposition follows by using 6.5, 6.6, 6.26, 6.28, 6.30 and 6.31. ■

From (6.21) it follows that if $\delta < \delta_1$ and $\sup_j \varepsilon_j \leqslant \varepsilon_0$

7. Corollary. *One has*[13]

$$\operatorname{Var}_\Delta(\log Df) \leqslant 2 \operatorname{Var}_\Delta(\log \phi) + (2l_2 + 1)pC$$

where p is the integer defined in 6.2 and by 6.3

$$p \leqslant 2(3d - 2) + 1.$$

By 6.22 $\operatorname{Var}_\Delta(\log \phi) \leqslant w$.

---

[12]tn: in the third term of the right hand side of 6.9, i.e. $\sum_{j'=1}^{l_2} 2p_j \Phi_{j'}(x)$, the positive terms require $|x - b_{j'}| < \varepsilon_{j'}$ by 6.11, and thus by 6.20, $I_i^k \subset \{|x - b_{j'}| < 10\varepsilon_{j'}\}$ and thus by 6.24, $\varepsilon_{j'} \geqslant \varepsilon_\nu$ whence by 6.12, $\Phi_{j'}(x) \leqslant \frac{1}{\varepsilon_\nu^2}$

[13]tn: the constant C is that of 6.21 and also depends on l_2, and on the p_j's defined in 6.7

8.1. Let us be given $(l_i)_{0\leqslant i\leqslant j-1} \in \mathcal{L}$, $l_i = (a_i, b_i, c_i, d_i)$ such that $[l_i] = [a_i, d_i] \subset [-3\delta, 3\delta]$ satisfying for all $x \in [-3\delta, 3\delta]$

$$\operatorname{card}\big\{j \in J,\ x \in [a_i, d_i]\big\} \leqslant 5 \text{ where } J = \{0, \dots, j-1\}. \tag{8.2}$$

We define

$$\underset{[-3\delta,3\delta]}{SD}(f) = \sup \prod_0^{j-1} D(l_i, f),$$

the sup being taken over all the $(l_i)_{i\in J}$ and J satisfying (8.2).

8.3. We consider the sequence $(f_k)_{k\geqslant k_0}$ given in 3.7. By using 3.6, 3.6.1 and 3.11 we may assume that 6.22 is satisfied[14].

8.4. Proposition. *One has, if $\delta < \delta_1$ where δ_1 is defined in 6.21*

$$\sup_k \underset{[-3\delta,3\delta]}{SD}(f_k) < +\infty.$$

Proof. Let us be given $(l_i)_{i\in J}$ satisfying 8.2. We want to bound from above

$$\prod_0^{j-1} D(f_k, l_i) \tag{8.6}$$

independently of $(l_i)_{i\in J}$, J and k. Let $\Delta^{(k)}$ be associated to f_k, as defined in 6.1. Let $J_1 = \big\{j \in J \,\big|\, [l_j] \cap \{c_1, \dots, c_{l_1}\} \neq \varnothing\big\}$. We have

$$\operatorname{card}(J_1) \leqslant 5 \operatorname{card}(\{c_1, \dots, c_{l_1}\}).$$

We bound from above

$$\prod_{j\in J_1} D(l_j, f_k) \leqslant c_1 \tag{8.7}$$

by using 5.6bis, where c_1, c_2, c_3 denote constants independent of J, $(l_i)_{i\in J}$ and k.

Let $J_2 \subset J - J_1$, $J_2 = \big\{j \in J - J_1,\ [l_j] \cap \partial\Delta^{(k)} \neq \varnothing\big\}$ where $\partial\Delta^{(k)}$ denotes the tips of $\Delta^{(k)}$. We have by 6.3

$$\operatorname{card}(J_2) \leqslant 5(2(3d-2)+1)$$

and by using 5.6bis

$$\prod_{j\in J_2} D(l_i, f_k) \leqslant c_2. \tag{8.8}$$

If $j \in J - J_1 \cup J_2$ then

$$[l_j] \subset [-3\delta, 3\delta] - \big(\{c_1, \dots, c_{l_1}\} \cup \partial\Delta^{(k)}\big).$$

Let $J_3 = \big\{j \in J - (J_1 \cup J_2),\ [l_j] \subset \operatorname{Int}\Delta^{(k)}\big\}$. We bound from above, using 7 ([15])

$$\prod_{j\in J_3} D(l_j, f_k) \leqslant e^{10\operatorname{Var}_{\Delta^{(k)}}(\log Df_k)} \leqslant c_3 \tag{8.9}$$

and finally

$$\prod_{j\in J - J_1\cup J_2\cup J_3} D(l_j, f_k) \leqslant 1 \tag{8.10}$$

since $Sf < 0$ on a neighborhood of l_j (cf. [H])

$$D(l_j, f_k) \leqslant 1.$$

The proposition follows by multiplying (8.7), …, (8.10). ∎

[14] tn: with a uniform ε'

[15] tn: see [H], section 9, part J_3 of the proof of theorem §2

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism that satisfies $R_p \circ f = f \circ R_p$, $p \in \mathbb{Z}$, $R_p : x \mapsto x + p$, i.e. $f \in \mathcal{D}^0(\mathbb{T}^1)$. We consider $(l_i)_{i\in J}$, $J = \{0, 1, \dots, j-1\}$ satisfying[16]

$$\forall x \in \mathbb{T}^1 \operatorname{card}\left\{j,\ x \in [l_j] \bmod 1\right\} \leqslant 5. \tag{9.1}$$

We define if $f \in \mathcal{D}^0(\mathbb{T}^1)$

$$SD(f) = \sup \prod_{j\in J} D(l_j, f) \in \mathbb{R} \cup \{+\infty\} \tag{9.2}$$

the sup being taken over all the $(l_j)_{j\in J}$ and J satisfying 9.1. We have if $\lambda \in \mathbb{R}$,

$$SD(f + \lambda) = SD(f) \tag{9.3}$$

and if $h \in \mathcal{D}^1(\mathbb{T}^1)$ such that $\operatorname{Var}_{\mathbb{T}^1}(\log Dh) = V < +\infty$ then[17]

$$SD(h \circ f \circ h^{-1}) \leqslant e^{20V} SD(f). \tag{9.4}$$

If K is an interval $[u, v]$, $u \neq v$ and if f is a homeomorphism to its image, then we define $\underset{K}{SD}(f)$ as we did in 9.2

$$\underset{K}{SD}(f + \lambda) = \underset{K}{SD}(f) \tag{9.5}$$

and if $g : K_1 \to K$ and $h : K \to K_2$ are $C^{1+\text{bounded variation}}$ diffeomorphisms

$$\underset{K_1}{SD}(h \circ f \circ g) \leqslant C(g, h)\ \underset{K}{SD}(f) \tag{9.6}$$

where $C(g, h) < +\infty$ is a constant independent of g and h.

10. Proof of theorem §2.

By 2.1 it is enough to prove

$$\sup_k SD(\widetilde{g}_{i_k}) < +\infty.$$

The proof is the same as that of [H, p.15 à 18][18]. We have a uniformity (c.f. 5.6bis) for $\sup_{l\in\mathcal{L}} D(l, \widetilde{g}_{i_k})$ and near critical points the inequality follows from 8.4, using 9.6 and 3.6[19]. The uniformity of the variation of $\log D\widetilde{g}_{i_k}$ on $[0, 1] - U_\varepsilon$ (notation from [H]) follows from 2.2 and 3.5. ■

11. Let α be a bounded type number and g_i a sequence satisfying the hypotheses of 1. We assume that $\rho(\widetilde{g}_i) = \alpha$. We have shown in [H] that

$$\widetilde{g}_i = \widetilde{h}_i \circ R_\alpha \circ \widetilde{h}_i^{-1},\ h_i \in \mathcal{D}^{\text{qs}}(\mathbb{T}^1),\ \widetilde{h}_i(0) = 0 \text{ et}$$

$$\left|\widetilde{h}_i\right|_{\text{qs}} \leqslant C(\alpha, SD(\widetilde{g}_i))$$

where C is a constant depending only on α and on $SD(\widetilde{g}_i)$. It follows from the proof in [H] and from 2 that

$$\sup_i C(\alpha, SD(g_i)) < +\infty.$$

[16]tn: We want in fact that $\operatorname{card}\left\{(j,k),\ x + k \in [l_j]\right\} \leqslant 5$. If we require that the length of the l_j is < 1, then it is equivalent.

[17]tn: Indeed, on one hand $\forall f, g$, $SD(f \circ g) \leqslant SD(f)SD(g)$, on the other hand $\forall h$, $SD(h) \leqslant e^{10 \operatorname{Var}_{\mathbb{T}^1}(\log Dh)}$, see the reference cited in note 15.

[18]tn: These page numbers refer to the manuscript. They correspond to the proof of theorem 2 of [H].

[19]tn: indeed we pass from $\widetilde{g}_{i_k}$ to f_k by a change of variable (exponential then Möbius map)

12. We extend $\widetilde{h}_i$ into a K_i-quasiconformal homeomorphism $\widetilde{H}_i : \mathbb{C} \to \mathbb{C}$ satisfying

$$K_i \leqslant 2\Big(C\big(\alpha(SD(\widetilde{g}_i))\big)\Big)^2$$

$$\begin{aligned} &\widetilde{H}_i(z+(1,0)) = (1,0) + \widetilde{H}_i(z) \quad \forall z; \\ &\widetilde{H}_i\big|_{\mathbb{R}} = \widetilde{h}_i. \end{aligned} \tag{12.1}$$

For this, it is enough to take the Beurling Ahlfors extension

$$\widetilde{H}_i(x+iy) = \frac{1}{2}\int_0^1 \Big(h(x+yt)+h(x-yt)\Big)dt + \frac{i}{2}\int_0^1 \Big(h(x+yt)-h(x-yt)\Big)dt.$$

Using (12.1), by $z \in \mathbb{C} \mapsto e^{2\pi iz} \in \mathbb{C}$ and by passing to the quotient, $\widetilde{h}_i$ projects to $h_i : \mathbb{S}^1 \to \mathbb{S}^1$

$$\text{and } \widetilde{H}_i \text{ to } H_i : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$$

where H_i is a K_i-quasiconformal homeomorphism satisfying

$$\begin{aligned} &H_i(0) = 0; \\ &H_i\big|_{\mathbb{S}^1} = h_i \\ \text{and} \quad &\sup_i K_i < +\infty. \end{aligned}$$

Moreover $H_i \circ r_\alpha \circ H_i\big|_{\mathbb{S}^1}^{-1} = g_i$ with $r_\alpha(z) = e^{2i\pi\alpha}z$.

13. Let $d \in \mathbb{N}$, $d \geqslant 2$. Let

$$\mathcal{H}_d = \left\{ g(z) = \lambda z^d \prod_{i=1}^{d-1} \frac{1-\overline{a}_i z}{z-a_i},\ 0<|a_i|<1,\ |\lambda|=1,\ g\big|_{\mathbb{S}^1} \text{ is a homeomorphism} \right\}.$$

14. Proposition. *One has*

$$\sup_{g\in\mathcal{H}_d} \big(SD(\widetilde{g}\big|_{\mathbb{S}^1})\big) < +\infty$$

where $\widetilde{g}\big|_{\mathbb{S}^1} : \mathbb{R} \to \mathbb{R}$ *denotes the lift of* g *to* $\mathbb{R}$.

15. Lemma. *Let* $(g_j)_{j\in\mathbb{N}} \subset \mathcal{H}_d$ *be a sequence and* $a^{(j)} = (a_1^{(j)}, \ldots, a_{d-1}^{(j)}) \in (\mathbb{D}^*)^{d-1}$ *where*

$$g_j(z) = \lambda_i z^d \prod_{k=1}^{d-1} \frac{1-\overline{a}_k^{(j)} z}{z-a_k^{(j)}}$$

then

$$\sup_j |a_k^{(j)}| < 1.$$

([20])
This lemma has been known to the author since december 1988.

Proof. Let us work by contradiction. Let $a^{(j_k)}$ be a subsequence such that

$$\sup_k |a^{(j_k)}| = 1,$$

[20]tn: The degree may drop at the limit if one of the $a_k \longrightarrow 0$.

up to another extraction and up to reordering the sequence $a_1^{(j_k)},\dots,a_{d-1}^{(j_k)}$ we may assume that $a^{(j_k)} \longrightarrow b \in \overline{\mathbb{D}}^{d-1}$,

$$\begin{aligned}&b=(b_1,\dots,b_q,\dots,b_{d-1}),\\&|b_j|=1 \text{ if } 1\leqslant j\leqslant q,\\&\sup|b_j|<1 \text{ if } q<j\leqslant d-1.\end{aligned}$$

We may well assume that $\lambda_j=1$. On $\widehat{\mathbb{C}}-\{b_1,\dots,b_q,0,\infty\}$, $g_{i_k} \longrightarrow g$ converges locally uniformly

$$g(z)=z^{d_1}\prod_{j=1}^{q}(-\bar b_j)\prod_{j\in J}\frac{1-\bar b_j z}{z-b_j}$$

$$J=\big\{k,\ b_k\neq 0,\ k>q\big\}$$

$$d_1=d-(d-1-\mathrm{card}(J)-q)=1+q+\mathrm{card}(J)>1+\mathrm{card}(J)$$

$g\big|_{\mathbb{S}^1}$ has degree $q+1>1$.

Let $z_0\in\mathbb{S}^1-\{b_1,\dots,b_q\}$ and λ_k be such that

$$\lambda_k g_{i_k}(z_0)=z_0,\quad |\lambda_k|=1.$$

Up to subsequences extractions we may assume that $\lambda_k \longrightarrow \lambda$ if $k\longrightarrow+\infty$. Since $G_k=\lambda_k g_{i_k}$ is a sequence of homeomorphisms satisfying $G_k(z_0)=z_0$ by Helly's theorem[21], since G_k converges on $\mathbb{S}^1-\{b_1,\dots,b_q\}$ we conclude that $\lambda g:\mathbb{S}^1-\{z_0\}\to\mathbb{S}^1-\{z_0\}$ is monotonic non-decreasing, and since λg is continuous, λg is a homeomorphism, which contradicts the fact that λg has degree $\geqslant 2$. ■

16. Proof of proposition 14.

If $\sup_{g\in\mathcal{H}_d} SD(\widetilde{g})=+\infty$ we can find a sequence $(g_i)_{i\in\mathbb{N}}$ such that

$$\sup_i SD(\widetilde{g}_i)=+\infty.$$

Up to extracting a subsequence g_{i_k} we may assume that $a^{(i_k)}$ converges to an element b of $\overline{\mathbb{D}}^{d-1}$. By the previous lemma $b\in\mathbb{D}^{d-1}$. The hypotheses of 1 are satisfied for the sequence (g_{i_k}) and by 2

$$\sup_i SD(\widetilde{g}_i)=+\infty\quad\text{is not possible.}$$

■

16.1. Remark. 14 and lemma 15 are false if we do not restrict to Blaschke products of the particular form that we considered in 13.

Example

$$g_t(z)=\frac{z-t}{1-\bar t z}\qquad |t|<1,\ t\longrightarrow 1.$$

If we had $\sup_t SD(\widetilde{g}_t)<+\infty$, $\widetilde{g}_t$ and $\widetilde{g}_t^{-1}$ would be uniformly k-quasisymmetric and thus if $t_i\longrightarrow 1$, $\widetilde{g}_{t_i}-\widetilde{g}_{t_i}(0)$ would have non constant limit values. For this example 17 is false if we require[22] that $H_{g_t}(0)=0$.

[21] tn: Helly's theorem: let I be an interval and $\mathcal{F}$ a family of functions from I to $\mathbb{R}$. Assume that $\exists M,N>0$ such that $\forall f\in\mathcal{F}$, $|f|\leqslant M$ and $\mathrm{Var}(f)\leqslant N$. Then we can extract from $\mathcal{F}$ a sequence f_n that converges at every point of I to a function f satisfying the same inequalities.

[22] tn: i.e. we do not take the Ahlfors-Beurling extension any more but we require instead that $H_{g_t}(0)=0$; note that on this example, which is a Möbius map, the conjugacy from $\lambda_t g_t$ to the rotation $z\mapsto e^{2i\pi\alpha}t$ is itself a Möbius map, thus admits an extension H_{g_t} which is conformal: $K(H_{g_t})=1\dots$

17. Corollary. *Let α be a bounded type number and $\mathcal{H}_{d,\alpha} = \{g \in \mathcal{H}_d,\ \rho(g) = \alpha \bmod 1\}$. If $H_g : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is the $K(H_g)$-quasiconformal homeomorphism that we defined in 12, then*

$$\sup_{g \in \mathcal{H}_{d,\alpha}} K(H_g) < +\infty.$$

This follows at once from 12.

References

[H] M.R. Herman. *Conjugaison quasi symétrique des homéomorphismes analytiques du cercle à des rotations*, manuscript, 1987.

[S] G. Świątek. *Rational rotation numbers for maps of the circle*, to appear in CMP.[23]

Notes

Structure of the article.

–§0 introduction
–§1 and §2 hypothese and statement of the main theorem: the Świątek distortion stays bounded; some remarks
–§3 extraction of a subsequence, expression into a local form at critical points of the limit
–§4 and §5 control of the crossratio distortion under one iterate
  §4 local problem
  §5 global problem
–§6 to §9 control of the Świątek distortion (product of the crossratio distortions on an almost disjoint set of intervals of $\mathbb{R}/\mathbb{Z}$)
  §6 and §7 statement and proof of a key proposition; this is the proposition that allows the adaptation of [H] to families of functions; it states that the total variation of the logarithm of the standard derivative on the union of the intervals where the Schwarzian derivative is non negative stays bounded
  §8 local problem
  §9 global problem
–§10 end of the proof of the main theorem
–§11 to §17 applications
  §11 link with the constant of quasisymmetry of the conjugacy to the rotation
  §12 link with the constant of quasiconformality of the Ahlfors-Beurling extension
  §13 definition of a class $\mathcal{H}_d$ of Blaschke products
  §15 proof that $\mathcal{H}_d$ is a compact class
  §14, §16 and §17 application of the main theorem to $\mathcal{H}_d$, remarks

Reminder of some definitions from [H]:

$$D(l, f) = \frac{b(f(l))}{b(l)}$$

where

$$l \in \mathcal{L} = \{(a, b, c, d) \in \mathbb{R}^4,\ a < b < c < d\}$$

and

$$b(l) = \frac{b-a}{c-a} \Big/ \frac{d-b}{d-c}$$

[23] tn: Published: Comm. Math. Phys., 119 (1988) 109–128.

is the crossratio. Also the following definitions are made there:

$$\mathcal{L}_1 = \big\{(a, b, c, +\infty) \in \mathbb{R}^4,\ a < b < c\big\},$$
$$\mathcal{L}_2 = \big\{(-\infty, b, c, d) \in \mathbb{R}^4,\ b < c < d\big\},$$
$$\forall l \in \mathcal{L}_1,\ b(l) = \frac{b-a}{c-a},$$
$$\forall l \in \mathcal{L}_2,\ b(l) = \frac{d-c}{d-b}.$$

Illustration of §3.4:

The critical points of g_k are either at distance $\geqslant \delta_1$ from $\mathbb{S}^1$, or sit within the gray disks. The latter are centered at the critical points of the limit map g.