About the Marmi Moussa Yoccoz conjecture

Xavier Buff and Arnaud Chéritat

Univ. Toulouse

Pisa, April 2007

The Marmi Moussa Yoccoz conjecture

This conjecture concerns a function that we'll call Υ (Upsilon) and whose definition we begin with.

$$P_\alpha(z) = e^{2i\pi\alpha} z + z^2$$

$r(\alpha)$ is the *conformal radius* of the Siegel disk of P_α. ▸ graph

$$\Phi(\alpha) = \sum_{n=0}^{+\infty} \alpha_0 \cdots \alpha_{n-1} \log \frac{1}{\alpha_n}$$

where $\alpha_{n+1} = \mathrm{Frac}(\alpha_n)$ and $\alpha_0 = \mathrm{Frac}(\alpha)$, is Yoccoz's variant of the *Brjuno sum*. It is an approximation of $-\log r(\alpha)$, and Υ is the *error term*:

$$\Upsilon(\alpha) = \Phi(\alpha) + \log r(\alpha)$$

The Marmi Moussa Yoccoz conjecture

This conjecture concerns a function that we'll call Υ (Upsilon) and whose definition we begin with.

$$P_\alpha(z) = e^{2i\pi\alpha} z + z^2$$

$r(\alpha)$ is the *conformal radius* of the Siegel disk of P_α. ▸ graph

$$\Phi(\alpha) = \sum_{n=0}^{+\infty} \alpha_0 \cdots \alpha_{n-1} \log \frac{1}{\alpha_n}$$

where $\alpha_{n+1} = \mathrm{Frac}(\alpha_n)$ and $\alpha_0 = \mathrm{Frac}(\alpha)$, is Yoccoz's variant of the *Brjuno sum*. It is an approximation of $-\log r(\alpha)$, and Υ is the *error term*:

$$\Upsilon(\alpha) = \Phi(\alpha) + \log r(\alpha)$$

- Yoccoz almost proved that Υ is bounded (around 1988).
- Marmi drew the graph.
- (let me show a version I drew in 2002)

The Marmi Moussa Yoccoz conjecture

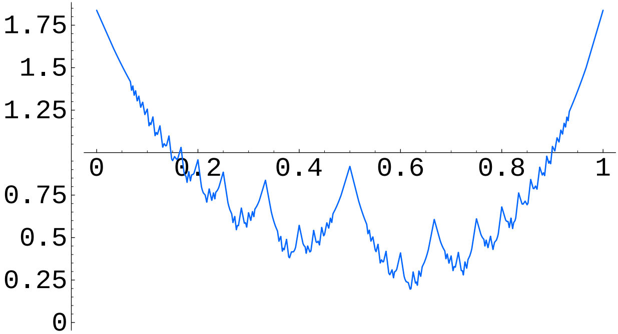
- Yoccoz almost proved that Υ is bounded (around 1988).
- Marmi drew the graph.
- (let me show a version I drew in 2002)

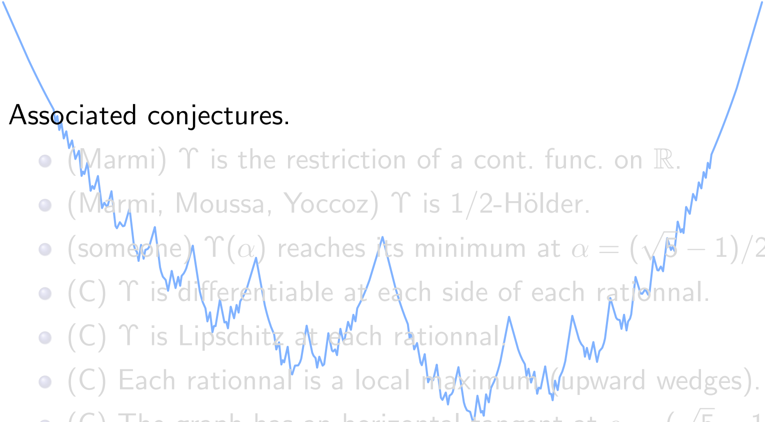
The Marmi Moussa Yoccoz conjecture

- Yoccoz almost proved that Υ is bounded (around 1988).
- Marmi drew the graph.
- (let me show a version I drew in 2002)

The Marmi Moussa Yoccoz conjecture

- Yoccoz almost proved that Υ is bounded (around 1988).
- Marmi drew the graph.
- (let me show a version I drew in 2002)

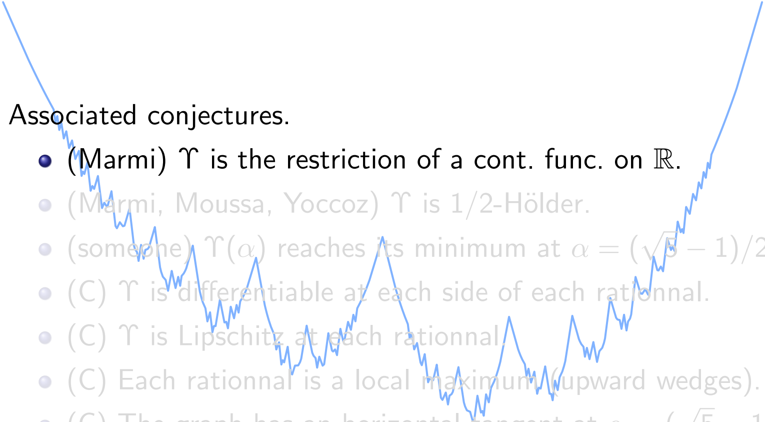




Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$.
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5}-1)/2$.
- (C) Υ is differentiable at each side of each rationnal.
- (C) Υ is Lipschitz at each rationnal.
- (C) Each rationnal is a local maximum (upward wedges).
- (C) The graph has an horizontal tangent at $\alpha = (\sqrt{5}-1)/2$.

better graph

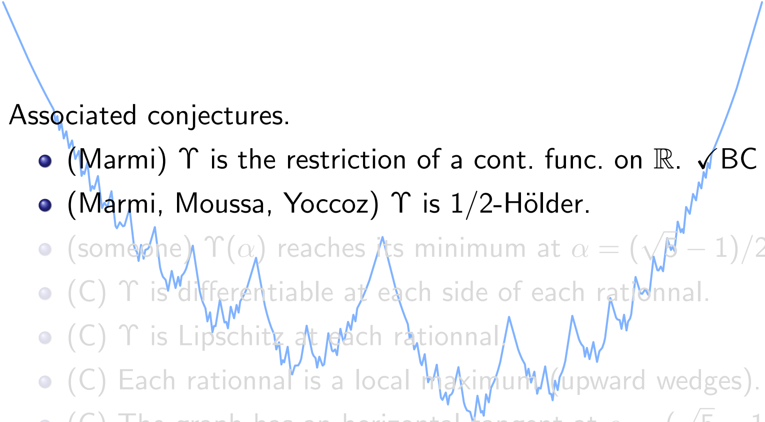


Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$.
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5}-1)/2$.
- (C) Υ is differentiable at each side of each rationnal.
- (C) Υ is Lipschitz at each rationnal.
- (C) Each rationnal is a local maximum (upward wedges).
- (C) The graph has an horizontal tangent at $\alpha = (\sqrt{5}-1)/2$.

better graph

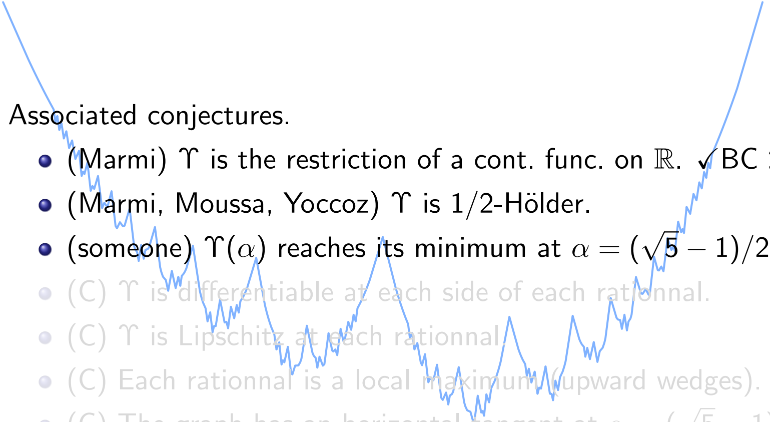
The Marmi Moussa Yoccoz conjecture



Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$. ✓BC 2003
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5} - 1)/2$.
- (C) Υ is differentiable at each side of each rationnal.
- (C) Υ is Lipschitz at each rationnal.
- (C) Each rationnal is a local maximum (upward wedges).
- (C) The graph has an horizontal tangent at $\alpha = (\sqrt{5} - 1)/2$.

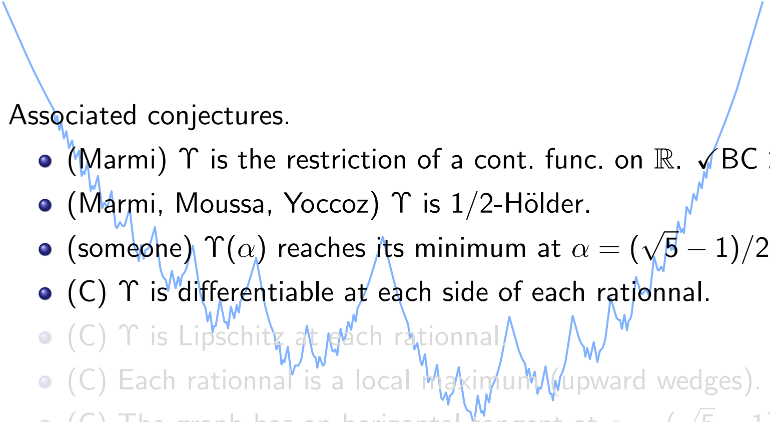
▸ better graph



Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$. ✓BC 2003
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5} - 1)/2$.
- (C) Υ is differentiable at each side of each rationnal.
- (C) Υ is Lipschitz at each rationnal.
- (C) Each rationnal is a local maximum (upward wedges).
- (C) The graph has an horizontal tangent at $\alpha = (\sqrt{5} - 1)/2$.

▸ better graph

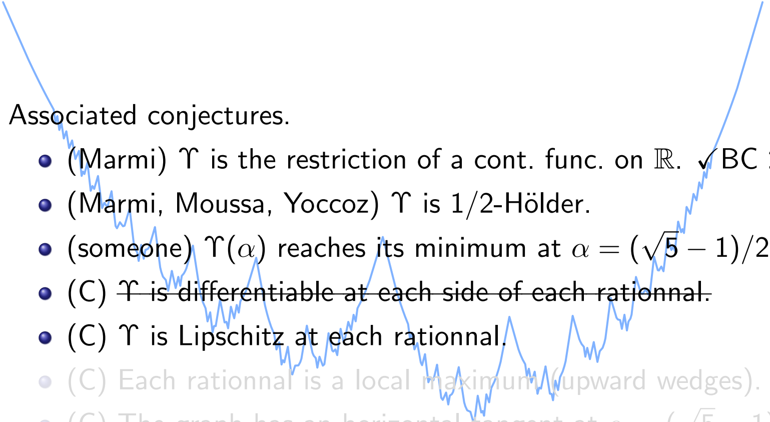


Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$. ✓BC 2003
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5} - 1)/2$.
- (C) Υ is differentiable at each side of each rationnal.
- (C) Υ is Lipschitz at each rationnal.
- (C) Each rationnal is a local maximum (upward wedges).
- (C) The graph has an horizontal tangent at $\alpha = (\sqrt{5} - 1)/2$.

better graph

The Marmi Moussa Yoccoz conjecture

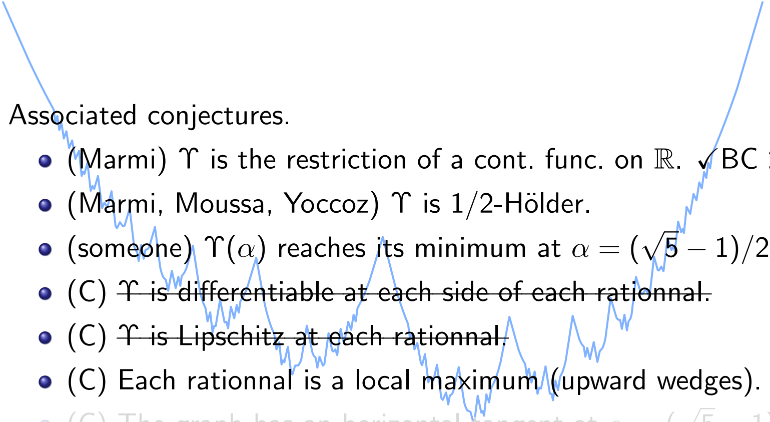


Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$. $\checkmark$BC 2003
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5} - 1)/2$.
- (C) ~~Υ is differentiable at each side of each rationnal.~~
- (C) Υ is Lipschitz at each rationnal.
- (C) Each rationnal is a local maximum (upward wedges).
- (C) The graph has an horizontal tangent at $\alpha = (\sqrt{5} - 1)/2$.

better graph

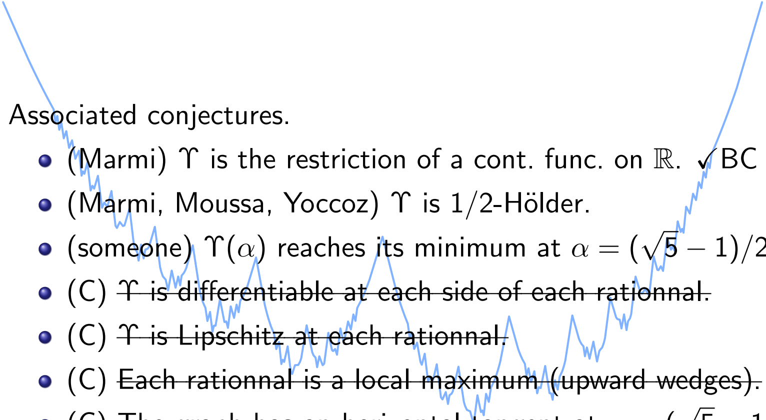
The Marmi Moussa Yoccoz conjecture



Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$. ✓BC 2003
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5}-1)/2$.
- (C) ~~Υ is differentiable at each side of each rationnal.~~
- (C) ~~Υ is Lipschitz at each rationnal.~~
- (C) Each rationnal is a local maximum (upward wedges).
- (C) The graph has an horizontal tangent at $\alpha = (\sqrt{5}-1)/2$.

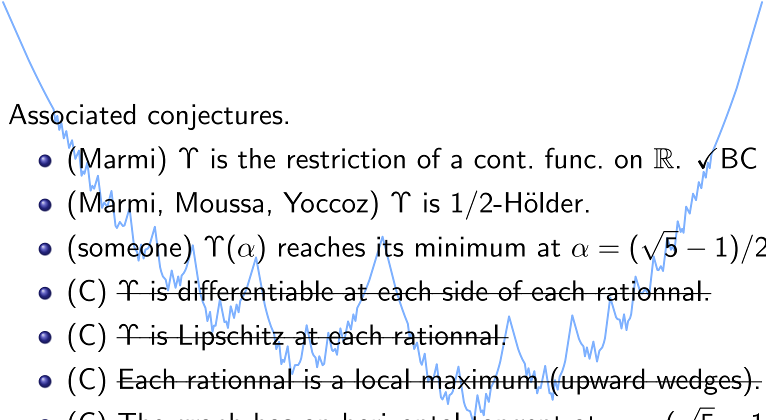
▸ better graph



Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$. ✓BC 2003
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5}-1)/2$.
- (C) ~~Υ is differentiable at each side of each rationnal.~~
- (C) ~~Υ is Lipschitz at each rationnal.~~
- (C) ~~Each rationnal is a local maximum (upward wedges).~~
- (C) The graph has an horizontal tangent at $\alpha = (\sqrt{5}-1)/2$.

better graph



Associated conjectures.

- (Marmi) Υ is the restriction of a cont. func. on $\mathbb{R}$. ✓BC 2003
- (Marmi, Moussa, Yoccoz) Υ is 1/2-Hölder.
- (someone) $\Upsilon(\alpha)$ reaches its minimum at $\alpha = (\sqrt{5} - 1)/2$.
- (C) ~~Υ is differentiable at each side of each rationnal.~~
- (C) ~~Υ is Lipschitz at each rationnal.~~
- (C) ~~Each rationnal is a local maximum (upward wedges).~~
- (C) ~~The graph has an horizontal tangent at $\alpha = (\sqrt{5} - 1)/2$.~~

▸ better graph ▸ zoom at gmean

The Marmi Moussa Yoccoz conjecture

Main theorem of this talk

Theorem (BC)

On every interval I, the function Υ is not δ-Hölder on I for any $\delta > 1/2$, and has unbounded variation on I.

In other words, if the MMY conjecture holds, then $1/2$ is the optimal exponent.

The functional equation

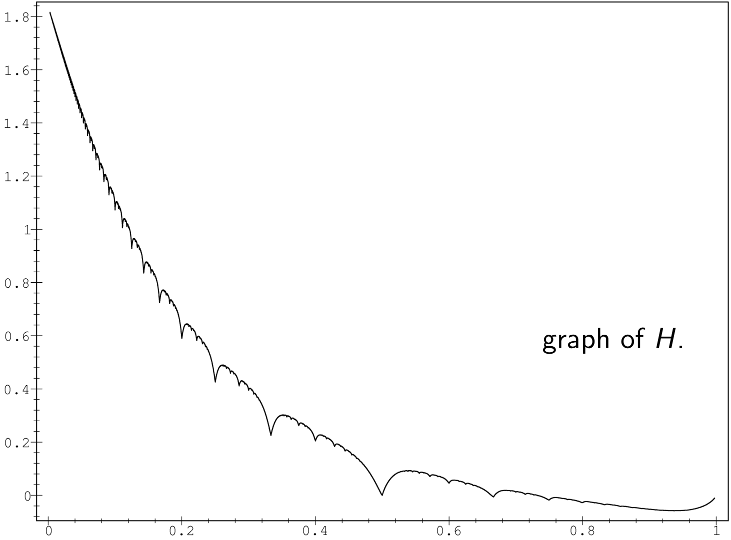
The arithmetical function Φ satisfies the following functional equation

$$\forall \alpha \in]0,1], \quad \Phi(\alpha) - \alpha\Phi(1/\alpha) = \log \frac{1}{\alpha}.$$

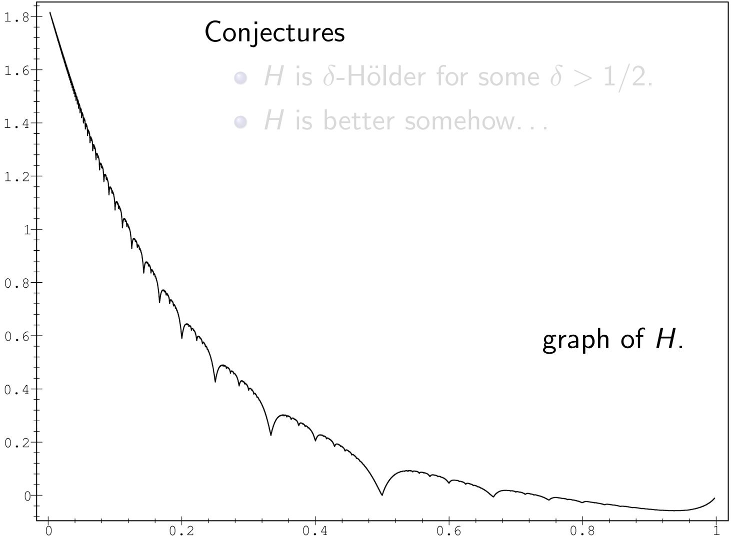
Let

$$H(\alpha) = \Upsilon(\alpha) - \alpha\Upsilon(1/\alpha).$$

The functional equation



The functional equation

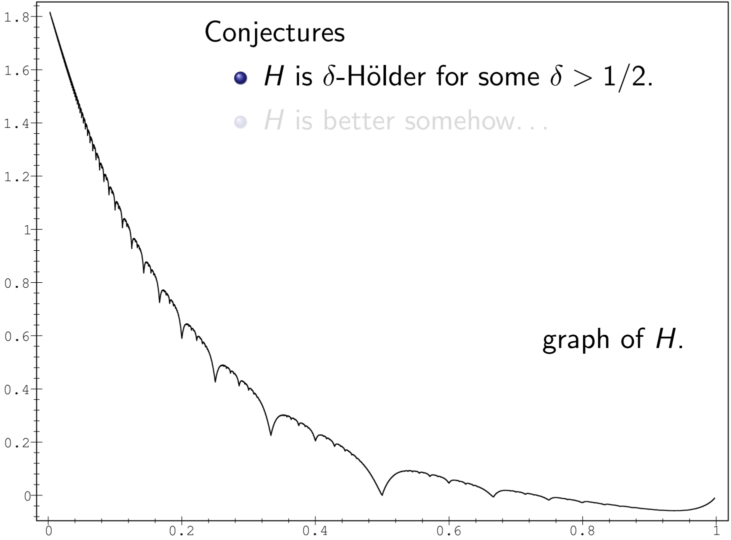


Conjectures

- H is δ-Hölder for some $\delta > 1/2$.
- H is better somehow...

graph of H.

The functional equation



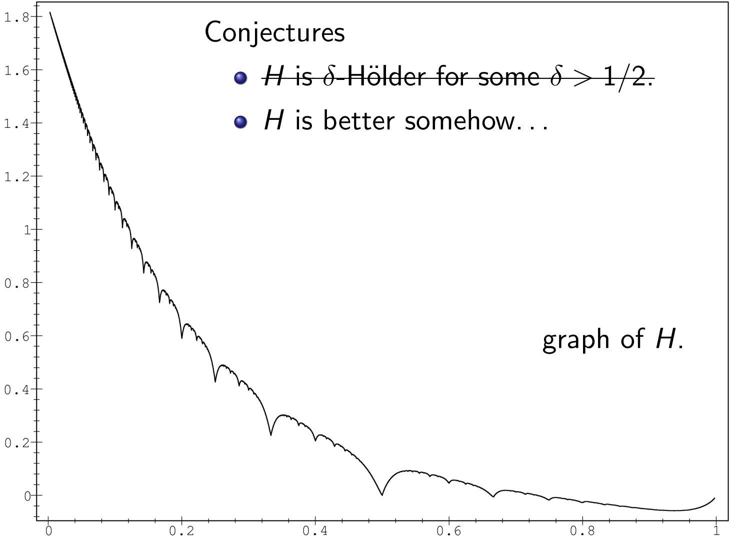
Conjectures

- H is δ-Hölder for some $\delta > 1/2$.
- H is better somehow. . .

graph of H.

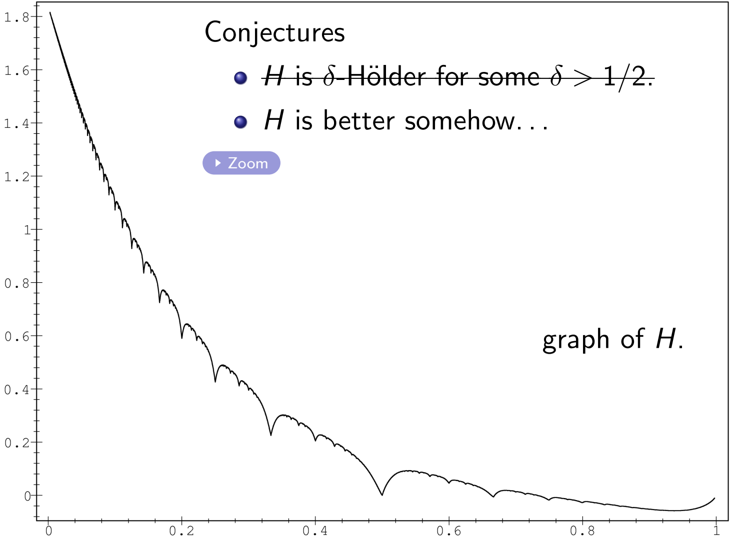
Conjectures

- ~~H is δ-Hölder for some $\delta > 1/2$.~~
- H is better somehow...



graph of H.

The functional equation



Conjectures

- ~~H is δ-Hölder for some $\delta > 1/2$.~~
- H is better somehow...

▸ Zoom

graph of H.

Theorem (BC 2002)

$$\Upsilon\left(\frac{p}{q}\right) = \frac{\log 2\pi}{q} + L_a\left(\frac{p}{q}\right) + \Phi_{\text{trunc}}\left(\frac{p}{q}\right)$$

where
L_a = asymptotic size of the parabolic fixed point of $P^q_{p/q}$,
Φ_{trunc} = truncated Yoccoz's Brjuno sum.

(Define these numbers on the blackboard.)

Theorem (BC 2002)

$$\Upsilon\left(\frac{p}{q}\right) = \frac{\log 2\pi}{q} + L_a\left(\frac{p}{q}\right) + \Phi_{\text{trunc}}\left(\frac{p}{q}\right)$$

where

L_a = asymptotic size of the parabolic fixed point of $P_{p/q}^q$,
Φ_{trunc} = truncated Yoccoz's Brjuno sum.

(Define these numbers on the blackboard.)

Example: $\boxed{\Upsilon(0) = \log 2\pi.}$

The expansion

on whom?

Consider one of the two continued fractions of

$$p/q = [a_0, \ldots, a_k] = a_0 + 1/(a_1 + \ldots).$$

Let $s \in \mathbb{R}$ and

$$x_n = [a_0, \ldots, a_k, n + s] = a_0 + 1/(\ldots + 1/(n + s)).$$

According to which continued fraction of p/q we chose, $x_n \longrightarrow p/q$ either from the right or the left.

The expansion

itself

$$x_n = [a_0, \dots, a_k, n+s] = a_0 + \cfrac{1}{\ddots + \cfrac{1}{n+s}}.$$

Theorem (BC 2006)

There exists constants A, $B_s \in \mathbb{R}$ such that if $s \in \mathbb{Q}$ then

$$\Upsilon(x_n) \underset{n\to\infty}{=} \Upsilon\left(\frac{p}{q}\right) + A\frac{\log n}{n} + B_s\frac{1}{n} + o\left(\frac{1}{n}\right)$$

The expansion

itself

$$x_n = [a_0, \dots, a_k, n+s] = a_0 + \cfrac{1}{\ddots + \cfrac{1}{n+s}}.$$

Theorem (BC 2006)

There exists constants A, $B_s \in \mathbb{R}$ such that if $s \in \mathbb{Q}$ then

$$\Upsilon(x_n) \underset{n\to\infty}{=} \Upsilon\left(\frac{p}{q}\right) + A\frac{\log n}{n} + B_s\frac{1}{n} + o\left(\frac{1}{n}\right)$$

Example: $$\boxed{\Upsilon\left(\frac{1}{n}\right) = \log 2\pi + 0 - \frac{7.052\dots}{n} + o\left(\frac{1}{n}\right).}$$

- Let I be the *holomorphic index* of $P^q_{p/q}$ at 0 and $\gamma = \frac{\frac{q+1}{2} - I}{q}$ be Écalle's *iterative residue* divided by q.
- Let $\mathcal{E}_\theta$ be the *parabolic renormalisation* (aka *horn map*). This is a family of maps such that $\mathcal{E}_\theta = e^{2i\pi\theta}\mathcal{E}_0$, $\mathcal{E}_0(0) = 0$ and $\mathcal{E}_0'(0) = 1$.
- For $s \in \mathbb{Q}$, let $\Upsilon_{\mathcal{E}}(s)$ be defined by analogy by $\Upsilon_{\mathcal{E}}(s) = \log(2\pi)/q + \log L_a(\mathcal{E}, s) + \Phi_{\text{trunc}}(s)$.

The familiy $\mathcal{E}$ depends up to conjugacy by a linear map, on choices made in defining the Fatou coordinates of $P^q_{p/q}$. Thus the value of $L_a(\mathcal{E}, s)$ and $\Upsilon_{\mathcal{E}}(s)$ depend on these choices.

The expansion

definitions

- Let I be the *holomorphic index* of $P^q_{p/q}$ at 0 and $\gamma = \frac{\frac{q+1}{2} - I}{q}$ be Écalle's *iterative residue* divided by q.
- Let $\mathcal{E}_\theta$ be the *parabolic renormalisation* (aka *horn map*). This is a family of maps such that $\mathcal{E}_\theta = e^{2i\pi\theta}\mathcal{E}_0$, $\mathcal{E}_0(0) = 0$ and $\mathcal{E}_0'(0) = 1$.
- For $s \in \mathbb{Q}$, let $\Upsilon_{\mathcal{E}}(s)$ be defined by analogy by $\Upsilon_{\mathcal{E}}(s) = \log(2\pi)/q + \log L_a(\mathcal{E}, s) + \Phi_{\text{trunc}}(s)$.

The familiy $\mathcal{E}$ depends up to conjugacy by a linear map, on choices made in defining the Fatou coordinates of $P^q_{p/q}$. Thus the value of $L_a(\mathcal{E}, s)$ and $\Upsilon_{\mathcal{E}}(s)$ depend on these choices.

Reminder

$$\Upsilon(x_n) \underset{n\to\infty}{=} \Upsilon\left(\frac{p}{q}\right) + A\frac{\log n}{n} + B_s\frac{1}{n} + o\left(\frac{1}{n}\right)$$

$$A = -\frac{q_{k-1}}{q^2} - \nu\frac{2\pi \operatorname{Im}\gamma\left(\frac{p}{q}\right)}{q}.$$

where

- $\nu = (-1)^k$ is the side from which $x_n \longrightarrow p/q$,
- q_{k-1} is the denominator of the last convergent p_{k-1}/q_{k-1} of p/q before p/q itself.

The numbers ν, q_{k-1} and ν all depend on which continued fraction of p/q we chose.

The expansion
definitions

- Let I be the *holomorphic index* of $P^q_{p/q}$ at 0 and $\gamma = \frac{\frac{q+1}{2} - I}{q}$ be Écalle's *iterative residue* divided by q.
- Let $\mathcal{E}_\theta$ be the *parabolic renormalisation* (aka *horn map*). This is a family of maps such that $\mathcal{E}_\theta = e^{2i\pi\theta}\mathcal{E}_0$, $\mathcal{E}_0(0) = 0$ and $\mathcal{E}_0'(0) = 1$.
- For $s \in \mathbb{Q}$, let $\Upsilon_{\mathcal{E}}(s)$ be defined by analogy by $\Upsilon_{\mathcal{E}}(s) = \log(2\pi)/q + \log L_a(\mathcal{E}, s) + \Phi_{\text{trunc}}(s)$.

The familiy $\mathcal{E}$ depends up to conjugacy by a linear map, on choices made in defining the Fatou coordinates of $P^q_{p/q}$. Thus the value of $L_a(\mathcal{E}, s)$ and $\Upsilon_{\mathcal{E}}(s)$ depend on these choices.

The expansion
definitions

- Let I be the *holomorphic index* of $P_{p/q}^{q}$ at 0 and $\gamma = \frac{\frac{q+1}{2}-I}{q}$ be Écalle's *iterative residue* divided by q.
- Let $\mathcal{E}_\theta$ be the *parabolic renormalisation* (aka *horn map*). This is a family of maps such that $\mathcal{E}_\theta = e^{2i\pi\theta}\mathcal{E}_0$, $\mathcal{E}_0(0) = 0$ and $\mathcal{E}_0'(0) = 1$.
- For $s \in \mathbb{Q}$, let $\Upsilon_{\mathcal{E}}(s)$ be defined by analogy by $\Upsilon_{\mathcal{E}}(s) = \log(2\pi)/q + \log L_a(\mathcal{E}, s) + \Phi_{\text{trunc}}(s)$.

The familiy $\mathcal{E}$ depends up to conjugacy by a linear map, on choices made in defining the Fatou coordinates of $P_{p/q}^{q}$. Thus the value of $L_a(\mathcal{E}, s)$ and $\Upsilon_{\mathcal{E}}(s)$ depend on these choices.

The expansion

The constant of the linear term

Reminder

$$\Upsilon(x_n) \underset{n\to\infty}{=} \Upsilon\left(\frac{p}{q}\right) + A\frac{\log n}{n} + B_s\frac{1}{n} + o\left(\frac{1}{n}\right)$$

$$B_s = \frac{\Upsilon_{\mathcal{E}}(-\nu s)}{q} \pm \frac{\pi^2}{q}\operatorname{Re}\gamma\left(\frac{p}{q}\right) + \nu c$$

Where c is a constant that depends on the choices in Fatou coordinates.

Hence, for our main theorem to hold near p/q, it is enough that $\Upsilon_{\mathcal{E}}$ be a *non constant function*.

We can prove it. ▸ Details

Reminder

$$\Upsilon(x_n) \underset{n\to\infty}{=} \Upsilon\left(\frac{p}{q}\right) + A\frac{\log n}{n} + B_s\frac{1}{n} + o\left(\frac{1}{n}\right)$$

$$B_s = \frac{\Upsilon_{\mathcal{E}}(-\nu s)}{q} \pm \frac{\pi^2}{q}\operatorname{Re}\gamma\left(\frac{p}{q}\right) + \nu c$$

Where c is a constant that depends on the choices in Fatou coordinates.

Hence, for our main theorem to hold near p/q, it is enough that $\Upsilon_{\mathcal{E}}$ be a *non constant function.*

We can prove it. Details

Reminder

$$\Upsilon(x_n) \underset{n\to\infty}{=} \Upsilon\left(\frac{p}{q}\right) + A\frac{\log n}{n} + B_s\frac{1}{n} + o\left(\frac{1}{n}\right)$$

$$B_s = \frac{\Upsilon_{\mathcal{E}}(-\nu s)}{q} \pm \frac{\pi^2}{q}\operatorname{Re}\gamma\left(\frac{p}{q}\right) + \nu c$$

Where c is a constant that depends on the choices in Fatou coordinates.

Hence, for our main theorem to hold near p/q, it is enough that $\Upsilon_{\mathcal{E}}$ be a *non constant function*.

We can prove it. Details

Reminder

$$\Upsilon(x_n) \underset{n\to\infty}{=} \Upsilon\left(\frac{p}{q}\right) + A\frac{\log n}{n} + B_s\frac{1}{n} + o\left(\frac{1}{n}\right)$$

$$B_s = \frac{\Upsilon_{\mathcal{E}}(-\nu s)}{q} \pm \frac{\pi^2}{q}\operatorname{Re}\gamma\left(\frac{p}{q}\right) + \nu c$$

Where c is a constant that depends on the choices in Fatou coordinates.

Hence, for our main theorem to hold near p/q, it is enough that $\Upsilon_{\mathcal{E}}$ be a *non constant function*.

We can prove it. Details

The expansion
a conjecture

Our expansion implies that the function

$$\frac{\Upsilon\left(\frac{p}{q}+\nu\varepsilon\right)-\Upsilon(p/q)}{\varepsilon}+\nu q^2 A\log\varepsilon.$$

where we substitute $\varepsilon=\frac{1}{n+x}$ converges simply for $x\in\mathbb{Q}$ to the function

$$\nu q^2(B_x - A\log q^2)=\nu q\Upsilon_{\mathcal{E}}(-\nu x)+\mathrm{cst}\,.$$

Conjecture

The convergence is uniform.

Illustration

The expansion
a conjecture

Our expansion implies that the function

$$\frac{\Upsilon\left(\frac{p}{q}+\nu\varepsilon\right)-\Upsilon(p/q)}{\varepsilon}+\nu q^2 A\log\varepsilon.$$

where we substitute $\varepsilon=\frac{1}{n+x}$ converges simply for $x\in\mathbb{Q}$ to the function

$$\nu q^2(B_x - A\log q^2)=\nu q\Upsilon_{\mathcal{E}}(-\nu x)+\mathrm{cst}\,.$$

Conjecture

The convergence is uniform.

▸ Illustration

Our expansion implies that the function

$$\frac{\Upsilon\left(\frac{p}{q}+\nu\varepsilon\right)-\Upsilon(p/q)}{\varepsilon}+\nu q^2 A\log\varepsilon.$$

where we substitute $\varepsilon=\frac{1}{n+x}$ converges simply for $x\in\mathbb{Q}$ to the function

$$\nu q^2(B_x - A\log q^2) = \nu q \Upsilon_{\mathcal{E}}(-\nu x) + \mathrm{cst}\,.$$

Conjecture

The convergence is uniform.

Illustration

The expansion
about the functional equation

These expansions yield expansions of H at rationnals. We are able to prove that for all $\delta > 1/2$:

- If Υ and $\Upsilon_{\mathcal{E}_0}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[0, \varepsilon]$ and has unbounded variation there.
- Let $p/q \notin \mathbb{Z}$ and $\nu = \pm 1$. If $\Upsilon_{\mathcal{E}_{p/q},\nu}$ and $\Upsilon_{\mathcal{E}_{q/p},\nu}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[p/q, p/q + \nu\varepsilon]$ and has unbounded variation there.

These differences do not depend on Yoccoz's Brjuno function Φ, which cancels out, leaving only the conformal radii/asymptotic sizes.

Therefore H cannot be better than $1/2$-Hölder on any $[0, \varepsilon]$ and it is very likely that it holds near every p/q.

These expansions yield expansions of H at rationnals. We are able to prove that for all $\delta > 1/2$:

- If Υ and $\Upsilon_{\mathcal{E}_0}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[0, \varepsilon]$ and has unbounded variation there.
- Let $p/q \notin \mathbb{Z}$ and $\nu = \pm 1$. If $\Upsilon_{\mathcal{E}_{p/q},\nu}$ and $\Upsilon_{\mathcal{E}_{q/p},\nu}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[p/q, p/q + \nu\varepsilon]$ and has unbounded variation there.

These differences do not depend on Yoccoz's Brjuno function Φ, which cancels out, leaving only the conformal radii/asymptotic sizes.

Therefore H cannot be better than $1/2$-Hölder on any $[0, \varepsilon]$ and it is very likely that it holds near every p/q.

The expansion

about the functional equation

These expansions yield expansions of H at rationnals. We are able to prove that for all $\delta > 1/2$:

- If Υ and $\Upsilon_{\mathcal{E}_0}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[0, \varepsilon]$ and has unbounded variation there.
- Let $p/q \notin \mathbb{Z}$ and $\nu = \pm 1$. If $\Upsilon_{\mathcal{E}_{p/q},\nu}$ and $\Upsilon_{\mathcal{E}_{q/p},\nu}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[p/q, p/q + \nu\varepsilon]$ and has unbounded variation there.

These differences do not depend on Yoccoz's Brjuno function Φ, which cancels out, leaving only the conformal radii/asymptotic sizes.

Therefore H cannot be better than $1/2$-Hölder on any $[0, \varepsilon]$ and it is very likely that it holds near every p/q.

The expansion

about the functional equation

These expansions yield expansions of H at rationnals. We are able to prove that for all $\delta > 1/2$:

- If Υ and $\Upsilon_{\mathcal{E}_0}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[0, \varepsilon]$ and has unbounded variation there.
- Let $p/q \notin \mathbb{Z}$ and $\nu = \pm 1$. If $\Upsilon_{\mathcal{E}_{p/q},\nu}$ and $\Upsilon_{\mathcal{E}_{q/p},\nu}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[p/q, p/q + \nu\varepsilon]$ and has unbounded variation there.

These differences do not depend on Yoccoz's Brjuno function Φ, which cancels out, leaving only the conformal radii/asymptotic sizes.

Therefore H cannot be better than $1/2$-Hölder on any $[0, \varepsilon]$ and it is very likely that it holds near every p/q.

These expansions yield expansions of H at rationnals. We are able to prove that for all $\delta > 1/2$:

- If Υ and $\Upsilon_{\mathcal{E}_0}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[0, \varepsilon]$ and has unbounded variation there.
- Let $p/q \notin \mathbb{Z}$ and $\nu = \pm 1$. If $\Upsilon_{\mathcal{E}_{p/q},\nu}$ and $\Upsilon_{\mathcal{E}_{q/p},\nu}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[p/q, p/q + \nu\varepsilon]$ and has unbounded variation there.

These differences do not depend on Yoccoz's Brjuno function Φ, which cancels out, leaving only the conformal radii/asymptotic sizes.

Therefore H cannot be better than $1/2$-Hölder on any $[0, \varepsilon]$ and it is very likely that it holds near every p/q.

The expansion

about the functional equation

These expansions yield expansions of H at rationnals. We are able to prove that for all $\delta > 1/2$:

- If Υ and $\Upsilon_{\mathcal{E}_0}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[0, \varepsilon]$ and has unbounded variation there.
- Let $p/q \notin \mathbb{Z}$ and $\nu = \pm 1$. If $\Upsilon_{\mathcal{E}_{p/q},\nu}$ and $\Upsilon_{\mathcal{E}_{q/p},\nu}$ do not differ on $\mathbb{Q}$ by a constant, then for all $\varepsilon > 0$, H is not δ-Hölder on $[p/q, p/q + \nu\varepsilon]$ and has unbounded variation there.

These differences do not depend on Yoccoz's Brjuno function Φ, which cancels out, leaving only the conformal radii/asymptotic sizes.

Therefore H cannot be better than $1/2$-Hölder on any $[0, \varepsilon]$ and it is very likely that it holds near every p/q.

The expansion

more constants

- We will describe, if time allows, a *normalization of Fatou coordinates*, which fixes choices.
- Start from $\theta = p/q$ and perturb it into $\theta = p/q + \varepsilon$. Then the parabolic point of $P_{p/q}$ explodes into a cycle $\langle z_1, \ldots, z_q$ of $P_\theta\rangle$. Let

$$\sigma(\varepsilon) = \prod z_i.$$

Then σ is an analytic function of ε and

$$\sigma = a\varepsilon + b\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

We define:

$$\boxed{\Gamma = \frac{-1}{4i\pi q^2} \cdot \frac{b}{a}}$$

The expansion

more constants

- We will describe, if time allows, a *normalization of Fatou coordinates*, which fixes choices.
- Start from $\theta = p/q$ and perturb it into $\theta = p/q + \varepsilon$. Then the parabolic point of $P_{p/q}$ explodes into a cycle $\langle z_1, \dots, z_q$ of $P_\theta \rangle$. Let

$$\sigma(\varepsilon) = \prod z_i.$$

  Then σ is an analytic function of ε and

$$\sigma = a\varepsilon + b\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

  We define:

$$\boxed{\Gamma = \frac{-1}{4i\pi q^2} \cdot \frac{b}{a}}$$

Reminder

$$B_s = \frac{\Upsilon_{\mathcal{E}}(-\nu s)}{q} \pm \frac{\pi^2}{q} \operatorname{Re} \gamma\left(\frac{p}{q}\right) + \nu c\left(\frac{p}{q}\right)$$

For the *normalized Fatou coordinates*:

$$c\left(\tfrac{p}{q}\right) = c_{\mathrm{arith}} + c_{\mathrm{geom}}$$

$$c_{\mathrm{arith}} = -\frac{1}{q^2}\left(\sum_{k=0}^{m-1}(-1)^k\left(q_{k-1}\log\frac{1}{\alpha_k} + \frac{1}{\alpha_0\cdots\alpha_k}\right) + (-1)^m q_{m-1}\right)$$

$$c_{\mathrm{geom}} = \tfrac{2\pi}{q}\operatorname{Im}\left(\Gamma + \gamma\log 2\pi\right).$$

The numbers c_{arith}, c_{geom} and $c(p/q)$ are independent of which continued fraction of p/q we chose (i.e. independent of the sign of ν).

A normalization of the Fatou Coordinates

Blackboard!

A normalization of the Fatou Coordinates

Consider the following 1-form:

$$\omega = \mathrm{polar}\left(\frac{1}{f(z)-z} + \frac{q+1}{2z}\right)dz = \mathrm{polar}\left(\frac{f'-1}{(f-z)\log f'}\right)dz$$

An α-*petal* is a petal which is mapped by the Fatou coordinates to a real symmetric sector with opening angle $= 2\alpha$.

Theorem

As $z \longrightarrow 0$ within an α-petal ($\alpha < \pi$):

$$\phi - \int \omega \longrightarrow \mathrm{cst}$$

A normalization of the Fatou Coordinates

Let $C \in \mathcal{C}$ be the constant in

$$P^q_{p/q}(z) = z + Cz^{q+1} + \mathcal{O}(z^{q+2}).$$

then one has $a_{-1} = q\gamma$ and $a_{-q+1} = C$ in

$$\omega = (\frac{a_{-q+1}}{z^{q+1}} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \ldots)dz.$$

On a given petal, we choose the primitive $\int_0 \omega$ such that

$$\int_0 \omega = \frac{1}{qCz^q} + \frac{a_{-q}}{z^{q-1}} + \cdots \frac{a_{-2}}{z} + \gamma \log(\pm qCz^q) + 0 + o(1)$$

for the branch of $\log(\ldots)$ that is real on the axis of the petal ($\pm qCz^q$ is real on the petal).

A normalization of the Fatou Coordinates

conjugacy

The behaviour under conjugacy is to be studied: for a parabolic map f fixing 0 and one of its petal $\mathcal{P}$ denote $\Phi^{\mathrm{nor}}_{f,\mathcal{P}}$ the normalized Fatou coordinates. Let g be an analytic change of variable that fixes 0. Then

$$\Phi^{\mathrm{nor}}_{g\circ f\circ g^{-1},g(\mathcal{P})} = b + \Phi^{\mathrm{nor}}_{f,\mathcal{P}} \circ g^{-1}$$

for some constant b that does not depend on the petal and can be explicitly computed in terms of the coefficients $a_{-q+1}, \ldots, a_{-2}$ of ω and the coefficients $b_1, \ldots, b_q$ in $g(z) = b_1 z + b_2 z^2 + \ldots$. Two features:

- The normalization is invariant under linear change of coordinates: if g is linear then $b = 0$.
- If $g = f$, then $b = -1$.

What does it give on an infinitesimal level (g close to id)?

A normalization of the Fatou Coordinates

conjugacy

The behaviour under conjugacy is to be studied: for a parabolic map f fixing 0 and one of its petal $\mathcal{P}$ denote $\Phi^{\mathrm{nor}}_{f,\mathcal{P}}$ the normalized Fatou coordinates. Let g be an analytic change of variable that fixes 0. Then

$$\Phi^{\mathrm{nor}}_{g\circ f\circ g^{-1},g(\mathcal{P})} = b + \Phi^{\mathrm{nor}}_{f,\mathcal{P}} \circ g^{-1}$$

for some constant b that does not depend on the petal and can be explicitly computed in terms of the coefficients $a_{-q+1}, \ldots, a_{-2}$ of ω and the coefficients $b_1, \ldots, b_q$ in $g(z) = b_1 z + b_2 z^2 + \ldots$. Two features:

- The normalization is invariant under linear change of coordinates: if g is linear then $b = 0$.
- If $g = f$, then $b = -1$.

What does it give on an infinitesimal level (g close to id)?

A normalization of the Fatou Coordinates

conjugacy

The behaviour under conjugacy is to be studied: for a parabolic map f fixing 0 and one of its petal $\mathcal{P}$ denote $\Phi^{\mathrm{nor}}_{f,\mathcal{P}}$ the normalized Fatou coordinates. Let g be an analytic change of variable that fixes 0. Then

$$\Phi^{\mathrm{nor}}_{g\circ f\circ g^{-1},g(\mathcal{P})} = b + \Phi^{\mathrm{nor}}_{f,\mathcal{P}} \circ g^{-1}$$

for some constant b that does not depend on the petal and can be explicitly computed in terms of the coefficients $a_{-q+1}, \ldots, a_{-2}$ of ω and the coefficients $b_1, \ldots, b_q$ in $g(z) = b_1 z + b_2 z^2 + \ldots$. Two features:

- The normalization is invariant under linear change of coordinates: if g is linear then $b = 0$.
- If $g = f$, then $b = -1$.

What does it give on an infinitesimal level (g close to id)?

A normalization of the Fatou Coordinates

conjugacy

The behaviour under conjugacy is to be studied: for a parabolic map f fixing 0 and one of its petal $\mathcal{P}$ denote $\Phi^{\mathrm{nor}}_{f,\mathcal{P}}$ the normalized Fatou coordinates. Let g be an analytic change of variable that fixes 0. Then

$$\Phi^{\mathrm{nor}}_{g\circ f\circ g^{-1},g(\mathcal{P})} = b + \Phi^{\mathrm{nor}}_{f,\mathcal{P}} \circ g^{-1}$$

for some constant b that does not depend on the petal and can be explicitly computed in terms of the coefficients $a_{-q+1}, \ldots, a_{-2}$ of ω and the coefficients $b_1, \ldots, b_q$ in $g(z) = b_1 z + b_2 z^2 + \ldots$. Two features:

- The normalization is invariant under linear change of coordinates: if g is linear then $b = 0$.
- If $g = f$, then $b = -1$.

What does it give on an infinitesimal level (g close to id)?

A normalization of the Fatou Coordinates

conjugacy

The behaviour under conjugacy is to be studied: for a parabolic map f fixing 0 and one of its petal $\mathcal{P}$ denote $\Phi^{\rm nor}_{f,\mathcal{P}}$ the normalized Fatou coordinates. Let g be an analytic change of variable that fixes 0. Then

$$\Phi^{\rm nor}_{g\circ f\circ g^{-1},g(\mathcal{P})} = b + \Phi^{\rm nor}_{f,\mathcal{P}} \circ g^{-1}$$

for some constant b that does not depend on the petal and can be explicitly computed in terms of the coefficients $a_{-q+1}, \ldots, a_{-2}$ of ω and the coefficients $b_1, \ldots, b_q$ in $g(z) = b_1 z + b_2 z^2 + \ldots$. Two features:

- The normalization is invariant under linear change of coordinates: if g is linear then $b = 0$.
- If $g = f$, then $b = -1$.

What does it give on an infinitesimal level (g close to id)?

A normalization of the Fatou Coordinates

conjugacy

The behaviour under conjugacy is to be studied: for a parabolic map f fixing 0 and one of its petal $\mathcal{P}$ denote $\Phi^{\mathrm{nor}}_{f,\mathcal{P}}$ the normalized Fatou coordinates. Let g be an analytic change of variable that fixes 0. Then

$$\Phi^{\mathrm{nor}}_{g\circ f\circ g^{-1},g(\mathcal{P})} = b + \Phi^{\mathrm{nor}}_{f,\mathcal{P}} \circ g^{-1}$$

for some constant b that does not depend on the petal and can be explicitly computed in terms of the coefficients $a_{-q+1}, \ldots, a_{-2}$ of ω and the coefficients $b_1, \ldots, b_q$ in $g(z) = b_1 z + b_2 z^2 + \ldots$. Two features:

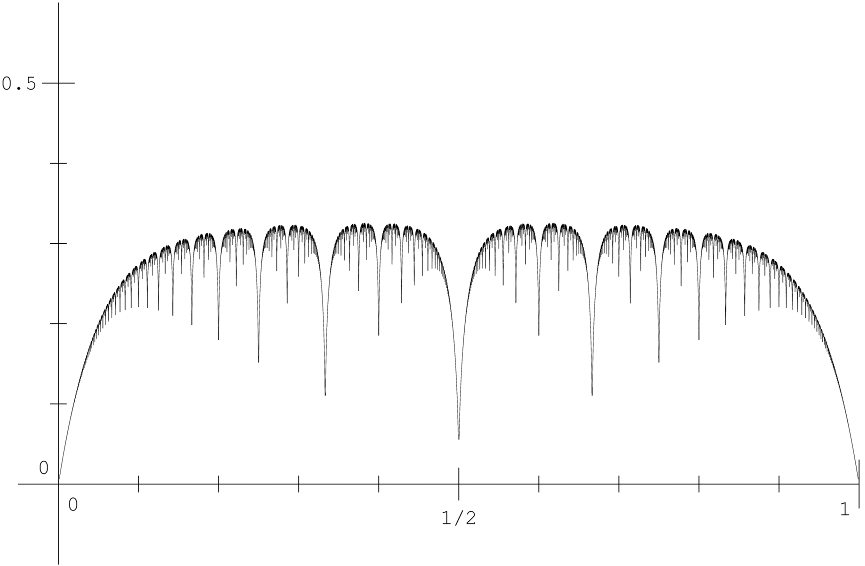
- The normalization is invariant under linear change of coordinates: if g is linear then $b = 0$.
- If $g = f$, then $b = -1$.

What does it give on an infinitesimal level (g close to id)?

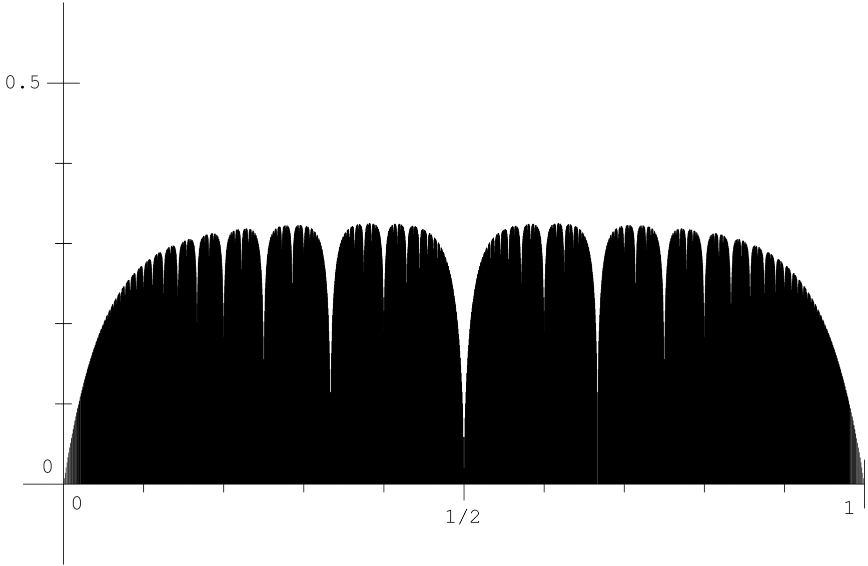
That's all

Thanks.

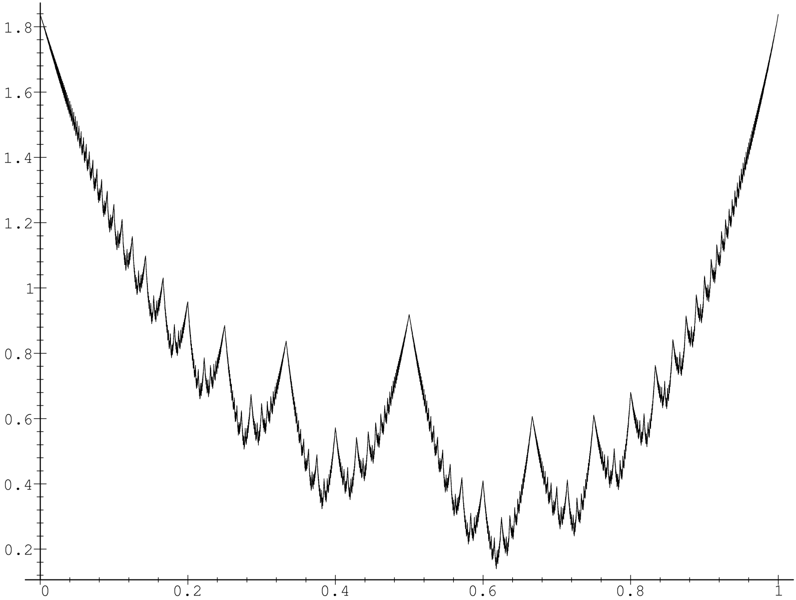
Back Closure



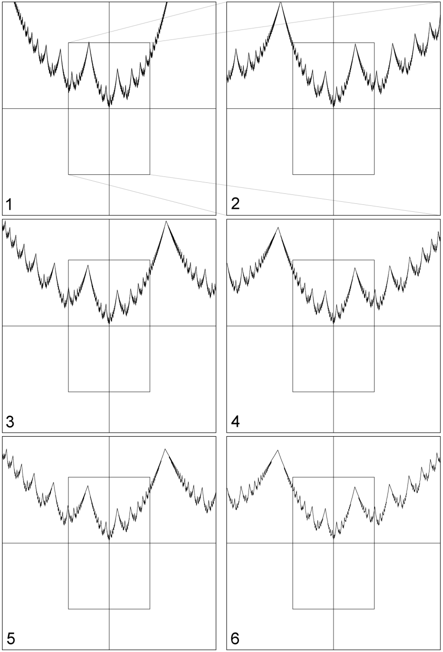
◂ Back

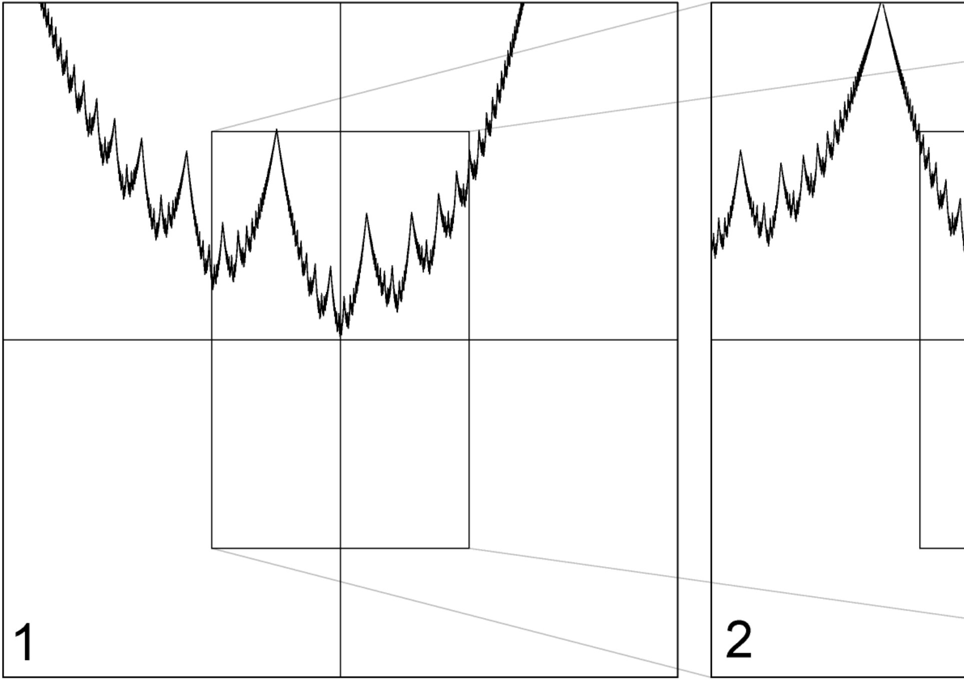


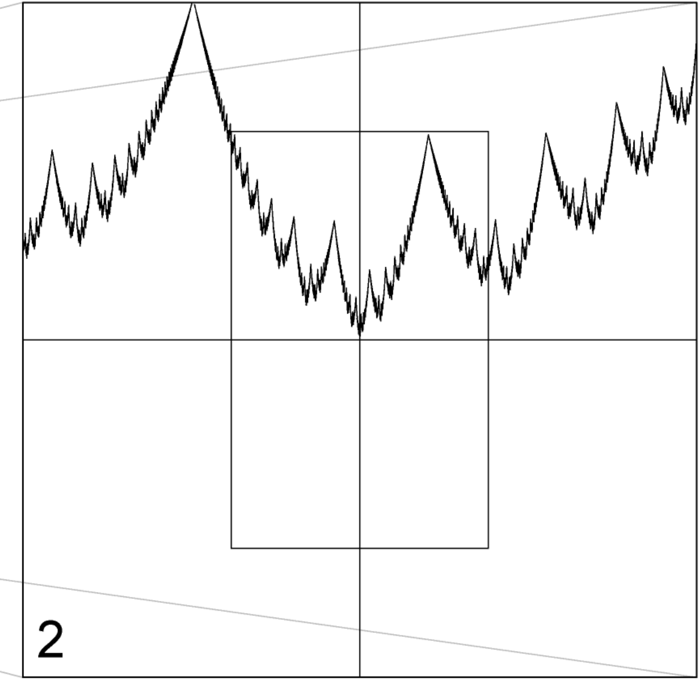
Back

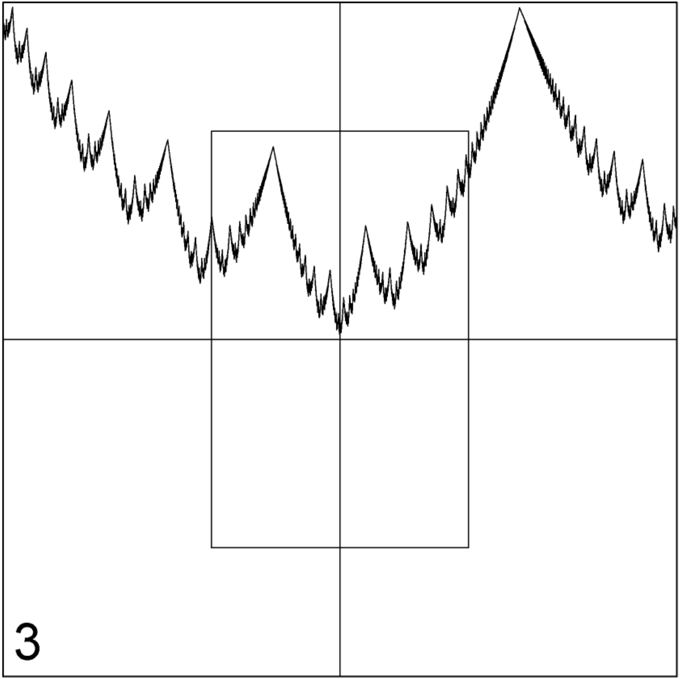


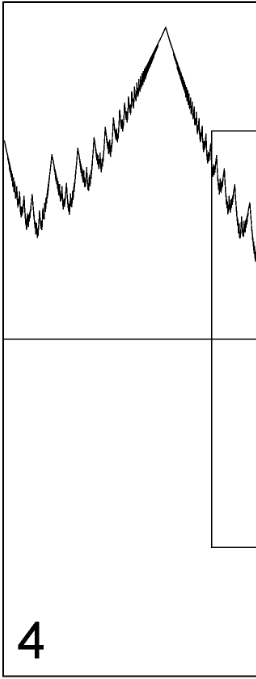
Back

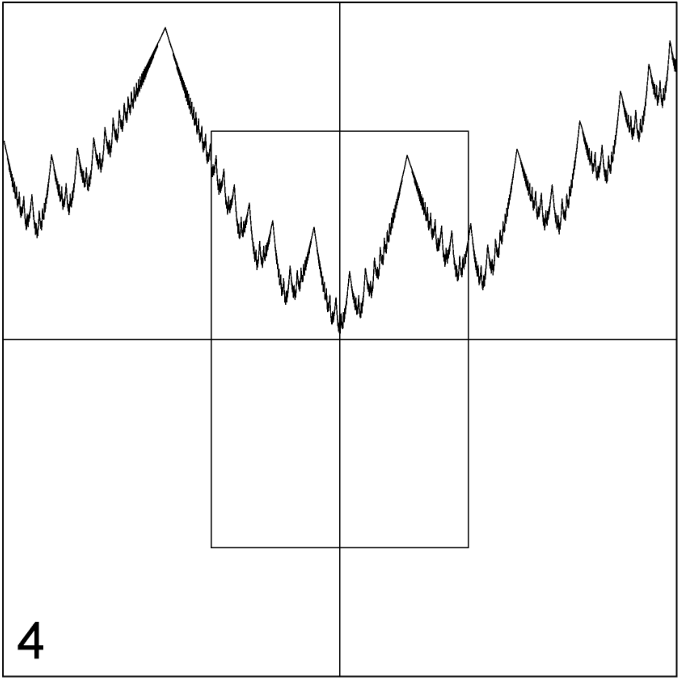


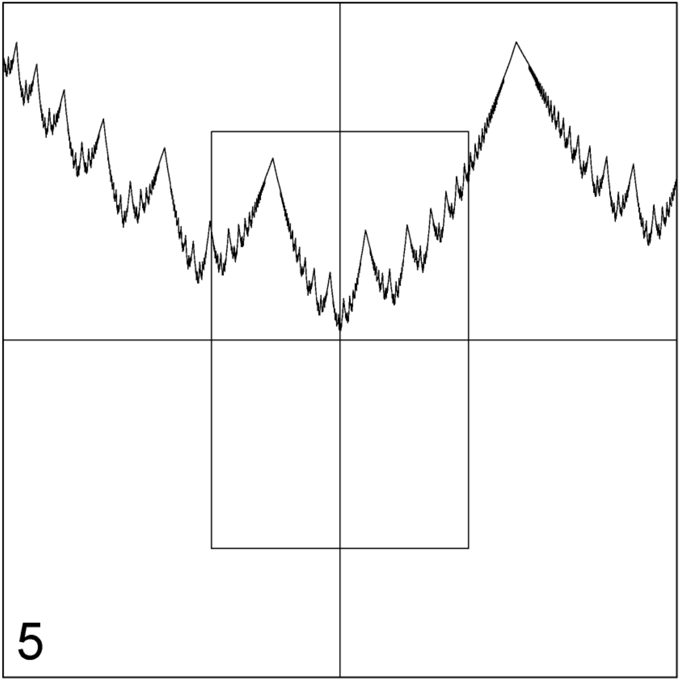

1
2

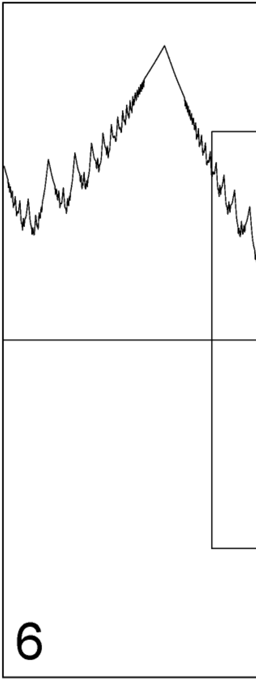

2

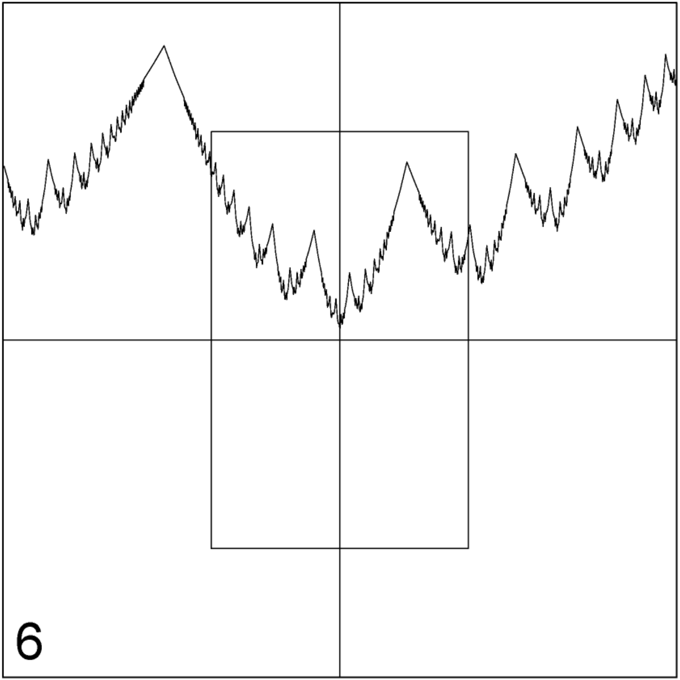

3

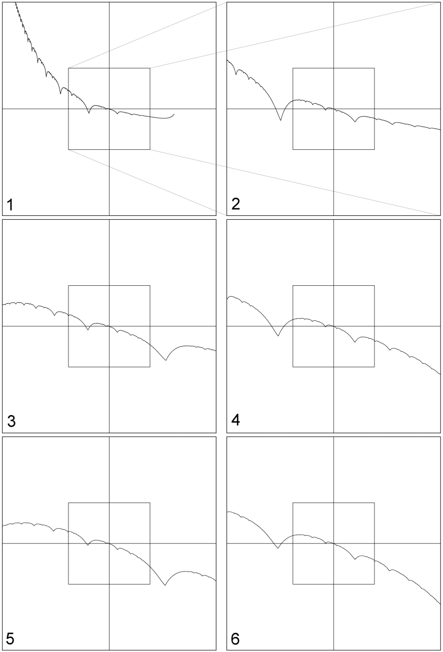

4

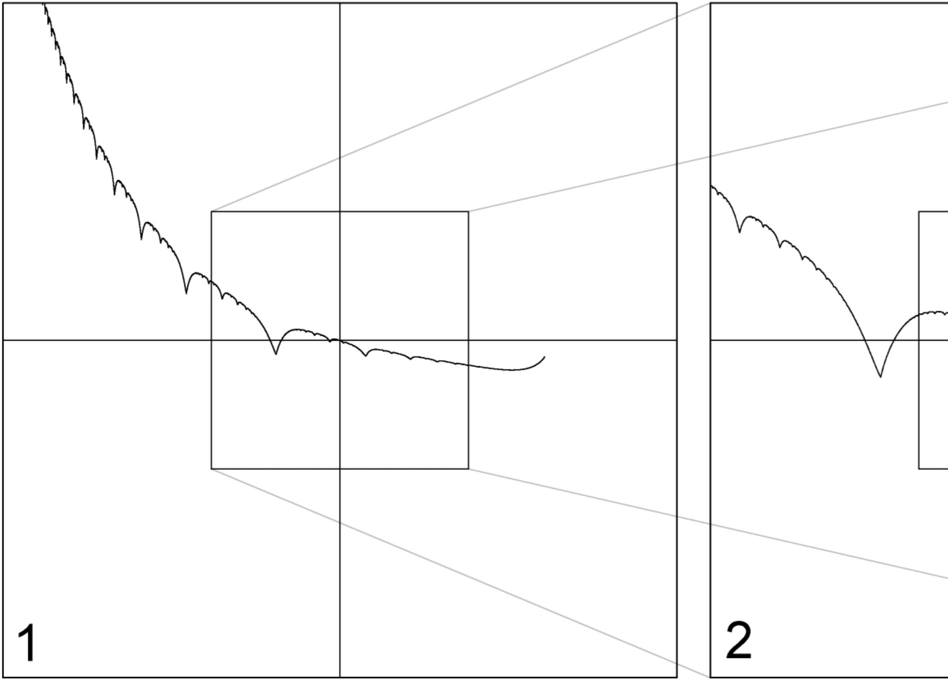

4

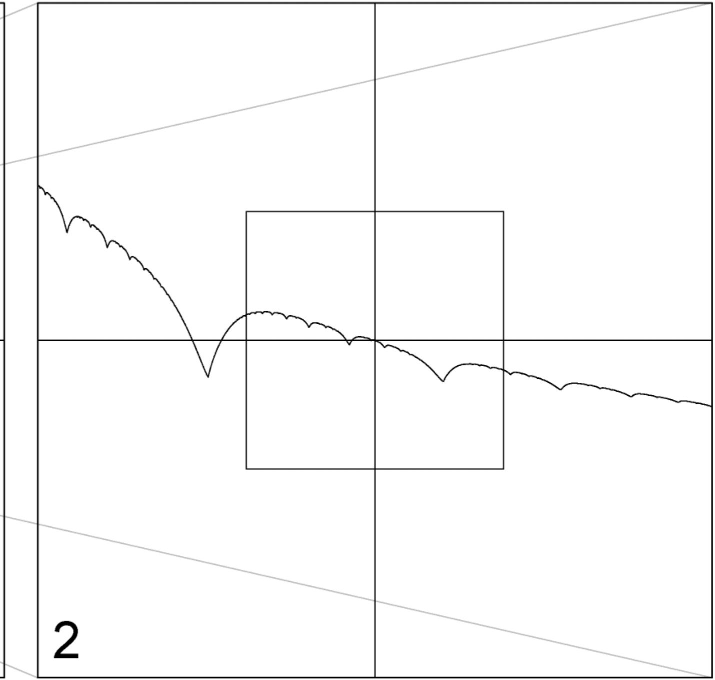

5

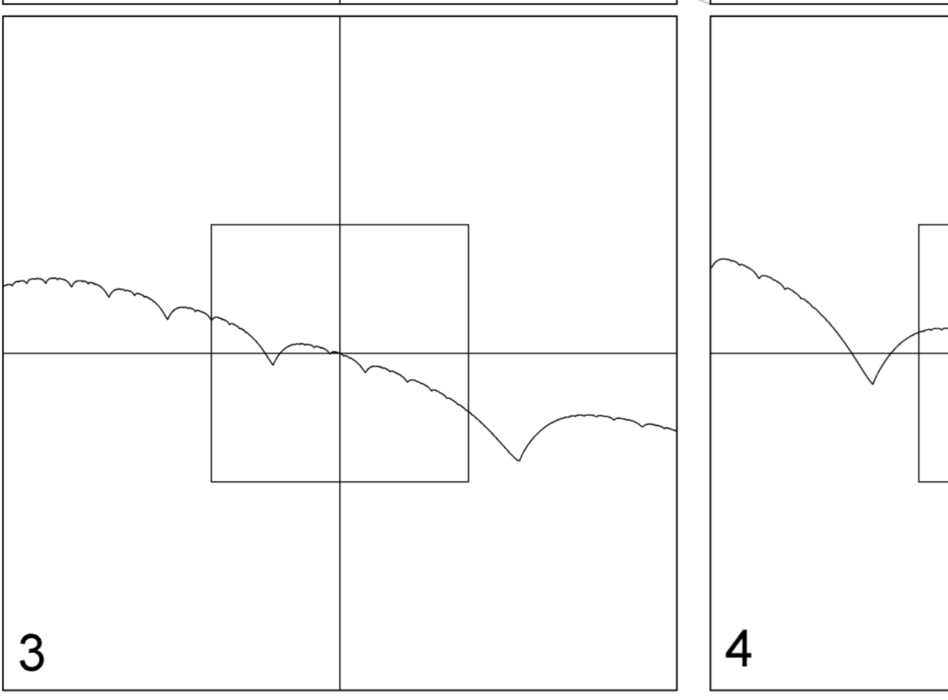

6

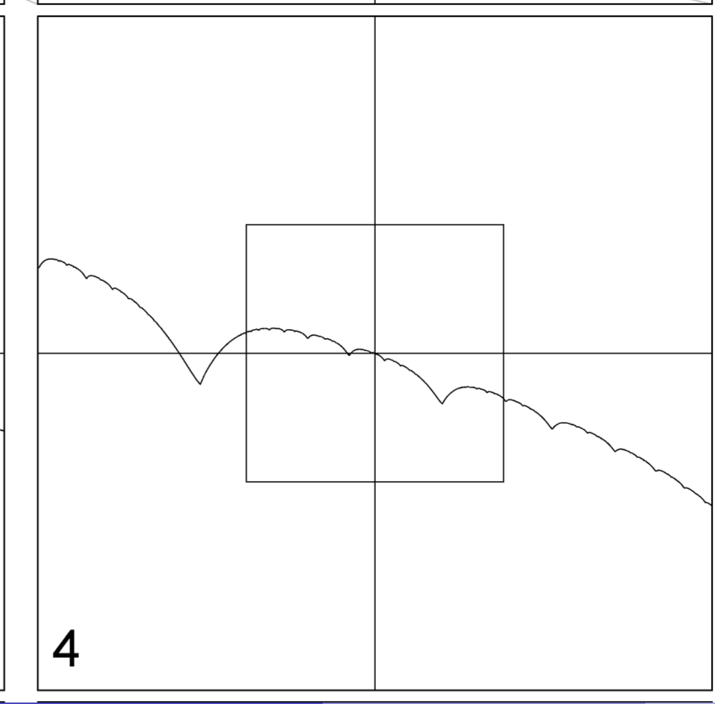

6

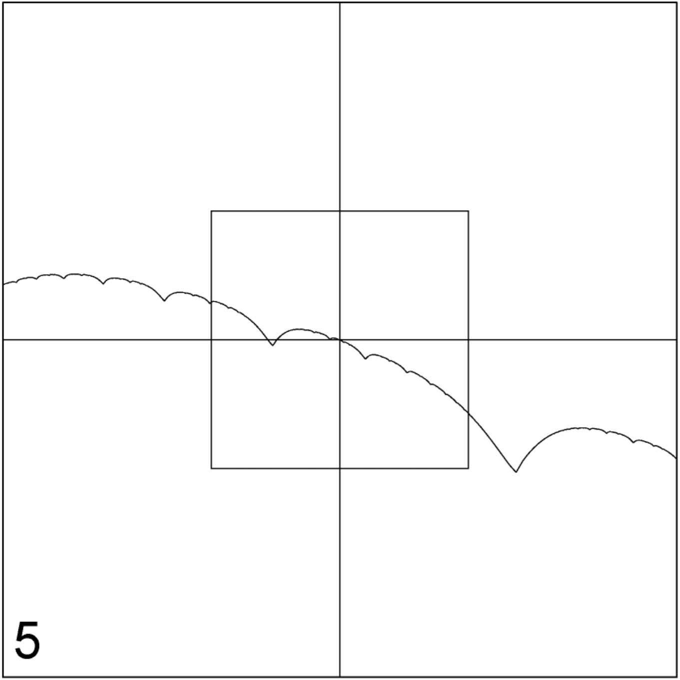

1
2
3
4
5
6

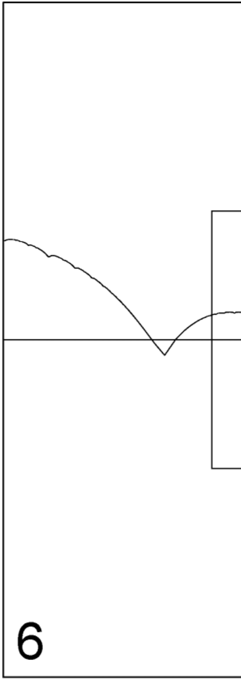

1
2

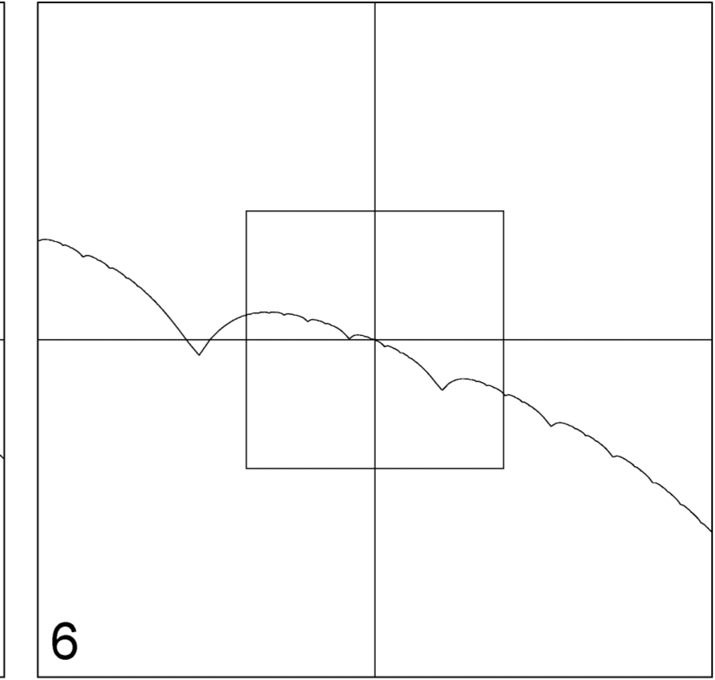

2

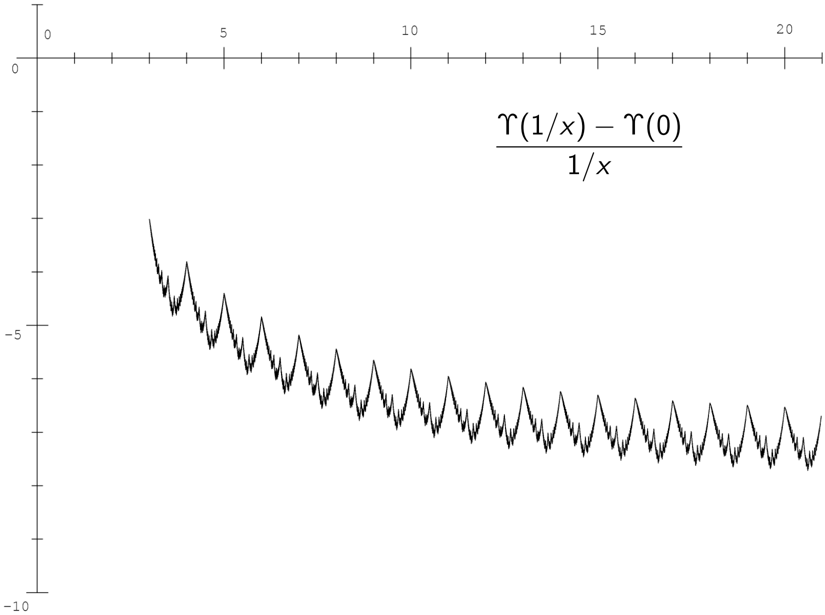

3
4


4

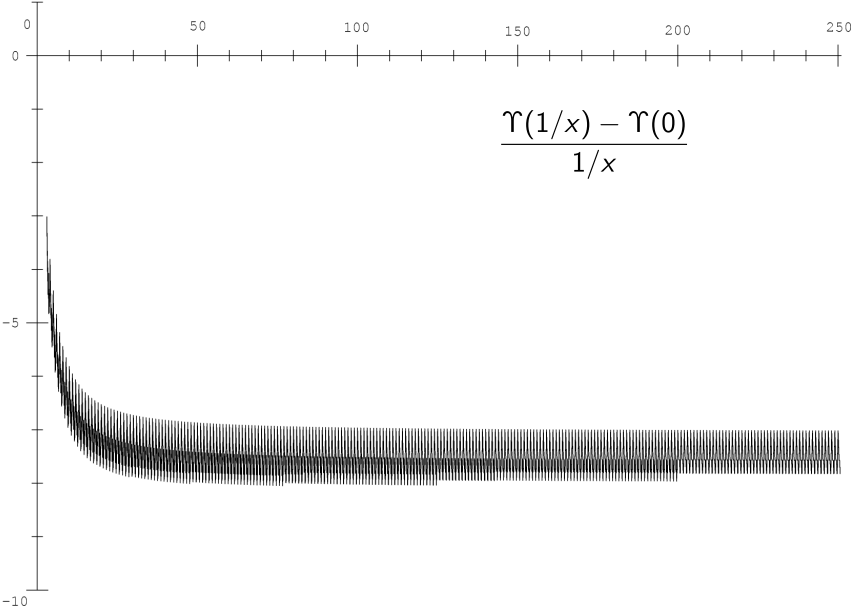

5


6


6


0
5
10
15
20
0
-5
-10
$\dfrac{\Upsilon(1/x) - \Upsilon(0)}{1/x}$

Back



$$\frac{\Upsilon(1/x) - \Upsilon(0)}{1/x}$$

$\Upsilon_{\mathcal{E}}$ is non constant

- For a family of the form $f_\theta = e^{2i\pi\theta} f_0$ with $f_0(z) = z + \mathcal{O}(z^2)$, we have

$$\Upsilon_f(0) - \Upsilon_f(1/2) = \frac{1}{2}\log|2\pi\gamma(f_0)|$$

- (Bergweiler Buff Epstein Shishikura) The horn map of a quadratic polynomial satisfies $\mathrm{Re}\,\gamma \geq 1/4$.

Back

$\Upsilon_{\mathcal{E}}$ is non constant

- For a family of the form $f_\theta = e^{2i\pi\theta} f_0$ with $f_0(z) = z + \mathcal{O}(z^2)$, we have

$$\Upsilon_f(0) - \Upsilon_f(1/2) = \frac{1}{2} \log |2\pi\gamma(f_0)|$$

- (Bergweiler Buff Epstein Shishikura) The horn map of a quadratic polynomial satisfies $\mathrm{Re}\,\gamma \geq 1/4$.

Back