A near parabolic renormalization invariant class for unicritical polynomials

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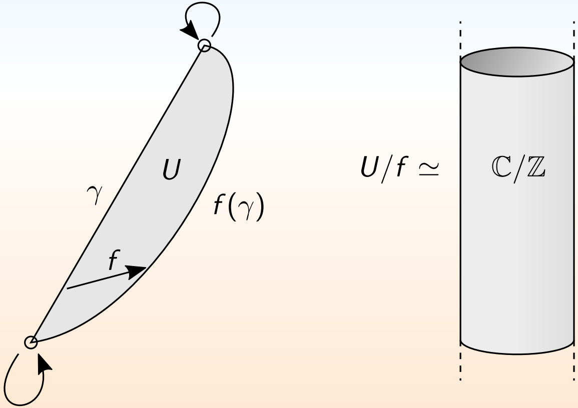
Renormalization in dynamics

Renormalization

=

First return map + Change of coordinate

f holomorphic
γ simple curve between 2 fixed points



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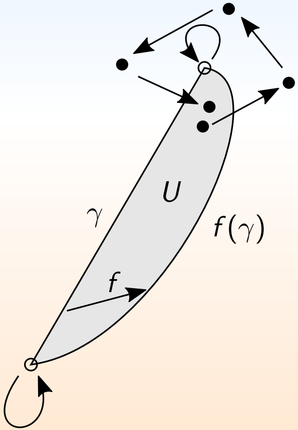
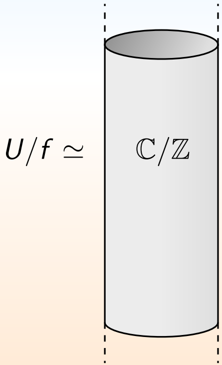


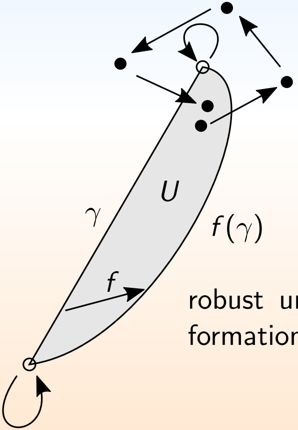


Cylinder renormalization

in complex dynamics

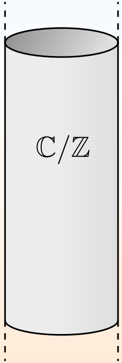
f holomorphic

γ simple curve between 2 fixed points



robust under deformations of γ

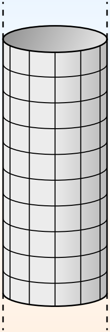
$U/f \simeq$



Cylinder

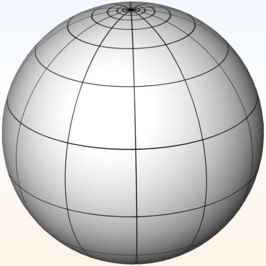
is $\mathbb{C}^*$

$+i\infty$



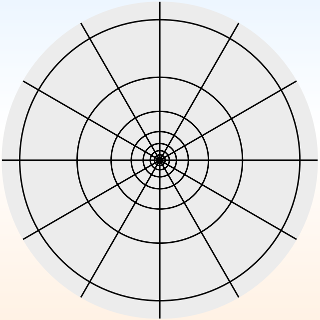
$\mathbb{C}/\mathbb{Z}$

P_N



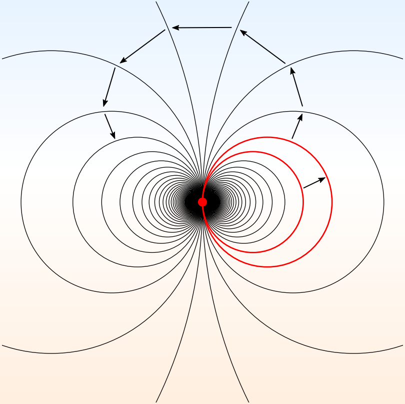
$\mathbb{S} \setminus \{P_S, P_N\}$

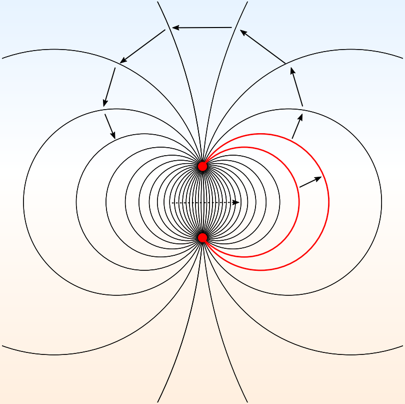
0



$\mathbb{C}^*$

Near parabolic situation





Near parabolic situation

Lavaurs, Douady, others: If $f_n \longrightarrow f$ and $f_n'(0) \longrightarrow 1$ in a controlled way, then

$$\mathcal{R}[f_n] \longrightarrow \mathcal{R}[f]$$

where

- $\mathcal{R}[f_n]$ is near parabolic cylinder renormalization and
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(I'm hiding details under the rug)

Invariant classes for renormalizations

Invariant classes usually have lots of consequences for the maps that can be infinitely renormalized, in particular:

- precise description of the long term dynamics,
- properties of invariant sets at microscopic scale.

When the renormalization operator is analytic, invariant classes often yield compact operators, so better bounds (spectral gaps, contraction up to a finite dimensional subspace, etc.).

High type numbers

Using near parabolic renormalization to study a neutral fixed point (placed at one end of γ) requires that the rotation number α be close to 0. It acts on the rotation number as the Gauss map: $\alpha \mapsto \mathrm{Frac}\,\frac{1}{\alpha}$.

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Iteration of $\mathcal{R}$ requires that all entries in the continued fraction of α be $\geq N$ for some N that depends on the invariant class under consideration.

Examples of consequences

Consequences of the invariant classes of Inou and Shishikura for near parabolic renormalization for high type numbers include:

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Consequences of the invariant classes of Inou and Shishikura for near parabolic renormalization for high type numbers include:

- Fact that the fixed point β of a quadratic polynomial is not in the boundary of the Siegel disk (Shishikura).
- Upper semi-continuity type control on the post-critical set (used in the proof of positive measure by Buff and Chéritat).
- Precise description of the postcritical set and hedgehogs, Herman's conjecture, Douady's conjecture (Cheraghi, Shishikura).
- MLC at some parameters (Cheraghi, Shishikura)
- ...

Parabolic renormalization

precise definition

For a parabolic map f fixing the origin 0, we now denote $\mathcal{R}[f]$ its *full parabolic renormalization at the upper end of the cylinder*, which we define at the end of the next few slides.

Parabolic renormalization

Fatou coordinates:

– ϕ_{att} on attracting petal P_{att} to right half plane

– ϕ_{rep} on repelling petal P_{rep} to left half plane

both are injective and satisfy $\phi(f(z)) = \phi(z) + 1$ wherever both hands are defined.

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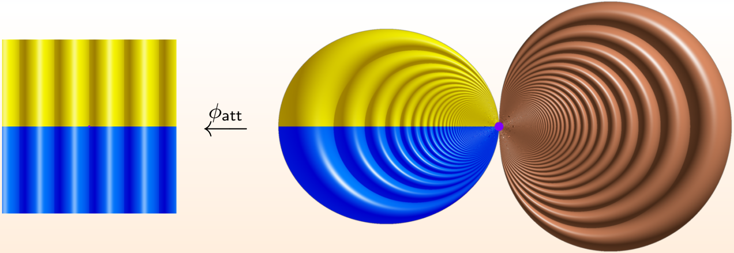

$f(z) = z + z^2$

Parabolic renormalization

Extended Fatou coordinates:

– ϕ_{att} extends into a unique function Φ_{att} such that:

$$\Phi_{\mathsf{att}} \circ f = T_1 \circ \Phi_{\mathsf{att}} \text{ (same domains)},$$

– ϕ_{rep}^{-1} extends to a unique function Ψ_{rep} such that

$$f \circ \Psi_{\mathsf{rep}} = \Psi_{\mathsf{rep}} \circ T_1 \text{ (same domains)}.$$

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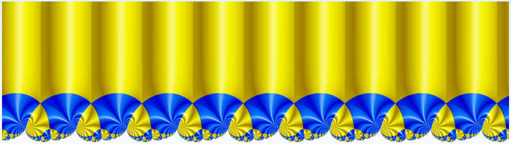
These extensions are holomorphic, not necessarily injective, the domain of Φ_{att} is the whole attracting basin of P_{att}.
If f maps its domain in itself then Ψ_{rep} is defined everywhere.

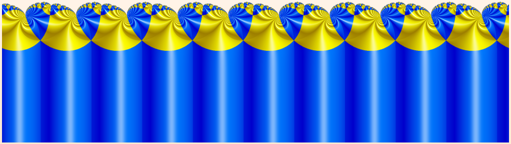
Parabolic renormalization

Dynamical chessboard

Parabolic renormalization

Structural chessboard





Parabolic renormalization

Translation : $T_\sigma(z) = z + \sigma$, $\sigma \in \mathbb{C}$.

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Parabolic renormalization: map $\mathcal{R}[f]$ such that

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completed by fixing 0, restricted to the c.c. containing 0 of its domain, with σ_0 such that $\mathcal{R}[f]'(0) = 1$.

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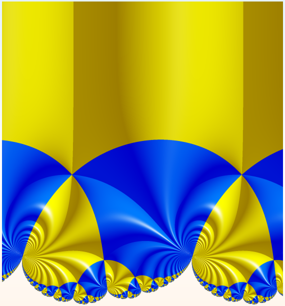
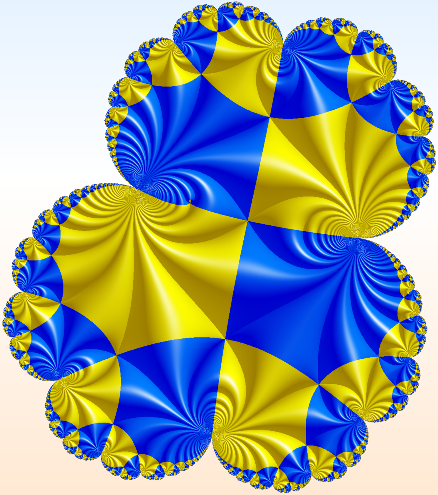
$\mathcal{R}[f]$ is the limit of cylinder renormalization $\mathcal{R}[f_n]$ of a carefully chosen sequence of perturbations f_n of f.

Structural chessboard of $\mathcal{R}[z \mapsto z + z^2]$





Unisingular parabolic Blaschke products

are unique up to Möbius conjugacy

We have

$$B_d(z) = \frac{z^d + a_d}{1 + a_d z^d}$$

with $a_d = \frac{d-1}{d+1}$, and

$$B_\infty(z) = \phi^{-1} \circ \tan \circ \phi$$

with $\phi : \mathbb{H} \to \mathbb{D}$, $z \mapsto \frac{i-z}{i+z}$.

Parabolic renormalization

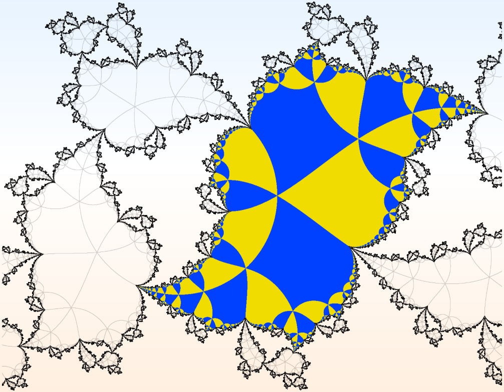
An invariant class

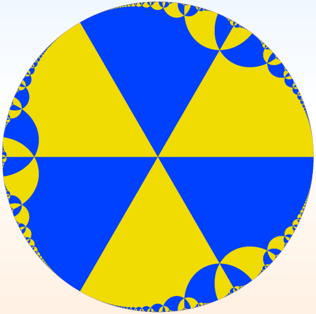
Theorem (folk?, Shishikura, Lanford-Yampolsky, others?)

Let $f : U \subset \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a holomorphic map with a parabolic petal of period one and such that one and only one singular value of f, as a map from U to $\widehat{\mathbb{C}}$, lies in the associated immediate basin A. Then the restriction of f to A is analytically conjugated to the restriction of B_d to $\mathbb{D}$ for some $d \in \{2, 3, \ldots\} \cup \{\infty\}$.

Parabolic renormalization

An invariant class





Parabolic renormalization

An invariant class

Theorem (Shishikura, Lanford-Yampolsky)

For a fixed d, all the maps in the previous situation have equivalent parabolic renormalizations in the following sense: $f_1 \sim f_2$ whenever there is a holomorphic bijection ϕ on domains such that $f_1 = f_2 \circ \phi$:

$$\begin{array}{ccc} \mathrm{dom}(f_1) & \overset{\phi}{\dashrightarrow} & \mathrm{dom}(f_2) \\ & {\scriptstyle f_1} \searrow \quad \swarrow {\scriptstyle f_2} & \\ & \widehat{\mathbb{C}} & \end{array}$$

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Let us call $\mathcal{S}_d$ the equivalence class of $\mathcal{R}[f]$ for any f as above.

Parabolic renormalization

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Maps in $\mathcal{S}_d$ as above have only one free singular value over $\widehat{\mathbb{C}}$.

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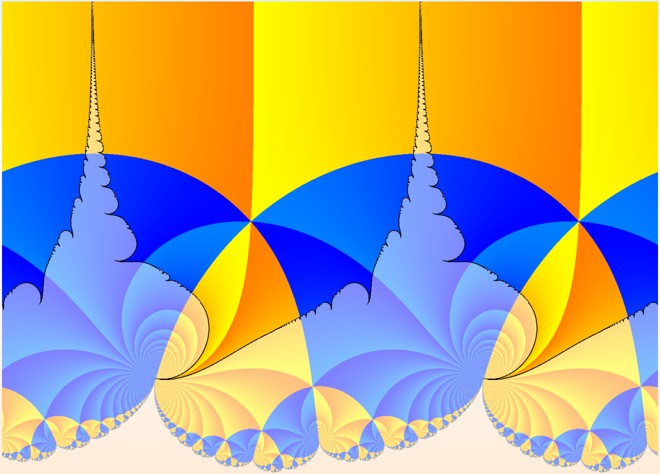
In other words:

Theorem (Shishikura, Lanford-Yampolsky)

$$\mathcal{R}[\mathcal{S}_d] \subset \mathcal{S}_d$$

Stroll



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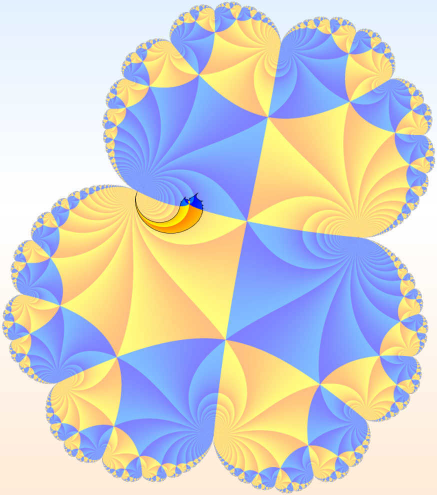

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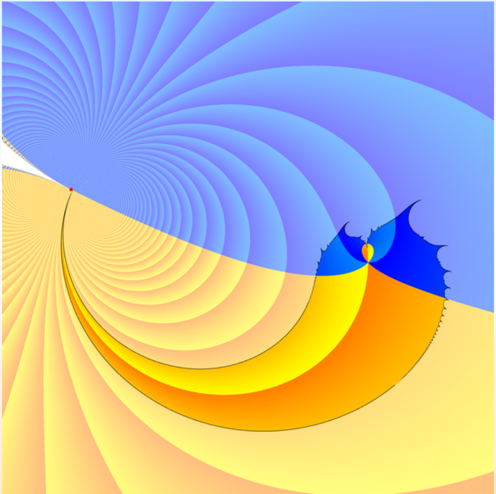

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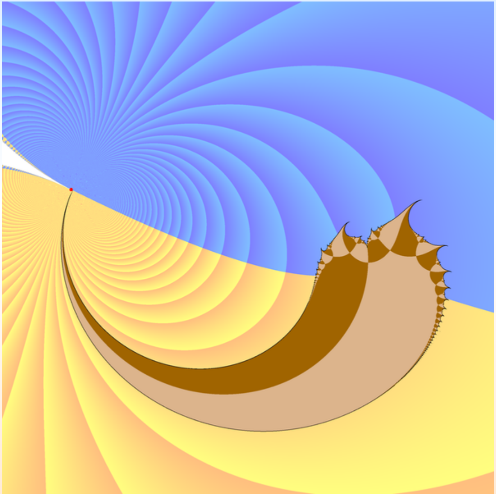

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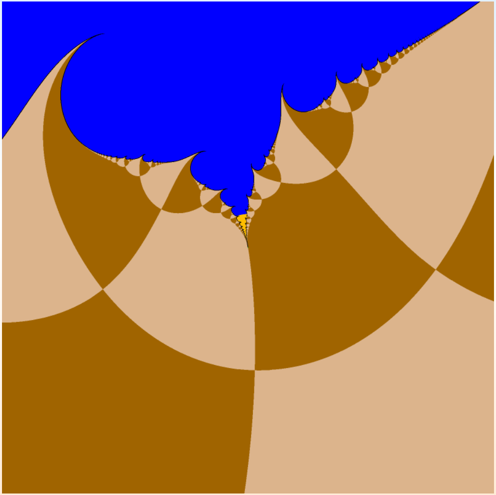

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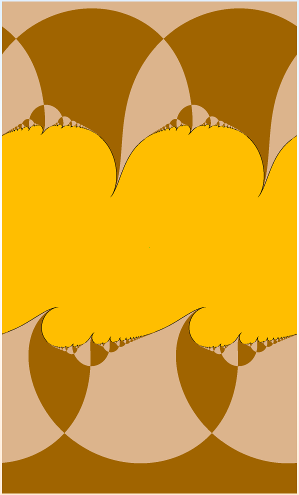

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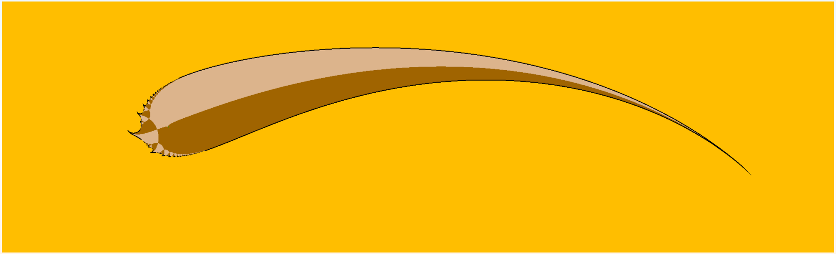

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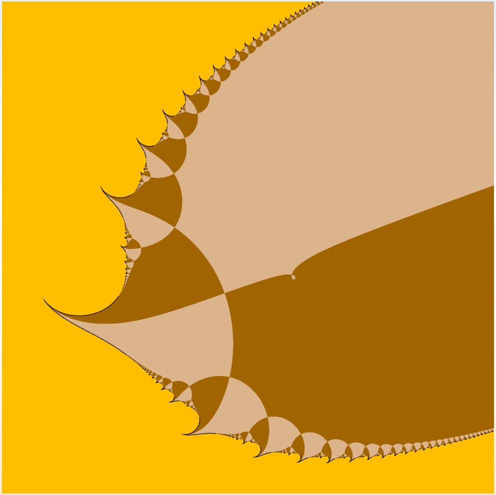
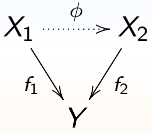


Structures

Let $f_1 : X_1 \to Y$ and $F_2 : X_2 \to Y$ be holomorphic. Let us say that the pairs f_1 and f_2 are structurally equivalent if there exists an analytic isomorphism $\phi : X_1 \to X_2$ such that $f_1 = f_2 \circ \phi$ i.e. such that the following diagram commutes:



(in the definition we should also add marked points but we do not mention them here to keep things simple).

The equivalence class of a map is called its structure.

On perturbability

Maps f in the $\mathcal{S}_d$ class have sort of a complete structure and the theorem says that parabolic renormalization of a map with the full structure also has the full structure.

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Unfortunately, this result does not withstand perturbation without modification:

If one perturbs an f that has a complete structure as f_n, for example composing with a rotation, and does near parabolic renormalization, it is not expected that the maps $\mathcal{R}[f_n]$ will have a complete structure.

Let $\mathcal{A}$ and $\mathcal{B}$ be structures and $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{B}$. If f_1 is structurally equivalent to a restriction of f_2, we say that $\mathcal{A}$ is a sub-structure of $\mathcal{B}$. If f_1 is structurally equivalent to a restriction of f_2 to a relatively compact subset of its domain, we say that $\mathcal{A}$ is relatively compact in $\mathcal{B}$.

Near parabolic renormalization

I.S. invariant class

Theorem (Inou, Shishikura)

There exists a relatively compact sub-structure $\mathcal{B}$ of $\mathcal{S}_2$ and a relatively compact sub-structure $\mathcal{A}$ of $\mathcal{B}$ such that:

- *$\forall f \in \mathcal{A}$, the map f is defined on a connected and simply connected Riemann surface and has exactly one critical point, of local degree two; the same holds for $\mathcal{B}$.*
- *For any map in $\mathcal{A}$ defined on a subset of $\mathbb{C}$ and that fixes the origin with multiplier one, its parabolic renormalization has at least structure $\mathcal{B}$.*

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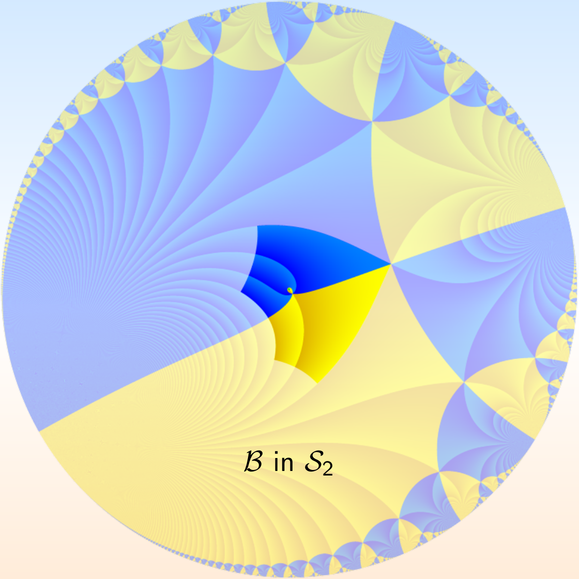
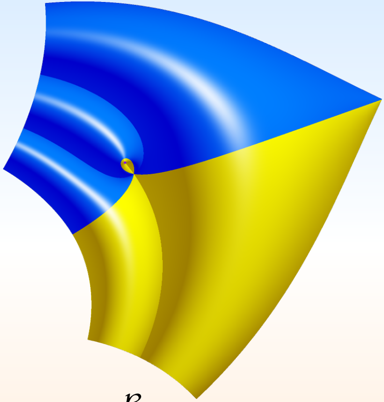
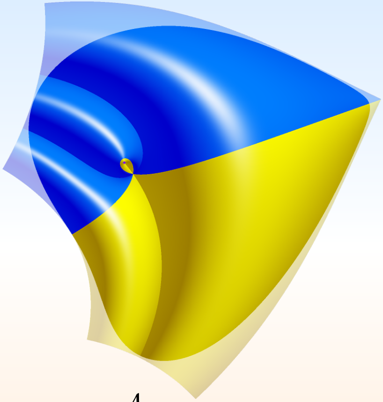
This result accommodates small perturbations, and can thus be applied to near parabolic renormalization as well.


$\mathcal{B}$ in $\mathcal{S}_2$



$\mathcal{B}$



$\mathcal{A}$

Near parabolic renormalization

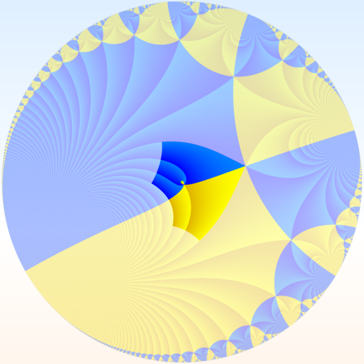
higher order critical points

Main Theorem (C.) (submitted)

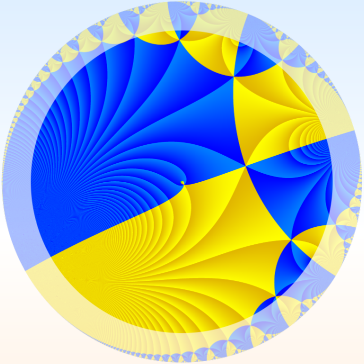
For all $1 < d < +\infty$ *here exists a relatively compact sub-structure* $\mathcal{B}$ *of* $\mathcal{S}_d$ *and a relatively compact sub-structure* $\mathcal{A}$ *of* $\mathcal{B}$ *such that:*

- $\forall f \in \mathcal{A}$*, the map* f *is defined on a connected and simply connected Riemann surface and has several critical points, all of local degree* d*, all mapping to the same point; the same holds for* $\mathcal{B}$*.*
- *For any map in* $\mathcal{A}$ *defined on a subset of* $\mathbb{C}$ *and that fixes the origin with with multiplier one, its parabolic renormalization has at least structure* $\mathcal{B}$*.*

Case $d = 2$



$\mathcal{B}$ for I.S.



$\mathcal{B}$ for us

Given $r \in]0,1[$ and a subset U of $\mathbb{C}$ conformally equivalent to $\mathbb{D}$ and containing 0, we will denote

$$U \circledcirc r = \phi(B(0,r))$$

where $\phi : \mathbb{D} \to U$ is a conformal isomorphism with $\phi(0) = 0$.

Strategy of the proof

Notations

Given $r \in]0,1[$ and a subset U of $\mathbb{C}$ conformally equivalent to $\mathbb{D}$ and containing 0, we will denote

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The domains U are bounded by inner equipotentials of U w.r.t. 0.

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Recall B_d is the unicritical parabolic Blaschke product of degree d.

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Define the classes of maps:

$$\mathcal{F}_0 = \left\{\mathcal{R}[B_d] \circ \phi^{-1} \,\middle|\, \phi : \mathbb{D} \to \mathbb{C} \text{ univalent, } \phi(z) = z + \mathcal{O}(z^2)\right\}$$

$$\mathcal{F}_\varepsilon = \left\{\mathcal{R}[B_d] \circ \phi^{-1} \,\middle|\, \phi : B(0, 1-\varepsilon) \to \mathbb{C} \text{ univalent, } \phi(z) = z + \mathcal{O}(z^2)\right\}$$

Morally, $\mathcal{F}_0 = \mathcal{S}_d$.

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Recall B_d is the unicritical parabolic Blaschke product of degree d.

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$$\mathcal{F}_0 = \left\{ \mathcal{R}[B_d] \circ \phi^{-1} \,\middle|\, \phi : \mathbb{D} \to \mathbb{C} \text{ univalent, } \phi(z) = z + \mathcal{O}(z^2) \right\}$$

$$\mathcal{F}_\varepsilon = \left\{ \mathcal{R}[B_d] \circ \phi^{-1} \,\middle|\, \phi : B(0, 1-\varepsilon) \to \mathbb{C} \text{ univalent, } \phi(z) = z + \mathcal{O}(z^2) \right\}$$

Morally, $\mathcal{F}_0 = \mathcal{S}_d$.

All maps in $\mathcal{F}_0$ are structurally equivalent.
All maps in $\mathcal{F}_\varepsilon$ are structurally equivalent.
Maps in $\mathcal{F}_0$ have the full $\mathcal{S}$-structure.
Maps in $\mathcal{F}_\varepsilon$ have less structure.

Strategy of the proof

Notations

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We will prove the main theorem with $\mathcal{A} = \mathcal{F}_{\varepsilon_0}$ and $\mathcal{B} = \mathcal{F}_{\varepsilon_1}$ for some pair $0 < \varepsilon_1 < \varepsilon_0 < 1$.

Strategy of the proof

Two steps:

- For a map in $f \in \mathcal{F}_0$, prove that the definition of $\mathcal{R}[f]$ on $\mathrm{dom}(\mathcal{R}[f]) \circledcirc (1-\varepsilon)$ uses only iteration of f on $\mathrm{dom}(f) \circledcirc (1-\varepsilon')$ where $\varepsilon' \gg \varepsilon$.
- For maps $f \in \mathcal{F}_0$, define a deformation $f_t \in \mathcal{F}_t$, $t < 1$, so that $f \mapsto f_t$ is a bijection from $\mathcal{F}_0$ to $\mathcal{F}_t$. As t increases from 0, $\mathcal{R}[f_t]$ looses structure. We prove that for $t \leq \varepsilon'/K$, $\mathcal{R}[f_t] \in \mathcal{F}_\varepsilon$, $(K > 1)$.

More precise statements below.

Step 1: how much structure is actually used

Let $\Phi[f]$ be the normalized extended attracting Fatou coordinate of f.
Let $\Psi[f]$ be the normalized extended inverse repelling Fatou coordinate.
Let $E(z) = \exp(2\pi i z)$.

Proposition (Step 1)

$\forall \varepsilon$, $\forall f \in \mathcal{F}_0$, $\Psi(E^{-1}(\operatorname{dom} \mathcal{R}[f] \circledcirc 1 - \varepsilon)) \subset \operatorname{dom}(f) \circledcirc 1 - \varepsilon'$ *with*

$$\log \frac{1}{\varepsilon'} \leq c' + c \log \left(1 + \frac{1}{\log \varepsilon}\right).$$

Reformulation

By definition

$$\mathcal{R}[f](z) = E \circ \phi_{\mathrm{att}} \circ f^n \circ \phi_{\mathrm{rep}}^{-1}(w)$$

for any $w \in E^{-1}(z)$ with $\mathrm{Re}(w)$ negative enough, and any n such that $f^n(u)$ maps $u := \psi_{\mathrm{rep}}^{-1}(w)$ from the repelling to the attracting petal of f.

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The attracting and repelling petals are both well-inside $\mathrm{dom}\, f$ and the proposition tells us that the rest of the orbit is not too close to $\partial\, \mathrm{dom}\, f$.

Reformulation

in terms of hyperbolic metric

$E^{-1}(z) = w + \mathbb{Z}$
$\Psi(E^{-1}(z)) = \left\{ u_n \,\middle|\, n \in \mathbb{Z} \right\}$ is a bidirectional orbit of f.

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The hyperbolic distance in $\mathbb{D}$ from 0 to $1 - \varepsilon$ in $\mathbb{D}$ is comparable to $\log \frac{1}{\varepsilon}$.
Hence the proposition can be reformulated as follows:

$$D' \leq c' + c \log D$$

with:
– D the $\operatorname{dom} \mathcal{R}[f]$-distance from 0 to z
– D' the biggest $\operatorname{dom} f$-distance from 0 to the orbit u_n.

Step 1

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Lemma: $\exists r_0 > 0$ *s.t.* $\forall f \in \mathcal{F}_0$, *the two main dynamical chessboard boxes of* f *in* A *are contained in* $B_{\mathrm{dom}(f)}(0, r_0)$.

Proof by compactness of the class $\mathcal{F}_0$.

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Lemma: *$\exists r_0 > 0$ s.t. $\forall f \in \mathcal{F}_0$, the two main dynamical chessboard boxes of f in A are contained in $B_{\operatorname{dom}(f)}(0, r_0)$.*

Proof by compactness of the class $\mathcal{F}_0$.

Lemma: *The orbit stays at A-hyperbolic distance $\leq L = c_1 + c_2 \log(1/\varepsilon)$ of the set of the previous lemma.*

Proof: Ψ is holomorphic from $E^{-1}(\operatorname{dom} \mathcal{R}[f])$ to A hence weakly contracts for respective hyperbolic metrics.

Step 1

Now

- the inclusion of A in $\operatorname{dom} f$ is contracting for the hyperbolic metric.
- The contraction factor is strong nearby $\partial \operatorname{dom} f$.

[see pictures]

Step 1

The actual bound is proved by introducing the box-Euclidean metric, pull-back of the cylinder metric by f.
Consider a path of A-length $\leq L$ from a point in the orbit to the set of the previous lemma.
The path is of box-Euclidean length $\mathcal{O}(L)$ because its image by f is still in A and has A-length $\leq L$ and A is a simply connected subset of the cylinder.

Step 1

Let $\mathcal{B}_m$ be the union of connected chains of length at most n of closed boxes starting from the box containing the origin.

Lemma: *$\exists m \in \mathbb{C}$ s.t. $\forall f \in \mathcal{F}_0$, the basin $A[f]$ is contained in $\mathcal{B}_m$.*

The proof is not so easy. [Picture]

On each box, there is a logarithmic gain:

Lemma: *Consider two points in a common box, the distance d_e between these two points for the box-Euclidean distance and the distance d_h between these two points for hyperbolic metric on U_1^*. Then*

$$d_h \leq c_2' + \log(1 + c_2 d_e).$$

From this we can conclude step 1.

Step 2: A perturbation

Putting back missing structure

For $f \in \mathcal{F}_0$, thus $f = \mathcal{R}[B_d] \circ \phi^{-1}$ for some Schlicht map $\phi : \mathbb{D} \to \mathbb{C}$, let

$$f_t = \mathcal{R}[B_d] \circ \phi_t^{-1}$$

with $\phi_t(z) = r_t \phi_0(z/r_t)$ and $r_t = 1 - t$. We have

$$\mathrm{dom}(f_t) = \mathrm{range}(\phi_t) = r_t \cdot \mathrm{dom}(f)$$

and $\phi_t^{-1}(z) = r_t \phi_0^{-1}(z/r_t)$ so

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Given a map $g \in \mathcal{F}_t$, there exists a unique $f \in \mathcal{F}_0$ such that $g = f_t$: f is a deformation of g with the totality of the structure. The domain of f is just the rescaled domain of g.

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Remark: Restricting is not enough: taking a map in $\mathcal{F}_0$ and restricting it to a sub-domain (and conjugating by a rescaling) would yield a *non-surjective* map from $\mathcal{F}_0$ to $\mathcal{F}_t$. In near parabolic renormalization, we need maps in $\mathcal{F}_t$ that *do not extend* to a map with the full structure.

Step 2: Following fibers

We work with a normalization of the Fatou coordinates that makes all renormalizations have the same critical value. Let

$$R : \begin{cases} \operatorname{dom} R & \to & \mathbb{C} \\ (t, z) & \mapsto & \mathcal{R}[f_t](z) \end{cases}$$

The domain of R is an open subset of $[0, 1[\times \mathbb{C}$ and R is continuous, analytic w.r.t. z for fixed values of t. (It is also analytic w.r.t. (t, z) but we will not use this fact.)

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$$R : \begin{cases} \operatorname{dom} R & \to & \mathbb{C} \\ (t,z) & \mapsto & \mathcal{R}[f_t](z) \end{cases}$$

To $z \in \operatorname{dom} \mathcal{R}[f]$, we associate a motion, which is defined using the connected component of the fiber of R that contains $(0, z)$:

Lemma: *This fiber is the graph, contained in* $\operatorname{dom} R \subset [0,1[\times\mathbb{C}$*, of a continuous map* $t \mapsto z\langle t\rangle$ *defined on* $[0,\omega(z)[$ *where* $\omega(z)$ *is called the survival time.*

This is because we work with a normalization of the Fatou coordinates so that all renormalizations have the same unique critical value. Fibers cannot undergo bifurcation, they can only disappear.

Step 2: Following fibers

To prove that $\mathcal{R}[f_t]$ has at least structure $\mathcal{F}_\varepsilon$, it is enough to prove that $\forall z \in \operatorname{dom}\mathcal{R}[f_0] \circledcirc (1-\varepsilon)$, $\omega(z) > t$.

Proposition

If the orbit associated to z is contained in $\operatorname{dom}(f) \circledcirc (1-\varepsilon')$ *then* $\omega(z) \geq \varepsilon'/K$.

(Provided ε' is small enough, independently of $f \in \mathcal{F}$ and of $z \in \operatorname{dom}\mathcal{R}[f]$.)

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The whole orbit u_n associated to z also undergoes a motion and becomes an orbit $u_n\langle t\rangle$ of f_t that still tends to 0 in the future and in the past: we are fixing its normalized attracting Fatou coordinate.

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To bound the motion of u_n we look at the *homotopic length* of the path $t \mapsto u_n\langle t\rangle$ for the hyperbolic metric on the set

$$W_0 = \mathbb{C} \setminus \overline{PC}(f_0).$$

or on the set

$$\mathbb{C} \setminus \{0, 1\}$$

where 1 is the critical value of f_0.

Step 2: Following fibers

The control on the homotophic length ℓ w.r.t W_0 is done by a backward induction on $n \in \mathbb{Z}$.

Each curve $u_{n-1}\langle[0, t_{\max}]\rangle$ is, under good conditions, homotopic to the concatenation $\gamma_1 \cdot \gamma_2$ where:
– γ_1 is the pull-back of $u_n\langle t\rangle$ by f_0 starting from $u_{n-1}\langle 0\rangle$,
– γ_2 is a correcting curve defined by $f_t(\gamma_2(t)) = f_0(\gamma_1(t_{\max}))$.

Under good conditions:
– $\ell(\gamma_1) \leq \ell(u_n)$ with $\lambda < 1$ independent of f_0 and n.
– $\ell(\gamma_2) \leq Kt$ for some $K > 0$.

Step 2: Following fibers

In reality it is a bit more complicated.

The orbit u_n for $t = 0$ is cut in chunks.

- 1st chunk: in the repelling petal for all n negative enough
- intermediate chunks: between n and $n + 1$ when u_{n+1} not in the repelling petal, between n and $n + k + 1$ when in the repelling petal from n to $n + k$,
- final chunk : when in the attracting petal or close to the critical orbit (here we replace the hyperbolic metric of W_0 by that of $\mathbb{C} \setminus \{0, 1\}$ in the attracting petal).