Siegel disks with non locally connected boundaries

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The definition of a Siegel disk may vary with the author.

For us, it will be the maximal domain of (analytic) conjugacy to an aperiodic rotation on a disk or on the plane, of a one complex dimensional holomorphic dynamical system $z_{n+1} = f(z_n)$.

So it depends on the germ of f but also on its domain of definition.

The dynamics in the Siegel disk is conjugated to a unique rotation $R_\theta(z) = e^{2\pi i\theta} z$.

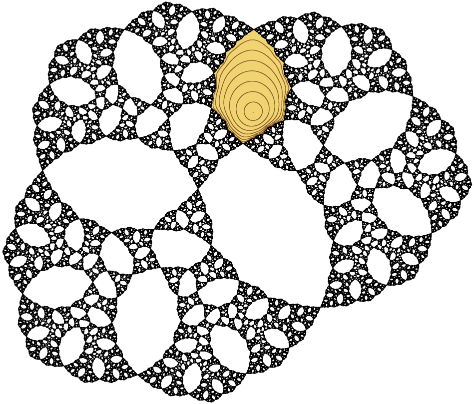
Aperiodicity means $\theta \notin \mathbb{Q}$.

The number θ, unique modulo $\mathbb{Z}$, is called the *rotation number*.

θ has *bounded type* if and only if its continued fraction has bounded entries.

Consider a rational map with a Siegel disk.

Then the Siegel disk is also the component of the Fatou set containing the center.



Siegel disks are often Jordan domains

Herman, Swiatek: a bounded type Siegel disk of a degree 2 polynomial is a Jordan domain (it is a quasidisk).

Shishikura (unpublished): it also holds for any degree.

Zhang, Gaofei (preprint): the same holds for rational maps.

Petersen and Zakeri: for a set of rotation numbers of full Lebesgue measure, the Siegel disk of degree 2 polynomials is a Jordan domain.

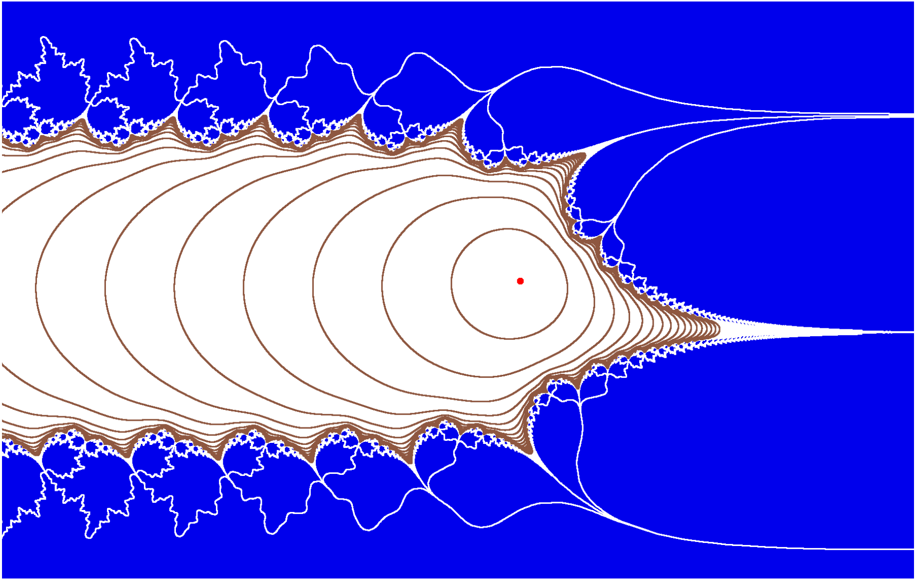
Inou and Shishikura: if all the entries of the continued fraction are bigger than some universal constant, then the Siegel disk of the degree 2 polynomial is a Jordan domain.

Conjecture: All rational Siegel disks are Jordan domains.

There are examples of Siegel disks that have non locally connected boundaries but they are not compactly contained in the domain:

On the next slide : the Siegel disk of $z \mapsto \exp(z) + \kappa$ with κ chosen so that there is an irrationally indifferent fixed point with rotation number equal to the golden mean.

An abysmal monster



We will now restrict to Siegel disks compactly contained in the domain of definition of the dynamical system.

Perez-Marco had constructed many examples of strange Siegel disks, like some with smooth boundaries. Kingshook Biswas continued his work.

All Siegel disk examples we were able to build using Perez-Marco's construction had a Jordan curve boundary, so I was thinking that maybe all compactly contained Siegel disks were Jordan domains:

Question: *Are all compactly contained Siegel disks Jordan domains?*

Remark: A compactly contained Siegel disk which is not a Jordan domain has necessarily a non locally connected boundary.

A *continuum* is a compact connected non empty metric space.

In 1921, Knaster introduced an interesting example of continuum in the plane. It is indecomposable and all subcontinua are indecomposable too. It was later called the *pseudo arc*.

A continuum X is *indecomposable* if whenever $X = C_1 \cup C_2$ with C_2 and C_1 continua (necessarily $C_1 \cap C_2 \neq \emptyset$) then $C_1 = X$ or $C_2 = X$.

It is not obvious to think of an indecomposable continua that be bigger than a single point. Such objects are necessarily non locally connected.

In the late 40's, Bing and Moise studied it further and proved it is uniquely characterized by some set of topological properties.

The *pseudocircle* is a variation on the pseudo arc. It is also indecomposable and topologically unique.

A *circular chain* is a family $\mathcal{C}$ of subsets U_k of $\mathbb{C}$, indexed by $k \in \mathbb{Z}/m\mathbb{Z}$ for some $m > 0$, and such that different links intersect if and only if they are consecutive.

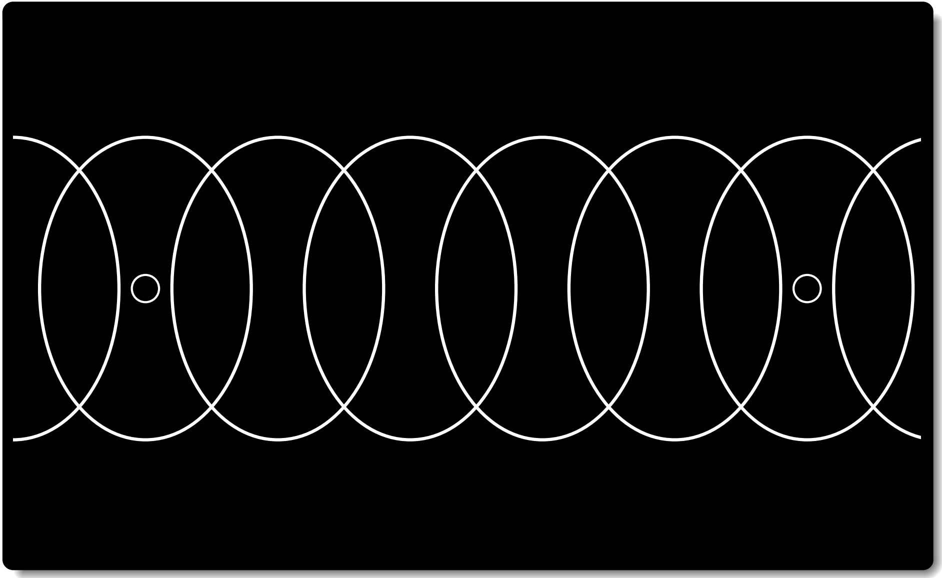
Its *support* is the union of the U_k.

A pseudocircle in the plane is the intersection of the support of a sequence of chains $\mathcal{C}_n$ such that $\mathcal{C}_{n+1}$ is *crookedly embedded* in $\mathcal{C}_n$ and such that the diameter of the links tends uniformly to 0 as n tends to $+\infty$.

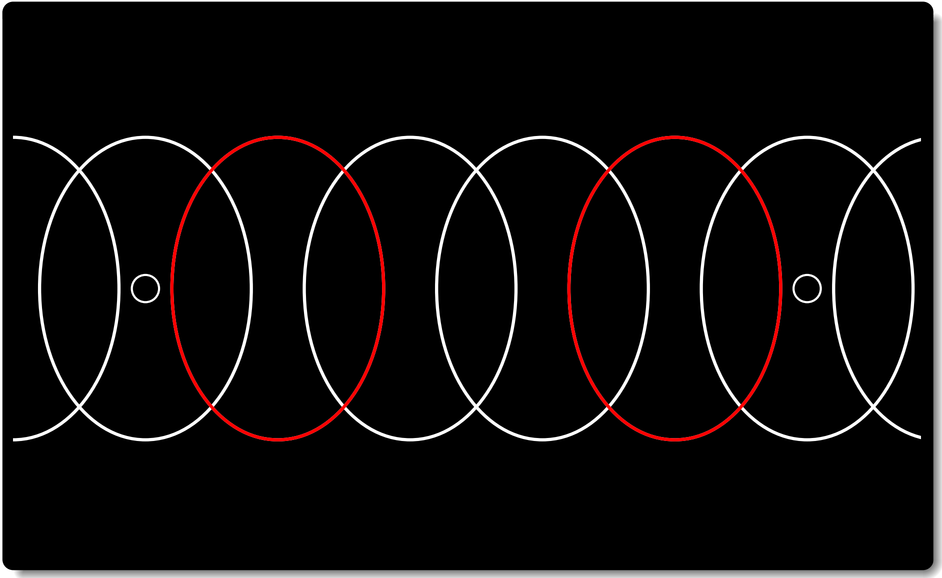
The definition of crookedly embedded is given in the next slide.

Note: A pseudocircle always separates the plane into two connected components.

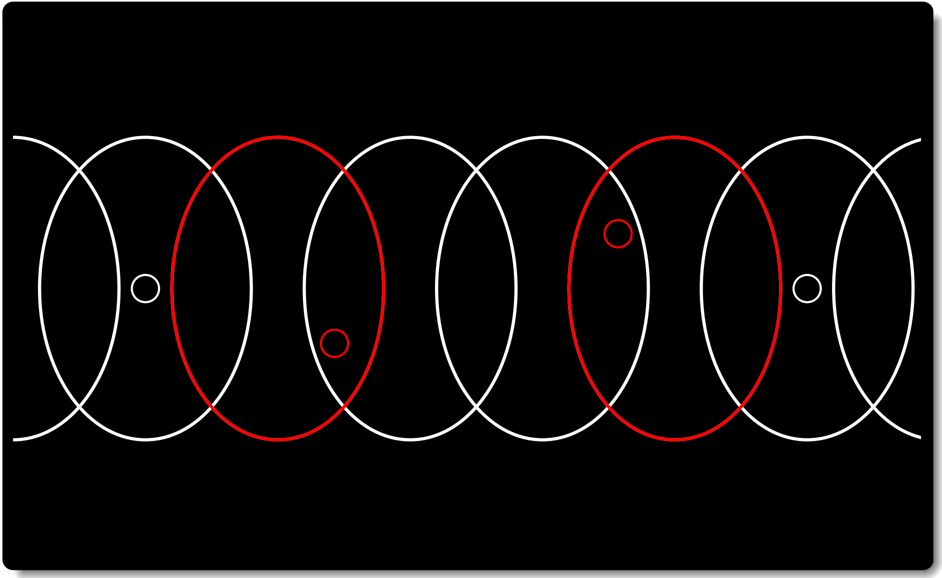
Crookedly embedded chain

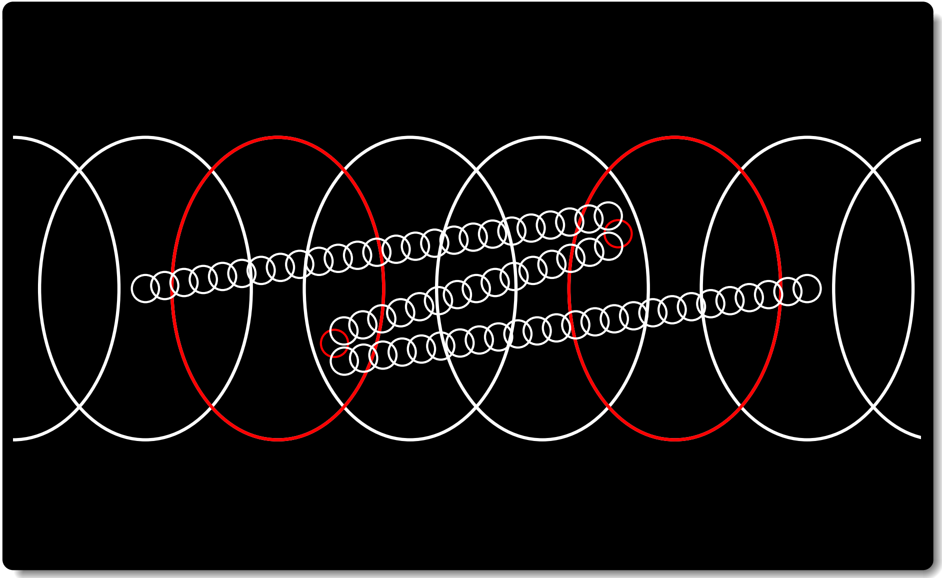


Crookedly embedded chain

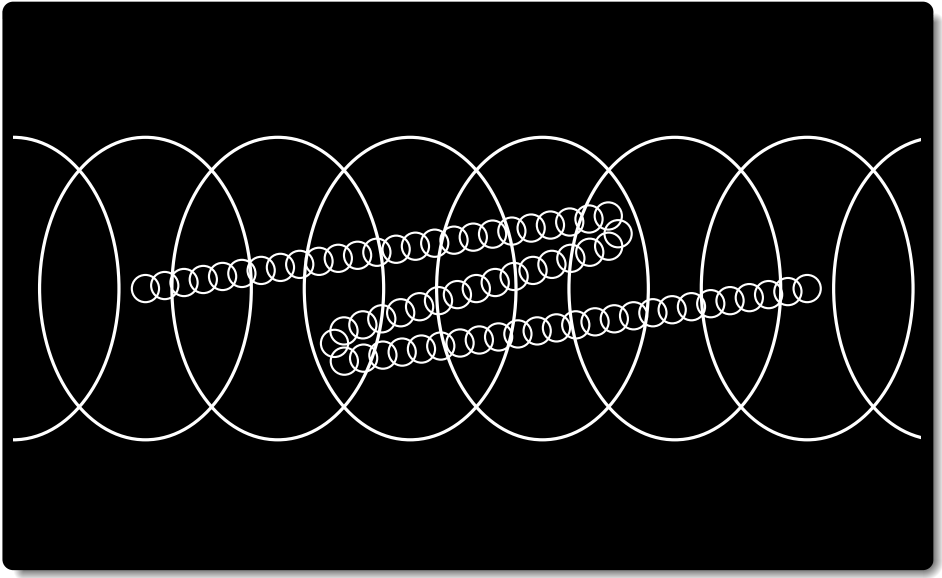


Crookedly embedded chain

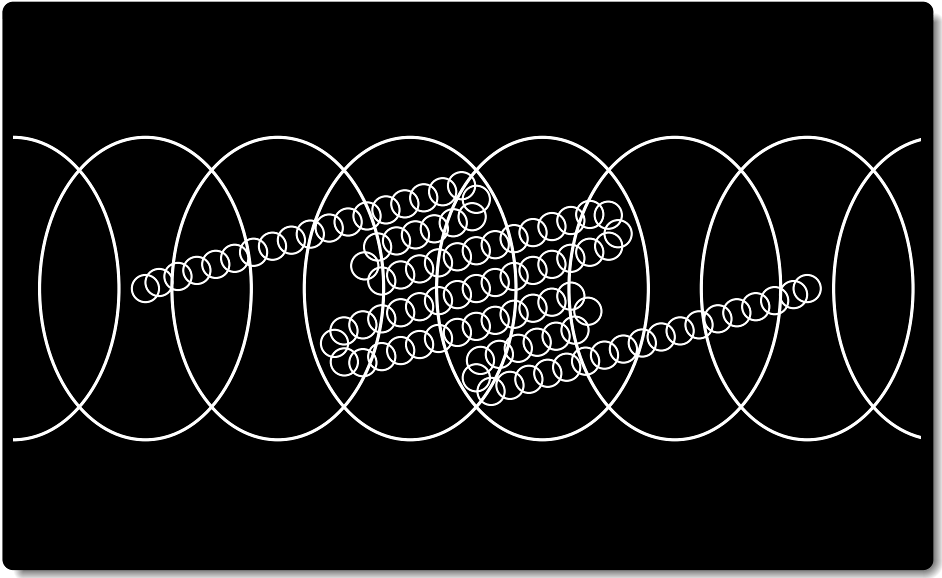


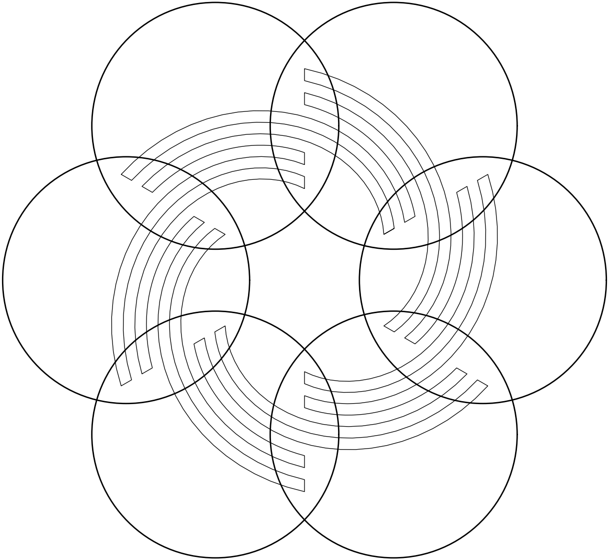


Crookedly embedded chain

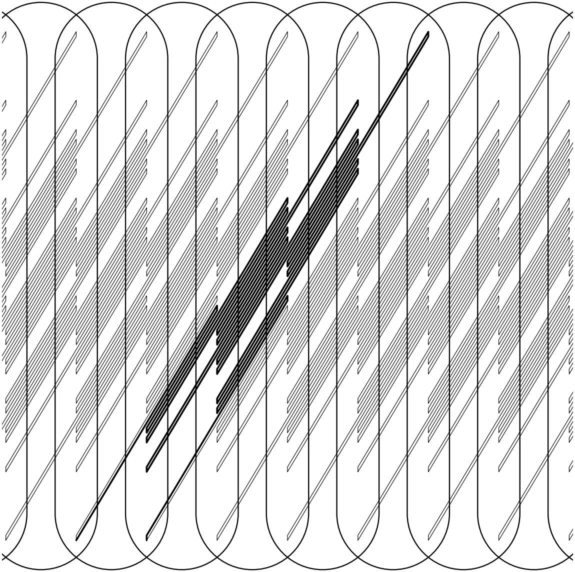


Crookedly embedded chain





Crooked in an 8 links circular chain, unfolded



Handel constructed in 1981 an example of an area preserving C^∞ diffeomorphism f of $\mathbb{R}^2$ with an invariant pseudocircle K on which f is minimal (there is no invariant compact subset of K besides K and $\emptyset$).

Herman constructed in 1984 an example of an C^∞ diffeomorphism f of the Riemann sphere $\mathbb{S}$ with an invariant pseudocircle K and $\mathbb{C}^\infty$ conjugate to rotations of opposite rotation numbers on the two components of $\mathbb{S} \setminus K$, and that is holomorphic on one of the two components of $\mathbb{S} \setminus K$.

Theorem

For all $R > 1$ there exists an injective holomorphic map $f : B(0, R) \to \mathbb{C}$ fixing the origin with a Siegel disk contained in $B(0, 1)$ and whose boundary is a pseudocircle.

(Work in progress)

Theorem

For all $R > 1$ there exists an injective holomorphic map $f : B(0,R) \to \mathbb{C}$ fixing the origin and whose restriction to $B(0,1)$ has a Hedgehog of empty interior and of positive measure.

(Work in progress)

Theorem

For all $R > 0$ there exists an injective holomorphic map on the annulus $f : B(1/R, R) \to \mathbb{C}$, with an invariant Jordan curve carrying an invariant line field.

Herman's construction is an instance of the Anosov-Katok method.

$$f = \lim_{n\to\infty} H_n^{-1} \circ R_{\alpha_n} \circ H_n$$

where $R_\alpha(z) = e^{2\pi i\theta} z$ and α_n is rationnal:

$$\alpha_n = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_2 \ldots q_n}$$

and H_n is a C^∞ diffeomorphism of the sphere of the form

$$H_n = h_n \circ \cdots \circ h_1$$

where h_n are C^∞ diffeomorphisms of the sphere such that

$$h_n \circ R_{1/q_1\cdots q_{n-1}} = R_{1/q_1\cdots q_{n-1}} \circ h_n$$

The q_n and the h_n are chosen in turns : first h_1, then q_1, then h_2, then q_2...The sequence H_n *does not converge*. h_n can be chosen as any diffeomorphism of the sphere commuting with $R_{\alpha_{n-1}}$, then by choosing q_n big enough, we can ensure that f_n is close to f_{n-1} in C^∞:

$$f_n = \left(H_n^{-1} R_{1/q_1\cdots q_n} H_n\right) \circ f_{n-1}$$

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Explain Herman's construction on a blackboard.

Theorem (Runge, polynomial version, 1885?)

If f is a holomorphic function defined in an open subset U of $\mathbb{C}$ with connected complement then there exists a sequence of polynomials $P_n \in \mathbb{C}[z]$ that converge to f uniformly on compact subsets of U.

We will use the following variant of the Anosov-Katok construction: let

$$f_n = g_n \circ f_{n-1}$$

where g_1 is the time $1/q_1 \cdots q_n$ map of a vector field X_n that is obtained as the pull-back of the rotation vector field $dz/dt = 2\pi iz$ by an *entire map* $H_n = h_n \circ \cdots \circ h_1$ with $h_n'(z) \neq 0$.

Runge's theorem will be used to get appropriate functions h_n.

Lemma

Let $\phi : U \to \mathbb{D}$ be a conformal isomorphism with $\phi(z) = z + \mathcal{O}(z^2)$ that commutes with $R_{1/q}$. Then there exists a sequence ϕ_n such that:

- $\phi_n \longrightarrow \phi$ *uniformly on compact subsets of* U
- ϕ_n *is an entire map*
- ϕ_n' *does not vanish*
- $\phi_n(z)$ *commute with* $R_{1/q}$
- $\phi_n(z) = z + \mathcal{O}(z^2)$

Proof on the blackboard.

What about the rotation number?

What about Siegel disks of entire maps?