Deepness of the boundary of bounded type quadratic Siegel disks following C. McMullen

At the advent of the 70' aniversary of Adrien Douady

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Introduction

This will be a talk in the Bourbaki tradition. All the results I will be presenting are due to Curt. T McMullen.
They can be found in the paper:

Self-similarity of Siegel disks
and Hausdorff dimension of Julia sets.
Acta Math. Vol 180, 1998.

Basic definitions

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- $P_\theta(z) = \mathrm{e}^{i2\pi\theta} z + z^2 : \mathbb{C} \to \mathbb{C}$, $\theta \in \mathbb{R}$.

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- For $\theta \in \mathcal{B}$ the ϵ-neighbourhood $\Delta_\theta(\epsilon)$ of Δ_θ and

$$K_\theta(\epsilon) = \{z \in \mathbb{C} | \forall n : P_\theta^n(z) \in \Delta_\theta(\epsilon)\}$$

Main Theorem

Theorem 1 (McMullen).

For every bounded type $\theta \in BT$ *and for every* $\epsilon > 0$ *the boundary* $\partial\Delta_\theta$ *is uniformly measureably deep in* $K_\theta(\epsilon)$.

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That is $\exists\ \alpha, C > 0$ depending on θ and ϵ such that

$$\forall\ z \in \partial\Delta_\theta, \forall r \leq 1 : \mathrm{Area}(B_r(z) \backslash K_\theta(\epsilon)) \leq C r^{2+\alpha},$$

where $B_r(z)$ denotes the euclidean ball of center z and radius r.

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where $B_r(z)$ denotes the euclidean ball of center z and radius r.

In particular every point of $\partial\Delta_\theta$ is a Lebesgue density point of $K_\theta(\epsilon)$.

Auxillary definitions 1

Definition 2. *A point* $z_0 \in \Lambda \subset \mathbb{C}$*, where* Λ *is a compact subset, is called a* deep *point of* Λ*, iff*
$\exists\, \delta > 0$ *and* $\exists\, C > 0$ *such that* $\forall r \leq 1$*:*

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Definition 3. *A compact subset* $\Lambda \subset \mathbb{C}$ *is called* porous *iff*
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We also say that Λ *is* C *porous.*

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The two notions deep and porous are in some sense opposite of each other.

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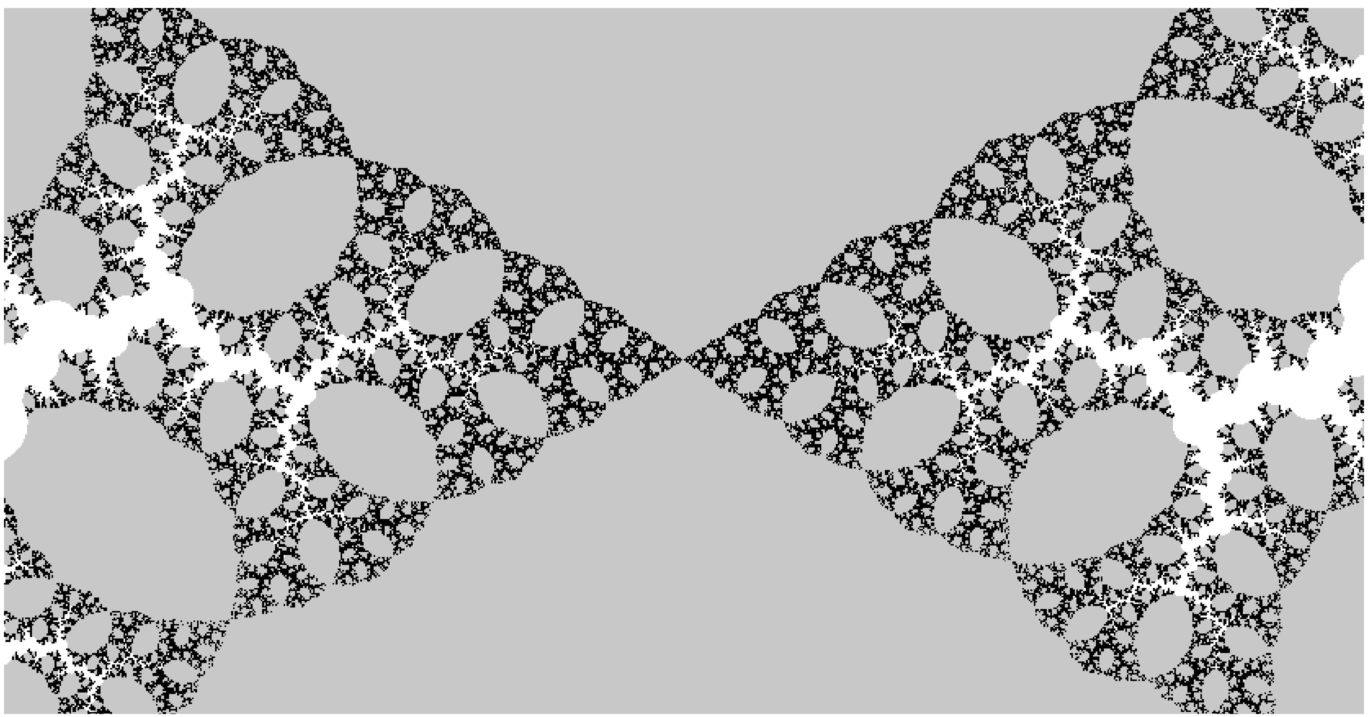

General definitions

Definition 4. *For $\Lambda \subset \mathbb{C}$ a compact subset. The upper box-dimension of Λ is the real number*

$$\overline{\dim}_{\text{box}}(\Lambda) := \limsup_{r \to 0} \frac{\log(N(\Lambda, r))}{\log 1/r},$$

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Trivially the Hausdorff dimension $\dim_H(\Lambda)$ satisfies:

$$\dim_H(\Lambda) \leq \overline{\dim}_{\text{box}}(\Lambda) \leq 2.$$

General poroussity results

Proposition 5. *For* $C > 0$ *let* $N = N(C) \in \mathbb{N}$ *satisfy* $N \geq \frac{\sqrt{2}}{C}$*.* *Then Any* C*-porous subset* $\Lambda \subset \mathbb{C}$ *satisfies:*

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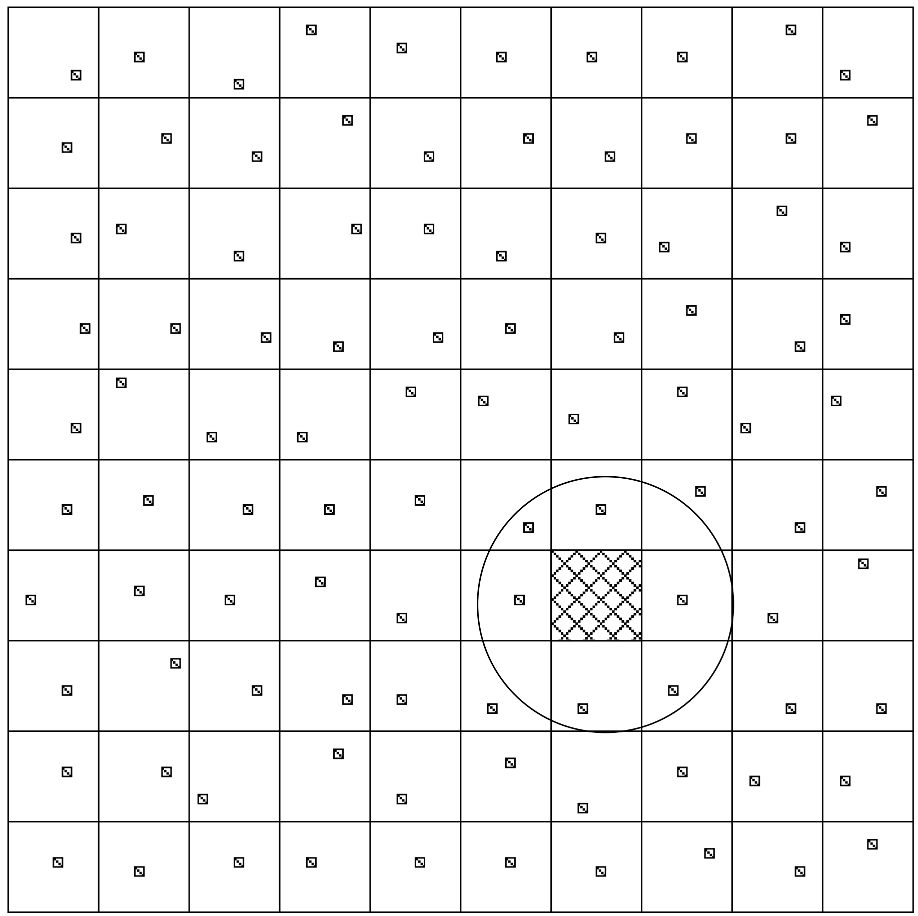
Moreover for $d_N < d < 2$ *there exists* $K_0 \geq N$ *such that for any* $K \geq K_0$*, for any square* Q *of side length* $r \leq 1$*: The partition of* Q *into sub squares* q *of equal side lengths* r/K *has the property that the number* $\widehat{N}((\Lambda \cap Q), K)$ *of small squares* q *needed to cover* $(\Lambda \cap Q)$ *satisfies*

$$\widehat{N}((\Lambda \cap Q), K) \leq K^d.$$

Proof

$$\lim_{n\to\infty} \frac{\log\left(N^2-1\right)^n}{\log N^n - \log r} = \frac{\log(N^2-1)}{\log N} = d_N$$



General results 2

Proposition 6. *Any deep point* z_0 *of a compact subset* $\Lambda \subset \mathbb{C}$ *with pourous boundary is a measureable deep point.*

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Moreover if z_0 is a $\delta, C_D > 0$ deep point and $\partial\Lambda$ is $C_P > 0$ porous. Then $\forall\, r \leq 1$

$$\mathrm{Area}(B_r(z_0)\backslash\Lambda) \leq C_{MD} \cdot r^{2+\alpha},$$

where $\alpha, C_{MD} > 0$ depends only on the constants $\delta, C_D, C_P > 0$.

Proof of Proposition 6

It suffices to prove it for small $r > 0$. Let $d_N < d < 2$ and K_0 be as in Proposition 5. For $r \leq \frac{1}{(2CK_0)^{1/\delta}}$ write

$K_0 \leq K \leq \frac{1}{2Cr^\delta} < K + 1 \in \mathbb{N}$.

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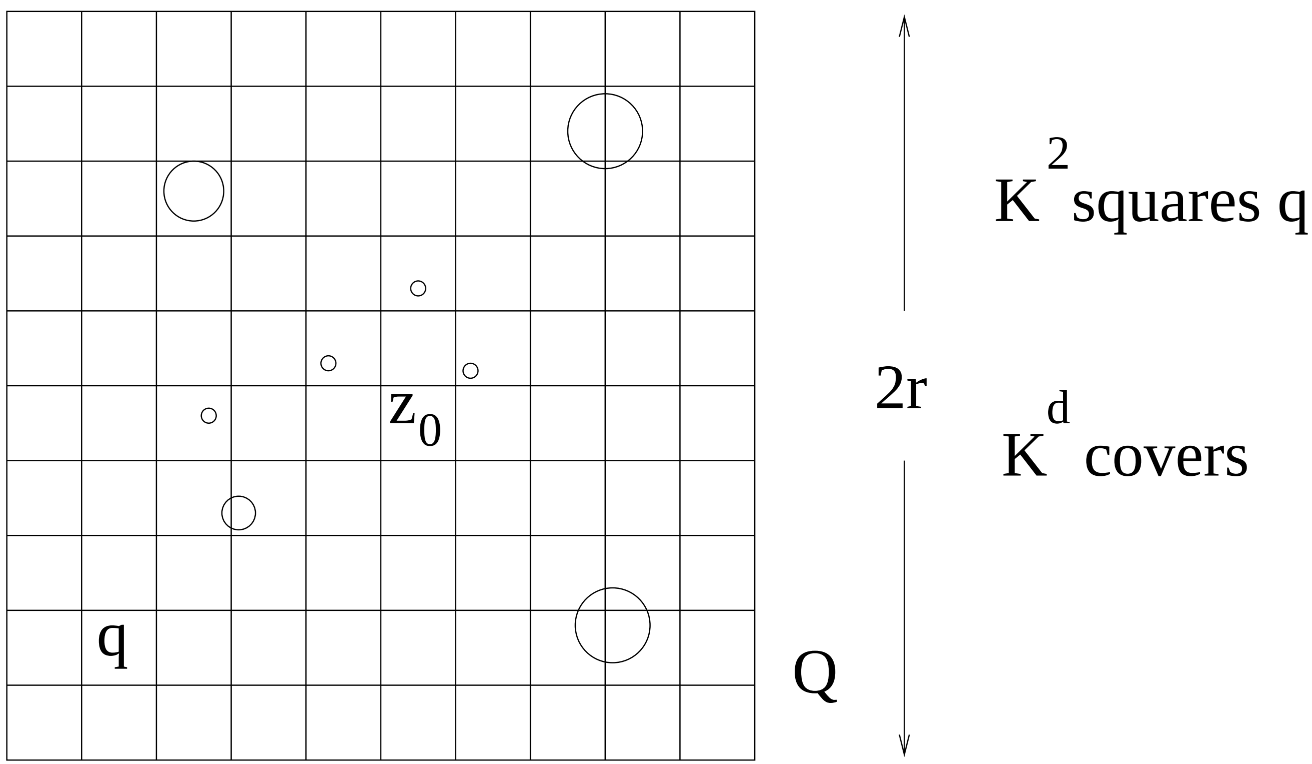

Proof of Proposition 6 cont.

Hence $B_r(z_0)\backslash\Lambda$ is contained in K^d small squares and we have the estimate:

$$\begin{aligned} \text{Area}(B_r(z_0)\backslash\Lambda) &\leq K^d\left(\frac{2r}{K}\right)^2 \\ \text{(as} \quad 1 \leq (K+1)\cdot 2Cr^\delta) \qquad &\leq 4r^2\left(\frac{K+1}{K}\right)^{2-d}(2Cr^\delta)^{2-d} \\ &\leq C_{MD}r^{2+\alpha} \end{aligned}$$

with $\mathbb{C}_{MD} = 4\left(\frac{K_0+1}{K_0}\right)^{2-d}(2C)^{2-d}$ and $\alpha = \delta(2-d)$.

Main Technical Theorems

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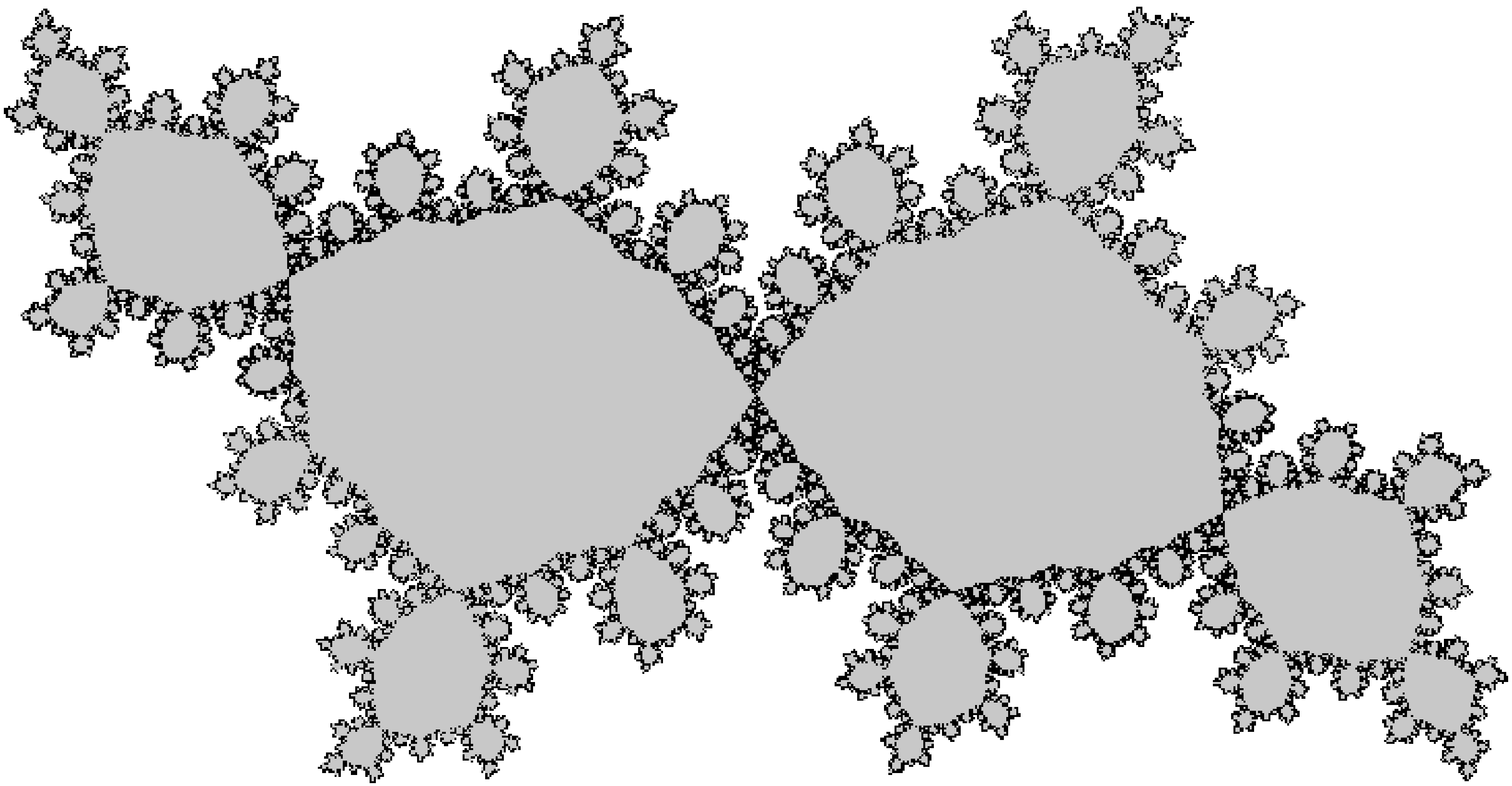
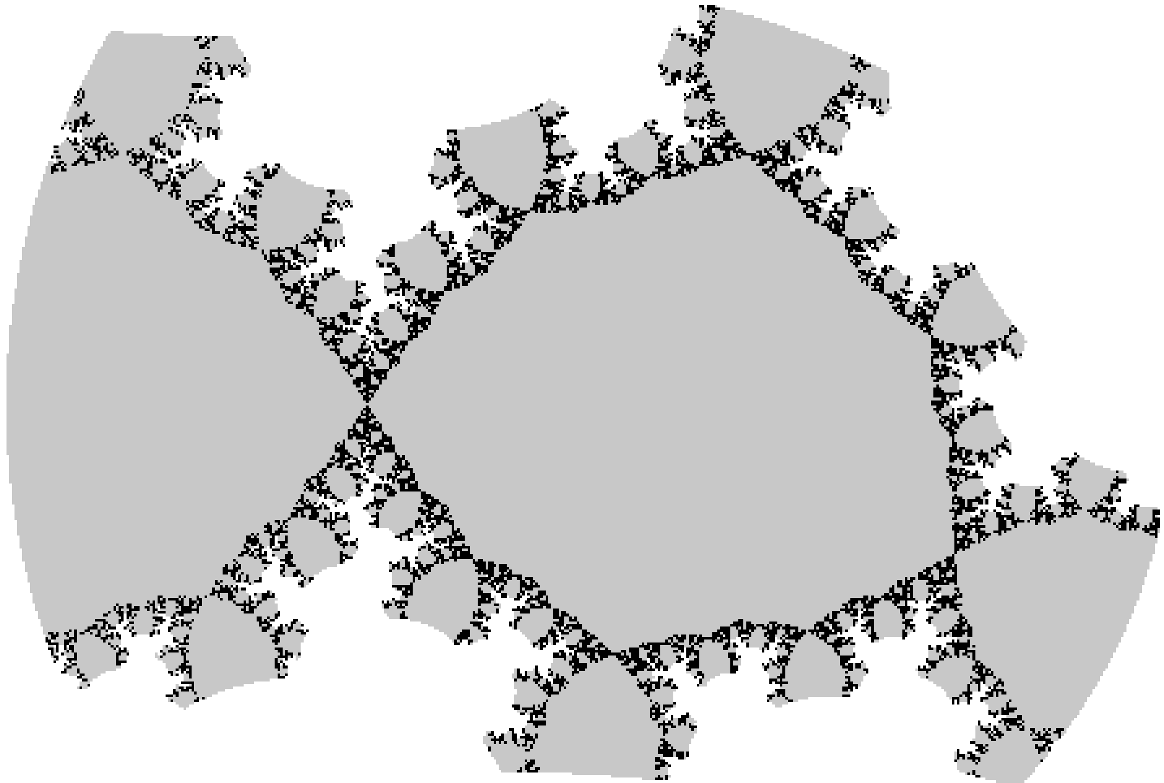
Combining Theorem 7 and Theorem 8 with Proposition 6 we obtain the Main Theorem.

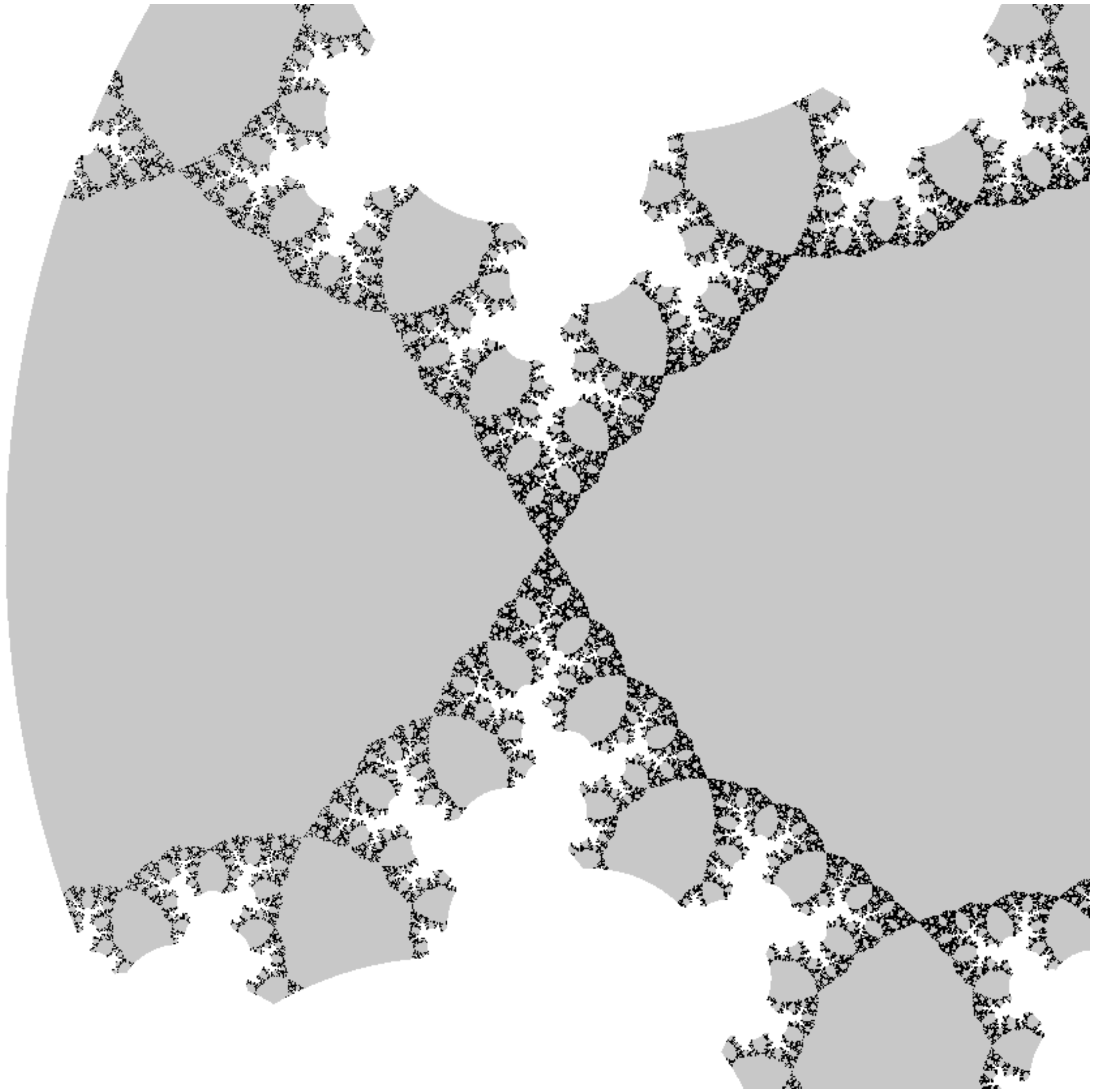
Golden Siegel disk



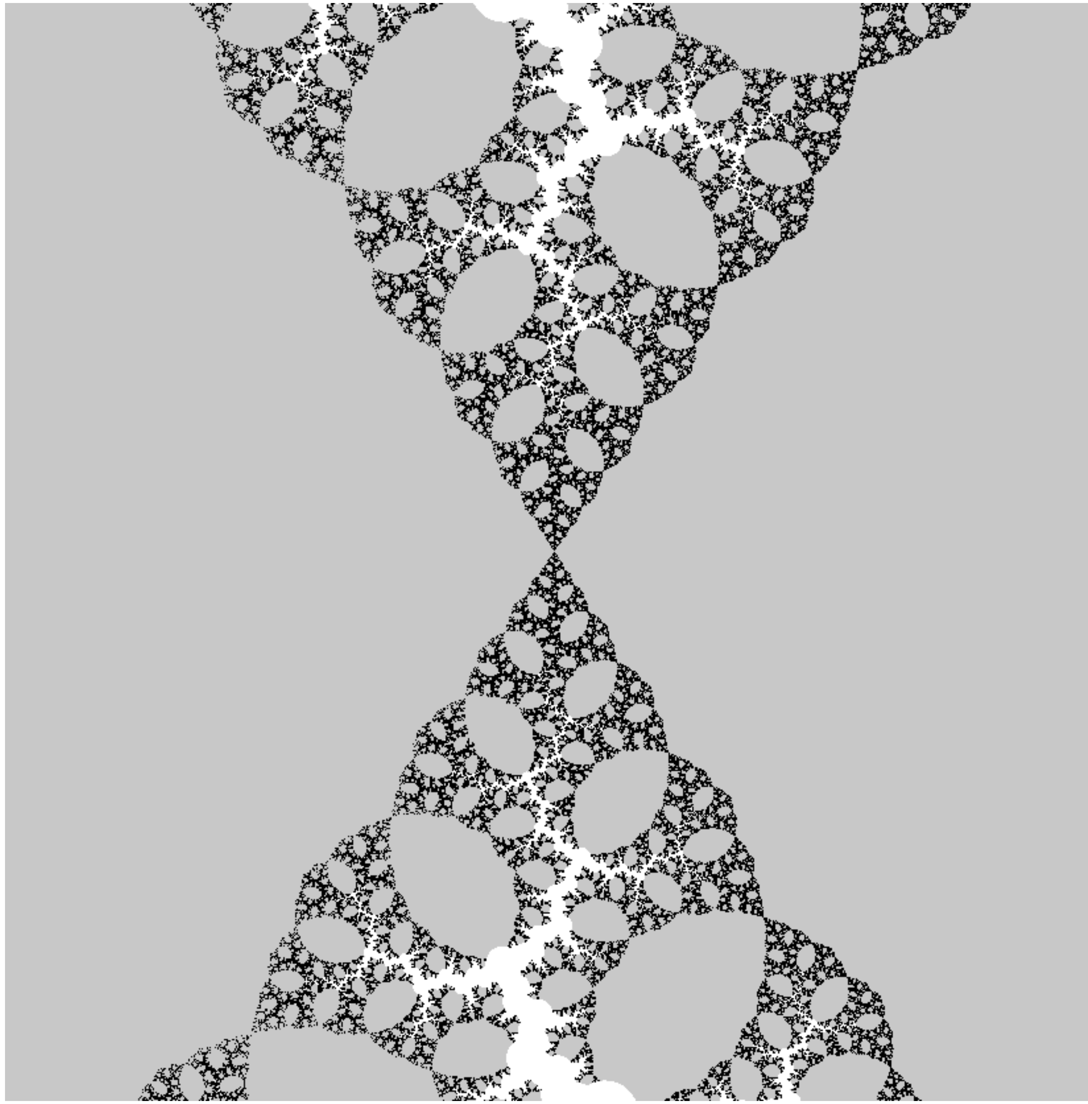
Approx $K_\theta(\epsilon)$



Zoom 1



Zoom 2



Reducing notation

In the following we shall fix a bounded type irrational θ and drop all subscripts θ to simplify writting and reading.

Recall that $\Omega = \mathbb{C} \backslash \Delta$ and that c denotes the critical point.

Nearby critical visits

Theorem 9 (McMullen). *There exists $C > 0$ such that for every $z \in J$ and every $r > 0$ there is a univalent iterate between pointed disks*

$$P^n : (U, y) \to (V, c), \qquad n \geq 0,$$

such that $|y - z| \leq r$ and $B_{Cr}(y) \subset U$.

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The Koebe distortion theorems for univalent maps and the fact that $\partial\Delta$ is a quasi disk easily implies that:

Corollary 10 (McMullen). *The Julia set J of P is porous.*

Elaborating a bit more one also obtain that $\partial K(\epsilon)$ is porous.

Small hyperbolic balls

Theorem 11. *For each* $\epsilon > 0$ *there exists* $\alpha, C > 0$ *depending on* ϵ *such that for any* $z_0 \in \mathbb{C} \backslash K(\epsilon)$ *with* $d = \mathrm{d}(z_0, \Delta) \leq 1$

$$\mathrm{d}_\Omega(z_0, K(\epsilon)) \leq C \cdot d^\alpha.$$

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The uniform deepness of $\partial\Delta$ in $K(\epsilon)$

Theorem 12. *McMullen*

For every bounded type $\theta \in BT$ and for every $\epsilon > 0$ the boundary $\partial\Delta_\theta$ is uniformly deep in $K_\theta(\epsilon)$.

is an easy Corollary as the coeficient function of the hyperbolic metric $\lambda_\Omega(z)$ is uniformly comparabel to $d = \mathrm{d}(z, \partial\Delta)$:

Hyperbolic expansion

Let $\Omega' = P^{-1}(\Omega)$ and let $\Delta' = \Omega' \cap P^{-1}(\Delta)$ denote the co-preimage of Δ. Then

$$P : \mathbb{C} \backslash (\overline{\Delta} \cup \overline{\Delta}') = \Omega' \to \Omega = \mathbb{C} \backslash \overline{\Delta}$$

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$$\forall\, z \in \Omega' : ||DP'(z)||_\Omega := \frac{|P'(z)| \cdot \lambda_\Omega(P(z))}{\lambda_\Omega(z)} > 1$$

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$$\forall\, z \in \Omega' : ||DP'(z)||_\Omega := \frac{|P'(z)| \cdot \lambda_\Omega(P(z))}{\lambda_\Omega(z)} > 1$$

and $\forall\, C > 0\ \exists \Lambda > 1$ such that

$$\forall\, z \in \Omega' \text{ with } \mathrm{d}_\Omega(z, \overline{\Delta}') \le C : ||DP'(z)||_\Omega \ge \Lambda.$$

Critical Visits

Proposition 13. $\exists C, C_1, C_2, 1/\beta > 1$ *s. t.* $\forall\ (z_k)_{k\geq 0}$, $P(z_k) = z_{k+1}$, $d = \mathrm{d}(z_0, \Delta) \leq 1$:

$$\exists k_0 : \qquad d^\beta / C_2 \leq |z_{k_0} - c| \leq C_2 d^{1/\beta}$$

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Moreover if $|z_0 - c| \leq 1$ *then:*

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Proposition 15. $\exists C, C_1, C_2, 1/\beta > 1$ *s. t.* $\forall\ (z_k)_{k\geq 0}$, $P(z_k) = z_{k+1}$, $d = \mathrm{d}(z_0, \Delta) \leq 1$*:*

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– *Or*

$$\exists k \geq 1 \qquad : |z_k - c| \leq C|z_0 - c| \quad \wedge \quad \mathrm{d}_\Omega(z_k, \Delta') \leq C_1.$$

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so that $$||DP(z_k)||_\Omega \geq \Lambda > 1.$$

where $\Lambda = \Lambda(C_1)$ *is the associated hyperbolic expansion coefficient.*

Sketch of proof of small hyperbolic balls.

Fix $0 < \epsilon \leq 1$, let $z_0 \in B_\epsilon(c) \backslash K(\epsilon)$, write $d = |z_0 - c| < \epsilon$ and denote by $(z_k)_{k \geq 0}$ the orbit of z_0.

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Let $k_1, \ldots k_{n+1}$ denote the critical visits until $|z_{k_{n+1}} - c| > \epsilon$.
Then

$$||DP^{k_{n+1}}(z_0)||_\Omega \geq \Lambda^n \quad \text{and} \quad \mathrm{d}_\Omega(z_{k_{n+1}}, \partial\Delta') \leq C_1$$

and in the worst case

$$C^n d \leq \epsilon < C^{n+1} d \quad \text{so that} \qquad n \leq \frac{\log(\epsilon/d)}{\log C} < n+1,$$

more small hyperbolic balls

Hence pulling back a geodesic arc of length at most C_1 from $z_{k_{n+1}}$ to $w_{k_{n+1}} \in \partial\Delta'$ we obtain a point $w_0 \in K(\epsilon)$ with

$$\begin{aligned}
\mathrm{d}_\Omega(z_0, w_0) \leq C_1 \Lambda^{-n} &\leq C_1 \Lambda \exp((\log d - \log \epsilon)\frac{\log \Lambda}{\log C}) \\
&= C' \cdot d^{\alpha'},
\end{aligned}$$

where $C' = \frac{C_1 \Lambda}{\epsilon^{\alpha'}}$ and $\alpha' = \frac{\log \Lambda}{\log C}$.

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As you may have noticed the sketch has a slight oversight. The closest point in $\partial\Delta'$ may bee outside $\Delta(\epsilon)$ or one of the iterates w_k may bee outside $\Delta(\epsilon)$.

Also what about the points near $\partial\Delta$, but not near c?

rectifying the sketch of proof.

For the first obstacle as the $\lambda_\Omega(z)$ is comparable to $\mathrm{d}(z, \overline{\Delta})$ on $\Delta(\epsilon)$. There exists a constant $C_3 > 1$ such that $\mathrm{d}_\Omega(\Delta(\epsilon/C_3), \partial\Delta(\epsilon)) > C_1$. Replacing ϵ by $\epsilon/(C \cdot C_3)$ in the estimates only changes C' and ensures that $w_{k_{n+1}} \in \partial(\Delta' \cap \Delta(\epsilon))$.

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Replacing further ϵ by ϵ^β/C_2 also only changes C' and ensures that the points z_j and their companions w_j with $k_n < j < k_{n+1}$ belongs to $\Delta(\epsilon)$ as

$$\mathrm{d}(z_j, \partial\Delta)^\beta/C_2 \leq \mathrm{d}(z_{k_{n+1}}, c) \leq \epsilon^\beta/C_2$$

Hence the full orbit of $w_0, \ldots, w_{k_{n+1}} \in \Delta(\epsilon)$ and thus in $K(\epsilon)$.

final estimate

Finally an arbitrary z near $\partial\Delta$ with $d = \mathrm{d}(z, \partial\Delta)$ has an iterate z_{k_0} with

$$d' = \mathrm{d}(z_{k_0}, c) \leq C_2 \cdot d^{1/\beta}.$$

Let $w_{k_0} \in K(\epsilon)$ be a point with $\mathrm{d}_\Omega(w_{k_0}, z_{k_0}) \leq C'' \cdot (d')^{\alpha'}$. Then there is a point $w_0 \in K(\epsilon)$ with

$$\mathrm{d}_\Omega(w_0, z_0) \leq C' \cdot (d')^{\alpha'} \leq C''' d^\alpha,$$

where $\alpha = \alpha'/\beta$ and $C''' = C'' \cdot C_2^{\alpha'} = C_1 \Lambda C_2^{2\alpha'} \left(\frac{CC_3}{\epsilon}\right)^{\beta\alpha'}$.

final comments

McMullen uses a very nice idea to prove the hyperbolic estimate and the theorem on nearby critical visits.
The idea being: The boundary of Δ is a quasi circle so the linearizer $\phi : \Delta \to \mathbb{D}$ of P extends to a quasi conformal homeomorphism also denoted $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

The conjugate degree 2 map $f = \phi \circ P \circ \phi^{-1}$ coincides on $\mathbb{D}$ with the corresponding rigid rotation R. Its iterates are uniformly quasi regular and tends to be close to corresponding iterates of R near $\partial\mathbb{D}$.

With the aid of this he maneges to prove all the above mentioned theorems without making explicit reference to the usual Blaschke model described by Douady in his 1987 Bourbaki seminar.