Construction of artificial boundary conditions for dispersive equations

by

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TBC for dispersive Eqs.

The Schrödinger Eq. in $\mathbb{R}$

$$(S) \quad \begin{cases} i\partial_t\psi + \partial_x^2\psi + V(x,t)\,\psi = 0, & (x,t) \in \mathbb{R}\times[0;T] \\ \lim\limits_{|x|\to+\infty} \psi(x,t) = 0, & t\in[0;T] \\ \psi(x,0) = \psi_0(x), & x\in\mathbb{R} \end{cases}$$

- $\psi(x,t)$: wave function, complex
- real potential, $\mathscr{V} = V(x,t) \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R})$
- ψ_0 compact support in Ω

- Laplace transform

$$\mathscr{L}_t(u)(x,\omega) = \int_0^\infty u(x,t)e^{-\omega t}dt$$

with the covariable $\omega = \sigma + i\tau$, $\sigma > 0$

- Fourier transform :

$$\mathscr{F}_t(u)(x,\tau) = \hat{u}(x,\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} u(x,t)e^{-it\tau}dt.$$

- $\mathscr{L}_t(\partial_t u)(x,\omega) = \omega\mathscr{L}_t(u)(x,\omega) - u(x,0)$

- $\mathscr{F}_t(\partial_t u)(x,\tau) = i\tau\mathscr{F}_t(u)(x,\tau)$

Pseudodifferential Operators in 1D

- A pseudodifferential operator $P(x,t,\partial_t)$ is described by its total symbol $p(x,t,\tau)$ in the Fourier space (τ is the covariable of t)

$$P(x,t,\partial_t)\,u(x,t) = \mathscr{F}_t^{-1}\big(\,p(x,t,\tau)\,\hat{u}(x,\tau)\big) = \int_{\mathbb{R}} p(x,t,\tau)\,\hat{u}(x,\tau)e^{it\tau}d\tau$$

Notations: $P = Op(p)$, $p(x,t,\tau) = \sigma(P(x,t,\partial_t))$

- Let $\alpha \in \mathbb{R}$ and the open set $\Xi \subset \mathbb{R}$. Symbol class: $S^\alpha(\Xi \times \Xi)$ vector space of functions $a(x,t,\tau) \in \mathcal{C}^\infty(\Xi \times \Xi \times \mathbb{R})$ s.t. $\forall K \subseteq \Xi \times \Xi$ and β, δ, γ, $\exists\ C_{\beta,\delta,\gamma}(K)$ s.t.

$$|\partial_\tau^\beta \partial_t^\delta \partial_x^\gamma a(x,t,\tau)| \leq C_{\beta,\delta,\gamma}(K)(1+|\tau|)^{\alpha-\beta},$$

$\forall (x,t) \in K$ and $\tau \in \mathbb{R}$.

- The order of P is the homogeneity order of its symbol w.r.t τ.

  $P(x,t,\partial_t)$ homogeneous of order m if and only if for $\mu > 0$, $p(x,t,\mu\tau) = \mu^m p(x,t,\tau)$.

Asymptotic expansion in homogeneous symbols

P is said to be of order M , $M \in \mathbb{Z}/2$, if:

$$p(x,t,\tau) \sim \sum_{j=0}^{+\infty} p_{M-j/2}(x,t,\tau),$$

p_M = principal symbol of P

$P \in OPS^m$ and $p \in S^m$

where $p_{M-j/2}$ is homogeneous of order $2M-j$ and $P_{M-j/2} : H^s \to H^{s+M-j/2}$.

Meaning of $\sim$: $\forall \widetilde{m} \in \mathbb{N}, \quad p - \sum_{j=0}^{\widetilde{m}} p_{M-j/2} \in S^{M-(\widetilde{m}+1)/2}$.

Symbolic calculus

Composition rule

$$\sigma(AB) = \sum_{\alpha=0}^{+\infty} \frac{(-i)^\alpha}{\alpha!} \partial_\tau^\alpha \sigma(A) \partial_t^\alpha \sigma(B)$$

If $A \in OPS^m$ and $B \in OPS^n$, then $AB \in OPS^{m+n}$.

Examples

The fractional operators $\partial_t^{1/2}$ and $I_t^{\alpha/2}$

$$\partial_t^{1/2} f(t) = \frac{1}{\sqrt{\pi}}\,\partial_t \int_0^t \frac{f(s)}{\sqrt{t-s}}\,ds$$

$$I_t^{\alpha/2} f(t) = \frac{1}{\Gamma(\alpha/2)} \int_0^t (t-s)^{\alpha/2-1} f(s)\,ds$$

Nonlocal w.r.t time convolution operator

Operator	∂_t	$\partial_t^{1/2}$	$I_t^{1/2}$	I_t
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Symbol	$i\tau$	$e^{-i\pi/4}\sqrt{-\tau}$	$\frac{e^{i\pi/4}}{\sqrt{-\tau}}$	$\frac{1}{i\tau}$
Class	OPS^1	$OPS^{1/2}$	$OPS^{-1/2}$	OPS^{-1}

Properties w.r.t derivatives

- Let $A \in OPS^m$: $\partial_\tau A \in OPS^{m-1}$, $\partial_{t,x} A \in OPS^m$
- $\partial_x P = Op(\partial_x p) + P\partial_x$, $\sigma(\partial_x P) = \partial_x p + \sigma(P\partial_x)$

Fractional operator $\partial_t^{1/2}$ et $I_t^{\alpha/2}$

$$\partial_t^{1/2} f(t) = \frac{1}{\sqrt{\pi}} \partial_t \int_0^t \frac{f(s)}{\sqrt{t-s}} ds, \qquad \sigma(\partial_t^{1/2}) = e^{-i\pi/4}\sqrt{-\tau} \in S^{1/2}$$

$$I_t^{\alpha/2} f(t) = \frac{1}{\Gamma(\alpha/2)} \int_0^t (t-s)^{\alpha/2-1} f(s) ds, \quad \sigma(I_t^{\alpha/2}) = \left(\frac{i}{\tau}\right)^{\alpha/2} \in S^{-\alpha/2}$$

- Case $\mathscr{V} = 0$
  TBC: $\partial_{\mathbf{n}}\psi + e^{-i\pi/4}\partial_t^{1/2}\psi = 0,$ on Σ_T.

- Case constant $\mathscr{V} = V$
  TBC: $\partial_{\mathbf{n}}\psi + e^{-i\pi/4}e^{itV}\partial_t^{1/2}\left(e^{-itV}\psi\right) = 0,$ on Σ_T.

- CASE $\mathscr{V} = 0$
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$$\partial_{\mathbf{n}}\psi - i\,Op\left(\sqrt{-\tau}\right)\psi = 0, \quad \text{on } \Sigma_T.$$

- CASE CONSTANT $\mathscr{V} = V$
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$$\partial_{\mathbf{n}}\psi - i\, Op\left(\sqrt{-\tau + V}\right)(\psi) = 0, \quad \text{on } \Sigma_T.$$

LEMMA

If a is a symbol belonging to S^m independent of t, and $V = V(x)$, then

$$Op\left(a(\tau - V(x))\right)\psi = e^{itV(x)} Op\left(a(\tau)\right)\left(e^{-itV(x)}\psi\right)$$

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$$Op\left(a(\tau - V(x))\right)\psi = e^{itV(x)}Op\left(a(\tau)\right)\left(e^{-itV(x)}\psi\right)$$

- CASE $\mathscr{V} = V(t)$: GAUGE CHANGE — Antoine, Besse et Descombes, 2006

$$\partial_{\mathbf{n}}\psi - i\,e^{i\mathcal{V}(t)}\,Op\left(\sqrt{-\tau}\right)\left(e^{-i\mathcal{V}(t)}\,\psi\right) = 0,\quad \text{on } \Sigma_T.$$

If $\mathscr{V} = V(x,t) = x$, by Fourier transform, the Eq. $i\partial_t u + \partial_x^2 u + xu = 0$ becomes the Airy Eq.

$$\partial_x^2 \hat{u} + (-\tau + x)\hat{u} = 0$$

So $\hat{u} = \mathrm{Ai}\left((x-\tau)e^{-i\pi/3}\right)$ and we have the TBC

$$\partial_{\mathbf{n}} u + e^{2i\pi/3}\, Op\left(\frac{\mathrm{Ai}'\left((x-\tau)e^{-i\pi/3}\right)}{\mathrm{Ai}\left((x-\tau)e^{-i\pi/3}\right)}\right)(u) = 0.$$

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In a first approximation

$$\frac{\mathrm{Ai}'\left((x-\tau)e^{-i\pi/3}\right)}{\mathrm{Ai}\left((x-\tau)e^{-i\pi/3}\right)} \approx -e^{-i\pi/6}\sqrt[+]{-\tau + x}$$

and one has the ABC

$$\partial_{\mathbf{n}} u + iOp\left(-\sqrt[+]{-\tau + x}\right)(u) = 0, \quad (x,t) \in \Sigma_T$$

which leads to

$$\partial_{\mathbf{n}} u + e^{itx} e^{-i\pi/4} \partial_t^{1/2}\left(e^{-itx} u\right) = 0, \quad (x,t) \in \Sigma_T$$

Remark there exists a change of unknown s.t. if v is solution to $i\partial_t v + \partial_x^2 v = 0$, then

$$u(x,t) = e^{-i(-\alpha t x + \frac{t^3}{3}|\alpha|^2)} v(x - t^2\alpha, t)$$

is solution to

$$i\partial_t u + \partial_x^2 u + \alpha x u = 0.$$

Therefore, one can work on the free Schrödinger equation.

Changes of unknown are also available for the cases $V(x) = \pm x^2$ by *lens transform* (R. Carles (05)).

Partial conclusion

- We have factorized the operator

$$i\partial_t + \partial_x^2 + V = \left(\partial_x + i\sqrt{i\partial_t + V}\right)\left(\partial_x - i\sqrt{i\partial_t + V}\right)$$

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- TBCs and ABCs are written through a DtN op.

$$\partial_{\mathbf{n}} u + iOp\left(-\sqrt[+]{-\tau}\right)(u) = 0 \quad \text{on } \Sigma_T$$

or

$$\partial_{\mathbf{n}} u + iOp\left(-\sqrt[+]{-\tau + V}\right)(u) = 0 \quad \text{on } \Sigma_T$$

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- if $\mathscr{V} = V(t)$, the change of unknowns $v(x,t) = e^{-i\mathcal{V}(t)}u(x,t)$ with $\mathcal{V}(t) = \int_0^t V(s)ds$ reduces the Schrödinger Eq. with potential to a free Schrödinger Eq. and the TBC is

$$\partial_{\mathbf{n}} u(x,t) + e^{-i\frac{\pi}{4}}\, e^{i\mathcal{V}(t)}\partial_t^{1/2}\left(e^{-i\mathcal{V}(t)}u\right)(x,t) = 0 \quad \text{on } \Sigma_T$$

1D Schrödinger Eq. $\mathscr{V} = V(x,t)$

$$\text{(Syst1)} \quad \begin{cases} i\partial_t \psi + \partial_x^2 \psi + \mathscr{V}\psi = 0\,, & (x,t) \in \mathbb{R}_x \times [0;T]\,, \\ \lim_{|x|\to\infty} \psi(x,t) = 0, & \\ \psi(x,0) = \psi_0(x)\,, & x \in \mathbb{R}_x\,. \end{cases}$$

In the general case $V(x,t)$, we can not expect to derive a TBC.

Use the symbolic calculus to determine ABCs.

High frequency solution: Engquist-Majda method

Admissible potentials class: repulsive potentials

Repulsive potential

V smooth and $x\partial_x V(x,t) > 0$ for $x \in \overline{\Omega}$, $t > 0$.

Ex : $V(x,t) = x^2$

Two strategies

Gauge change (strategy 1)

- Change of unknown (this solves the case $\mathscr{V} = V(t)$): $v = e^{-i\mathcal{V}}u$ with $\mathcal{V}(x,t) = \int_0^t V(x,s)ds$ where $f = 2i\partial_x \mathcal{V}$ et $g = i\partial_x^2 \mathcal{V} - (\partial_x \mathcal{V})^2$.
- We work with the equation for v :

$$i\partial_t v + \partial_x^2 v + f\,\partial_x v + g\,v = 0$$

Direct method (strategy 2)

- One works directly on the original equation

$$i\partial_t u + \partial_x^2 u + V\,u = 0$$

$\Rightarrow$ Boundary conditions for $i\partial_t w + \partial_x^2 w + A\,\partial_x w + B\,w = 0$, with

- $A = 0$ and $B = V(x,t)$ if $w = u$,
- $A = f(x,t)$ and $B = g(x,t)$ if $w = ve^{-i\mathcal{V}}u$.

General Schrödinger operator: $L = i\partial_t + \partial_x^2 + A\,\partial_x + B$

Factorization of the operator L (Nirenberg)

$$L = i\partial_t + \partial_x^2 + A\,\partial_x + B = (\partial_x + i\Lambda^-)(\partial_x + i\Lambda^+) + R$$

$\Lambda^\pm \in \mathrm{OPS}^{1/2}$ and $R \in \mathrm{OPS}^{-\infty}$
Λ^+ has an asymptotic expansion in homogeneous symbols:

$$\sigma(\Lambda^+) = \lambda^+ \sim \sum_{j=0}^{+\infty} \lambda^+_{1/2-j/2} = \lambda^+_{1/2} + \lambda^+_0 + \lambda^+_{-1/2} + \lambda^+_{-1} + \cdots$$

with $\lambda^+_{1/2-j/2}$ homogeneous of order $1/2 - j/2$.

Artificial condition : $\partial_{\mathbf{n}} w + i\Lambda^+ w = 0$

$$\partial_{\mathbf{n}} w + i \sum_{j=0}^{+\infty} Op\left(\lambda^+_{1/2-j/2}\right) w = 0, \quad \text{on } \Sigma_T$$

Approximated condition of order M:

$$\partial_{\mathbf{n}} w_M + i \sum_{j=0}^{M-1} Op\left(\lambda^+_{1/2-j/2}\right) w_M = 0, \quad \text{on } \Sigma_T$$

Identification of the involved terms

Thanks to $\partial_x \Lambda^+ = Op(\partial_x \Lambda^+) + \Lambda^+ \partial_x$, we have

- $L = \partial_x^2 \; + A\partial_x \; + i\partial_t + B$
- $(\partial_x + i\Lambda^-)(\partial_x + i\Lambda^+) =$

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- $L = \partial_x^2 + A\partial_x + i\partial_t + B$
- $(\partial_x + i\Lambda^-)(\partial_x + i\Lambda^+) = \partial_x^2$

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- $(\partial_x + i\Lambda^-)(\partial_x + i\Lambda^+) = \partial_x^2 \ + i(\Lambda^+ + \Lambda^-)\partial_x \ + iOp(\partial_x \lambda^+) - \Lambda^- \Lambda^+$

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Thanks to $\partial_x \Lambda^+ = Op(\partial_x \Lambda^+) + \Lambda^+ \partial_x$, we have

- $L = \partial_x^2 \; + A\partial_x \; + i\partial_t + B$
- $(\partial_x + i\Lambda^-)(\partial_x + i\Lambda^+) = \partial_x^2 \; + i(\Lambda^+ + \Lambda^-)\partial_x \; + iOp(\partial_x \lambda^+) - \Lambda^- \Lambda^+$
- Identification of the coefficients (up to R)

$$\begin{aligned} & i(\Lambda^- + \Lambda^+) = A \\ & iOp(\partial_x \lambda^+) - \Lambda^- \Lambda^+ = i\partial_t + B \end{aligned}$$

- Symbolic system

$$\begin{aligned} & i(\lambda^- + \lambda^+) = a \\ & i\partial_x \lambda^+ - \sum_{\alpha=0}^{\infty} \frac{(-i)^\alpha}{\alpha!} \partial_\tau^\alpha \partial_t^\alpha \lambda^+ = -\tau + b \end{aligned}$$

- Since $\lambda^\pm \sim \sum_{j=0}^{+\infty} \lambda^\pm_{1/2-j/2}$, we compute $\lambda^\pm_{1/2-j/2}$ by identification of the terms of same order in the system.

The principal symbol with negative real part characterizes the outgoing wave of u

- STRATEGY 1: Case $A = f$ and $B = g$. We choose

$$\lambda_{1/2}^{+} = -\sqrt{-\tau}\ (S1)$$

- STRATEGY 2: Case $A = 0$ and $B = V$. We choose

$$\lambda_{1/2}^{+} = -\sqrt{-\tau + V}\ (S2).$$

Remark: for the second strategy, we could also have chosen $\lambda_{1/2}^{+} = -\sqrt{-\tau}$. This choice would lead to a less accurate ABC since it would give some symbols which are approx. of $-\sqrt{-\tau + V}$ when $|\tau| \to +\infty$.

Strategy 1: Gauge change $v = e^{-i\mathcal{V}}u$

- $$\lambda_{1/2}^+ = -\sqrt{-\tau}, \quad \lambda_0^+ = \partial_x \mathcal{V}, \quad \lambda_{-1/2}^+ = 0, \quad \lambda_{-1}^+ = \frac{i\partial_x V}{4\tau}$$
- Interpretation of symbols

$$Op\left(-\sqrt{-\tau}\right) = e^{-3i\pi/4}\partial_t^{1/2}$$

$$Op\left(\frac{i\partial_x V}{4\tau}\right) = \frac{\partial_{\mathbf{n}} V}{4} I_t \quad \text{or} \quad \mathsf{sg}(\partial_{\mathbf{n}} V)\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2}\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} I_t$$

STRATEGY 2: DIRECT METHOD $A = 0$, $B = V$

- $$\lambda^+_{1/2} = -\sqrt{-\tau + V}, \quad \lambda^+_0 = 0, \quad \lambda^+_{-1/2} = 0, \quad \lambda^+_{-1} = -\frac{i}{4}\frac{\partial_x V}{-\tau + V}$$
- Interpretation of symbols

$$Op\left(-\sqrt{-\tau+V}\right) = \sqrt{i\partial_t + V} \text{ mod } OPS^{-3/2}$$
$$Op\left(\frac{\partial_x V}{-\tau+V}\right) = \partial_{\mathbf{n}} V (i\partial_t + V)^{-1} \text{ mod } OPS^{-5/2}$$

Comparison for $V(x,t) = x$

$$\lambda^+ = e^{2i\pi/3} \frac{\mathrm{Ai}'\left((x-\tau)e^{-i\pi/3}\right)}{\mathrm{Ai}\left((x-\tau)e^{-i\pi/3}\right)}.$$

with

$$\lambda = i\lambda^+_{1/2} + i\lambda^+_{-1} = -i\sqrt{-\tau + x} + \frac{1}{4}\frac{1}{-\tau + x}$$

Abramowitz-Stegun :$\lambda^{+,(2)}$ is actually the asymptotic expansion of λ^+ for large enough τ.

- Strategy 1: ABC is $\partial_{\mathbf{n}} v + i\Lambda^{+} v = 0$ on Σ_T. But $v(x,t) = e^{-i\mathcal{V}(x,t)} u(x,t)$. Therefore, for u, retaining the M first symbols, we have

$$\partial_{\mathbf{n}} u - i(\partial_x \mathcal{V})\, u + ie^{i\mathcal{V}} \sum_{j=0}^{M-1} Op\left(\lambda^{+,(1)}_{1/2-j/2}\right)\left(e^{-i\mathcal{V}} u\right) = 0, \quad \text{on } \Sigma_T,$$

- Strategy 2: ABC is $\partial_{\mathbf{n}} u + i\Lambda^{+} u = 0$

$$\partial_{\mathbf{n}} u + i \sum_{j=0}^{M-1} Op\left(\lambda^{+,(2)}_{1/2-j/2}\right) u = 0 \quad \text{on } \Sigma_T.$$

- Strategy 1: For reason of symmetry and to get adequate estimates, the ABC of 4^{th} order ABC_1^4 is

$$\partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}(x,t)} \partial_t^{1/2} \left(e^{-i\mathcal{V}(x,t)} u \right) \quad (\text{ABC}_1^2)$$
$$+ i \,\mathrm{sg}(\partial_{\mathbf{n}} V) \frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} e^{i\mathcal{V}(x,t)} I_t \left(\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} e^{-i\mathcal{V}(x,t)} u \right) = 0$$

- Strategy 2: The ABC of 4^{th} order ABC_2^4 is

$$\partial_{\mathbf{n}} u + i\sqrt{i\partial_t + V} u \quad (\text{ABC}_2^2) \quad + \frac{i}{4} \partial_{\mathbf{n}} V (i\partial_t + V)^{-1} u = 0$$

Proposition Let $u_0 \in L^2(\Omega)$ s.t. $\mathrm{Supp}(u_0) \subset \Omega$. Let $V \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$ and u a solution of

$$\begin{cases} i\partial_t u + \partial_x^2 u + Vu = 0, & \text{in } \Omega_T, \\ \partial_{\mathbf{n}} u + \Lambda_1^M u = 0, & \text{on } \Sigma_T, \\ u(x,0) = u_0(x), & \forall x \in \Omega, \end{cases}$$

where

$$\Lambda_1^2(x,t,\partial_t)\,u = e^{-i\pi/4} e^{i\mathcal{V}(x,t)} \partial_t^{1/2} \left(e^{-i\mathcal{V}(x,t)} u \right)$$

and

$$\Lambda_1^4(x,t,\partial_t)\,u = \Lambda_1^2(x,t,\partial_t)\,u + i\,\mathrm{sg}(\partial_{\mathbf{n}} V) \frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} e^{i\mathcal{V}(x,t)} I_t \left(\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} e^{-i\mathcal{V}(x,t)} u \right)$$

Then, u fulfils the following energy bound

$$\forall t > 0, \quad ||u(t)||_{L^2(\Omega)} \leq ||u_0||_{L^2(\Omega)},$$

for $M = 2$ and for $M = 4$ if $\mathrm{sg}(\partial_{\mathbf{n}} V)$ is constant on Σ_T, which implies the uniqueness of the solution.

In the case of strategy 2, we have

$$\partial_{\mathbf{n}} u + \Lambda_2^M u = 0, \quad \text{on } \Sigma_T,$$

with

$$\Lambda_2^2(x,t,\partial_t)\, u = Op\left(-i\sqrt{-\tau + V}\right) u$$

and

$$\Lambda_2^4(x,t,\partial_t)\, u = \Lambda_2^2(x,t,\partial_t)\, u + \frac{1}{4} Op\left(\frac{\partial_x V}{-\tau + V}\right) u$$

If $V(x,t) = V(x)$, ABC_2^M and ABC_1^M are strictly equivalent.

$$\begin{cases} i\partial_t u + \partial_x^2 u + V\,u = 0, & \text{in } \Omega_T \\ u(\cdot,0) = u_0, & \text{in } \Omega \end{cases}$$

with ABC on Σ_T, for $M = 2$ or 4

Strategy 1

$$ABC_1^2 \quad \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2}(e^{-i\mathcal{V}} u) = 0,$$

$$ABC_1^4 \quad \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2}(e^{-i\mathcal{V}} u) + i\,\mathrm{sg}(\partial_{\mathbf{n}} V) \frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} e^{i\mathcal{V}(x,t)} I_t \left(\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} e^{-i\mathcal{V}(x,t)} u \right) = 0.$$

or

Strategy 2

$$ABC_2^2 \quad \partial_{\mathbf{n}} u - i\sqrt{i\partial_t + V}\,u = 0,$$

$$ABC_2^4 \quad \partial_{\mathbf{n}} u - i\sqrt{i\partial_t + V}\,u + \mathrm{sg}(\partial_{\mathbf{n}} V) \frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} (i\partial_t + V)^{-1} \left(\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} u \right) = 0.$$

Let $\Delta t = T/N$ be the time step and let us set $t_n = n\Delta t$ and u^n stands for an approximation of $u(t_n)$.

- Time approximation Semi-discrete Crank-Nicolson *symmetrical* scheme

$$i\,\frac{u^{n+1}-u^n}{\Delta t} + \partial_x^2\left(\frac{u^{n+1}+u^n}{2}\right) + \frac{V^{n+1}+V^n}{2}\,\frac{u^{n+1}+u^n}{2} = 0,$$

for $n = 0, \ldots, N-1$.
Implementation

$$2i\,\frac{v^{n+1}}{\Delta t} + \partial_x^2 v^{n+1} + W^{n+1} v^{n+1} = 2i\,\frac{u^n}{\Delta t}$$

with $v^{n+1} = (u^{n+1}+u^n)/2 = u^{n+1/2}$, $W^{n+1} = (V^{n+1}+V^n)/2 = V^{n+1/2}$.

The symmetry is fundamental to guarantee the stability of the numerical scheme.

- Space approximation Finite Element Method
  - ABC_1^M: discrete convolutions
  - ABC_2^M: rational approximation of the square root (Padé)

Strategy 1

$$\begin{cases} i\partial_t u + \partial_x^2 u + V\,u = 0, & \text{in } \Omega_T \\ \partial_{\mathbf{n}} u + \Lambda_1^M u = 0, & \text{on } \Sigma_T, \text{ for } M = 2 \text{ ou } 4 \\ u(\cdot,0) = u_0, & \text{in } \Omega \end{cases}$$

with

$$\begin{aligned} ABC_1^2 \quad & \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2} (e^{-i\mathcal{V}} u) = 0, \\ ABC_1^4 \quad & \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2} (e^{-i\mathcal{V}} u) \\ & + i\,\mathrm{sg}(\partial_{\mathbf{n}} V) \frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} e^{i\mathcal{V}(x,t)} I_t \left(\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} e^{-i\mathcal{V}(x,t)} u \right) = 0. \end{aligned}$$

Numerical scheme for ABC_1^M: discrete convolutions

Approximations of $\partial_t^{1/2}$, $I_t^{1/2}$ and I_t in agreement with the Crank-Nicolson scheme
⇒ trapezoidal formula [Schmidt-Yevick (97), Antoine-Besse (03)]

$$\begin{aligned}
\partial_t^{1/2} f(t^n) &\approx \sqrt{\frac{2}{\Delta t}} \sum_{k=0}^{n} \beta_{n-k} f^k \\
I_t^{1/2} f(t^n) &\approx \sqrt{\frac{\Delta t}{2}} \sum_{k=0}^{n} \alpha_{n-k} f^k \\
I_t\, f(t^n) &\approx \frac{\Delta t}{2} \sum_{k=0}^{n} \gamma_{n-k} f^k
\end{aligned}
\qquad
\begin{cases}
(\alpha_0, \alpha_1, \alpha_2, \ldots) = (1, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \ldots) \\
\beta_k = (-1)^k \alpha_k, \quad \forall k \geq 0, \\
(\gamma_0, \gamma_1, \gamma_2, \ldots) = (1, 2, 2, \ldots)
\end{cases}$$

Proposition

Let u^n be the solution to the problem with the boundary conditions ABC_1^M discretized with discrete convolutions. For $M = 2$, we have

$$\forall n \in \{0, \cdots, N\}, \quad \|u^n\|_{L^2(\Omega)} \leq \|u^0\|_{L^2(\Omega)},$$

and if $\partial_{\mathbf{n}} W^k$ has a constant sign, also true for $M = 4$.

⇒ The unconditional stability of the scheme is preserved.

Strategy 2

$$\begin{cases} i\partial_t u + \partial_x^2 u + V\,u = 0, & \text{in } \Omega_T \\ \partial_{\mathbf{n}} u + \Lambda_2^M u = 0, & \text{on } \Sigma_T, \text{ for } M = 2 \text{ or } 4 \\ u(\cdot, 0) = u_0, & \text{in } \Omega \end{cases}$$

with

$$\begin{aligned} ABC_2^2 \quad & \partial_{\mathbf{n}} u - i\sqrt{i\partial_t + V}u = 0, \\ ABC_2^4 \quad & \partial_{\mathbf{n}} u - i\sqrt{i\partial_t + V}u \\ & + \mathrm{sg}(\partial_{\mathbf{n}} V)\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2}(i\partial_t + V)^{-1}\left(\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2}u\right) = 0. \end{aligned}$$

Numerical scheme for ABC_2^M

The square root is approximated by Padé approximants of order m:

$$\sqrt{z} \approx R_m(z) = \sum_{k=0}^{m} a_k^m - \sum_{k=1}^{m} \frac{a_k^m\, d_k^m}{z + d_k^m},$$

with $a_0^m = 0 \quad , \quad a_k^m = \dfrac{1}{m \cos^2\left(\frac{(2k+1)\pi}{4m}\right)} \quad , \quad d_k^m = \tan^2\left(\dfrac{(2k+1)\pi}{4m}\right).$

For the conditions ABC_2^M :

$$\sqrt{i\partial_t + V} \rightsquigarrow R_m(i\partial_t + V)$$

$$\Rightarrow \sqrt{i\partial_t + V} \approx \left(\sum_{k=0}^{m} a_k^m\right) u - \sum_{k=1}^{m} a_k^m\, d_k^m \underbrace{(i\partial_t + V + d_k^m)^{-1} u}_{\varphi_k}$$

Lindmann's trick (85) : introduction of auxiliary functions

$$i\partial_t \varphi_k + (V + d_k^m)\varphi_k = u, \quad \text{pour } 1 \leq k \leq m, \text{ in } x = x_{l,r},$$

with $\varphi_k(x, 0) = 0$.

The ABC becomes for the semi-discrete scheme

$$\begin{cases} \partial_{\mathbf{n}} v^{n+1} - i \sum_{k=0}^{m} a_k^m v^{n+1} + i \sum_{k=1}^{m} a_k^m \, d_k^m \, \varphi_k^{n+1/2} = 0, \\ i \dfrac{\varphi_k^{n+1} - \varphi_k^n}{\Delta t} + (W^{n+1} + d_k^m) \varphi_k^{n+1/2} = v^{n+1}, \\ \varphi_k^0 = 0. \end{cases}$$

For ABC_2^4

$$\partial_{\mathbf{n}} u - i\sqrt{i\partial_t + V} u + \mathrm{sg}(\partial_{\mathbf{n}} V) \frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} (i\partial_t + V)^{-1} \left(\frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} u \right) = 0,$$

we introduce an auxiliary function ψ s.t.

$$(i\partial_t + V)\psi = \frac{\sqrt{|\partial_{\mathbf{n}} V|}}{2} u.$$

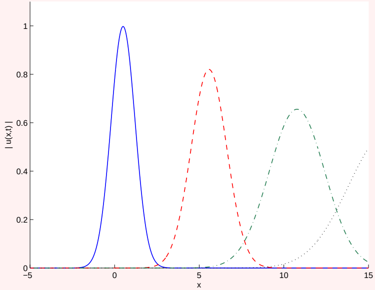
No stability results

Exact solutions profile

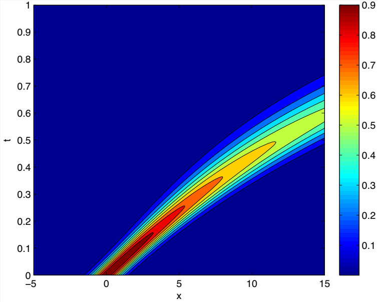
Gaussian initial data $u_0(x) = e^{-x^2+ik_0x}$, $k_0 = 10$.

$$V(x) = x^2$$
$$\Omega_T = [-5; 15] \times [0, 1]$$

repulsive potential

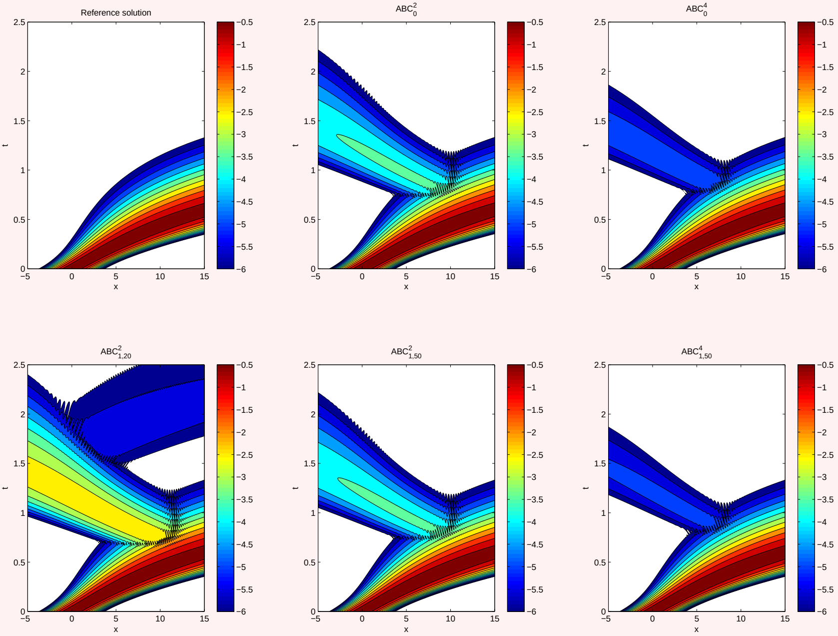


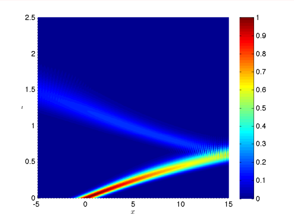
Evolution for different times



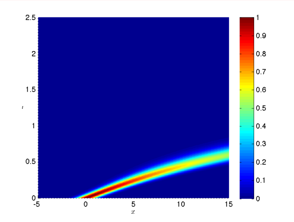
Plot of $|u|$ in plane (x, t)

Applications $\mathscr{V} = x^2$

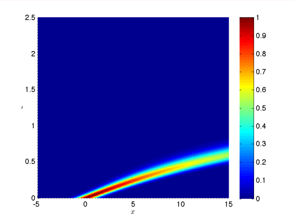




ABC^0



ABC^2

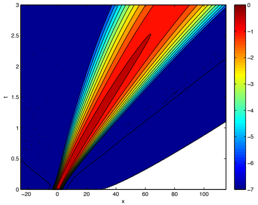


ABC^4

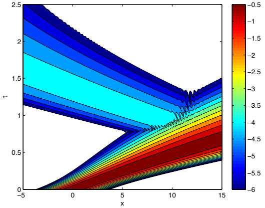
Application to a potential $V(x,t)$: $\mathscr{V} = x(2+\cos(2t))$

Computational domain $\Omega_T = [-5; 15] \times [0; 2.5]$

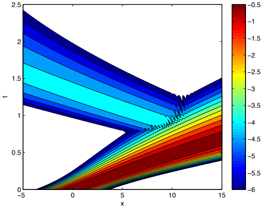
$\Delta x = 2.5 \cdot 10^{-3}$, $\Delta t = 10^{-3}$, 50 Padé functions, **Logarithmic scale**



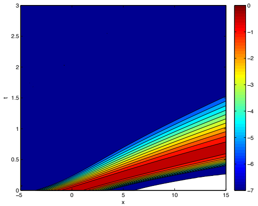
Reference solution computed on a wide domain $[-25; 115]$



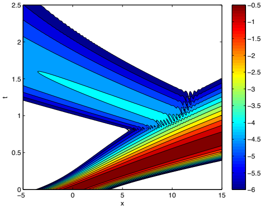
ABC_1^2 10^{-4}



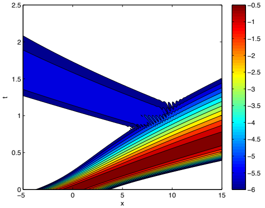
ABC_1^4 10^{-4}



Truncated reference solution



ABC_2^2 10^{-4}

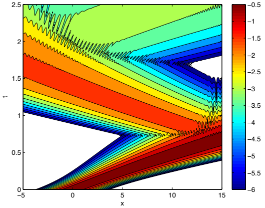


ABC_2^4 $10^{-5.5}$

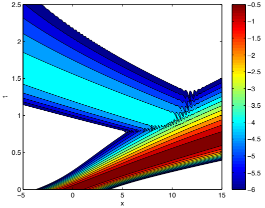
Application to a potential $V(x,t)$: $\mathscr{V} = x(2 + \cos(2t))$

Computational domain $\Omega_T = [-5; 15] \times [0; 2.5]$

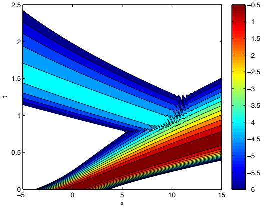
$\Delta x = 2.5 \cdot 10^{-3}$, $\Delta t = 10^{-3}$, 50 Padé functions, **Logarithmic scale**



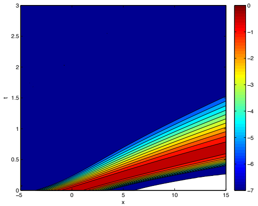
TBC $10^{-1.5}$



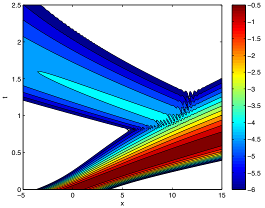
ABC_1^2 10^{-4}



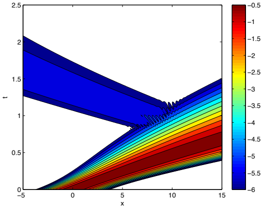
ABC_1^4 10^{-4}



Truncated reference solution



ABC_2^2 10^{-4}



ABC_2^4 $10^{-5.5}$

- $\mathscr{V} = 0$ without potential

$$\partial_{\mathbf{n}} u + e^{-i\pi/4} \partial_t^{1/2} u = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = V$ constant

$$\partial_{\mathbf{n}} u + e^{-i\pi/4}\, e^{itV}\, \partial_t^{1/2} \left(e^{-itV} u \right) = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = V(t)$: Gauge change

Setting $v(x,t) = u(x,t) e^{-i\mathcal{V}(t)}$ with $\mathcal{V}(t) = \int_0^t V(s)\, ds$,

then v is solution of the free-potential equation.

$$\partial_{\mathbf{n}} u + e^{-i\pi/4}\, e^{i\mathcal{V}(t)}\, \partial_t^{1/2} \left(e^{-i\mathcal{V}(t)}\, u \right) = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = 0$ without potential

$$\partial_{\mathbf{n}} u + e^{-i\pi/4} \partial_t^{1/2} u = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = V$ constant

$$\partial_{\mathbf{n}} u + e^{-i\pi/4}\, e^{itV}\, \partial_t^{1/2} \left(e^{-itV} u \right) = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = V(t)$: Gauge change

Setting $v(x,t) = u(x,t) e^{-i\mathcal{V}(t)}$ with $\mathcal{V}(t) = \int_0^t V(s)\, ds$,

then v is solution of the free-potential equation.

$$\partial_{\mathbf{n}} u + e^{-i\pi/4}\, e^{i\mathcal{V}(t)}\, \partial_t^{1/2} \left(e^{-i\mathcal{V}(t)}\, u \right) = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = 0$ without potential

$$\partial_{\mathbf{n}} u - i\, Op\left(\sqrt{-\tau}\right) u = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = V$ constant

$$\partial_{\mathbf{n}} u - i\, e^{itV} Op\left(\sqrt{-\tau}\right) \left(e^{-itV} u\right) = 0 \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = V(t)$: Gauge change

Setting $v(x,t) = u(x,t)e^{-i\mathcal{V}(t)}$ with $\mathcal{V}(t) = \int_0^t V(s)\, ds$,

then v is solution of the free-potential equation.

$$\partial_{\mathbf{n}} u - i\, e^{i\mathcal{V}(t)} Op\left(\sqrt{-\tau}\right) \left(e^{-i\mathcal{V}(t)} u\right) = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = 0$ WITHOUT POTENTIAL

$$\partial_{\mathbf{n}} u - i\, Op\left(\sqrt{-\tau}\right) u = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = V$ CONSTANT

$$\partial_{\mathbf{n}} u - i\, e^{itV} Op\left(\sqrt{-\tau}\right) \left(e^{-itV} u\right) = 0 \quad \text{on } \Sigma_T.$$

$$\partial_{\mathbf{n}} u - i\, Op\left(\sqrt{-\tau + V}\right)(u) = 0, \quad \text{on } \Sigma_T.$$

- $\mathscr{V} = V(t)$: GAUGE CHANGE

Setting $v(x,t) = u(x,t)e^{-i\mathcal{V}(t)}$ with $\mathcal{V}(t) = \int_0^t V(s)\, ds$,

then v is solution of the free-potential equation.

$$\partial_{\mathbf{n}} u - i\, e^{i\mathcal{V}(t)}\, Op\left(\sqrt{-\tau}\right) \left(e^{-i\mathcal{V}(t)}\, u\right) = 0, \quad \text{on } \Sigma_T.$$

1) Gauge change

- $v(x,t) = e^{-i\mathcal{V}(x,t)}u(x,t), \qquad$ with $\quad \mathcal{V}(x,t) = \int_0^t V(x,s)\,ds.$
- No longer exact
- Involves operators $\quad e^{i\mathcal{V}(x,t)} Op\left(\sqrt{-\tau}\right)\left(e^{-i\mathcal{V}(x,t)}\,u\right)$

$$ABC_1^4: \quad \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2}\left(e^{-i\mathcal{V}}u\right) + i\,\frac{\partial_{\mathbf{n}} V}{4} e^{i\mathcal{V}} I_t\left(e^{-i\mathcal{V}}u\right) = 0$$

2) Direct method

- No gauge change
- Involves operators $\quad Op\left(\sqrt{-\tau + V(x,t)}\right)(u)$

$$\widetilde{ABC_2^4}: \quad \partial_{\mathbf{n}} u - i\sqrt{i\partial_t + V}\,u + \frac{1}{4}\,\partial_{\mathbf{n}} V\,(i\partial_t + V)^{-1}\,u = 0$$

- Strategies equivalent for $V = V(x)$, non equivalent for $V = V(x,t)$
- In both cases, approximate boundary conditions, of different orders M.

Nonlinearity $f(u) = g(|u|^2)$

- Cubic $f(u) = q|u|^2$ / quintic $f(u) = q|u|^4$
- $f(u) = n_2|u|^2 + n_4|u|^4$, $f(u) = \frac{|u|^2}{1+\sigma|u|^2}$
- Mixed: $\mathscr{V} = \alpha x^2 + \beta|u|^2$

ABCs for a potential $V(x,t)$

$$ABC_1^4: \quad \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2}\left(e^{-i\mathcal{V}} u\right) + i\,\frac{\partial_{\mathbf{n}} V}{4} e^{i\mathcal{V}} I_t \left(e^{-i\mathcal{V}} u\right) = 0$$

$$\widetilde{ABC_2^4}: \quad \partial_{\mathbf{n}} u - i\sqrt{i\partial_t + V}\, u + \frac{1}{4}\,\partial_{\mathbf{n}} V \left(i\partial_t + V\right)^{-1} u = 0$$

with the phase function: $\mathcal{V}(x,t) = \int_0^t V(x,s)\,ds$

NONLINEARITY $f(u) = g(|u|^2)$

- Cubic $f(u) = q|u|^2$ / quintic $f(u) = q|u|^4$
- $f(u) = n_2|u|^2 + n_4|u|^4$, $f(u) = \frac{|u|^2}{1+\sigma|u|^2}$
- Mixed: $\mathscr{V} = \alpha x^2 + \beta|u|^2$

ABCs FOR A NONLINEARITY

$$NLABC_1^4 : \quad \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2} \left(e^{-i\mathcal{V}} u \right) + i \frac{\partial_{\mathbf{n}} f(u)}{4} e^{i\mathcal{V}} I_t \left(e^{-i\mathcal{V}} u \right) = 0$$

$$\widetilde{NLABC}_2^4 : \quad \partial_{\mathbf{n}} u - i\sqrt{i\partial_t + f(u)}\, u + \frac{1}{4}\, \partial_{\mathbf{n}} f(u) \left(i\partial_t + f(u) \right)^{-1} u = 0$$

New phase function: $\mathcal{V}(x,t,u) = \int_0^t f(x, u(x,s))\, ds$

PROPOSITION ($NLABC_1^2$)

Let $u_0 \in L^2(\Omega)$ be compactly supported in Ω, and let $f \in C(\mathbb{R};\mathbb{R})$.
Assume that there exists a solution $u \in C^1(]0;T[;H^1(\Omega))$ of the problem:

$$\begin{cases} i\partial_t u + \partial_x^2 u + f(u)u = 0, & \text{in } \Omega_T, \\ \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2} \left(e^{-i\mathcal{V}} u \right) = 0, & \text{on } \Sigma_T, \\ u(x,0) = u_0(x), & \text{on } \Omega, \end{cases} \tag{1}$$

where $\mathcal{V}(x,t,u) = \displaystyle\int_0^t f(x,u)(x,s)\, ds.$

Then, u satisfies:

$$\forall t > 0, \quad \|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}.$$

Duràn-Sanz-Serna scheme

$$i\,\frac{u^{n+1}-u^n}{\Delta t}+\partial_x^2\,\frac{u^{n+1}+u^n}{2}+f\left(\frac{u^{n+1}+u^n}{2}\right)\frac{u^{n+1}+u^n}{2}=0$$

Scheme

$$\begin{cases} \dfrac{2i}{\Delta t}v^{n+1}+\partial_x^2 v^{n+1}+f(v^{n+1})\,v^{n+1}=\dfrac{2i}{\Delta t}\,u^n, & \text{on } \Omega_T,\\ \partial_{\mathbf{n}}v^{n+1}+\Lambda_p^{M,n+1}v^{n+1}=0, & \text{on } \Sigma_T, \quad p=1,2,\\ +\ \text{I.C.} \end{cases}$$

with $v^{n+1}=u^{n+1/2}=\dfrac{u^{n+1}+u^n}{2}$.

Discretized ABC

- discrete convolution (gauge change)
- or Padé approximants (direct method)

- PRINCIPLE: Solve the equation $i\partial_t u + \Delta u + f(u)u = 0$ through the resolution of the system:

$$\begin{cases} i\partial_t u + \Delta u + \Upsilon u = 0, & \text{on } \Omega_T, \\ \Upsilon = f(u), & \text{on } \Omega_T. \end{cases}$$

- SEMI DISCRETIZATION

$$\begin{cases} i\,\dfrac{u^{n+1}-u^n}{\Delta t} + \Delta u^{n+1/2} + \Upsilon^{n+1/2} u^{n+1/2} = 0, \\ \dfrac{\Upsilon^{n+3/2}+\Upsilon^{n+1/2}}{2} = f(u^{n+1}), \end{cases} \quad \text{for } 0 \le n \le N$$

where $\Upsilon^{n+1/2} = \dfrac{\Upsilon^{n+1}+\Upsilon^n}{2}$, $\Upsilon^{-1/2} = \Upsilon^{1/2} = f(u^0)$.

- INTERESTS : Speed: equivalent to one fixed point iteration

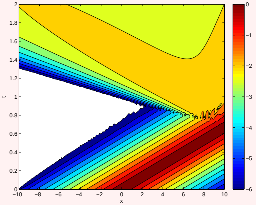
  Simplicity: same code as for a space- and time-depending potential $V(\mathbf{x},t)$

  Preservation of the invariants: mass, energy

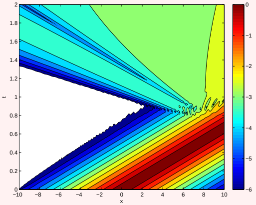
Besse, 2004

Initial datum $u_0 = \sqrt{\frac{2a}{q}} \cdot \operatorname{sech}(\sqrt{a}x) \exp(i\frac{c}{2}x)$ (soliton) with $q = 1$, $a = 2$, $c = 15$

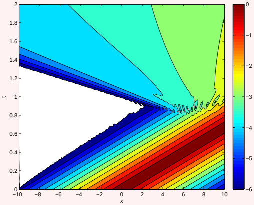
$\Omega_T = [-10; 10] \times [0; 2]$, $\Delta x = 5 \cdot 10^{-3}$, $\Delta t = 10^{-3}$, 50 Padé functions



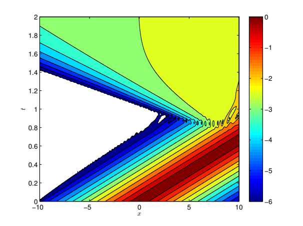
ABC_0



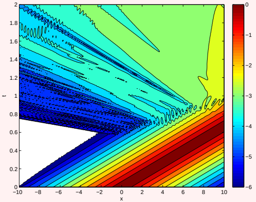
$NLABC_1^2$



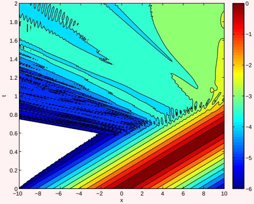
$NLABC_1^4$



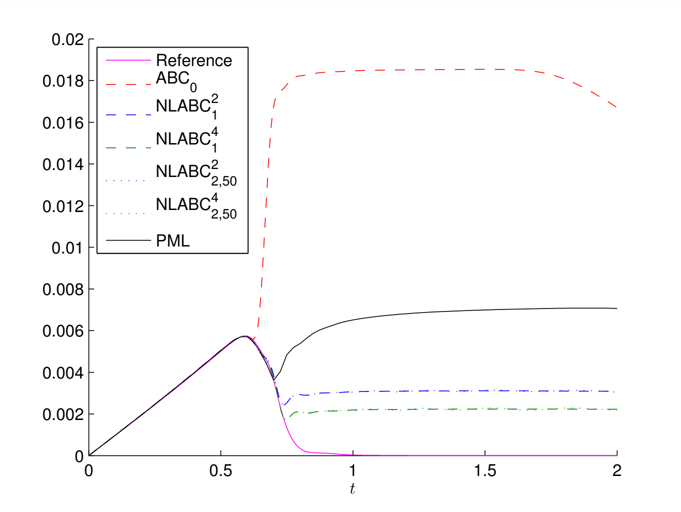
PML



$NLABC_2^2$



$NLABC_2^4$

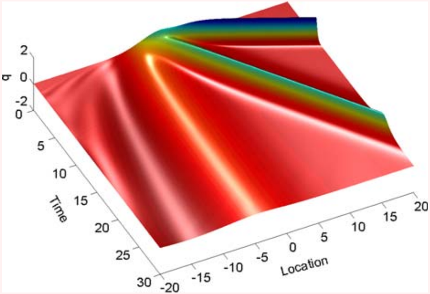


Relative L^2 error $\dfrac{\|u(t) - u_{ex}(t)\|_{L^2(\Omega)}}{\|u_{ex}(t)\|_{L^2(\Omega)}}$ for linear and nonlinear ABCs

- Schrödinger : Szeftel (06) (paradifferential technique), Zheng (06) : use of inverse scattering for cubic NLS, exact TBC
- modified KdV : Zheng (06)

$$u_t \pm 6u^2 u_x + u_{xxx} = 0$$

Use of inverse scattering to get exact TBC.
Example (Zheng) : solitary waves generated by an initial Gaussian profile $u_0(x) = \exp(-1.5x^2)$.



Schrödinger 2D

$$\begin{cases} i\partial_t u + \partial_x^2 u + \partial_y^2 u + V(x,y,t)\, u = 0, & (x,y) \in \mathbb{R}^2,\ t > 0 \\ u(x,y,0) = u_0(x,y), & (x,y) \in \mathbb{R}^2 \end{cases}$$

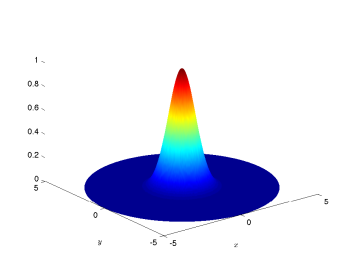
with $\mathrm{Supp}(u_0) \subset \Omega$.

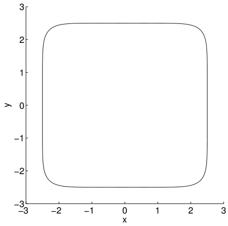
$\mathscr{V} = V(t)$: Gauge change

Setting $\mathcal{V}(t) = \int_0^t V(s)ds$ and $v(x,y,t) = e^{-i\mathcal{V}(t)} u(x,y,t)$,

then v is solution of $i\partial_t v + \partial_x^2 v + \partial_y^2 v = 0$.

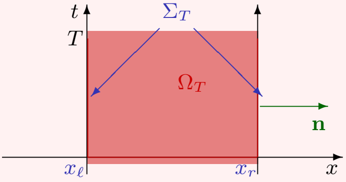
Profile of solutions





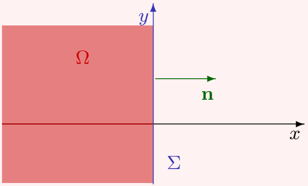
In one dimension of space

- Domain $\Omega_T = [x_\ell; x_r] \times [0; T]$
- Boundary $\Sigma = \{x_\ell; x_r\}$
- Outwardly directed normal $\mathbf{n}$ directed according to x
- Fourier transform w.r.t. t (x fixed)



In dimension two with straight boundary

- Domain: half-plane $\Omega = \{x < 0\}$
- Normal $\mathbf{n}$ directed according to x
- Partial Fourier transform w.r.t. (t, y) (x fixed)



$$\partial_x^2 + i\partial_t + \partial_y^2 = 0$$

$i\partial_t + \partial_y^2$ plays the role of $i\partial_t$ in 1D

∂_x^2 plays the role of ∂_x^2

Factorization

- 1D without potential: $\partial_x^2 + i\partial_t = (\partial_{\mathbf{n}} + i\sqrt{i\partial_t})(\partial_{\mathbf{n}} - i\sqrt{i\partial_t})$
- 1D with variable potential: $\partial_x^2 + i\partial_t + V = (\partial_{\mathbf{n}} + i\sqrt{i\partial_t + V})(\partial_{\mathbf{n}} - i\sqrt{i\partial_t + V}) + R$
- 2D with straight boundary:

$$\partial_x^2 + i\partial_t + \partial_y^2 + V = \left(\partial_{\mathbf{n}} + i\sqrt{i\partial_t + \partial_y^2 + V}\right)\left(\partial_{\mathbf{n}} - i\sqrt{i\partial_t + \partial_y^2 + V}\right) + R$$

Transparent boundary condition when $V|_{\{x\geq 0\}} = 0$:

$$\partial_{\mathbf{n}} u - i\sqrt{i\partial_t + \partial_y^2}\, u = 0, \quad \text{on } \Sigma_T.$$

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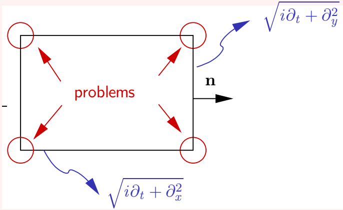
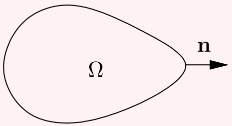
Bounded domain with straight boundary:

Singularities caused by corners



- Consideration of the geometry: convex domain of general, smooth boundary; curvature κ.



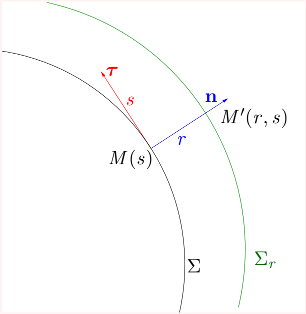
- Local parametrization of the boundary
  normal variable r, curvilinear abscissa s

$$\Delta = \partial_r^2 + \kappa_r \partial_r + h^{-1}\partial_s\left(h^{-1}\partial_s\right)$$

$\kappa_r = h^{-1}\kappa$: curvature on the parallel surface Σ_r to Σ
$h(r,s) = 1 + r\kappa$

$$L = i\partial_t + \Delta + V$$

$$\Rightarrow \quad L = \partial_r^2 + i\partial_t + \kappa_r\partial_r + h^{-1}\partial_s\left(h^{-1}\partial_s\right) + V$$



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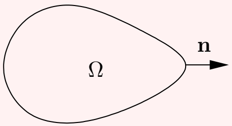

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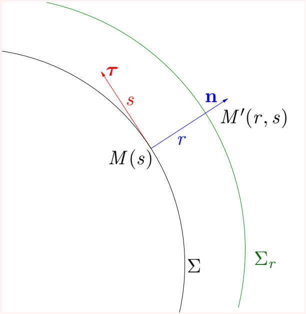
$$L = \partial_x^2 + i\partial_t + \partial_y^2 + V$$



- Partial Fourier transform w.r.t. (s,t); covariables (ξ,τ)

$u(r, s, t)$

- Pseudodifferential operator $P(r, s, t, \partial_s, \partial_t)$ defined through its total symbol $p(r, s, t, \xi, \tau)$ in Fourier space for $\mathscr{F}_{(s,t)}$ (ξ and τ covariables of s and t)

$$\begin{aligned} P(r, s, t, \partial_s, \partial_t)u(r, s, t) &= \mathscr{F}^{-1}_{(s,t)}\Big(p(r, s, t, \xi, \tau)\hat{u}(r, \xi, \tau)\Big) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} p(r, s, t, \xi, \tau)\hat{u}(r, \xi, \tau)e^{is\xi}e^{it\tau}d\xi d\tau \end{aligned}$$

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$$= \int_{\mathbb{R}}\int_{\mathbb{R}} p(r, s, t, \xi, \tau)\hat{u}(r, \xi, \tau)e^{is\xi}e^{it\tau}d\xi d\tau$$

- Composition rule: $\sigma(AB) \sim \displaystyle\sum_{|\alpha|=0}^{+\infty} \frac{(-i)^{|\alpha|}}{\alpha!}\,\partial^{\alpha}_{(\xi,\tau)}\sigma(A)\,\partial^{\alpha}_{(s,t)}\sigma(B)$

- Homogeneity according to the couple (ξ^2, τ) : $\sqrt{-\tau - \xi^2}$ is of order 1

  Order m : $f(r, s, t, \lambda\xi, \lambda^2\tau) = \lambda^m f(r, s, t, \xi, \tau)$

- Asymptotic expansion in homogeneous symbols: $P \in OPS^m$ if:

$$p(r, s, t, \xi, \tau) \sim \sum_{j=0}^{+\infty} p_{m-j}(r, s, t, \xi, \tau),$$

where p_{m-j} is homogeneous of order $m - j$; p_m is the principal symbol.

1 - Gauge change

- Change of unknown (which solve the case $\mathscr{V} = V(t)$)

$$v = e^{-i\mathcal{V}} u \quad \text{avec} \quad \mathcal{V}(r,s,t) = \int_0^t V(r,s,\sigma)\, d\sigma$$

- We work on the equation written for v:

$$i\partial_t v + \partial_r^2 v + (\kappa_r + F)\partial_r v + h^{-1}\partial_s(h^{-1}\partial_s v) + G\, v = 0$$

2 - Direct method

- We work directly on the original equation (with local coordinates)

$$i\partial_t u + \partial_r^2 u + \kappa_r \partial_r u + h^{-1}\partial_s(h^{-1}\partial_s u) + V\, u = 0$$

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$\Rightarrow$ Absorbing boundary condition for

$$Lw = i\partial_t w + \partial_r^2 w + (\kappa_r + A)\partial_r w + h^{-1}\partial_s(h^{-1}\partial_s w) + B\, w = 0, \qquad \text{with}$$

- $A = F(r,s,t)$ and $B = G(r,s,t)$ if $w = v = e^{-i\mathcal{V}} u$
- $A = 0$ and $B = V(r,s,t)$ if $w = u$,

Unification of both strategies: General Schrödinger operator:

$$L = i\partial_t + \partial_r^2 + (\kappa_r + A)\partial_r + h^{-1}\partial_s(h^{-1}\partial_s) + B$$

Factorization of Nirenberg-type of operator L

$$L = (\partial_r + i\Lambda^-)(\partial_r + i\Lambda^+) + R \qquad \text{on } \Sigma_r,$$

where :
$\Lambda^{\pm}(r, s, t, \partial_s, \partial_t) \in OPS^1$ is a pseudodifferential operator of order 1,
$R \in OPS^{-\infty}$,
and Λ^+ admits the asymptotic expansion in homogeneous symbols:

$$\sigma(\Lambda^+) = \lambda^+ \sim \sum_{j=0}^{+\infty} \lambda^+_{1-j} = \lambda^+_1 + \lambda^+_0 + \lambda^+_{-1} + \lambda^+_{-2} + \dots$$

with λ^+_{1-j} homogeneous of order $1-j$ according to the couple (ξ^2, τ).

The knowledge of the symbols (λ^+_j) describes entirely the operator Λ^+.

Back on the surface Σ:

$$\widetilde{\Lambda^+} = \Lambda^+{}_{|r=0}$$

$$\widetilde{\lambda}_j = \left(\lambda^+_j\right)_{|r=0}$$

Absorbing boundary condition which expresses that the wave is outgoing:

$$\partial_{\mathbf{n}} w + i\,\widetilde{\Lambda^+}\,w = 0 \qquad \text{where} \quad \widetilde{\Lambda^+} = Op\left(\sum_{j=0}^{+\infty} \widetilde{\lambda}_{1-j}\right)$$

$$\partial_{\mathbf{n}} w \quad + i \sum_{j=0}^{+\infty} Op\left(\widetilde{\lambda}_{1-j}\right) w \quad = 0\ , \quad \text{on } \Sigma_T$$

Identification of the principal symbol λ_1^+

Outgoing wave $\quad \mathrm{Im}(\lambda_1^+(s,t,\xi,\tau)) \leq 0, \quad \text{for } |\tau| \gg 1$

Strategy 1 $\quad \lambda_1^+ = -\sqrt{-\tau - h^{-2}\xi^2}$

Strategy 2 $\quad \lambda_1^+ = -\sqrt{-\tau - h^{-2}\xi^2 + ih^{-1}(\partial_s h^{-1})\xi + V}$

Asymptotic expansion: $\widetilde{\lambda}_j$ are functions of $\sqrt{-\tau - \xi^2}$ (resp. $\sqrt{-\tau - \xi^2 + V}$).

$\Longrightarrow$ non local operators w.r.t to time AND space

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$$\partial_{\mathbf{n}} w_M + i \sum_{j=0}^{M-1} Op\left(\widetilde{\lambda}_{1-j}\right) w_M = 0\ , \quad \text{on } \Sigma_T$$

Approximate condition of order M

IDENTIFICATION OF THE PRINCIPAL SYMBOL λ_1^+

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$\Longrightarrow$ non local operators w.r.t to time AND space

Approach valid for both strategies.

- Taylor expansion of the symbols, under the assumption $|\tau| \gg \xi^2$.

$$-\tau - \xi^2 + b = -\tau \left(1 + \frac{\xi^2}{\tau} - \frac{b}{\tau}\right)$$

Thereby:

$$\sqrt{-\tau - \xi^2 + b} \approx \sqrt{-\tau}\left(1 + \frac{\xi^2}{2\tau} - \frac{b}{2\tau}\right) = \sqrt{-\tau} - \frac{\xi^2}{2}\frac{1}{\sqrt{-\tau}} + \frac{b}{2}\frac{1}{\sqrt{-\tau}}$$

- Then

$$Op\left(\sqrt{-\tau}\right) = e^{i\pi/4}\partial_t^{1/2}, \qquad Op\left(\xi\right) = -i\partial_s,$$

$$Op\left(\frac{1}{\sqrt{-\tau}}\right) = e^{-i\pi/4} I_t^{1/2}, \qquad Op\left(\xi^2\right) = -\partial_s^2 = -\Delta_\Sigma,$$

$$Op\left(\frac{1}{\tau}\right) = i\, I_t$$

$\Longrightarrow$ The operators are localized in space only

We approximate ABC_2^1 (direct method) : $\partial_{\mathbf{n}} u - i\sqrt{i\partial_t + \Delta_\Sigma + V}\, u = 0, \quad$ on Σ_T.

Padé approximants of order m

$$\sqrt{z} \approx R_m(z) = \sum_{k=1}^{m} \frac{a_k^m z}{z + d_k^m} = \sum_{k=0}^{m} a_k^m - \sum_{k=1}^{m} \frac{a_k^m d_k^m}{z + d_k^m}$$

ABC_2^1 becomes $ABC_{2,P}^1$: $\qquad \partial_{\mathbf{n}} u - i\, R_m(i\partial_t + \Delta_\Sigma + V)\, u = 0$

$$\partial_{\mathbf{n}} u - i\left(\sum_{k=0}^{m} a_k^m\right) u + i \sum_{k=1}^{m} a_k^m d_k^m \underbrace{(i\partial_t + \Delta_\Sigma + V + d_k^m)^{-1} u}_{\varphi_k} = 0$$

We introduce m auxiliary functions defined on Σ

$$(i\partial_t + \Delta_\Sigma + V + d_k^m)\varphi_k = u, \quad 1 \leq k \leq m$$

We get a coupling between u and $(\varphi_k)_{1\leq k\leq m}$ on Σ

$$\begin{cases} \left(\sum_{k=0}^{m} a_k^m\right) u - \sum_{k=1}^{m} a_k^m d_k^m \varphi_k \\ i\partial_t \varphi_k + \Delta_\Sigma \varphi_k + (V + d_k^m)\varphi_k = u, \qquad 1 \leq k \leq m. \end{cases}$$

$\Longrightarrow$ The operators are localized in space AND time

- Gauge change

$$\mathsf{ABC}^2_{1,T} \qquad \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2}\left(e^{-i\mathcal{V}} u\right) + \frac{\kappa}{2} u$$

$$\mathsf{ABC}^3_{1,T} \qquad - e^{i\pi/4} e^{i\mathcal{V}} \left(\frac{\kappa^2}{8} + \frac{\Delta_\Sigma}{2} + i\partial_s \mathcal{V} \partial_s + \frac{1}{2}\left(i\partial_s^2 \mathcal{V} - (\partial_s \mathcal{V})^2\right)\right) I_t^{1/2}\left(e^{-i\mathcal{V}} u\right)$$

$$\mathsf{ABC}^4_{1,T} \qquad + i e^{i\mathcal{V}} \left(\frac{\partial_s(\kappa \partial_s)}{2} + \frac{\kappa^3 + \partial_s^2 \kappa}{8} + \frac{i\partial_s \kappa \partial_s \mathcal{V}}{2}\right) I_t \left(e^{-i\mathcal{V}} u\right)$$

$$- i \frac{\mathrm{sg}(\partial_{\mathbf{n}} V)}{4} \sqrt{|\partial_{\mathbf{n}} V|}\, e^{i\mathcal{V}} I_t \left(\sqrt{|\partial_{\mathbf{n}} V|}\, e^{-i\mathcal{V}} u\right) = 0$$

- Direct method

$$\mathsf{ABC}^2_{2,T} \qquad \partial_{\mathbf{n}} u + e^{-i\pi/4} \partial_t^{1/2} u + \frac{\kappa}{2} u$$

$$\mathsf{ABC}^3_{2,T} \qquad - e^{i\pi/4} \left(\frac{\kappa^2}{8} + \frac{\Delta_\Sigma}{2} \right) I_t^{1/2} u - e^{i\pi/4} \frac{\mathrm{sg}(V)}{2} \sqrt{|V|}\, I_t^{1/2} \left(\sqrt{|V|}\, u \right)$$

$$\mathsf{ABC}^4_{2,T} \qquad + i \left(\frac{\partial_s(\kappa \partial_s)}{2} + \frac{\kappa^3 + \partial_s^2 \kappa}{8} \right) I_t u - i \frac{\mathrm{sg}(\partial_{\mathbf{n}} V)}{4} \sqrt{|\partial_{\mathbf{n}} V|}\, I_t \left(\sqrt{|\partial_{\mathbf{n}} V|}\, u \right) = 0$$

- Gauge change

$$\mathsf{ABC}^1_{1,P} \qquad \partial_{\mathbf{n}} u - i e^{i\mathcal{V}} \sqrt{i\partial_t + \Delta_\Sigma} \left(e^{-i\mathcal{V}} u \right)$$

$$\mathsf{ABC}^2_{1,P} \qquad + \frac{\kappa}{2} u + \partial_s \mathcal{V} e^{i\mathcal{V}} \partial_s \left(i\partial_t + \Delta_\Sigma \right)^{-1/2} \left(e^{-i\mathcal{V}} u \right)$$

$$- \frac{\kappa}{2} e^{i\mathcal{V}} \left(i\partial_t + \Delta_\Sigma \right)^{-1} \Delta_\Sigma \left(e^{-i\mathcal{V}} u \right) = 0$$

- Direct method

$$\mathsf{ABC}^1_{2,P} \qquad \partial_{\mathbf{n}} u - i \sqrt{i\partial_t + \Delta_\Sigma + V}\, u$$

$$\mathsf{ABC}^2_{2,P} \qquad + \frac{\kappa}{2} u - \frac{\kappa}{2} \left(i\partial_t + \Delta_\Sigma + V \right)^{-1} \Delta_\Sigma u = 0$$

For conditions $ABC_{1,T}^{M}$ and $ABC_{2,T}^{M}$ (Taylor)

Proposition

Let $u_0 \in L^2(\Omega)$ s.t. $\mathrm{Supp}(u_0) \subset \Omega$. Let $V \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+, \mathbb{R})$ and u a solution of

$$\begin{cases} i\partial_t u + \partial_x^2 u + Vu = 0, & \text{in } \Omega_T, \\ \partial_{\mathbf{n}} u + \Lambda_{j,T}^{M} u = 0, & \text{on } \Sigma_T, \quad j = 1, 2, \\ u(x,0) = u_0(x), & \forall x \in \Omega. \end{cases}$$

We assume that we are in the quasi-hyperbolic area $\mathcal{H} = \{-\tau - \xi^2 > 0\}$.

Then, u fulfills the following energy bound

$$\forall t > 0, \quad \|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}, \tag{EI}$$

- for $M = 2$,
- for $M = 3$ if V is positive on Σ ($ABC_{2,T}^{3}$) or if V and Ω are radially symmetrical ($ABC_{1,T}^{3}$),
- for $M = 4$ if $\partial_{\mathbf{n}} V$ is of constant sign on Σ, and if furthermore $\partial_{\mathbf{n}} V$ is positive on Σ ($ABC_{2,T}^{4}$) or if the problem is radially symmetrical ($ABC_{1,T}^{4}$),

which implies the uniqueness of the solution.

Time step $\Delta t = T/N$, $\quad t_n = n\Delta t$,

$u^n(r,s) \approx u(r,s,t_n)$ for $0 \le n \le N$.

Interior equation:

Semi discrete Crank-Nicolson scheme: symmetrical, inconditionally stable

$$i\,\frac{u^{n+1}-u^n}{\Delta t} + \Delta\,\frac{u^{n+1}+u^n}{2} + \frac{V^{n+1}+V^n}{2}\,\frac{u^{n+1}+u^n}{2} = 0$$

for $n = 0,\ldots,N-1$.

Implementation:

$$\frac{2i}{\Delta t}\,v^{n+1} + \Delta\,v^{n+1} + V^{n+1/2}\,v^{n+1} = \frac{2i}{\Delta t}\,u^n$$

with $v^{n+1} = u^{n+1/2} = \dfrac{u^{n+1}+u^n}{2}$, $\;V^{n+1/2} = \dfrac{V^{n+1}+V^n}{2}$.

Space discretization : Finite Element Method

$$ABC^2_{1,T}: \qquad \partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2} \left(e^{-i\mathcal{V}} u \right) + \frac{\kappa}{2} u = 0, \quad \text{on } \Sigma_T$$

$$\partial_{\mathbf{n}} v^{n+1} + e^{-i\pi/4} e^{i\mathscr{W}^{n+1}} \sqrt{\frac{2}{\Delta t}} \sum_{k=0}^{n+1} \beta_{n+1-k}\, e^{-i\mathscr{W}^{k}} v^k + \frac{\kappa}{2} v^{n+1} = 0, \quad n \geq 0.$$

$$\partial_{\mathbf{n}} v^{n+1} + \left(e^{-i\pi/4} \beta_0 + \frac{\kappa}{2} \right) v^{n+1} + e^{-i\pi/4} e^{i\mathscr{W}^{n+1}} \sqrt{\frac{2}{\Delta t}} \sum_{k=0}^{n} \beta_{n+1-k}\, e^{-i\mathscr{W}^{k}} v^k = 0.$$

Proposition

For discretized boundary conditions $ABC^M_{1,T}$ or $ABC^M_{2,T}$, we have

$$\forall n \in \{0, \ldots, N\}, \quad \|u^n\|_{L^2(\Omega)} \leq \|u^0\|_{L^2(\Omega)}, \tag{2}$$

under the semi discrete assumptions equivalent to those of the continuous case.

The inconditional stability of the scheme is preserved.

System associated to the boundary condition $ABC^1_{2,P}$:
$\partial_{\mathbf{n}} u - i\sqrt{i\partial_t + \Delta_\Sigma + V}\, u = 0$

$$\begin{cases} i\partial_t u + \Delta u + V\, u = 0, & \text{on } \Omega_T, \\ \partial_{\mathbf{n}} u - i\left(\sum_{k=0}^{m} a_k^m\right) u + i \sum_{k=0}^{m} a_k^m d_k^m \varphi_k = 0, & \text{on } \Sigma_T, \\ (i\partial_t + \Delta_\Sigma + V + d_k^m)\, \varphi_k = u, & \text{on } \Sigma_T, \quad \text{pour } 1 \leq k \leq m. \end{cases}$$

Crank-Nicolson scheme ($m+1$ equations):

$$\begin{cases} \frac{2i}{\Delta t} u^{n+1/2} + \Delta u^{n+1/2} + V^{n+1/2} u^{n+1/2} = \frac{2i}{\Delta t} u^n, & \text{on } \Omega, \\ \partial_{\mathbf{n}} u^{n+1/2} - i\left(\sum_{k=0}^{m} a_k^m\right) u^{n+1/2} + i \sum_{k=0}^{m} a_k^m d_k^m \varphi_k^{n+1/2} = 0, & \text{on } \Sigma, \\ \frac{2i}{\Delta t} \varphi_k^{n+1/2} + \Delta_\Sigma \varphi_k^{n+1/2} + V^{n+1/2} \varphi_k^{n+1/2} + d_k^m \varphi_k^{n+1/2} = u^{n+1/2} + \frac{2i}{\Delta t} \varphi_k^n, & \text{on } \Sigma. \end{cases}$$

- System in $(u^{n+1/2}, \varphi_1^{n+1/2}, \ldots, \varphi_m^{n+1/2})$, coupled through the boundary Σ.
- Entirely local / No stability result

Initial datum: $u_0(x,y) = e^{-(x^2+y^2)-ik_0 x}$, with $k_0 = 10$

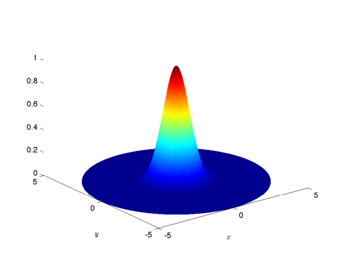
$\Delta t = 10^{-3}$, $T = 1$

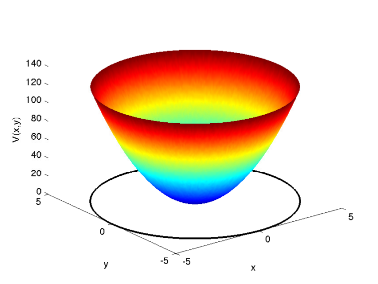
Disk meshed with $1\,700\,000$ triangles

25 Padé functions

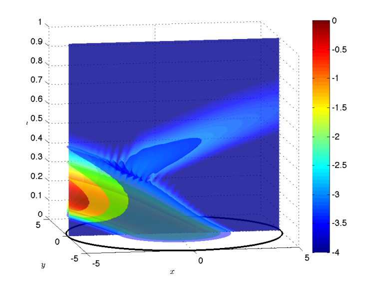
Logarithmic levels, threshold 10^{-4}

DOMAINS: disk / mediator / "smoothed square"





POTENTIAL ON DISK



PROFILE OF AN APPROXIMATE SOLUTION

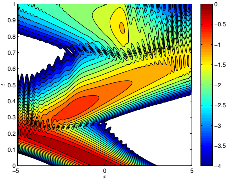
$V(r) = 5r^2$

$r = \sqrt{x^2 + y^2}$

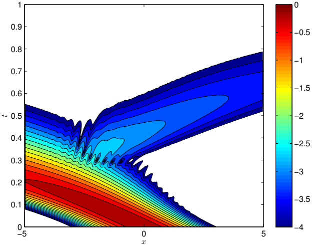
Gauge change

Direct method

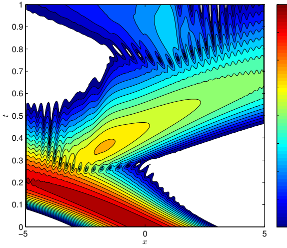
Taylor approach



Condition without potential

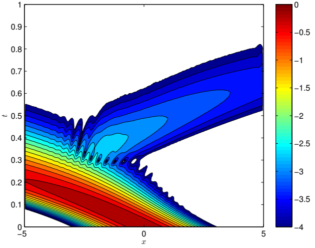


$ABC^4_{1,T}$

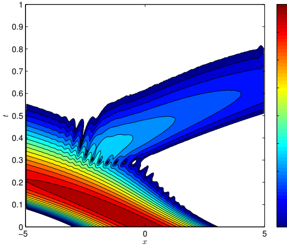


$ABC^4_{2,T}$

Padé approach

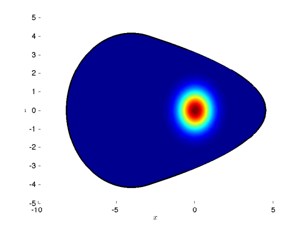


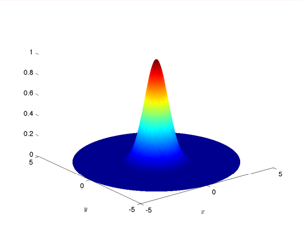
$ABC^2_{1,P}$



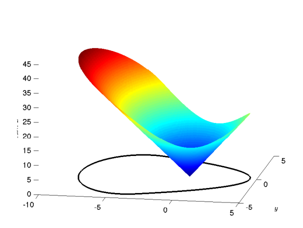
$ABC^2_{2,P}$

Initial datum

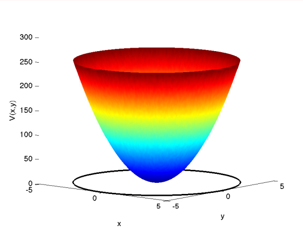




Profile of the potential



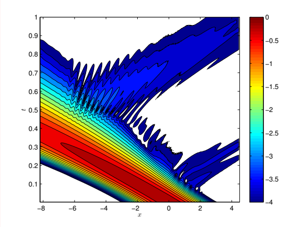
$$V(x,y) = 5\sqrt{x^2+y^2}$$



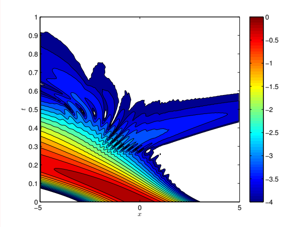
Potential at $t = 0$

$$V(x,y,t) = 5(x^2+y^2)(1+\cos(4\pi t))$$

ABC



$ABC^2_{2,P}$



$ABC^4_{1,T}$

Extension to nonlinearities (2D case)

$$\mathcal{V}(x,y,t) = \int_0^t f(u)$$

- Taylor approach, gauge change

$NLABC^2_{1,T}$ $$\partial_{\mathbf{n}} u + e^{-i\pi/4} e^{i\mathcal{V}} \partial_t^{1/2} \left(e^{-i\mathcal{V}} u \right) + \frac{\kappa}{2} u = 0$$

- Taylor approach, direct method

$NLABC^2_{2,T}$ $$\partial_{\mathbf{n}} u + e^{-i\pi/4} \partial_t^{1/2} u + \frac{\kappa}{2} u$$

$NLABC^3_{2,T}$ $$- e^{i\pi/4} \left(\frac{\kappa^2}{8} + \frac{\Delta_\Sigma}{2} \right) I_t^{1/2} u - e^{i\pi/4} \frac{\text{sg}(f(u))}{2} \sqrt{|f(u)|}\, I_t^{1/2} \left(\sqrt{|f(u)|}\, u \right)$$

$NLABC^4_{2,T}$ $$+ i \left(\frac{\partial_s(\kappa \partial_s)}{2} + \frac{\kappa^3 + \partial_s^2 \kappa}{8} \right) I_t u$$

$$-i \frac{\text{sg}(\partial_{\mathbf{n}} f(u))}{4} \sqrt{|\partial_{\mathbf{n}} f(u)|}\, I_t \left(\sqrt{|\partial_{\mathbf{n}} f(u)|}\, u \right) = 0$$

- Padé approach, direct method

$NLABC^1_{2,P}$ $$\partial_{\mathbf{n}} u - i \sqrt{i\partial_t + \Delta_\Sigma + f(u)}\, u$$

$NLABC^2_{2,P}$ $$+ \frac{\kappa}{2} u - \frac{\kappa}{2} \left(i\partial_t + \Delta_\Sigma + f(u) \right)^{-1} \Delta_\Sigma u = 0$$

Fixed point method

- Interior equation: Duràn - Sanz-Serna scheme

$$i\,\frac{u^{n+1}-u^n}{\Delta t}+\Delta\,\frac{u^{n+1}+u^n}{2}+f\left(\frac{u^{n+1}+u^n}{2}\right)\frac{u^{n+1}+u^n}{2}=0$$

- Energy bound: $\|u^n\|_{L^2(\Omega)}\leq\|u^0\|_{L^2(\Omega)}$
  - for $NLABC^2_{2,T}$ and $NLABC^2_{1,T}$
  - for $NLABC^3_{2,T}$ when $f(u)\geq 0$.

Relaxation method

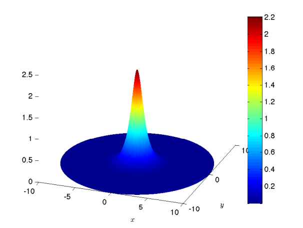
- $$\begin{cases} i\,\dfrac{u^{n+1}-u^n}{\Delta t}+\Delta u^{n+1/2}+\Upsilon^{n+1/2}u^{n+1/2}=0,\\ \dfrac{\Upsilon^{n+3/2}+\Upsilon^{n+1/2}}{2}=f(u^{n+1}),\end{cases}\qquad \text{for } 0\leq n\leq N$$

with $\Upsilon^{n+1/2}=\dfrac{\Upsilon^{n+1}+\Upsilon^n}{2},\ \ \Upsilon^{-1/2}=f(u^0).$

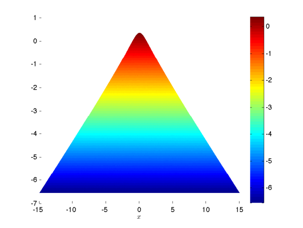
- Cubic equation $i\partial_t + \Delta u + q|u|^2 u = 0$
- Numerical construction of the soliton: the search of stationary solutions leads to

$$\begin{cases} \partial_r^2 \psi + \frac{1}{r}\partial_r \psi - \psi + q|\psi|^2\psi = 0, \quad 0 < r < R, \\ \psi'(0) = 0, \quad \psi(0) = \beta, \end{cases}$$

solved by a shooting method [Di Menza 09]

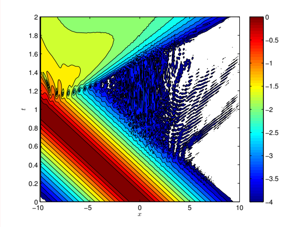


$|u_0|$ on $R = 10$

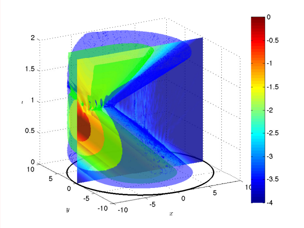


$|u_0|$ in logarithmic scale on $R = 15$

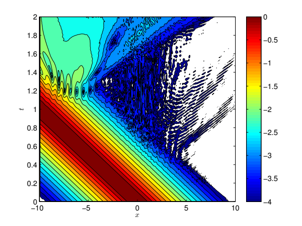
WITHOUT POTENTIAL



ABC_0



GAUGE CHANGE



$NLABC^2_{1,T}$

Initial datum: soliton

Domain: disc of radius 10

$\Delta t = 2 \times 10^{-3}$

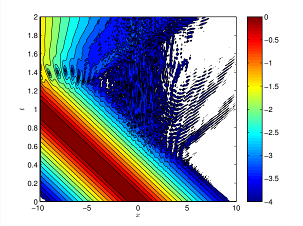
1 700 000 triangles

$k_0 = 5$

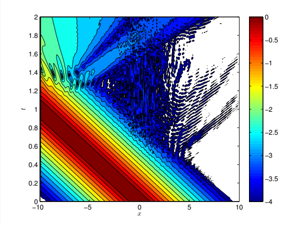
$T = 2$

Logarithmic scale

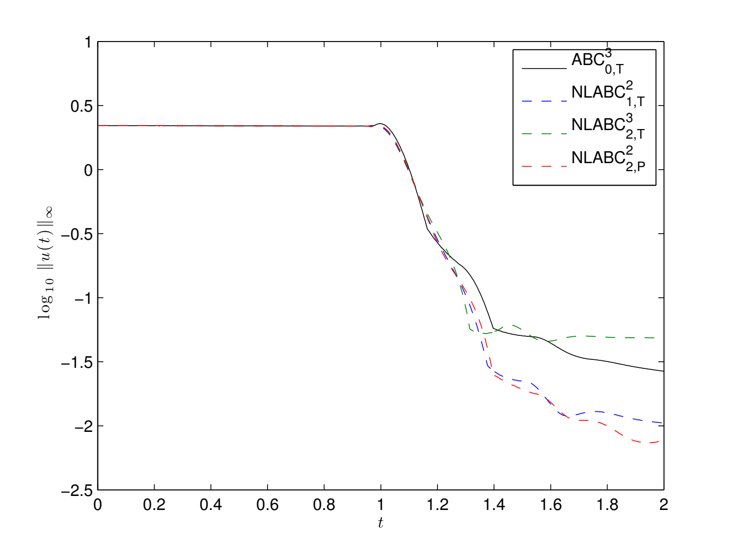
DIRECT METHOD



$NLABC^3_{2,T}$



$NLABC^2_{2,P}$

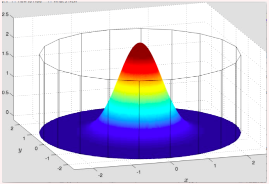


L^∞ norm associated to different ABCs (logarithmic scale)

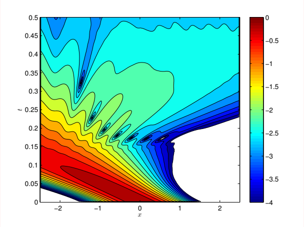
$$u_0(x,y) = e^{-\frac{x^2+y^2}{0.5^2} - 10ix}$$

Circular domain, $R = 2.5$

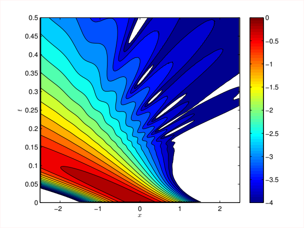
$\mathscr{V} = |u|^2$



$\mathscr{V} = x^2 + y^2 + |u|^2$



ABC_0



$NLABC^2_{2,P}$