# TOTALLY REAL POINTS IN THE MANDELBROT SET

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Abstract. The *postcritically finite* parameters in the Mandelbrot set $M$ are special algebraic integers $c \in M$ for which $z = 0$ has finite orbit under the polynomial $f_c(z) = z^2 + c$. Recently, Noytaptim and Petsche proved that the only totally real parameters that are postcritically finite are $0, -1$, and $-2$. In this note, we study another distinguished algebraic subset of $M$, the *parabolic* parameters. A parameter $c \in M$ is parabolic provided that $f_c(z) = z^2 + c$ possesses a parabolic cycle. We show that the only totally real parameters that are parabolic are $\frac{1}{4}$, $-\frac{3}{4}$, $-\frac{5}{4}$ and $-\frac{7}{4}$.

## Introduction

Consider the family of quadratic polynomials defined by

$$f_c(z) := z^2 + c, \quad c \in \mathbb{C}.$$

Each polynomial $f_c : \mathbb{C} \to \mathbb{C}$ is a dynamical system that can display rich and complicated behavior, depending on the value of the parameter $c$. There is a natural invariant subset of $\mathbb{C}$ associated to iterating $f_c$.

**Definition 1.** The *filled Julia set* of $f_c$ is the set of all $z_0 \in \mathbb{C}$ for which the *orbit* $z_n := f_c(z_{n-1})$ is bounded.

**Definition 2.** The *Mandelbrot set* $M \subseteq \mathbb{C}$ is the set of all parameters $c \in \mathbb{C}$ for which the filled Julia set of $f_c$ is connected.

The Mandelbrot set has been extensively studied for decades; it is well known that $M$ is connected, compact, and full [DH]. (Recall that a subset of $\mathbb{C}$ is *full* provided that its complement is connected). In this note, we study special algebraic points in $M$ that we will discuss after introducing a few more dynamical concepts.

**Definition 3.** The distinct points $z_0, \ldots, z_{n-1}$ comprise a *periodic cycle of period* $n$ for $f_c$ provided that $z_i = f_c(z_{i-1})$, where arithmetic on the indices is taken modulo $n$. The *multiplier* of the cycle is the product of the derivative of $f_c$ along the points in the cycle

$$\lambda = (f_c)'(z_0) \cdots (f_c)'(z_{n-1}).$$

The multiplier encodes the behavior of $f_c$ near the cycle. The cycle is said to be

- *attracting* provided that $|\lambda| < 1$,
- *repelling* provided that $|\lambda| > 1$, and
- *indifferent* provided that $|\lambda| = 1$. An indifferent cycle is *parabolic* provided that $\lambda$ is a root of unity.

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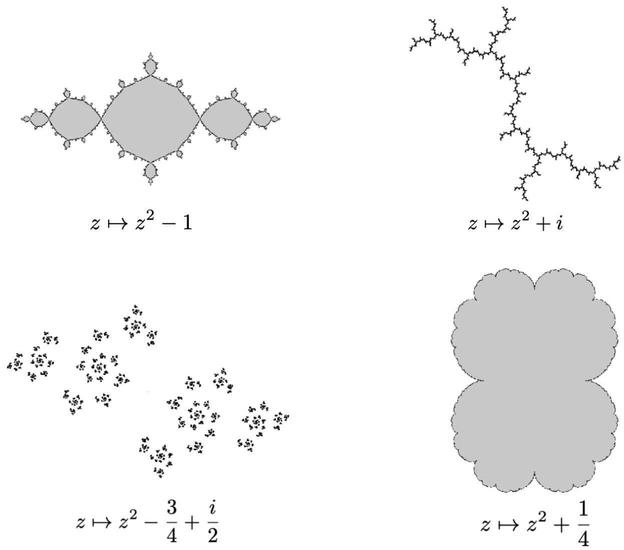


FIGURE 1. Filled Julia sets of four polynomials $z \mapsto z^2 + c$. The parameters $c = -1$ and $c = i$ belong to $M$, and they are postcritically finite. The parameter $c = 1/4$ belongs to $M$, and it is parabolic. The parameter $c = -3/4 + i/2$ does not belong to $M$.

In Figure 1, the filled Julia set of the polynomial $z \mapsto z^2 - 1$ is drawn in the upper left. This polynomial has an attracting cycle of period 2. The filled Julia set of the polynomial $z \mapsto z^2 + 1/4$ is drawn in the lower right. This polynomial has a parabolic fixed point.

A main principle in complex dynamics asserts that in order to understand the global behavior of a polynomial under iteration, one should study the orbits of its critical points. Indeed, the parameter $c \in M$ if and only if the orbit of $z = 0$ (the critical point of $f_c$) is bounded. Furthermore, if the polynomial $f_c$ has an attracting cycle or a parabolic cycle, then orbit of the critical point will converge to it. Consequently, $f_c$ can have at most one attracting or one parabolic cycle.

**Definition 4.** A parameter $c \in \mathbb{C}$ is *parabolic* provided that $f_c$ has a parabolic cycle.

As an example, the parameter $c = 1/4$ is parabolic; it is located at the cusp of the main cardiod of $M$ (see Figure 2). The parabolic parameters form a countably infinite collection of algebraic numbers contained in the boundary of $M$. In particular, if $c \in \mathbb{C}$ is a parabolic parameter, then $4c$ is an algebraic integer (see [Bo]); moreover, $4c$ is an algebraic unit in $\overline{\mathbb{Z}}/2\overline{\mathbb{Z}}$ (see [M, Remark 3.2]).

The following parameters form another distinguished algebraic subset of $M$.

**Definition 5.** A parameter $c \in \mathbb{C}$ is *postcritically finite* provided that the orbit of the critical point of $f_c$ is finite.

The parameter $c$ can be postcritically finite in exactly one of two ways:

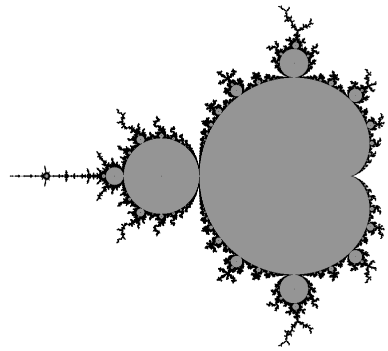

FIGURE 2. The Mandelbrot set $M$. The parabolic parameters and the postcritically finite parameters are special algebraic subsets of $M$.

(1) the critical point 0 is part of a periodic cycle of $f_c$; in this case, the parameter $c$ is contained in the interior of $M$, or
(2) the critical point 0 is strictly preperiodic; that is, 0 is eventually mapped into a periodic cycle, but 0 is not part of the cycle itself. In this case, the parameter $c$ is contained in the boundary of $M$.

For example, the parameters $\{-1, i\}$ are postcritically finite: $c = -1$ is contained in the interior of $M$, and $c = i$ is contained in the boundary of $M$. The postcritically finite parameters form a countably infinite collection of algebraic integers contained in $M$. More precisely, $c \in \mathbb{C}$ is a postcritically finite parameter if and only if $c$ is an algebraic integer whose Galois conjugates all belong to $M$ (see [M] and [Bu]).

The arithmetic properties of the postcritically finite parameters and the parabolic parameters are a bit mysterious. Recently, Noytaptim and Petsche [NP] completely determined which postcritically finite parameters are totally real.

**Definition 6.** An algebraic number $c \in \overline{\mathbb{Q}}$ is *totally real* provided that all of its Galois conjugates are in $\overline{\mathbb{Q}} \cap \mathbb{R}$.

**Proposition 1** (Noytaptim-Petsche)**.** *The only totally real parameters $c \in \overline{\mathbb{Q}}$ for which $z \mapsto z^2 + c$ is postcritically finite are $-2$, $-1$ and $0$.*

Their proof relies on the fact that the Galois conjugates of a postcritically finite parameter are also postcritically finite parameters, thus contained in $M$, and on the fact that $M \cap \mathbb{R} = [-2, \frac{1}{4}]$ has small arithmetic capacity. We recount their proof and then determine the totally real parabolic parameters as in the following result.

**Proposition 2.** *The only totally real parameters $c \in \overline{\mathbb{Q}}$ for which $z \mapsto z^2 + c$ has a parabolic cycle are $\frac{1}{4}$, $-\frac{3}{4}$, $-\frac{5}{4}$ and $-\frac{7}{4}$.*

## 1. Postcritically finite parameters

We revisit the proof of Noytaptim and Petsche [NP].

*Proof of Proposition 1.* Assume that $c$ is a totally real postcritically finite parameter. Then, $c$ is an algebraic integer and all of its Galois conjugates are real postcritically finite parameters, so they must all lie in the interval $[-2,0]$. Indeed,

- for $c \in (0, \frac{1}{4})$, the orbit of 0 is infinite, converging to an attracting fixed point of $f_c$;
- for $c = \frac{1}{4}$ the orbit of 0 under iteration of $f_c$ is infinite, converging to a parabolic fixed point of $f_c$ at $z = \frac{1}{2}$;
- for $c \in (-\infty, -2) \cup (\frac{1}{4}, +\infty)$ the orbit of 0 under iteration of $f_c$ is infinite, converging to $\infty$.

We would now like to apply Kronecker's theorem, which states that if $\alpha \in \overline{\mathbb{Q}}$ is an algebraic integer that is contained in the unit circle $\mathbb{U}$, and all Galois conjugates of $\alpha$ are also contained in $\mathbb{U}$, then $\alpha$ is a root of unity [G].

To this end, consider the surjective map $\Psi : \mathbb{U} \to [-2,2]$ given by $\Psi(z) = z + 1/z$. Let $a \in \Psi^{-1}(\{c\})$; that is, let $a$ be a solution of $a^2 - ca + 1 = 0$. Then $a$ is an algebraic integer contained in the unit circle, and $a$ has nonpositive real part (since $c \in [-2,0]$). Moreover, all Galois conjugates of $a$ are also contained in the unit circle, and they all have nonpositive real part. By Kronecker's theorem, $a$ is a root of unity. And since the Galois conjugates of $a$ all have nonpositive real part, the only possibilities are the following:

- $a = -1$, and $\Psi(a) = c = -2$;
- $a = \mathrm{e}^{\pm 2\pi i/3}$, and $\Psi(a) = c = -1$;
- $a = \pm i$, and $\Psi(a) = c = 0$.

Therefore, the only postcritically finite parameters that are totally real are $-2$, $-1$, and 0. □

## 2. Parabolic parameters

We now present the proof of Proposition 2. First note that $\left\{\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, -\frac{7}{4}\right\}$ are all parabolic parameters. Indeed,

- $z \mapsto z^2 + 1/4$ has a fixed point, at $z = 1/2$, with multiplier 1
- $z \mapsto z^2 - 3/4$ has a fixed point, at $z = -1/2$, with multiplier $-1$
- $z \mapsto z^2 - 5/4$ has a cycle of period 2, at the two roots of $4z^2 + 4z - 1$, with multiplier $-1$
- $z \mapsto z^2 - 7/4$ has a cycle of period 3, at the three roots of $8z^3 + 4z^2 - 18z - 1$, with multiplier 1

*Proof of Proposition 2.* Assume that $c$ is a totally real parabolic parameter. Then, the Galois conjugates of $c$ are also parabolic parameters. Either $c \in \{\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}\}$, or $c$ and all of its Galois conjugates lie in the interval $[-2, -\frac{5}{4})$. Indeed, a parabolic cycle must attract the orbit of the critical point 0. However,

- for $c \in (-\frac{3}{4}, \frac{1}{4})$, the orbit of 0 converges to an attracting fixed point of $f_c$;
- for $c \in (-\frac{5}{4}, -\frac{3}{4})$ the orbit of 0 converges to an attracting cycle of period 2 of $f_c$;
- for $c \in (-\infty, -2) \cup (\frac{1}{4}, +\infty)$ the orbit of 0 converges to $\infty$.

Let us assume that $c \in [-2, -\frac{5}{4})$. Then, $b := 4c + 6$ and all of its Galois conjugates lie in the interval $[-2, 1)$. Let $a \in \Psi^{-1}(\{b\})$; that is, let $a$ be a solution of $a^2 - ba + 1 = 0$. Then $a$ is an algebraic integer contained in the unit circle with real part less than $\frac{1}{2}$, and all of its Galois conjugates are also contained in the unit circle, and they all have real part less than $\frac{1}{2}$.

By Kronecker's theorem, $a$ is a root of unity. And since the Galois conjugates of $a$ all have real part less than $\frac{1}{2}$, the only possibilities are the following:

- $a = -1, b = -2, c = -2$, and $c$ is not a parabolic parameter,
- $a = -1, b = -2, c = -2$, and $c$ is not a parabolic parameter,
- $a = \mathrm{e}^{\pm 2\pi i/3}, b = -1, c = -\frac{7}{4}$, and $c$ **is** a parabolic parameter,
- $a = \pm i, b = 0, c = -\frac{3}{2}$; in this case $4c = -6$, so $c$ is not an algebraic unit in $\overline{\mathbb{Z}}/2\overline{\mathbb{Z}}$, and $c$ is not a parabolic parameter,
- $a = \mathrm{e}^{\pm 2\pi i/5}, b = 2\cos(\frac{2\pi}{5}), c = \frac{\sqrt{5}-13}{8}$; in this case, $f_c$ has an attracting cycle of period 4, so $c$ is not a parabolic parameter,
- $a = \mathrm{e}^{\pm 4\pi i/5}, b = 2\cos(\frac{4\pi}{5}), c = \frac{-\sqrt{5}-13}{8}$; then the Galois conjugate $\frac{\sqrt{5}-13}{8}$ is not a parabolic parameter, so $c$ is not a parabolic parameter.

This completes the proof of the proposition. □

Remark: the following proof that $c = -3/2$ is not a parabolic parameter is due to Valentin Huguin. We will use the notation $(f_c)^{\circ n} := f_c \circ \cdots \circ f_c$, the composition of $f_c$ with itself $n$ times. It follows from [Bo] that for all $n \geq 1$,

$$\operatorname{discriminant}\big((f_c)^{\circ n}(z) - z, z\big) = P_n(4c) \quad \text{with} \quad P_n(b) \in \mathbb{Z}[b] \quad \text{and} \quad \pm P_n \text{ monic.}$$

As an example,

$$P_1(b) = -b + 1, \quad P_2(b) = (b-1)(b+3)^3, \quad P_3(z) = (b-1)(b+7)^3(b^2+b+7)^4,$$

and

$$P_4(z) = (b-1)(b+3)^3(b+5)^6(b^3+9b^2+27b+135)^4(b^2-2b+5)^5.$$

Note that this yields an alternate proof that $c = \frac{1}{4}$, $c = -\frac{3}{4}$, $c = -\frac{5}{4}$ and $c = -\frac{7}{4}$ are parabolic parameters. In addition,

$$P_n(0) = \operatorname{discriminant}\big(z^{2^n} - z, z\big) \equiv 1 \pmod 2.$$

As a consequence

$$P_n(-6) \equiv 1 \pmod 2.$$

Thus, for all $n \geq 1$, the roots of

$$\Big(f_{-\frac{3}{2}}\Big)^{\circ n}(z) - z \text{ are simple, which shows that } f_{-\frac{3}{2}} \text{ has no parabolic cycle.}$$

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