ON THE STABILITY OF HOLOMORPHIC FAMILIES OF ENDOMORPHISMS OF $\mathbb{P}^k$

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ABSTRACT. In the context of holomorphic families of $\mathbb{P}^k$ endomorphisms, we show that various notions of stability are equivalent. This allows us to both extend and simplify the architecture of the proof of certain results of [**BBD**].

Keywords: holomorphic dynamics, dynamical stability.

1. INTRODUCTION AND RESULTS

A *holomorphic family of degree* $d \geq 2$ *endomorphisms on* $\mathbb{P}^k$, parametrized by a complex manifold M, is a holomorphic map $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$, of the form $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$ such that $d \geq 2$ is the common algebraic degree of the maps f_λ as endomorphisms of $\mathbb{P}^k$.

For such families, the simplest notion of dynamical stability concerns repelling cycles and is referred to as *weak stability*. We are going to define various versions of it. For every integer n and every parameter λ, we denote by $\mathscr{R}_n(\lambda)$ the set of n-periodic repelling points of f_λ which belong to the Julia set of f_λ. Let us recall that, by the Briend-Duval equidistribution theorem [**BrDu**], $|\mathscr{R}_n(\lambda)| \sim d^{kn}$ for every $\lambda \in M$.

Definition 1.1. A family f is said *partially weakly stable* if there exists a sequence of trivial holomorphic laminations $(\mathscr{L}_n^r)_{n\in\mathbb{N}}$ in $M \times \mathbb{P}^k$ such that $\mathscr{L}_n^r \cap (\{\lambda\} \times \mathbb{P}^k) \subset \mathscr{R}_n(\lambda)$ for every $\lambda \in M$, $f(\mathscr{L}_n^r) \subset \mathscr{L}_n^r$ and $\limsup_n |\mathscr{L}_n^r| d^{-kn} > 0$. Such a family is said *asymptotically weakly stable* if moreover $\lim_n |\mathscr{L}_n^r| d^{-kn} = 1$, and *weakly stable* if $\mathscr{L}_n^r \cap (\{\lambda\} \times \mathbb{P}^k) = \mathscr{R}_n(\lambda)$ for every $\lambda \in M$.

Roughly speaking, weak stability means that all repelling cycles move holomorphically. In dimension $k = 1$, Mañé-Sad-Sullivan [**MSS**] and, independently, Lyubich [**Lyu**] have shown that weak stability is equivalent to the existence of a holomorphic motion of the full Julia sets. One of their main tool is the so-called λ-lemma. Combining this lemma with the ergodicity of the equilibrium measure of f_λ, one may see that partial weak stability is actually sufficient [**Ber**].

In higher dimensions, the classical approach fails and new tools have to be introduced. This has been done in [**BBD**] where, in particular, a probabilistic notion of stability called μ-stability has been defined (see Section 2). It is proved in [**BBD**] that the notions of weak stability and μ-stability are equivalent within the full family $\mathscr{H}_d(\mathbb{P}^k)$ of degree d holomorphic endomorphisms of $\mathbb{P}^k$, or within arbitrary families of endomorphisms of $\mathbb{P}^2$. Using further techniques, Bianchi [**Bia**] proved the equivalence of asymptotic weak stability and μ-stability in any dimension and in the more general setting of holomorphic families of polynomial like mappings with large topological degree.

Our aim in this note is to prove the equivalence of the notions of weak stability and μ-stability. Our main results are the following. In section 3 we prove the

Proposition 1.2. *Let M be a connected complex manifold and $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a degree $d \geq 2$ holomorphic family of endomorphisms of $\mathbb{P}^k$. If f is μ-stable then f is weakly stable.*

Combining the above result with some of [**BBD**] and [**BB**], we then show in Section 4 that all the notions of stability are equivalent when the parameter space is simply connected (see Theorem 4.1 for a complete set of equivalences).

Theorem 1.3. *Let M be a simply connected complex manifold and $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a degree $d \geq 2$ holomorphic family of endomorphisms of $\mathbb{P}^k$.*
Then: f is μ-stable $\Leftrightarrow$ f is weakly stable $\Leftrightarrow$ f is partially weakly stable.

Finally, we should mention that the above result also covers the fact that the bifurcation of repelling cycles is a chain reaction in the sense of [**Ber**].

2. Equilibrium webs, laminations, and μ-stability

In all this section f is a given degree d holomorphic family of endomorphisms of $\mathbb{P}^k$. The critical set of f is denoted C_f and, for each parameter $\lambda \in M$, the equilibrium measure of f_λ is denoted μ_λ while its support, the (small) Julia set of f_λ, is denoted J_λ.

We endow $\mathbb{P}^k$ with the spherical distance $d_{\mathbb{P}^k}$ and the space $\mathscr{O}(M, \mathbb{P}^k)$ of holomorphic maps from M to $\mathbb{P}^k$ with the distance d_{uloc} of local uniform convergence, which makes $(\mathscr{O}(M, \mathbb{P}^k), d_{uloc})$ a complete separable metric space. The graph of $\gamma \in \mathscr{O}(M, \mathbb{P}^k)$ is denoted Γ_γ. The holomorphic map f clearly induces a continuous selfmap

$$\mathscr{F} : \mathscr{O}(M, \mathbb{P}^k) \to \mathscr{O}(M, \mathbb{P}^k)$$

which is defined by $\mathscr{F} \cdot \gamma(\lambda) := f_\lambda(\gamma(\lambda))$ for any $\gamma \in \mathscr{O}(M, \mathbb{P}^k)$ and $\lambda \in M$. We will also use the evaluation maps

$$e_\lambda : \mathscr{O}(M, \mathbb{P}^k) \to \mathbb{P}^k$$

defined by $e_\lambda(\gamma) := \gamma(\lambda)$ for any $\gamma \in \mathscr{O}(M, \mathbb{P}^k)$ and $\lambda \in M$.

An important role is played by the two following (possibly empty) closed $\mathscr{F}$-invariant subspaces of $\mathscr{O}(M, \mathbb{P}^k)$

$$\mathscr{J} := \{\gamma \in \mathscr{O}(M, \mathbb{P}^k) \, : \, \gamma(\lambda) \in J_\lambda, \; \forall \lambda \in M\},$$

$$\mathscr{J}_s := \{\gamma \in \mathscr{J} \, : \, \Gamma_\gamma \cap (\cup_{m \geq 0} f^{-m}(\cup_{n \geq 0} f^n(C_f))) \neq \emptyset\}.$$

As we shall see, the μ-stability of the family f amounts to say that there exists an ergodic dynamical system of the form $(\mathscr{J}, \mathscr{M}, \mathscr{F})$ which does not interact with the critical dynamics of f.

Definition 2.1. An *equilibrium web* for the family $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ is a probability measure $\mathscr{M}$ on $\mathscr{O}(M, \mathbb{P}^k)$ such that:

1. $\mathscr{F}_* \mathscr{M} = \mathscr{M}$,
2. $\mathrm{supp}\mathscr{M}$ is compact in $(\mathscr{O}(M, \mathbb{P}^k), d_{uloc})$,
3. $(e_\lambda)_* \mathscr{M} = \mu_\lambda, \; \forall \lambda \in M$.

An equilibrium web $\mathscr{M}$ is said *acritical* if $\mathscr{M}(\mathscr{J}_s) = 0$, and *ergodic* if the dynamical system $(\mathscr{O}(M, \mathbb{P}^k), \mathscr{F}, \mathscr{M})$ is ergodic.

An *equilibrium lamination* for the family $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ is a relatively compact subset $\mathscr{L}$ of $\mathscr{J}$ such that

1. $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ for every distinct $\gamma, \gamma' \in \mathscr{L}$,
2. $\mu_\lambda\{\gamma(\lambda) : \gamma \in \mathscr{L}\} = 1$ for every $\lambda \in M$,
3. Γ_γ does not meet the grand orbit of the critical set of f for every $\gamma \in \mathscr{L}$,
4. the map $\mathscr{F} : \mathscr{L} \to \mathscr{L}$ is d^k to 1.

An equilibrium lamination is said *subordinated to an equilibrium web $\mathscr{M}$* if $\mathscr{M}(\mathscr{L}) = 1$. A family f is said to be *μ-stable* if it admits an equilibrium lamination.

It turns out that every equilibrium web is supported in $\mathscr{J}$ (see the comment after Proposition 2.3 in [**BB**]). Let us also mention that Bianchi and Rakhimov [**BR**] have recently shown that the μ-stability, i.e. the stability of the maximal entropy measure, implies a similar property for all measures with entropy strictly bigger than $(k-1)\ln d$.

The interplay between equilibrium webs and laminations is given by the following fundamental result from [**BBD**].

Theorem 2.2. *Let M be a simply connected complex manifold and $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a degree $d \geq 2$ holomorphic family of endomorphisms of $\mathbb{P}^k$.*

1) *If f admits an acritical and ergodic equilibrium web $\mathscr{M}$ then there exists an equilibrium lamination $\mathscr{L}$ for f which is subordinated to $\mathscr{M}$, moreover $\mathscr{M}(\mathscr{L}\Delta\mathscr{L}') = 0$ for any other equilibrium lamination $\mathscr{L}'$ of f,*
2) *if f admits an equilibrium lamination $\mathscr{L}$ then there exists an acritical and ergodic equilibrium web $\mathscr{M}$ of f to which $\mathscr{L}$ is subordinated.*

The first assertion is Theorem 4.1 of [**BBD**], the second one is only implicit there and we thus prove it below.

Proof. Pick $\gamma_0 \in \mathscr{L}$ and, for every $n \in \mathbb{N}$, set $\mathscr{M}_n := \frac{1}{n}\sum_{i=1}^n d^{-ki}\sum_{\mathscr{F}^{\circ i}\sigma=\gamma_0}\delta_\sigma$. Then any weak limit $\mathscr{M}$ of $(\mathscr{M}_n)_n$ is an equilibrium web for f (see [**BBD**] Proposition 2.2). We will show that $\mathscr{M}$ yields an acritical and ergodic equilibrium web for f to which $\mathscr{L}$ is subordinated. Let us first check that for every $k \in \mathbb{N}$ and every $\gamma \in \mathrm{supp}\mathscr{M}$ one has:

$$\Gamma_\gamma \cap f^{\circ k}(C_f) \neq \emptyset \Rightarrow \Gamma_\gamma \subset f^{\circ k}(C_f).$$

Indeed, if this were not the case, by Hurwitz theorem, we could find some $n \in \mathbb{N}$ and some $\sigma \in \mathrm{supp}\mathscr{M}_n$ such that $\Gamma_\sigma \cap f^{\circ k}(C_f) \neq \emptyset$, and therefore $f^{\circ(i+k)}(C_f) \cap \Gamma_{\gamma_0} \neq \emptyset$ for some $i \leq n$ which, as $\gamma_0 \in \mathscr{L}$, is impossible.

Now, for any fixed $\lambda_0 \in M$ we get

$$\mathscr{M}(\{\gamma \in \mathscr{J} : \Gamma_\gamma \cap (\cup_{k\geq 0} f^{\circ k}(C_f)) \neq \emptyset\}) = \mathscr{M}(\{\gamma \in \mathscr{J} : \Gamma_\gamma \subset (\cup_{k\geq 0} f^{\circ k}(C_f))\}) \leq$$
$$\mathscr{M}(\{\gamma \in \mathscr{J} : (\lambda_0, \gamma(\lambda_0)) \in (\cup_{k\geq 0} f^{\circ k}(C_f))\}) = (e_{\lambda_0})_*\mathscr{M} \cup_{k\geq 0} f_{\lambda_0}^{\circ k}(C_{f_{\lambda_0}}) = 0$$

where the last equality follows from the fact that $(e_{\lambda_0})_*\mathscr{M} = \mu_{\lambda_0}$ gives no mass to pluripolar subsets of $\mathbb{P}^k$. Then $\mathscr{M}(\mathscr{J}_s) = 0$ follows from the $\mathscr{F}$-invariance of $\mathscr{M}$ and therefore, according to the Proposition 2.4 of [**BBD**], $\mathscr{M}$ can be replaced by an acritical and ergodic equilibrium web, which we still note $\mathscr{M}$.

By the first assertion of Theorem 2.2, there exists an equilibrium lamination $\mathscr{L}'$ of f such that $\mathscr{M}(\mathscr{L}') = 1$ and $\mathscr{M}(\mathscr{L}\Delta\mathscr{L}') = 0$. It follows that $\mathscr{M}(\mathscr{L}) = 1$. □

3. Proof of Proposition 1.2

Let $\mathscr{L}$ be an equilibrium lamination, subordinated to an equilibrium web $\mathscr{M}$, for f. Let $\lambda_0 \in M$ and $z_0 \in J_{\lambda_0}$ be such that z_0 is p-periodic and repelling for f_{λ_0}. Without any loss of generality we may assume that $p = 1$. According to [**BBD**] Lemma 2.5, there exists $\gamma_0 \in \mathscr{J}$ such that $\gamma_0(\lambda_0) = z_0$, $\mathscr{F}(\gamma_0) = \gamma_0$ and $\gamma_0(\lambda)$ is a fixed repelling point of f_λ for λ sufficiently close to λ_0. We must show that $\gamma_0(\lambda)$ is repelling for every $\lambda \in M$.

We proceed by contradiction. If this is not the case, then there exists $\lambda_1 \in M$ such that at least one of the eigenvalues of $f'_{\lambda_1}(\gamma_0(\lambda_1))$ has a modulus smaller or equal to 1.

The fixed point $\gamma_0(\lambda_1)$ is non repelling, let us first show that after possibly moving λ_1 a little, it becomes hyperbolic. Let $Z := \{(\lambda, w) \in M\times\mathbb{C} : \det\left(f'_\lambda(\gamma_0(\lambda)) - w \text{ id}\right) = 0\}$. The canonical projection $\pi_M : Z \to M$ has degree d for some $d \leq k$ and $\min_{\pi_M^{-1}(\lambda_1)} |w| \leq 1$. If $\min_{\pi_M^{-1}(\lambda_1)} |w| = 1$, we cannot have $\min_{\pi_M^{-1}(\lambda)} |w| \geq 1$ on some neighbourhood of λ_1 since otherwise the maximum modulus principle, applied to the restriction of the function $\frac{1}{w}$ to Z, would imply that $\min_{\pi_M^{-1}(\lambda_0)} |w| \leq 1$. We may thus slightly move λ_1 so that $\min_{\pi_M^{-1}(\lambda_1)} |w| < 1$ and π_M has exactly d distinct preimages at λ_1. Now, $\pi_M^{-1}(\lambda)$ is given by the graphs of d holomorphic functions $w_j(\lambda)$ above a small neighbourhood of λ_1 and, since $\min_{\pi_M^{-1}(\lambda_0)} |w| > 1$, any function w_j is not constant if $|w_j(\lambda_1)| = 1$. Thus, after possibly moving λ_1 again, $\gamma_0(\lambda_1)$ stays non repelling and becomes hyperbolic.

Let W be the local unstable manifold of f_{λ_1} at $\gamma_0(\lambda_1)$. It is a proper analytic subset of some neighbourhood of $\gamma_0(\lambda_1)$ in $\mathbb{P}^k$. Let $\Omega_0 := \{z \in \mathbb{P}^k : d_{\mathbb{P}^k}(z, z_0) < \alpha_0\}$ be a neighbourhood of z_0 in $\mathbb{P}^k$ and $\mathscr{L}_0$, $\mathscr{L}_1$ be the following subsets of $\mathscr{L}$

$$\mathscr{L}_0 := \{\gamma \in \mathscr{L} : \gamma(\lambda_0) \in \Omega_0\} = e_{\lambda_0}^{-1}(\Omega_0) \cap \mathscr{L}$$

$$\mathscr{L}_1 := \{\gamma \in \mathscr{L} : \exists n \in \mathbb{N} \text{ such that } \gamma(\lambda_1) \in f_{\lambda_1}^{\circ n}(W)\} = \bigcup_{n\in\mathbb{N}} e_{\lambda_1}^{-1}(f_{\lambda_1}^{\circ n}(W)) \cap \mathscr{L}.$$

Let us show that

$$\mathscr{M}(\mathscr{L}_0) > 0 \text{ and } \mathscr{M}(\mathscr{L}_1) = 0.$$

As $\mathscr{M}(\mathscr{L}) = 1$ we have $\mathscr{M}(\mathscr{L}_0) = \mathscr{M}(e_{\lambda_0}^{-1}(\Omega_0))$ and $\mathscr{M}(\mathscr{L}_1) \leq \sum_{n\in\mathbb{N}} \mathscr{M}(e_{\lambda_1}^{-1}(f_{\lambda_1}^{\circ n}(W))$. Since $z_0 \in J_{\lambda_0}$ we have $\mathscr{M}(\mathscr{L}_0) = (e_{\lambda_0})_*\mathscr{M}(\Omega_0) = \mu_{\lambda_0}(\Omega_0) > 0$. Since μ_{λ_1} gives no mass to pluripolar sets, $\mathscr{M}(e_{\lambda_1}^{-1}(f_{\lambda_1}^{\circ n}(W))) = (e_{\lambda_1})_*\mathscr{M}(f_{\lambda_1}^{\circ n}(W)) = \mu_{\lambda_1}(f_{\lambda_1}^{\circ n}(W)) = 0$ for every $n \in \mathbb{N}$ and thus $\mathscr{M}(\mathscr{L}_1) = 0$.

Now, the expected contradiction will be obtained by showing that, for Ω_0 suitably small,

$$\mathscr{L}_0 \subset \mathscr{L}_1.$$

As f_{λ_0} is repelling at z_0, there exists $0 < a < 1$, $r_0 > 0$, a neighbourhood V_0 of λ_0 in M, and an inverse branch $\varphi : T_0 \to T_0$ of f which is defined on the neighbourhood $T_0 := \{(\lambda, z) \in V_0 \times \mathbb{P}^k : d_{\mathbb{P}^k}(z, \gamma_0(\lambda)) < r_0\}$ of (λ_0, z_0), such that

$$d_{\mathbb{P}^k}(\gamma_0(\lambda), \varphi(\lambda, z)) \leq a\, d_{\mathbb{P}^k}(\gamma_0(\lambda), z);\ \forall(\lambda, z) \in T_0. \tag{3.1}$$

Owing to the equicontinuity of the family $\mathscr{L}$, we may take the neighbourhood V_0 of λ_0 and the neighbourhood Ω_0 of z_0 small enough so that

$$\forall \gamma \in \mathscr{L},\ \forall \lambda \in V_0 :\ \gamma(\lambda_0) \in \Omega_0 \Rightarrow (\lambda, \gamma(\lambda)) \in T_0. \tag{3.2}$$

Let $\gamma \in \mathscr{L}_0$. It follows from (3.2) and the definition of $\mathscr{L}_0$ that the map $\lambda \mapsto \varphi(\lambda, \gamma(\lambda))$ is well defined on V_0. Now, since the map $\mathscr{F} : \mathscr{L} \to \mathscr{L}$ is d^k to 1, this implies that there exists $\gamma_{-1} \in \mathscr{L}$ such that $\varphi(\lambda, \gamma(\lambda)) = \gamma_{-1}(\lambda)$ on V_0. By (3.1), one sees that γ_{-1} actually belongs to $\mathscr{L}_0$. We may thus iterate this construction and find a sequence $(\gamma_{-n})_n$ in $\mathscr{L}_0$ such that $\mathscr{F}^{\circ n}(\gamma_{-n}) = \gamma$ and $\lim_n \gamma_{-n}(\lambda) = \gamma_0(\lambda)$ for every $\lambda \in V_0$. By analyticity, and equicontinuity of $\mathscr{L}$, this implies that $\lim_n \gamma_{-n}(\lambda) = \gamma_0(\lambda)$ for every $\lambda \in M$ and, in particular, that $\lim_n \gamma_{-n}(\lambda_1) = \gamma_0(\lambda_1)$. This is only possible if $\gamma_{-n}(\lambda_1) \in W$ for n big enough, and thus $\gamma \in \mathscr{L}_1$. We have shown that $\mathscr{L}_0 \subset \mathscr{L}_1$. □

4. Proof of Theorem 1.3

We shall actually present the following more complete result which combines the above Proposition 1.2 with results of [**BBD**] and [**BB**].

Theorem 4.1. *Let $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of degree $d \geq 2$ on $\mathbb{P}^k$ parametrized by a simply connected complex manifold M. Then the following assertions are equivalent:*

1) *L is pluriharmonic on M,*
2) *$\|f_*^{\circ n}[C_f]\|_{U \times \mathbb{P}^k} = O(d^{(k-1)n})$ for every $U \Subset M$,*
3) *the ramification current $R_f := \sum_{n \geq 0} d^{-kn} (f^{\circ n})_* [f(C_f)]$ converges on $M \times \mathbb{P}^k$,*
4) *f is μ-stable,*
5) *f is weakly stable,*
6) *f is asymptotically weakly stable,*
7) *f is partially weakly stable.*

Let us recall that $[C_f]$ denotes the current of integration on the critical set C_f of f, and that the Lyapunov function $L(\lambda) := \int_{\mathbb{P}^k} \ln |\mathrm{Jac} f_\lambda|\ \mu_\lambda$ is $p.s.h$ on M ([**BaBe**]) and coincides with the sum of the Lyapunov exponents of the ergodic dynamical system $(J_\lambda, f_\lambda, \mu_\lambda)$.

One of the main threads in the proof of the Theorem 4.1 is that the stability properties of f are encoded by the function L, from which they can be extracted using the following two formulas

***$dd^c L$*-formula**: $dd^c L = \pi_{M*} \left((dd^c_{\lambda,z} g(\lambda, z) + \omega_{\mathbb{P}^k})^k \wedge [C_f] \right)$,
Approximation formula: $L(\lambda) = \lim_n d^{-kn} \sum_{z \in \mathscr{R}_n(\lambda)} \ln |\mathrm{Jac} f_\lambda(z)|$.

The first formula has been obtained by Bassanelli and Berteloot in [**BaBe**], it generalizes similar formulas in dimension one due to Przytycki [**Prz**] and Manning [**Man**] for polynomials and DeMarco [**DMa**] for rational functions. It might be useful to stress that $g(\lambda, \cdot)$ is the Green function of f_λ and that $\mu_\lambda = (dd^c_z g(\lambda, z) + \omega_{\mathbb{P}^k})^k$ (see [**DS**] page 176).

The second formula was proved by Berteloot, Dupont and Molino in [**BDM**] and a simplified proof, avoiding difficulties due to the possible resonances between the Lyapunov exponents, has been given by Berteloot and Dupont in [**BD**].

Let us now enter into details. The implications $2) \Rightarrow 3)$ and $5) \Rightarrow 6) \Rightarrow 7)$ are obvious. Note that the convergence of the positive current R_f means that every point in $M \times \mathbb{P}^k$ admits a neighbourhood U such that the series $\sum_{n\geq 0} \|\mathbb{1}_U d^{-kn}(f^{\circ n})_*[f(C_f)]\|$ converges.

$1) \Rightarrow 2)$ follows immediately from the following estimate which is a direct consequence of the dd^cL-formula ([**BBD**] Lemma 3.13). There exists a positive constant α, only depending on k and $\dim_{\mathbb{C}} M$, such that $\|f_*^n[C_f]\|_{U\times\mathbb{P}^k} = \alpha d^{kn}\|dd^cL\|_U + O(d^{(k-1)n})$ for every relatively compact open subset U of M.

$3) \Rightarrow 4)$ This is based on the key result of [**BB**]. Namely, the convergence of the ramification current implies that any $\lambda_0 \in M$ has a neighbourhood D_0 such that the restricted family $f|_{D_0\times\mathbb{P}^k}$ admits an acritical equilibrium web $\mathscr{M}$ ([**BB**] Theorem 1.4 and Lemma 2.4). Then, by the Proposition 2.4 in [**BBD**] this web can be assumed to be ergodic, and thus $f|_{D_0\times\mathbb{P}^k}$ admits an equilibrium lamination (Theorem 2.2.) As M is simply connected, f itself admits an equilibrium lamination.

$4) \Rightarrow 5)$ is Proposition 1.2.

$7) \Rightarrow 1)$ This has been proved in [**Ber**], we reproduce here the argument. By Banach-Alaoglu theorem, there exists a subsequence $(n_q)_q$ and a compactly supported positive measure $\mathscr{M}$ of mass τ on $\mathscr{O}(M,\mathbb{P}^k)$, such that $\lim_q \frac{1}{d^{kn_q}} \sum_{\gamma\in\mathscr{L}^r_{n_q}} \delta_\gamma = \mathscr{M}$ and $0 < \tau \leq 1$.

We now fix $\lambda \in M$. By Briend-Duval theorem ([**BrDu**]), $\lim_n d^{-kn} \sum_{z\in\mathscr{R}_n(\lambda)} \delta_z = \mu_\lambda$ and, setting $\sigma_\lambda := e_{\lambda *}\mathscr{M}$, we have $\lim_q \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{L}^r_{n_q}(\lambda)} \delta_z = \sigma_\lambda$. The positive measure σ_λ is f_λ-invariant, of mass τ, and $\sigma_\lambda \leq \mu_\lambda$. Writing $\mu_\lambda = \tau(\frac{\sigma_\lambda}{\tau}) + (1-\tau)\left(\frac{\mu_\lambda - \sigma_\lambda}{1-\tau}\right)$, we deduce from the ergodicity of μ_λ that $\sigma_\lambda = \tau\mu_\lambda$. Now, setting $\mathscr{L}^r_n(\lambda) := \mathscr{L}^r_n \cap (\{\lambda\}\times\mathbb{P}^k)$ and $\mathscr{R}'_n(\lambda) := \mathscr{R}_n(\lambda) \setminus \mathscr{L}^r_n(\lambda)$, we get $\lim_q \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{R}'_{n_q}(\lambda)} \delta_z = (1-\tau)\mu_\lambda$.

Let $\ln_\varepsilon$ be a family of smooth functions on $[0,+\infty[$ which converges pointwise to $\ln$ and such that $\ln_\varepsilon \geq \ln$. Then

$$\begin{aligned}\tau\int_{\mathbb{P}^k} \ln_\varepsilon |\mathrm{Jac} f_\lambda|\,\mu_\lambda = \int_{\mathbb{P}^k} \ln_\varepsilon |\mathrm{Jac} f'_\lambda|\,\sigma_\lambda &= \lim_q \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{L}^r_{n_q}(\lambda)} \ln_\varepsilon |\mathrm{Jac} f_\lambda(z)| \\ &\geq \limsup_q \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{L}^r_{n_q}(\lambda)} \ln |\mathrm{Jac} f_\lambda(z)|\end{aligned}$$

and, making $\varepsilon \to 0$,

$$\tau L(\lambda) \geq \limsup_q \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{L}^r_{n_q}(\lambda)} \ln |\mathrm{Jac} f_\lambda(z)|.$$

Similarly, we have

$$(1-\tau)L(\lambda) \geq \limsup_q \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{R}'_{n_q}(\lambda)} \ln |\mathrm{Jac} f_\lambda(z)|.$$

On the other hand, $\lim_q \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{L}^r_{n_q}(\lambda)} \ln |\mathrm{Jac} f_\lambda(z)| + \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{R}'_{n_q}(\lambda)} \ln |\mathrm{Jac} f_\lambda(z)| = L(\lambda)$ by the approximation formula. Thus, $\tau L(\lambda) = \lim_q \frac{1}{d^{kn_q}} \sum_{z\in\mathscr{L}^r_{n_q}(\lambda)} \ln |\mathrm{Jac} f_\lambda(z)|$, which makes L appearing as the pointwise limit of a locally uniformly bounded sequence of pluriharmonic functions. The function L is therefore pluriharmonic on M. □

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