QUADRATIC SIEGEL DISKS WITH SMOOTH BOUNDARIES.

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Abstract. We show the existence of angles $\alpha \in \mathbb{R}/\mathbb{Z}$ such that the quadratic polynomial $P_\alpha(z) = e^{2i\pi\alpha}z + z^2$ has a Siegel disk with smooth (C^∞) boundary. This result was first announced by R. Pérez-Marco in 1993.

Introduction.

Assume U is an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a holomorphic map which fixes 0 with derivative $e^{2i\pi\alpha}$, $\alpha \in \mathbb{R}/\mathbb{Z}$. We say that f is *linearizable* if it is topologically conjugate to the rotation $z \mapsto e^{2i\pi\alpha}z$ in a neighborhood of 0. If $f : U \to \mathbb{C}$ is linearizable, there is a largest f-invariant connected domain $\Delta \subset U$ containing 0 on which f is conjugate to the rotation $z \mapsto e^{2i\pi\alpha}z$. This domain is simply connected and is called the *Siegel disk* of f. A basic but remarkable fact, is that the conjugacy can be taken holomorphic.

In this article, we are mainly concerned with the dynamics of the quadratic polynomials $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. They have $z = 0$ as an indifferent fixed point. In part II, we will extend our results to *periodic* points of quadratic-*like* maps.

For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists a unique formal power series ϕ_α of the form $\phi_\alpha(z) = z + b_2z^2 + b_3z^3 + \ldots$, such that

$$\phi_\alpha(e^{2i\pi\alpha}z) = P_\alpha(\phi_\alpha(z)).$$

We denote by $r_\alpha \geq 0$ the radius of convergence of the series ϕ_α. It is known (see [Y] for example) that $r_\alpha > 0$ for Lebesgue almost every $\alpha \in \mathbb{R}$, and more precisely, $r_\alpha > 0$ if and only if α satisfies the Bruno condition (see definition 3 below).

From now on, we assume that $r_\alpha > 0$. In that case, the map $\phi_\alpha : B(0, r_\alpha) \to \mathbb{C}$ is univalent, and it is well known that its image Δ_α coincides with the Siegel disk of P_α associated to the point 0. The number r_α is called the *conformal radius* of the Siegel disk. The Siegel disk is also the connected component of $\mathbb{C} \setminus J(P_\alpha)$ which contains 0, where $J(P_\alpha)$ is the Julia set of P_α, i.e., the closure of the set of repelling periodic points. Figure 1 shows the Julia sets of the quadratic polynomials P_α, for $\alpha = \sqrt{2}$ and $\alpha = \sqrt{10}$. Both polynomials have a Siegel disk colored grey.

In this article, we investigate the structure of the boundary of the Siegel disk. It is known since Fatou that this boundary is contained in the closure of the forward orbit of the critical point $\omega_\alpha = -e^{2i\pi\alpha}/2$ (for example, see [Mi] Theorem 11.17 or [Mi] Corollary 14.4). By plotting a large number of points in the forward orbit of ω_α, we should therefore get a good idea of what those boundaries look like. In practice, that works only when α is sufficiently well-behaved, the number of iterations needed being otherwise enormous.

In 1983, Herman [He1] proved that when α satisfies the Herman condition (this is the optimal arithmetic condition to ensure that all analytic circle diffeomorphisms

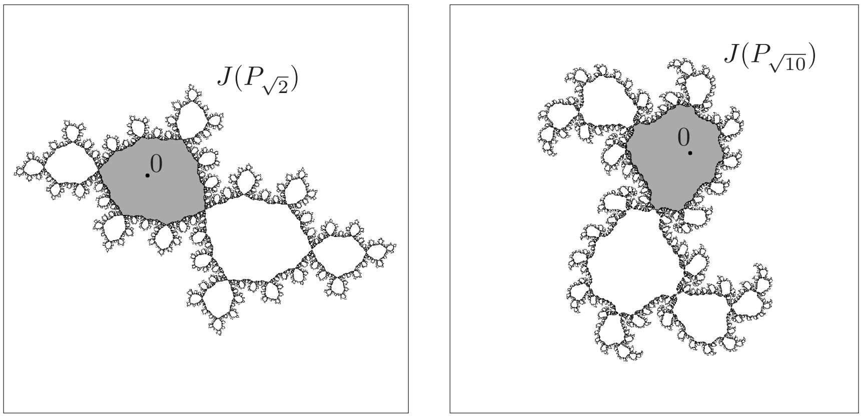

FIGURE 1. Left: the Julia set of the polynomial $z \mapsto e^{2i\pi\sqrt{2}}z + z^2$. Right: the Julia set of the polynomial $z \mapsto e^{2i\pi\sqrt{10}}z + z^2$. In both cases, there is a Siegel disk.

with rotation number α are analytically linearizable in a neighborhood of the circle), then the critical point actually belongs to the boundary of the Siegel disk. We will not define this condition in this article. Using a construction due to Ghys, Herman [He2] also proved the existence of quadratic polynomials P_α for which the boundary of the Siegel disk is a quasi-circle which does not contain the critical point. Afterwards, following an idea by Douady [D] and using work by Światek [S] (see also [P2]), he proved that when α is Diophantine of exponent 2, the boundary of the Siegel disk is a quasi-circle containing the critical point. In [McM1], McMullen shows that the corresponding Julia sets have Hausdorff dimension less than 2 and that when α is a quadratic irrational, the boundary of the Siegel disk is self-similar about the critical point. More recently, Petersen and Zakeri [PZ] proved that for Lebesgue almost every $\alpha \in \mathbb{R}/\mathbb{Z}$, the boundary is a Jordan curve containing the critical point. Moreover, when α is not Diophantine of exponent 2, this Jordan curve is not a quasi-circle (see [PZ]).

In [PM], Pérez-Marco proves that there exist maps univalent in $\mathbb{D}$ having Siegel disks compactly contained in $\mathbb{D}$ whose boundaries are smooth (C^∞) Jordan curves. This result is very surprising and very few people suspected that such a result could be true. As Pérez-Marco notices in [PM], the boundary cannot be an analytic Jordan curve since the linearizing map would extend across the boundary by Schwarz reflection. Pérez-Marco even produces examples where an uncountable number of intrinsic rotations extend univalently in a neighborhood of the closure of the Siegel disk. Pérez-Marco's results have several nice corollaries (see [PM]). For example, it follows that there exist analytic circle diffeomorphisms which are C^∞ linearizable but not analytically linearizable. This answers a question posed by Katok in 1970.

In 1993, in a seminar at Orsay, Pérez-Marco announced the existence of quadratic polynomials having Siegel disks with smooth boundaries. According to Pérez-Marco, his proof is rather technical. In this article, we will present a different approach to the existence of such quadratic polynomials.

Definition 1. *We say that the boundary of a Siegel disk Δ_α is accumulated by small cycles if any neighborhood of $\overline{\Delta}_\alpha$ contains a (whole) periodic cycle of P_α.*

Theorem A. *Assume $\alpha \in \mathbb{R}$ is a Bruno number, $\ell \geq 0$ is an integer, $r \in]0, r_\alpha[$, $\eta > 0$ and $\varepsilon > 0$ are real numbers. Let $u_\alpha : \mathbb{R} \to \mathbb{C}$ be the $\mathbb{Z}$-periodic function $t \mapsto \phi_\alpha(re^{2i\pi t})$. Then there exists a Bruno number α' such that the following properties hold:*

- $|\alpha' - \alpha| < \eta$,
- $r_{\alpha'} = r$,
- *the linearizing map $\phi_{\alpha'} : B(0,r) \to \Delta_{\alpha'}$ extends continuously to a function $\phi_{\alpha'} : \overline{B(0,r)} \to \overline{\Delta_{\alpha'}}$ mapping the boundary of $B(0,r)$ to the boundary of $\Delta_{\alpha'}$,*
- *the $\mathbb{Z}$-periodic function $u_{\alpha'} : \mathbb{R} \to \mathbb{C}$ defined by $u_{\alpha'}(t) = \phi_{\alpha'}(re^{2i\pi t})$ is C^∞, injective and its derivative does not vanish (thus the boundary of the Siegel disk is a smooth Jordan curve) and*
- *the functions u_α and $u_{\alpha'}$ are close in the following sense:*
$$\sup_{j=0,\dots,\ell} \sup_{t\in\mathbb{R}} \left| \frac{\partial^j u_{\alpha'}}{\partial t^j}(t) - \frac{\partial^j u_\alpha}{\partial t^j}(t) \right| < \varepsilon.$$

Finally, we may choose α' so that the boundary of the Siegel disk $\Delta_{\alpha'}$ is accumulated by small cycles.

Remark. When the polynomial P_α is not linearizable, i.e., $r_\alpha = 0$, it is known that 0 is accumulated by small cycles. It may be the case that the boundary of the Siegel disk of a quadratic polynomial is always accumulated by small cycles.

Corollary 1. *There exist quadratic polynomials with Siegel disks whose boundaries do not contain the critical point.*

Proof. Let α be any Bruno number, choose $r \in]0, r_\alpha[$ sufficiently small so that $\phi_\alpha(\partial B(0,r)) \subset B(0, 1/10)$ and then choose $\varepsilon < 1/10$. Then, the boundary of the Siegel disk $\Delta_{\alpha'}$ given by Theorem A is contained in $B(0, 1/5)$. Therefore, the critical point $\omega_{\alpha'} = -e^{2i\pi\alpha'}/2$ cannot belong to the boundary of the Siegel disk $\Delta_{\alpha'}$.

One may also argue that Theorem A asserts that there exist quadratic polynomials with Siegel disks whose boundaries are smooth Jordan curves. But an invariant Jordan curve cannot be smooth at both the critical point and the critical value. ■

Remark. Our proof of existence of quadratic Siegel disks whose boundaries do not contain critical points is totally different from Herman's proof.

Corollary 2. *The set $\mathcal{S} \subset \mathbb{R}$ of real numbers α for which P_α has a Siegel disk with smooth boundary is dense in $\mathbb{R}$ and has uncountable intersection with any open subset of $\mathbb{R}$.*

Proof. Given any Bruno number α and any $\eta > 0$, the conformal radius $r_{\alpha'}$ is an arbitrary real number in the interval $]0, r_\alpha[$ and so, the intersection of $\mathcal{S}$ with the interval $]\alpha - \eta, \alpha + \eta[$ is uncountable. The proof is completed since the set of Bruno numbers is dense in $\mathbb{R}$. ■

Moreover, a theorem of Mañé asserts that the boundary of a Siegel disk of any rational map is contained in the accumulation set of some recurrent critical point (see for example [ST]). Thus, a critical point with orbit falling on the boundary of a fixed Siegel disk must belong itself to this boundary. As a consequence, if P_α has a Siegel disk Δ_α with smooth boundary, the orbit of the critical point avoids $\overline{\Delta_\alpha}$ and thus all the preimages of the Siegel disk also have smooth boundaries.

The following theorem will be proved in part II.

Theorem B. *There exists a dense uncountable set of angles $\alpha \in \mathbb{R}/\mathbb{Z}$ such that every quadratic-like map with a fixed point of multiplier $e^{2i\pi\alpha}$ has a Siegel disk with smooth (C^∞) boundary accumulated by small cycles.*

Remark. This implies that quadratic-like maps having a *cycle* with multiplier $e^{2i\pi\alpha}$ also have Siegel disks with smooth boundaries. Indeed, if we note n the smallest period, Douady and Hubbard (see [P3] appendix A for a detailed proof) have proved that there are bounded domains U and V containing some point z_0 in the cycle such that $\overline{U} \subset V$, $f^{\circ n}(U) = V$ and $f^{\circ n} : U \to V$ is a quadratic-like map. This applies in particular to all quadratic polynomials.

The straightening theorem of Douady and Hubbard (see [DH2]) states that quadratic-like maps are topologically conjugated to polynomials on a neighborhood of the non-escaping set so that Mañé's theorem applies. Thus in this case also, every preimage of the Siegel disk has smooth boundary.

Let us now give an informal description of the construction in theorem A. The main tool is a perturbation lemma.

Lemma 1. *Given any Bruno number α with Siegel disk Δ_α of conformal radius r_α and for any radius r_1 such that $r < r_1 < r_\alpha$, there exists a sequence $\alpha[n] \longrightarrow \alpha$ such that*

(a) *the Siegel disk associated to $\alpha[n]$ tends to the P_α-invariant subdisk of Δ_α with conformal radius r_1, for the Carathéodory topology on simply connected domains containing* 0 *and*
(b) *there is a cycle of $P_{\alpha[n]}$ with period depending on n which tends to the boundary of this subdisk for the Hausdorff topology on compact sets.*

The proof of (a) consists in estimating the size of the Siegel disk of a map which is close to a rotation as done in [C] part 2, and is given in sections 3 and 4. The proof of (b) is based on the work performed in [C] part 1, and is given in section 5.

Now, the conformal map $\phi_{\alpha[n]}$ tends to ϕ_α on the closed ball of radius r for all C^ℓ norms. We fix a n big enough, and do the same procedure again with α replaced by $\alpha(1) = \alpha[n]$, and so on to define $\alpha(2)$, $\alpha(3)$, ... The angle α' in theorem A is produced by a diagonal argument: we arrange so that the restriction of $\phi_{\alpha(j)}$ to the closed ball of radius r is a Cauchy sequence for all C^ℓ norms. We also take perturbations $\alpha(j+1)$ of $\alpha(j)$ small enough so that the cycles we created when defining $\alpha(1)$, ..., $\alpha(j)$ almost do not move. As a consequence, $\phi_{\alpha'}(\partial B(0,r))$ is accumulated by these cycles, which proves it *is* the boundary of $\Delta_{\alpha'}$.

The proof of theorem B goes along the same lines. We first restrict to compact classes $\mathcal{C}_m$ of quadratic-like maps. They are chosen so that $\mathcal{C}_m \subset \mathcal{C}_{m+1}$ and so that the union covers all *analytic* conjugacy classes. The only new difficulty consists in getting a uniform control on the construction. We then do a diagonal argument as for Theorem A. Let us mention that controlling the cycles is also a bit more subtle.

The purpose of Figure 2 is to illustrate this construction. We have drawn the boundary of three quadratic Siegel disks, for

$$\alpha = (\sqrt{5}+1)/2 = [1,1,1,1,\ldots], \quad \alpha(1) = [1,1,1,1,1,1,25,1,1,1,\ldots]$$

and

$$\alpha(2) = [1, 1, 1, 1, 1, 1, 25, 10^{10}, 1, 1, 1, \ldots].$$

For $\alpha(1)$, there is a periodic cycle of period 8 that forces the boundary of the Siegel disk to oscillate slightly. For $\alpha(2)$, there is an additional cycle (of period 205) that forces the boundary to oscillate much more. We have not been able to produce a picture for a possible choice of $\alpha(3)$. The number of iterates of the critical point required to get a relevant picture was much too large. Note also that to be honest, we should have put a big amount of 1's between 25 and 10^{10} in $\alpha(2)$.

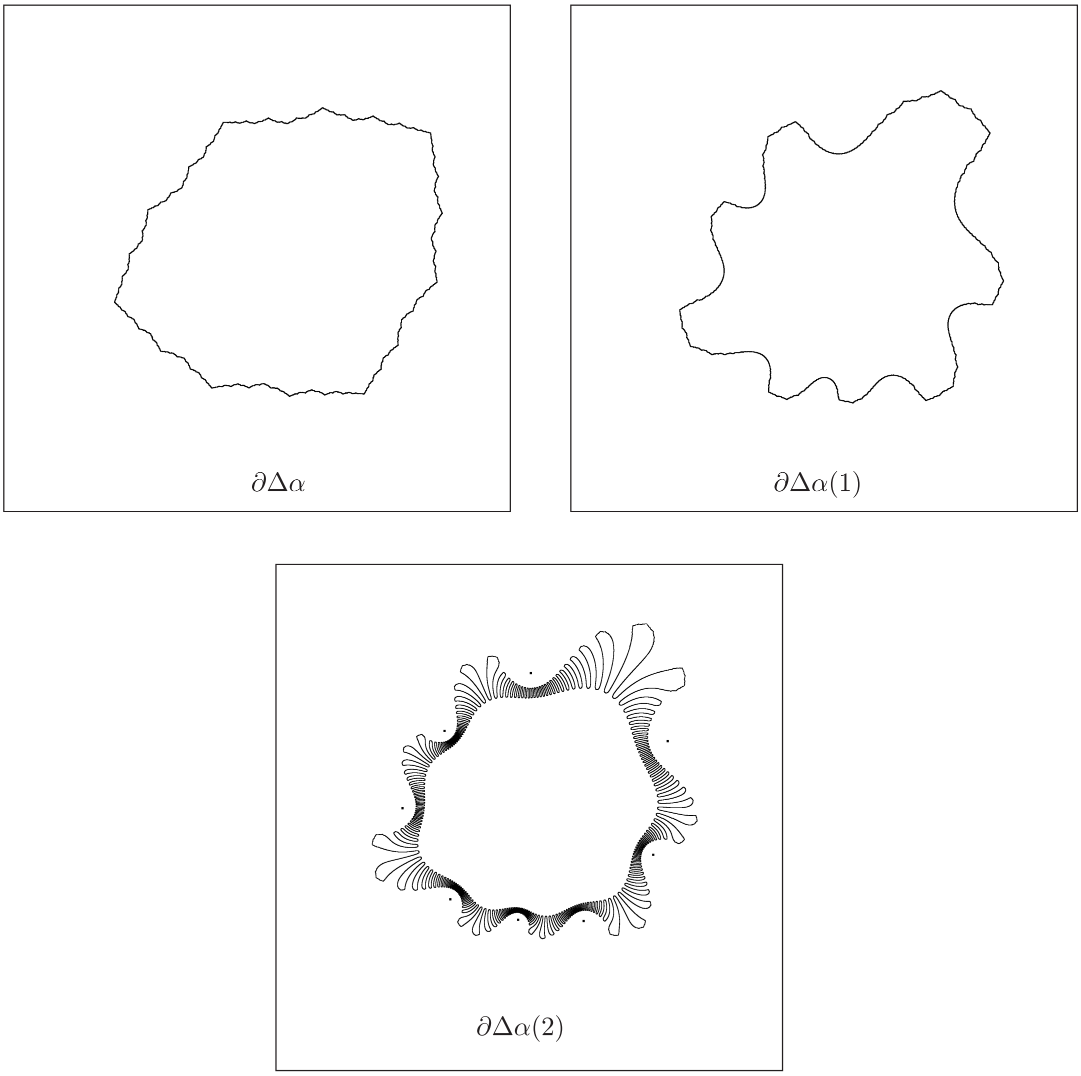

FIGURE 2. The first steps in the construction of a Siegel disk with smooth boundary. In the last frame, we plotted the period 8 cycle that creates the first order oscillation. The period 205 cycle that creates the stronger oscillation is too close to $\partial\Delta\alpha(2)$ to be clearly represented here.

We would like to make an observation. In the same way one produces C^∞ functions which are nowhere analytic with lacunary Fourier series, our Siegel disks are produced with rotation numbers whose continued fractions have large coefficients (in a certain sense) which are more and more spaced out. This is not completely surprising. Indeed, if $\phi : \mathbb{D} \to \Delta$ is a linearizing map, then the coefficients b_k of the power series of ϕ are the Fourier coefficients of the angular parametrization of the boundary of Δ. Those coefficients are also related to the arithmetic nature of α. Indeed, they are defined by the recursive formula

$$b_1 = r_\alpha \quad \text{and} \quad b_{n+1} = \frac{1}{e^{2i\pi\alpha}(e^{2i\pi n\alpha} - 1)} \sum_{i=1}^{n} b_i b_{n+1-i}.$$

If the entry $k+1$ in the continued fraction of α is large, $e^{2i\pi q_k \alpha} - 1$ is close to 0 and b_{1+q_k} is large.

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1. Arithmetical preliminaries.

This section gives a short account of a very classical theory. See for instance [HW] or [Mi].

If $(a_k)_{k\geq 0}$ are integers, we use the notation $[a_0, a_1, \ldots, a_k, \ldots]$ for the continued fraction

$$[a_0, a_1, \ldots, a_k, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k + \ddots}}}.$$

We will say that a_0 is entry 0 of the continued fraction, a_1 is entry 1 of the continued fraction, and a_k is the entry k of the continued fraction. The entry 0 may be any integer in $\mathbb{Z}$, but we require the other to be positive. Then the sequence of finite fractions converges, and the notation refers to it's limit. We define two sequences $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$ recursively by

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = a_0, \quad p_k = a_k p_{k-1} + p_{k-2}, \\ q_{-1} &= 0, \quad q_0 = 1, \quad\;\; q_k = a_k q_{k-1} + q_{k-2}. \end{aligned}$$

The numbers p_k, q_k satisfy

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

In particular, p_k and q_k are coprime. Moreover, if $a_1, a_2, \ldots$ are positive integers, then for all $k \geq 0$, we have

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k].$$

The number p_k/q_k is called the k-th *approximant* to α.

For any irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we denote by $\lfloor \alpha \rfloor \in \mathbb{Z}$ the integer part of α, i.e., the largest integer $n \leq \alpha$, by $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ the fractional part of α, and we define two sequences $(a_k)_{k\geq 0}$ and $(\alpha_k)_{k\geq 0}$ recursively by setting

$$a_0 = \lfloor \alpha \rfloor, \quad \alpha_0 = \{\alpha\}, \quad a_{k+1} = \left\lfloor \frac{1}{\alpha_k} \right\rfloor \quad \text{and} \quad \alpha_{k+1} = \left\{ \frac{1}{\alpha_k} \right\},$$

so that

$$\frac{1}{\alpha_k} = a_{k+1} + \alpha_{k+1}.$$

We then set $\beta_{-1} = 1$ and $\beta_k = \alpha_0\alpha_1 \ldots \alpha_k$.

It is well known that

$$\alpha = [a_0, a_1, \ldots, a_k, \ldots].$$

More precisely, we have the following formulas.

Proposition 1. *Let α be an irrational number and define the sequences $(a_k)_{k\geq 0}$, $(\alpha_k)_{k\geq 0}$, $(\beta_k)_{k\geq -1}$, $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$ as above, so that*

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k].$$

Then, for $k \geq 0$, we have the formulas

$$\alpha = \frac{p_k + p_{k-1}\alpha_k}{q_k + q_{k-1}\alpha_k}, \quad q_k\alpha - p_k = (-1)^k \beta_k,$$

$$q_{k+1}\beta_k + q_k\beta_{k+1} = 1 \quad \textit{and} \quad \frac{1}{q_{k+1} + q_k} < \beta_k < \frac{1}{q_{k+1}}.$$

The last inequalities imply, for $k \geq 0$,

$$\frac{1}{2q_kq_{k+1}} < \left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_kq_{k+1}}.$$

Moreover, for all $k \geq 0$,

$$\alpha_k = [0, a_{k+1}, a_{k+2}, \ldots]$$

2. The proof of Theorem A.

In this section, we give a proof of Theorem A assuming two propositions. The first proposition corresponds to assertion (a) of lemma 1. The second proposition corresponds to assertion (b).

2.1. Two key propositions.

Definition 2. *Given any Bruno number $\alpha = [a_0, a_1, \ldots]$, any real number $A > 1$ and any integer $n \geq 1$, we set*

$$\mathcal{T}(\alpha, A, n) = [a_0, a_1, \ldots, a_n, A_n, 1, 1, \ldots]$$

where $A_n = \lfloor A^{q_n} \rfloor$ is the integer part of A^{q_n}.

We can now state the two propositions.

Proposition 2. *Let $\alpha \in \mathbb{R}$ be a Bruno number and $A > 1$ be a real number. For each $n \geq 0$, set $\alpha[n] = \mathcal{T}(\alpha, A, n)$. Then,*

$$\liminf_{n\to\infty} r_{\alpha[n]} \geq \frac{r_\alpha}{A},$$

and the linearizing parametrizations $\phi_{\alpha[n]}$ converge uniformly to ϕ_α on every compact subset of $B(0, r_\alpha/A)$.

This proposition is proved as proposition 13 in section 6.

Proposition 3. *Under the same assumptions as in proposition 2, $P_{\alpha[n]}$ has a cycle O_n such that we have the following limit for the Hausdorff topology on compact sets:*

$$\lim_{n\to\infty} O_n = \phi_\alpha\left(\partial B\left(0, \frac{r_\alpha}{A}\right)\right).$$

This proposition is proved as proposition 14 in section 6. The next statement will not be used *stricto sensu* in the proof of theorem A, but it is worth mentioning.

Corollary 3. *Under the same assumptions,*

$$r_{\alpha[n]} \underset{n\to\infty}{\longrightarrow} \frac{r_\alpha}{A}$$

and $\Delta_{\alpha_{[n]}}$ tends to $\phi_\alpha\Big(B\big(0, \dfrac{r_\alpha}{A}\big)\Big)$ for the Carathéodory topology on simply connected domains with basepoint 0.

2.2. **Some C^ℓ norms.** Now, and in the next section, we fix α an arbitrary Bruno number, r a real number in $]0, r_\alpha[$, $\eta > 0$ and $\varepsilon > 0$ small real numbers and $\ell_0 \geq 1$ an integer. We want to exhibit a Bruno number α' such that $|\alpha' - \alpha| < \eta$, the conformal radius of the Siegel disk $\Delta_{\alpha'}$ of $P_{\alpha'}$ is $r_{\alpha'} = r$, and the linearizing parametrization $\phi_{\alpha'} : B(0,r) \to \Delta_{\alpha'}$ has a continuous extension to the boundary, such that the $\mathbb{Z}$-periodic function $u_{\alpha'} : \mathbb{R} \to \mathbb{C}$ defined by $u_{\alpha'}(t) = \lim_{z\to re^{2i\pi t}} \phi_{\alpha'}(z)$ is C^∞ and satisfies

$$\sup_{j=0,\ldots,\ell_0} \sup_{t\in\mathbb{R}} \left| \frac{\partial^j u_{\alpha'}}{\partial t^j}(t) - \frac{\partial^j u_\alpha}{\partial t^j}(t) \right| < \varepsilon,$$

where $u_\alpha : \mathbb{R} \to \mathbb{C}$ is the $\mathbb{Z}$-periodic analytic curve $t \mapsto \phi_\alpha(re^{2i\pi t})$.

In order to prove this result, let us introduce the following definition. If $\phi : \partial B(0,r) \to \mathbb{C}$ is a C^∞ function and $\ell \geq 0$ is an integer, we denote by

$$\|\phi\|_{C^\ell} = \sup_{j=0,\ldots,\ell} \sup_{t\in\mathbb{R}} \left| \frac{\partial^j u}{\partial t^j}(t) \right|,$$

where $u : \mathbb{R} \to \mathbb{C}$ is the function $t \mapsto \phi(re^{2i\pi t})$. The same definition holds when ϕ is defined on a bigger set. Note that if a sequence of functions ϕ_j, defined and holomorphic on $B(0,r')$ with $r' > r$, converge uniformly to a limit ϕ, then, for all $\ell > 0$, we have

$$\|\phi_j - \phi\|_{C^\ell} \underset{j\to\infty}{\longrightarrow} 0.$$

Note also that when a sequence of functions ϕ_j, holomorphic in a possibly varying neighborhood of $\overline{B(0,r)}$, is a Cauchy sequence for the norm $\|.\|_{C^0}$ we just defined, then by the maximum modulus principle, it is a Cauchy sequence for the norm $\|.\|_{\sup}$ as a sequence of functions from $\overline{B(0,r)} \to \mathbb{C}$ and so, it has a limit $\phi : \overline{B(0,r)} \to \mathbb{C}$ which is holomorphic in $B(0,r)$ and continuous in $\overline{B(0,r)}$.

Finally, we recall the definition of the Hausdorff metric: if K and L are two compact subsets of $\mathbb{C}$, then

$$d(K,L) = \max\big(d^+(K,L), d^+(L,K)\big)$$

where

$$d^+(K,L) = \sup_{x\in K} \operatorname{dist}(x,L)$$

2.3. **The diagonal argument.** Now, set $\alpha(0) = \alpha$ and let $[a_0, a_1, a_2, \ldots]$ be the continued fraction of α. Moreover, let $r'_0 = r_{\alpha(0)}$ be the conformal radius of the Siegel disk of P_α. We are about to construct by induction sequences $a(j) \in \mathbb{R} \setminus \mathbb{Q}$ for $j \geq 0$, $r'_j \in]r, \infty[$ for $j \geq 0$, $r_j \in]r, \infty[$ for $j \geq 1$, $n_j \in \mathbb{N}$ for $j \geq 0$, C_j a cycle of $P_{\alpha(j)}$ for $j \geq 1$, and C_i^j a cycle of $P_{\alpha(j)}$ for $0 < i \leq j$, of the same period as C_i, with $C_j^j = C_j$.

Choose a radius $r_1 \in]r, r'_0[$ sufficiently close to r so that the Hausdorff distance between the Jordan curves $\phi_{\alpha(0)}(\partial B(0, r_1))$ and $\phi_{\alpha(0)}(\partial B(0, r))$ is less than $\varepsilon/2$. Define the sequence

$$\alpha(0)[n] = \mathcal{T}\left(\alpha(0), \frac{r'_0}{r_1}, n\right).$$

By propositions 2 and 3, we know that choosing n_0 sufficiently large, we have

$$\left|\alpha(0) - \alpha(0)[n_0]\right| < \frac{\eta}{2}, \quad r_{\alpha(0)[n_0]} > r, \quad \left\|\phi_{\alpha(0)[n_0]} - \phi_{\alpha(0)}\right\|_{C^{\ell_0}} < \frac{\varepsilon}{2}$$

and $P_{\alpha(0)[n_0]}$ has a periodic cycle $C_1 = C_1^1 = O_{n_0}$ whose Hausdorff distance to the Jordan curve $\phi_{\alpha(0)}(\partial B(0, r_1))$ is less than $\varepsilon/2$. We then define $\alpha(1) = \alpha(0)[n_0]$, $r'_1 = r_{\alpha(1)}$ and we let A'_1 be the entry $n_0 + 1$ in the continued fraction of $\alpha(1)$:

$$\alpha(1) = [a_0, a_1, a_2, \ldots, a_{n_0-1}, a_{n_0}, A'_1, 1, 1, 1, \ldots].$$

Inductively, we construct a sequence $\alpha(j)$. At each step, we choose a radius $r_{j+1} \in]r, r'_j[$ sufficiently close to r so that the Hausdorff distance between the Jordan curves $\phi_{\alpha(j)}(\partial B(0, r_{j+1}))$ and $\phi_{\alpha(j)}(\partial B(0, r))$ is less than $\varepsilon/2^{j+1}$. We define the sequence

$$\alpha(j)[n] = \mathcal{T}\left(\alpha(j), \frac{r'_j}{r_{j+1}}, n\right).$$

We choose n_j sufficiently large so that

$$\left|\alpha(j) - \alpha(j)[n_j]\right| < \frac{\eta}{2^{j+1}}, \quad r_{\alpha(j)[n_j]} > r, \quad \left\|\phi_{\alpha(j)[n_j]} - \phi_{\alpha(j)}\right\|_{C^{\ell_0+j}} < \frac{\varepsilon}{2^{j+1}},$$

$P_{\alpha(j)[n_j]}$ has a periodic cycle $C_{j+1} = C_{j+1}^{j+1} = O_{n_j}$ whose Hausdorff distance to the Jordan curve $\phi_{\alpha(j)}(\partial B(0, r_{j+1}))$ is less than $\varepsilon/2^{j+1}$ and for all $i \leq j$, $P_{\alpha(j)[n_j]}$ has a periodic cycle C_i^{j+1} whose Hausdorff distance to C_i^j is less than $\varepsilon/2^{j+1}$ and whose period is equal to the one of C_i^j. We then define $\alpha(j+1) = \alpha(j)[n_j]$, $r'_{j+1} = r_{\alpha(j+1)}$ and we let A'_{j+1} be the entry $n_j + 1$ in the continued fraction of $\alpha(j+1)$:

$$\alpha(j+1) = [a_0, a_1, \ldots, a_{n_0-1}, a_{n_0}, A'_1, 1, 1, \ldots, 1, 1, A'_{j+1}, 1, 1, 1, \ldots].$$

By construction, the sequence $\alpha(j)$ is a Cauchy sequence and the limit α' satisfies $|\alpha' - \alpha| < \eta$. Let p_k/q_k be its approximants. The continued fraction expansion of α' is

$$\alpha' = [a_0, \ldots, a_{n_0}, A'_1, 1, 1, \ldots, 1, 1, A'_2, 1, 1, \ldots, 1, 1, A'_3, 1, 1, \ldots]$$

where A'_{j+1} is the integer part of $(r'_j/r_{j+1})^{q_{n_j}}$ and is the entry n_j+1 of the continued fraction.

Also by construction, the sequence $\phi_{\alpha(j)} : \overline{B(0, r)} \to \mathbb{C}$ is a Cauchy sequence for all the norms $\|.\|_{C^\ell}$, $\ell \geq 0$. In particular, it has a limit $\phi : \overline{B(0, r)} \to \mathbb{C}$

which is holomorphic in $B(0,r)$ and continuous in $\overline{B(0,r)}$. Moreover, the function $u : t \mapsto \phi(re^{2i\pi t})$ is C^∞. By construction and as required, we have

$$\|\phi - \phi_\alpha\|_{C^{\ell_0}} = \sup_{j=0,\dots,\ell_0} \sup_{t\in\mathbb{R}} \left| \frac{\partial^j u}{\partial t^j}(t) - \frac{\partial^j u_\alpha}{\partial t^j}(t) \right| < \varepsilon.$$

Since the map $\phi : B(0,r) \to \mathbb{C}$ is the limit of the linearizing parametrizations $\phi_{\alpha(j)}$ for $P_{\alpha(j)}$, it is a linearizing parametrization for $P_{\alpha'}$. Thus, $P_{\alpha'}$ has a Siegel disk $\Delta_{\alpha'}$ which contains $\phi(B(0,r))$.

Let us now prove that $\phi(\partial B(0,r))$ is accumulated by small cycles, which shows that $\Delta_{\alpha'} = \phi(B(0,r))$. For any given $i > 0$, we have constructed a periodic cycle C_i for $P_{\alpha(i)}$, and sequence of periodic cycles C_i^j, $j > i$, whose periods are equal to the one of C_i. By construction, those cycles converge to a periodic cycle C_i' for $P_{\alpha'}$ whose Hausdorff distance to C_i is less than $\varepsilon/2^i$. Now, the Hausdorff distance between C_i and $\phi_{\alpha(i-1)}(\partial B(0,r_i))$ is less than $\varepsilon/2^i$ and the Hausdorff distance between $\phi_{\alpha(i-1)}(\partial B(0,r_i))$ and $\phi_{\alpha(i-1)}(\partial B(0,r))$ is also less than $\varepsilon/2^i$. Finally, by construction, the Hausdorff distance between $\phi_{\alpha(i-1)}(\partial B(0,r))$ and $\phi(\partial B(0,r))$ is less than $\varepsilon/2^{i-1}$. Therefore, for any $i > 0$, the Hausdorff distance between C_i' and $\phi(\partial B(0,r))$ is less than $5\varepsilon/2^i$.

Finally, since $\phi_{\alpha'}$ is a homeomorphism between $\mathbb{D}(0,r)$ and $\Delta_{\alpha'}$, and since it extends continuously by ϕ to the closure $\overline{\mathbb{D}(0,r)}$ which is compact, it follows that $\phi(\partial\mathbb{D}(0,r)) = \partial\Delta_{\alpha'}$. Thus, the boundary of the Siegel disk is locally connected. It follows that it must be a Jordan curve (see [Mi], Lemma 18.7).

Remark. We did not require that $r_j \longrightarrow r$ but this is true, as can be deduced, for instance from the condition that the Hausdorff distance between $\phi_{\alpha(j)}(\partial B(0,r_{j+1}))$ and $\phi_{\alpha(j)}(\partial B(0,r))$ is less than $\varepsilon/2^{j+1}$, the latter converging to $\phi(\partial B(0,r))$.

Now, the conjugacy $u(t+\alpha') = P_{\alpha'}(u(t))$ implies that if u' vanished at some point $t \in \mathbb{R}/\mathbb{Z}$, it would vanish at all the points $t + n\alpha$ for $n \in \mathbb{N}$, which forms a dense set in $\mathbb{R}/\mathbb{Z}$, and thus u would be constant, thus ϕ would be constant on $\overline{B(0,r)}$ by the maximum principle, which is absurd since $\phi'(0) = 1$. Thus $\partial\Delta_{\alpha'}$ is a smooth curve.

Remark. The same conjugacy implies that if $u(t)$ were a critical point, then $u'(t+\alpha)$ would cancel, which we just saw was impossible. Thus there is no critical point on $\partial\Delta_{\alpha'}$. This gives another proof of corollary 1.

Remark. Alternately, we could prove directly, by controlling the norms $\|.\|_{C^0}$ and $\|.\|_{C^1}$ of $\phi - \phi_\alpha$, that u' does not vanish, u is injective and the critical point does not belong to $u(\mathbb{R}/\mathbb{Z})$.

3. The Yoccoz function.

Definition 3 (The Bruno function)**.** *If α is an irrational number, we set*

$$B(\alpha) = \sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k}.$$

If α is a rational number we set $B(\alpha) = +\infty$. The Bruno numbers are those $\alpha \in \mathbb{R}$ for which $B(\alpha) < +\infty$.

Remark. The set of Bruno numbers has full measure in $\mathbb{R}$. It contains the set of all Diophantine numbers, i.e., numbers for which $\log q_{k+1} = \mathcal{O}(\log q_k)$.

We can give an equivalent definition of Bruno numbers via the Yoccoz function.

Definition 4 (The Yoccoz function)**.** *If α is an irrational number, we set*

$$\Phi(\alpha) = \sum_{k=0}^{+\infty} \beta_{k-1} \log \frac{1}{\alpha_k}.$$

If α is a rational number we set $\Phi(\alpha) = +\infty$. The Bruno numbers are those $\alpha \in \mathbb{R}$ for which $\Phi(\alpha) < +\infty$.

Remark. Observe that for any $k_0 \geq 1$, and all α irrational, we have

$$\Phi(\alpha) = \sum_{k=0}^{k_0-1} \beta_{k-1} \log \frac{1}{\alpha_k} + \beta_{k_0-1}\Phi(\alpha_{k_0}).$$

In [Y], Yoccoz uses a modified version of continued fractions but we won't need those modifications. The function Φ we will use is not exactly the same as the one introduced by Yoccoz, but the difference between the two functions is bounded (see [Y], page 14).

The reason why the two definitions of Bruno numbers are equivalent is contained in the following proposition. The proof is essentially given in [Y], page 14.

Proposition 4. *There exists a universal constant C such that for all $k_0 \geq 0$, and all α irrational, we have*

$$\left| \sum_{k=0}^{k_0} \beta_{k-1} \log \frac{1}{\alpha_k} - \sum_{k=0}^{k_0} \frac{\log q_{k+1}}{q_k} \right| < C.$$

The Yoccoz function is highly discontinuous since it takes the value $+\infty$ on a dense set containing $\mathbb{Q}$ and takes finite values on a set of full measure. Let us now see what we can control.

Proposition 5. *The Yoccoz function is lower semi-continuous.*

Proof. We must show that

$$\liminf_{\alpha' \to \alpha} \Phi(\alpha') \geq \Phi(\alpha).$$

Without loss of generality, we may assume that α' is irrational, so that the sequence α'_k is well defined for all $k \geq 0$. Let us first assume that α is an integer. On the one hand, as $\alpha' \to \alpha$ with $\alpha' > \alpha$, we have $\alpha'_0 \to 0$. Thus,

$$\Phi(\alpha') > \log \frac{1}{\alpha'_0} \to +\infty.$$

On the other hand, as $\alpha' \to \alpha$ with $\alpha' < \alpha$, we have $\alpha'_0 \to 1$ with $\alpha'_0 < 1$, and $\alpha'_1 = \{1/\alpha'_0\} \to 0$. Thus,

$$\Phi(\alpha') > \log \frac{1}{\alpha'_0} + \alpha'_0 \log \frac{1}{\alpha'_1} \to +\infty.$$

Now, if α belongs to $\mathbb{Q} \setminus \mathbb{Z}$, we may define $\alpha_0 = \{\alpha\}$, $\alpha_1 = \{1/\alpha_0\}$ and so on, until we get to $\alpha_{k_0+1} = \{1/\alpha_{k_0}\} = 0$. Then, we cannot define the sequence α_k

further. When $\alpha' \to \alpha$, we may also define $\alpha'_0 = \{\alpha'\}$, $\alpha'_1 = \{1/\alpha'_0\}$ up to at least $\alpha'_{k_0+1} = \{1/\alpha'_{k_0}\}$. Then, $\alpha'_0 \to \alpha_0$, $\alpha'_1 \to \alpha_1$, and so on, until we get to $\alpha'_{k_0} \to \alpha_{k_0}$. Depending on whether $\alpha'_{k_0} < \alpha_{k_0}$ or $\alpha'_{k_0} > \alpha_{k_0}$ we have $\alpha'_{k_0+1} \to 1$ or $\alpha'_{k_0+1} \to 0$. In both cases, $\Phi(\alpha'_{k_0+1}) \to +\infty$ and

$$\Phi(\alpha') > \alpha'_0 \cdots \alpha'_{k_0} \Phi(\alpha'_{k_0+1}) \to +\infty.$$

Now, assume α is irrational. Since $\alpha' \to \alpha$, it follows that $\alpha'_k \to \alpha_k$ and $\beta'_k \to \beta_k$. As a consequence, for all $k \geq 0$, we have

$$\beta'_{k-1} \log \frac{1}{\alpha'_k} \underset{\alpha' \to \alpha}{\longrightarrow} \beta_{k-1} \log \frac{1}{\alpha_k}.$$

Since those terms are positive, we get

$$\Phi(\alpha) = \sum_{k=0}^{+\infty} \liminf_{\alpha' \to \alpha} \left(\beta'_{k-1} \log \frac{1}{\alpha'_k} \right) \leq \liminf_{\alpha' \to \alpha} \left(\sum_{k=0}^{+\infty} \beta'_{k-1} \log \frac{1}{\alpha'_k} \right) = \liminf_{\alpha' \to \alpha} \Phi(\alpha').$$

■

For the next proposition, we will have to approximate α by sequences of irrational numbers. In order to avoid the confusion between such a sequence and the sequence $(\alpha_k)_{k\geq 0}$ introduced previously, we will denote the new sequence by $(\alpha[n])_{n\geq 0}$. One corollary of the following proposition is that the graph of Φ contains all the points (α, t) with $t \geq \Phi(\alpha)$.

Proposition 6. *Let $\alpha \in \mathbb{R}$ be a Bruno number and $A > 1$ be a real number. For each integer $n \geq 0$, set $\alpha[n] = \mathcal{T}(\alpha, A, n)$. Then, $\alpha[n] \longrightarrow \alpha$ and*

$$\Phi(\alpha[n]) \underset{n \to \infty}{\longrightarrow} \Phi(\alpha) + \log A.$$

Proof. Simple convergence of the entries in the continued fraction ensures convergence of the number, because the difference between a number and it's k-th approximant p_k/q_k is less than $1/q_k q_{k+1}$ which is less than $1/F_k F_{k+1}$ where F_k is the k-th Fibonacci number ($F_0 = 1$, $F_1 = 1$, $F_{k+2} = F_{k+1} + F_k$). For each integer $n \geq 0$, let us denote by $(\alpha_k[n])_{k\geq 0}$ and $(\beta_k[n])_{k\geq -1}$ the sequences associated to $\alpha[n]$. For each fixed k, we have

$$\lim_{n\to\infty} \alpha_k[n] = \alpha_k \quad \text{and} \quad \lim_{n\to\infty} \beta_k[n] = \beta_k.$$

In particular,

$$\lim_{n\to\infty} \beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} = \beta_{k-1} \log \frac{1}{\alpha_k}.$$

Observe that for $k \leq n$, the approximants of $\alpha[n]$ and α are the same: p_k/q_k. Hence, if $0 < k \leq n-1$, we have

$$\beta_{k-1}[n] < \frac{1}{q_k} \quad \text{and} \quad \frac{1}{\alpha_k[n]} \leq \frac{1}{\beta_k[n]} < 2q_{k+1}.$$

It follows that when $k \leq n-1$, we have

$$\beta_{k-1}[n] \log \frac{1}{\alpha_k[n]} < \frac{\log 2}{q_k} + \frac{\log q_{k+1}}{q_k}.$$

The right terms form a convergent series since α is a Bruno number. Thus, as a function of k, the pointwise convergence with respect to n of the summand

$\beta_{k-1}[n]\log 1/\alpha_k[n]$ is dominated. Therefore we have

$$\sum_{k=0}^{n-1}\beta_{k-1}[n]\log\frac{1}{\alpha_k[n]}\underset{n\to\infty}{\longrightarrow}\sum_{k=0}^{+\infty}\beta_{k-1}\log\frac{1}{\alpha_k}=\Phi(\alpha). \tag{1}$$

We will now estimate the term $\beta_{n-1}[n]\log(1/\alpha_n[n])$ in the Yoccoz function. First, observe that

$$\frac{1}{\alpha_n[n]}=A_n+\frac{1}{\theta},$$

where $\theta=(\sqrt{5}+1)/2$ is the golden mean. As $n\to\infty$, we have $\log A_n\sim q_n\log A$ and thus,

$$\beta_{n-1}[n]\log\frac{1}{\alpha_n[n]}\underset{n\to\infty}{\sim}\beta_{n-1}[n]q_n\log A.$$

We know that $\beta_{n-1}[n]q_n\in]1/2,1[$ and we would like to prove that in our case, this sequence tends to 1. Observe that

$$\beta_{n-1}[n]q_n=1-\beta_n[n]q_{n-1}.$$

Moreover,

$$\begin{aligned}\alpha[n] &= \frac{p_n\left(A_n+\dfrac{1}{\theta}\right)+p_{n-1}}{q_n\left(A_n+\dfrac{1}{\theta}\right)+q_{n-1}}\\ &= \frac{p_n}{q_n}\left(1+\frac{p_{n-1}}{p_n}\frac{1}{A_n}-\frac{q_{n-1}}{q_n}\frac{1}{A_n}+\mathcal{O}\left(\frac{1}{A_n^2}\right)\right)\\ &= \frac{p_n}{q_n}+\frac{(-1)^n}{q_n^2A_n}+\mathcal{O}\left(\frac{1}{A_n^2}\right).\end{aligned}$$

It follows that

$$\beta_n[n]=(-1)^n(q_n\alpha[n]-p_n)=\frac{1}{q_nA_n}\left(1+\mathcal{O}\left(\frac{q_n^2}{A_n}\right)\right).$$

Since $q_n\sim\log A_n/\log A$, we have $q_n^2=o(A_n)$ and

$$\beta_n[n]\sim\frac{1}{q_nA_n}, \tag{2}$$

and since $q_{n-1}<q_n$, we get $\beta_n[n]q_{n-1}\to 0$ as $n\to\infty$. As a consequence, $\beta_{n-1}[n]q_n\longrightarrow 1$ and thus,

$$\beta_{n-1}[n]\log\frac{1}{\alpha_n[n]}\underset{n\to\infty}{\longrightarrow}\log A. \tag{3}$$

Finally, we have $\alpha_{n+1}[n]=1/\theta$ and thus

$$\sum_{k=n+1}^{+\infty}\beta_{k-1}[n]\log\frac{1}{\alpha_k[n]}=\beta_n[n]\cdot\Phi\left(\frac{1}{\theta}\right)\underset{n\to\infty}{\longrightarrow}0. \tag{4}$$

Combining inequalities (1), (3) and (4) gives the required result. ■

Remark. Instead of using a sequence of the form $\alpha[n]=\mathcal{T}(\alpha,A,n)$, we could have taken any sequence $\alpha[n]=[a_0,\dots,a_n,A_n,\theta(n)]$, where A_n are positive integers such that

$$A_n^{1/q_n} \longrightarrow A$$

and $\theta(n) > 1$ are Bruno numbers such that

$$\Phi(1/\theta(n)) = o(q_n A_n).$$

4. Semi-continuity with loss for Siegel disks.

4.1. **Normalized statements.** We will first study perturbations of rotations on the unit disk. We will generalize two theorems due to Yoccoz [Y] for the first and independently to Risler [R] and Chéritat [C] for the second.

Definition 5. *For any real number α, we denote by R_α the rotation $z \mapsto e^{2i\pi\alpha}z$.*

Definition 6. *For any irrational $\alpha \in]0,1[$, we denote by $\mathcal{O}_\alpha$ the set of holomorphic functions f defined in an open subset of $\mathbb{D}$ containing 0, which satisfy $f(0) = 0$ and $f'(0) = e^{2i\pi\alpha}$. We denote by $\mathcal{S}_\alpha$ the set of functions $f \in \mathcal{O}_\alpha$ which are defined and univalent on $\mathbb{D}$.*

Given a map f in $\mathcal{O}_\alpha$, we can define K_f to be the set of points $z \in \mathbb{D}$ whose infinite forward orbit under iteration of f is well defined. The map f is linearizable at 0 if and only if 0 belongs to the interior of K_f, and in that case, the connected component of the interior of K_f which contains 0 is the Siegel disk Δ_f for f (whose definition was given at the beginning of the introduction). We denote by $\mathrm{inrad}(\Delta_f)$ the radius of the largest disk centered at 0 and contained in Δ_f.

Theorem 1 (Yoccoz). *There exists a universal constant C_0 such that for any Bruno number α and any function $f \in \mathcal{S}_\alpha$,*

$$\mathrm{inrad}(\Delta_f) \geq \exp(-\Phi(\alpha) - C_0).$$

Remark. As mentioned earlier, the function Φ defined by Yoccoz in [Y] is not exactly the same as the one we defined in this article, but the difference of the two functions is bounded by a universal constant, so that the result is true as stated here.

In the following, when we say that a sequence of functions f_n converges uniformly on every compact subset of $\mathbb{D}$ to a function f, we do not require the maps f_n to be defined on $\mathbb{D}$. We only ask that any compact set $K \subset \mathbb{D}$ is contained in the domain of f_n for n large enough. We will use the notation $f_n \rightrightarrows f$ on $\mathbb{D}$.

Theorem 2 (Risler-Chéritat). *Assume α is a Bruno number and let $f_n \in \mathcal{O}_\alpha$ be a sequence of functions which converges uniformly to the rotation R_α on every compact subset of $\mathbb{D}$. Then,*

$$\lim_{n\to\infty} \mathrm{inrad}(\Delta_{f_n}) = 1.$$

Our goal is to generalize those results as follows.

Theorem 3. *Assume $(\alpha[n])_{n\geq 0}$ is a sequence of Bruno numbers converging to a Bruno number α such that*

$$\limsup_{n\to\infty} \Phi(\alpha[n]) \leq \Phi(\alpha) + C$$

for some constant $C \geq 0$. Assume $f_n \in \mathcal{O}_{\alpha[n]}$ is a sequence of functions which converges uniformly to the rotation R_α on every compact subset of $\mathbb{D}$. Then,

$$\liminf_{n\to\infty} \operatorname{inrad}(\Delta_f) \geq e^{-C}.$$

The proof will be given in section 4.3.

4.2. **Douady-Ghys's renormalization.** In this section, we describe a renormalization construction introduced by Douady [D] and Ghys. This construction is the heart of Yoccoz's proof of theorem 1. We adapt this construction to our settings, i.e, to maps which are univalent on $\mathbb{D}$ and close to a rotation.

Step 1. Construction of a Riemann surface. Consider a map $f \in \mathcal{S}_\alpha$. Let $\mathbb{H}$ be the upper half plane. There exists a unique lift $F : \mathbb{H} \to \mathbb{C}$ of f such that

$$e^{2i\pi F(Z)} = f(e^{2i\pi Z}) \quad \text{and} \quad F(Z) = Z + \alpha + u(Z),$$

where u is holomorphic, $\mathbb{Z}$-periodic and $|u(Z)| \to 0$ as $\operatorname{Im}(Z) \to \infty$.

Definition 7. *For $\delta > 0$, and $0 < \alpha < 1$ we denote by $\mathcal{S}_\alpha^\delta$ the set of functions $f \in \mathcal{S}_\alpha$ such that for all $Z \in \mathbb{H}$,*

$$|u(Z)| < \delta\alpha \quad \text{and} \quad |u'(Z)| < \delta.$$

Remark. If $\delta < 1$, the condition $|u'(Z)| < \delta$ implies that F has a continuous and injective extension to $\overline{\mathbb{H}}$, and so, f has a continuous and injective extension to $\overline{\mathbb{D}}$.

Remark. Given $\delta > 0$, if $f \in \mathcal{S}_\alpha$, and if $|f(z) - e^{2i\pi\alpha}z|$ and $|f'(z) - e^{2i\pi\alpha}|$ are sufficiently small on $\mathbb{D}$, then $f \in \mathcal{S}_\alpha^\delta$.

We now assume that $0 < \delta < 1/2$ and $f \in \mathcal{S}_\alpha^\delta$. Then, we set $L_0 = i\mathbb{R}^+$ and $L_0' = F(L_0)$. Note that for all $Z \in \mathbb{H}$, $F(Z)$ belongs to the disk centered at $Z + \alpha$ with radius $\delta\alpha$. It follows that the angle between the horizontal and the segment $[Z, F(Z)]$ is less than $\arcsin(\delta) < \pi/6$. Moreover, for all $Z \in \mathbb{H}$, we have $|\arg(F'(Z))| < \arcsin(\delta)$. So L_0' is a smooth curve and its tangents make an angle less than $\pi/6$ with the vertical. This implies that the union $L_0 \cup [0, F(0)] \cup L_0' \cup \{\infty\}$ forms a Jordan curve in the Riemann sphere $\mathbb{P}^1$ bounding a region U such that for $Y > 0$, the segment $[iY, F(iY)]$ is contained in $\overline{U}$. We set $\mathcal{U} = U \cup L_0$.

Denote by B_0 the half-strip

$$B_0 = \{Z \in \mathbb{H} \mid 0 < \operatorname{Re}(Z) < 1\}$$

and consider the map $H : \overline{B_0} \to \overline{\mathcal{U}}$ defined by

$$H(Z) = (1 - X)i\alpha Y + XF(i\alpha Y)$$

where $Z = X + iY$, $(X, Y) \in [0, 1] \times [0, +\infty[$. Then,

$$H(Z) = \alpha Z + Xu(i\alpha Y)$$

Thus,

$$\frac{\partial H}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial H}{\partial X} + i\frac{\partial H}{\partial Y}\right) = \frac{1}{2}\big(u(i\alpha Y) - \alpha X u'(i\alpha Y)\big)$$

and

$$\frac{\partial H}{\partial Z} = \frac{1}{2}\left(\frac{\partial H}{\partial X} - i\frac{\partial H}{\partial Y}\right) = \alpha + \frac{1}{2}\big(u(i\alpha Y) + \alpha X u'(i\alpha Y)\big).$$

It follows that
$$\left|\frac{\partial H}{\partial \overline{Z}}\right| < \alpha\delta \quad \text{and} \quad \left|\frac{\partial H}{\partial Z}\right| > \alpha(1-\delta)$$
and since $\delta < 1/2$, H is a K_δ-quasiconformal homeomorphism between $\overline{B_0}$ and $\overline{\mathcal{U}}$, with $K_\delta = 1/(1-2\delta)$.

If we glue the sides L_0 and L_0' of $\overline{\mathcal{U}}$ via F, we obtain a topological surface $\overline{\mathcal{V}}$. We denote by $\iota : \overline{\mathcal{U}} \to \overline{\mathcal{V}}$ the canonical projection. The space $\overline{\mathcal{V}}$ is a topological surface with boundary, whose boundary $\iota([0, F(0)])$ is denoted $\partial\mathcal{V}$. We set $\mathcal{V} = \overline{\mathcal{V}} \setminus \partial\mathcal{V}$. Since the gluing map F is analytic, the surface $\mathcal{V}$ has a canonical analytic structure induced by the one of $\mathcal{U}$ (see [C] page 70 or [Y] for details).

When $Z \in i\mathbb{R}^+$, $H(Z+1) = F(H(Z))$, and so, the quasiconformal homeomorphism $H : \overline{B}_0 \to \overline{\mathcal{U}}$ induces a homeomorphism between the half cylinder $\mathbb{H}/\mathbb{Z}$ and the Riemann surface $\mathcal{V}$. This homeomorphism is clearly quasiconformal on the image of B_0 in $\mathbb{H}/\mathbb{Z}$, i.e., outside an $\mathbb{R}$-analytic curve. It is therefore quasiconformal in the whole half cylinder ($\mathbb{R}$-analytic curves are removable for quasiconformal homeomorphisms). Therefore, there exists an isomorphism between $\mathcal{V}$ and $\mathbb{D}^*$, which, by a theorem of Carathéodory, extends to a homeomorphism between $\partial\mathcal{V}$ and $\partial\mathbb{D}$. Let $\phi : \overline{\mathcal{V}} \to \overline{\mathbb{D}}^*$ be such an isomorphism and let $K : \overline{\mathcal{U}} \to \overline{\mathbb{H}}$ be a lift of $\phi \circ \iota$ by the exponential map $Z \mapsto \exp(2i\pi Z) : \overline{\mathbb{H}} \to \overline{\mathbb{D}}^*$. The map K is unique up to post-composition with a real translation. We choose ϕ and K such that $K(0) = 0$. By construction, if $Z \in L_0$, then
$$K(F(Z)) = K(Z) + 1.$$

Step 2. The renormalized map. Let us now set
$$\mathcal{U}' = \{Z \in \mathcal{U} \mid \mathrm{Im}(Z) > 5\delta\}, \quad \overline{\mathcal{V}'} = \iota(\overline{\mathcal{U}'})$$
and let $\mathcal{V}'$ be the interior of $\overline{\mathcal{V}'}$.

Let us consider a point $Z \in \mathcal{U}'$. The segment $[Z-1, Z]$ intersects neither L_0' nor $[0, F(0)]$. So either $Z - 1 \in \mathcal{U}$ or $\mathrm{Re}(Z-1) < 0$. For $m \geq 0$, the iterates
$$Z_m \overset{\text{def}}{=} F^{\circ m}(Z-1)$$
stay above the line starting at $Z-1$ and going down with a slope $\tan(\arcsin(\delta))$ ($< 2\delta$ when $\delta < 1/2$), as long as $Z_m \in \mathbb{H}$. Since $\mathrm{Re}(Z-1) \geq -1$ and $\mathrm{Im}(Z-1) > 5\delta$, there exists a least integer $n \geq 0$ such that $\mathrm{Re}(Z_n) \geq 0$ and Z_n is defined.

Let us show that $Z_n \in \mathcal{U}$. If $Z - 1 \in \mathcal{U}$, then $n = 0$ and there is nothing to prove. Otherwise, $n \geq 1$ and $\mathrm{Re}(Z_{n-1}) < 0$. Since Z_{n-1} is above the line starting at $Z-1$ and going down with a slope 2δ, we have $\mathrm{Im}(Z_{n-1}) > 3\delta$. Consider the horizontal segment I joining Z_{n-1} to L_0. Let J be its image. Since $|F'(Z) - 1| < \delta < 1/2$, J is a curve whose tangents make an angle less than $\pi/6$ with the horizontal. Thus, J is to the right of Z_n and in particular, to the right of L_0. Moreover, the tangents of L_0' make an angle less than $\pi/6$ with the vertical. So, J joins Z_n to L_0' and remains to the left of L_0'. Finally, points in I have imaginary part greater than 3δ, and since $|F(Z) - Z - \alpha| < \delta\alpha < \delta$, points in J have imaginary part greater than 2δ. Thus, J does not hit the segment $[0, F(0)]$. It follows that $Z_n \in \mathcal{U}$ and we set $G(Z) = Z_n$. This defines a "first-return map" $G : \mathcal{U}' \to \mathcal{U}$. Note that G is discontinuous. Figure 3 shows the construction of the map G.

The map $G : \mathcal{U}' \to \mathcal{U}$ induces a univalent map $g : \phi(\mathcal{V}') \to \mathbb{D}^*$ such that $g \circ \phi \circ \iota = \phi \circ \iota \circ G$. The fact that g is univalent is not completely obvious (see [Y]

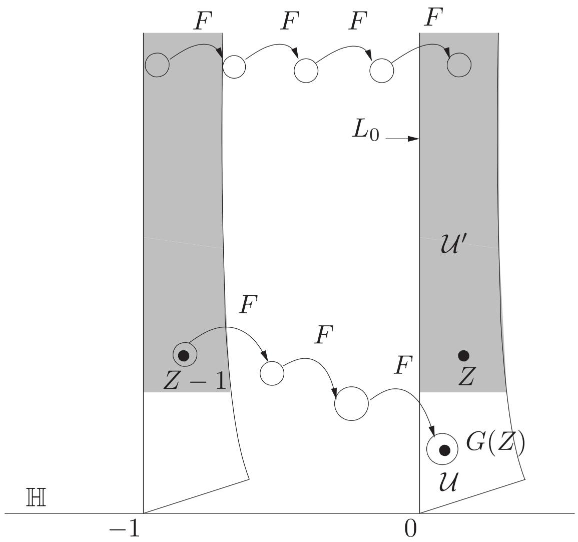

FIGURE 3. The regions $\mathcal{U}$, $\mathcal{U}'$ and the map $G : \mathcal{U}' \to \mathcal{U}$.

for details). The renormalized map of f is the map

$$\mathcal{R}(f) : z \mapsto \overline{g(\overline{z})}.$$

By the removable singularity theorem, this map extends holomorphically to the origin by $\mathcal{R}(f)(0) = 0$, and it is possible to show that $[\mathcal{R}(f)]'(0) = e^{2i\pi/\alpha}$ (again, see [Y] for details). Thus, $\mathcal{R}(f) \in \mathcal{O}_{\alpha'}$, where α' denotes the fractional part of $1/\alpha$. This completes the description of the renormalization operator.

4.3. **The proof of Theorem 3.** Let us now assume that $(\alpha[n])_{n\geq 0}$ is a sequence of Bruno numbers converging to a Bruno number α such that

$$\limsup_{n\to\infty} \Phi(\alpha[n]) \leq \Phi(\alpha) + C$$

for some constant $C \geq 0$. We define the sequences $(\alpha_k)_{k\geq 0}$, $(\beta_k)_{k\geq -1}$, $(\alpha_k[n])_{k\geq 0}$ and $(\beta_k[n])_{k\geq -1}$ as in section 1.

Lemma 2. *For all $k \geq 0$, we have*

$$\limsup_{n\to\infty} \Phi(\alpha_k[n]) \leq \Phi(\alpha_k) + \frac{C}{\beta_{k-1}}.$$

Proof. We have

$$\begin{aligned} \Phi(\alpha[n]) - \Phi(\alpha) &= \sum_{j=0}^{k-1} \left(\beta_{j-1}[n] \log \frac{1}{\alpha_j[n]} - \beta_{j-1} \log \frac{1}{\alpha_j} \right) \\ &\quad + \beta_{k-1}[n]\Phi(\alpha_k[n]) - \beta_{k-1}\Phi(\alpha_k). \end{aligned}$$

For each fixed $j \geq 0$, we have $\alpha_j[n] \to \alpha_j$ as $n \to \infty$, and so,

$$\lim_{n\to\infty} \beta_{j-1}[n] \log \frac{1}{\alpha_j[n]} = \beta_{j-1} \log \frac{1}{\alpha_j}.$$

Thus,

$$\begin{aligned} C &\geq \limsup_{n\to\infty} \Phi(\alpha[n]) - \Phi(\alpha) \\ &= \limsup_{n\to\infty} \beta_{k-1}[n]\Phi(\alpha_k[n]) - \beta_{k-1}\Phi(\alpha_k) \\ &= \beta_{k-1}\left(\limsup_{n\to\infty} \Phi(\alpha_k[n]) - \Phi(\alpha_k)\right). \end{aligned}$$

■

Now, for all $k \geq 0$, we set

$$\rho_k = \inf\left\{\liminf_{n\to\infty} \operatorname{inrad}(\Delta_{f_n})\right\},$$

where the infimum is taken over all sequences $\left(f_n \in \mathcal{O}_{\alpha_k[n]}\right)_{n\geq 0}$ such that $f_n \rightrightarrows R_{\alpha_k}$ on $\mathbb{D}$. Similarly, we set

$$\rho'_k = \inf\left\{\liminf_{n\to\infty} \operatorname{inrad}(\Delta_{f_n})\right\},$$

where the infimum is taken over all sequences $\left(f_n \in \mathcal{S}^{\delta_n}_{\alpha_k[n]}\right)_{n\geq 0}$ such that $\delta_n \to 0$ (which implies $f_n \rightrightarrows R_{\alpha_k}$ on $\mathbb{D}$). It is easy to check that each infimum is realized for some sequence f_n. We will show that

$$\log \rho_0 \geq -C,$$

which is a restatement of Theorem 3.

Lemma 3. *For all $k \geq 0$, we have $\rho_k = \rho'_k$.*

Proof. We clearly have $\rho'_k \geq \rho_k$ since there are fewer functions in $\mathcal{S}_{\alpha_k[n]}$ than in $\mathcal{O}_{\alpha_k[n]}$. Now, assume $(\delta_n)_{n\geq 0}$ and $(f_n \in \mathcal{O}_{\alpha_k[n]})_{n\geq 0}$ are sequences such that $\delta_n \to 0$ and $f_n \rightrightarrows R_{\alpha_k}$ on $\mathbb{D}$. Then, we can find a sequence of real numbers $\lambda_n < 1$ such that $\lambda_n \to 1$ and

$$g_n : z \mapsto \frac{1}{\lambda_n} f_n(\lambda_n z)$$

belongs to $\mathcal{S}^{\delta_n}_{\alpha_k[n]}$. The Siegel disk Δ_{f_n} contains $\lambda_n \Delta_{g_n}$. Therefore

$$\liminf_{n\to\infty} \operatorname{inrad}(\Delta_{f_n}) \geq \liminf_{n\to\infty} \lambda_n \operatorname{inrad}(\Delta_{g_n}) \geq \rho'_k.$$

This shows that $\rho_k \geq \rho'_k$. ■

Lemma 4. *For all $k \geq 0$, we have*

$$\log \rho_k \geq -\Phi(\alpha_k) - \frac{C}{\beta_{k-1}} - C_0,$$

where C_0 is the universal constant provided by theorem 1.

Proof. Indeed, theorem 1 implies that when $f_n \in \mathcal{S}_{\alpha_k[n]}$, then

$$\log \operatorname{inrad}(\Delta_{f_n}) \geq -\Phi(\alpha_k[n]) - C_0.$$

Since

$$\limsup_{n\to\infty} \Phi(\alpha_k[n]) \leq \Phi(\alpha_k) + \frac{C}{\beta_{k-1}},$$

the lemma follows. ■

Let us now fix some $k \geq 0$. Assume that $\left(f_n \in \mathcal{S}^{\delta_n}_{\alpha_k[n]}\right)_{n\geq 0}$ is a sequence of functions such that $\delta_n \to 0$. Then, $f_n \rightrightarrows R_{\alpha_k}$ on $\mathbb{D}$. Moreover, when n is large enough, $\delta_n < 1/2$. So, we can perform Douady-Ghys renormalization. We lift $f_n : \mathbb{D} \to \mathbb{C}$ to a map $F_n : \mathbb{H} \to \mathbb{C}$. We define $\mathcal{U}_n$, $\mathcal{U}'_n$, $\mathcal{V}_n$, $\iota_n : \overline{\mathcal{U}_n} \to \overline{\mathcal{V}_n}$, $H_n : \overline{B_0} \to \overline{\mathcal{U}_n}$, $\phi_n : \overline{\mathcal{V}_n} \to \overline{\mathbb{D}}^*$ and $K_n : \overline{\mathcal{U}_n} \to \overline{\mathbb{H}}$. Remind that $\phi_n \circ \iota_n = \pi \circ K_n$ where $\pi(Z) = \exp(2i\pi Z)$. Then, we define a "first-return map" $G_n : \mathcal{U}'_n \to \mathcal{U}_n$ which induces a univalent map g_n such that $g_n \circ \phi_n \circ \iota_n = \phi_n \circ \iota_n \circ G_n$. The renormalized map is

$$\mathcal{R}(f_n) : z \mapsto \overline{g_n(\overline{z})}.$$

Remember that $\mathcal{R}(f_n)$ belongs to $\mathcal{O}_{\alpha_{k+1}[n]}$, not to $\mathcal{S}_{\alpha_{k+1}[n]}$.

Lemma 5. *The maps $\phi_n \circ \iota_n : \mathcal{U}_n \to \overline{\mathbb{D}}^*$ converge to $Z \mapsto e^{2i\pi Z/\alpha_k}$ uniformly on every compact subset of $B_{\alpha_k} = \{Z \in \mathbb{H} \mid 0 \leq \mathrm{Re}(Z) < \alpha_k\}$.*

Proof. The lifts F_n converge to the translation $Z \mapsto Z + \alpha_k$. It follows that the K_{δ_n}-quasiconformal homeomorphisms $H_n : \overline{B_0} \to \overline{\mathcal{U}_n}$ converge to the map $Z \mapsto \alpha_k Z$ uniformly on $\overline{B_0}$. Moreover, $K_n \circ H_n : \overline{B_0} \to K_n(\overline{\mathcal{U}_n})$ is a K_{δ_n}-quasiconformal homeomorphism which satisfies $K_n \circ H_n(Z+1) = K_n \circ H_n(Z) + 1$ for $Z \in i\mathbb{R}^+$ and sends 0 to 0. Therefore, it extends by periodicity to a K_{δ_n}-quasiconformal automorphism of $\overline{\mathbb{H}}$ fixing 0, 1 and ∞ (we know that the extension is quasiconformal outside $\mathbb{Z} + i\mathbb{R}^+$, and thus, it is quasiconformal on $\mathbb{H}$ since $\mathbb{R}$-analytic curves are removable for quasiconformal homeomorphisms). Since $K_{\delta_n} = 1/(1-2\delta_n) \to 1$ as $n \to \infty$, we see that $K_n \circ H_n$ converges uniformly on every compact subset of $\overline{\mathbb{H}}$ to the identity as $n \to \infty$. As a consequence, the maps K_n converge to $Z \mapsto Z/\alpha_k$ uniformly on every compact subset of B_{α_k}. So, the maps $\phi_n \circ \iota_n : \mathcal{U}_n \to \overline{\mathbb{D}}^*$ converge to $Z \mapsto e^{2i\pi Z/\alpha_k}$ uniformly on every compact subset of B_{α_k}. ■

Lemma 6. *For all $k \geq 0$, we have*

$$\log \rho_k \geq \alpha_k \log \rho_{k+1}.$$

Proof. Let us assume that $\rho_k < 1$ since otherwise, the result is obvious. Let us choose a sequence $\delta_n \to 0$ and a sequence of functions $f_n \in \mathcal{S}^{\delta_n}_{\alpha_k[n]}$ which converges to the rotation $z \mapsto e^{2i\pi\alpha_k}$ and such that

$$\rho_k = \lim_{n\to\infty} \mathrm{inrad}(\Delta_{f_n}).$$

Then, we can find a sequence of points $z_n \in \mathbb{D}$ with $|z_n| \to \rho_k$ and the orbit of z_n under iteration of f_n escapes from $\mathbb{D}$. Without loss of generality, conjugating f_n with a rotation fixing 0 if necessary, we may assume that $z_n \in]0,1[$. Let us consider the points $Z_n \in i\mathbb{R}^+$ such that $e^{2i\pi Z_n} = z_n$. Then, $\mathrm{Im}(Z_n) \to -\log(\rho_k)/2\pi$ and for n large enough, $Z_n \in \mathcal{U}'_n$.

Now, note that by lemma 4, we have $\rho_k > 0$. So Z_n remains in a compact subset of $B_{\alpha_k} = \{Z \in \mathbb{H} \mid 0 \leq \mathrm{Re}(Z) < \alpha_k\}$. Thus, lemma 5 implies that for n large enough, the point $z'_n = \phi_n \circ \iota_n(Z_n)$ is close to $e^{2i\pi Z_n/\alpha_k}$. In particular, we see that for n large enough, $|z'_n| \to \rho_k^{1/\alpha_k}$. Moreover, since the orbit of z_n escapes from $\mathbb{D}$ under iteration of f_n, the orbit of Z_n under iteration of F_n escapes from $\mathbb{H}$ and thus the orbit of z'_n under iteration of $\mathcal{R}(f_n)$ escapes from $\mathbb{D}$. It follows that

$$\log \rho_k = \lim_{n\to\infty} \log |z_n| = \alpha_k \lim_{n\to\infty} \log |z'_n| \geq \alpha_k \liminf_{n\to\infty} \log \mathrm{inrad}\left(\Delta_{\mathcal{R}(f_n)}\right).$$

Finally, lemma 5 also implies that the sequence $(\mathcal{R}(f_n))_{n\geq 0}$ converges to the rotation $z \mapsto e^{2i\pi\alpha_{k+1}}z$ uniformly on every compact subset of $\mathbb{D}$. The definition of ρ_{k+1} implies that

$$\liminf_{n\to\infty} \log \operatorname{inrad}(\Delta_{\mathcal{R}(f_n)}) \geq \log \rho_{k+1}.$$

■

The proof of theorem 3 is now completed easily. Indeed, we see by induction that for all $k \geq 0$, we have

$$\log \rho_0 \geq \alpha_0 \cdots \alpha_k \log \rho_{k+1} = \beta_k \log \rho_{k+1}.$$

And by lemma 4, we get

$$\log \rho_0 \geq -\beta_k \Phi(\alpha_{k+1}) - C - \beta_k C_0.$$

We immediately have

$$\lim_{k\to\infty} \beta_k C_0 = 0.$$

Moreover, the first term in the right side is the rest of the series defining $\Phi(\alpha)$ (see the remark following definition 4). This series converges, and so,

$$\lim_{k\to\infty} \beta_k \Phi(\alpha_{k+1}) = 0.$$

4.4. **Application to quadratic polynomials.** Let us now see how to apply this result to the family of quadratic polynomials $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$. Remember that P_α has a Siegel disk Δ_α if and only if α is a Bruno number and in that case, the conformal isomorphism $\phi_\alpha : B(0, r_\alpha) \to \Delta_\alpha$ fixing 0 with derivative 1 linearizes P_α:

$$\phi_\alpha(e^{2i\pi\alpha}z) = P_\alpha \circ \phi_\alpha(z).$$

The map $\phi_\alpha : B(0, r_\alpha) \to \Delta_\alpha$ is called a *linearizing parametrization.*

Proposition 7. *Assume α is a Bruno number and $\alpha[n]$ is a sequence of Bruno numbers such that*

$$\limsup_{n\to\infty} \Phi(\alpha[n]) \leq \Phi(\alpha) + C,$$

for some constant C. Then,

$$\liminf_{n\to\infty} r_{\alpha[n]} \geq r_\alpha e^{-C},$$

and the linearizing parametrizations $\phi_{\alpha[n]}$ converge uniformly to ϕ_α on every compact subset of $B(0, r_\alpha e^{-C})$.

Proof. Since $\alpha[n] \to \alpha$, we have $P_{\alpha[n]} \to P_\alpha$ uniformly on every compact subset of $\mathbb{C}$. Let us consider the maps

$$f_n(z) = \frac{1}{r_\alpha}\phi_\alpha^{-1} \circ P_{\alpha[n]} \circ \phi_\alpha(r_\alpha z).$$

Then, $f_n \in \mathcal{O}_{\alpha[n]}$ and $f_n \rightrightarrows R_\alpha$. It therefore follows from Theorem 3 that for any $\eta > 0$, if n is sufficiently large, the Siegel disk of f_n contains the disk $B(0, e^{-C-\eta})$.

Thus, for n sufficiently large, $\Delta_{\alpha[n]}$ contains $\phi_\alpha(B(0, r_\alpha e^{-C-\eta}))$. As a consequence, the map $\phi_{\alpha[n]}^{-1} \circ \phi_\alpha$ is defined on the whole disk $B(0, r_\alpha e^{-C-\eta})$ and takes its values in the disk $B(0, r_{\alpha[n]})$. Since $[\phi_{\alpha[n]}^{-1} \circ \phi_\alpha]'(0) = 1$, it follows from Schwarz's lemma that for n large enough, $r_{\alpha[n]} \geq r_\alpha e^{-C-\eta}$, and thus

$$\liminf_{n\to\infty} r_{\alpha[n]} \geq r_\alpha e^{-C-\eta}.$$

Letting η tend to 0 shows that $\liminf r_{\alpha[n]} \geq r_\alpha e^{-C}$.

Let us now deal with the linearizing parametrizations. The map $\phi_{\alpha[n]}$ is defined on the disk $B(0, r_{\alpha[n]})$. Since $\liminf r_{\alpha[n]} \geq r_\alpha e^{-C}$, we see that for any $r < r_\alpha e^{-C}$, the map $\phi_{\alpha[n]}$ is defined on $B(0,r)$ as soon as n is greater than some integer n_r. For $n \geq n_r$, the map $\phi_{\alpha[n]} : B(0,r) \to \mathbb{C}$ takes its values in the filled-in Julia set $K(P_{\alpha[n]}) \subset \overline{B(0,2)}$ and so, the family $\{\phi_{\alpha[n]} : B(0,r) \to \mathbb{C}\}_{n \geq n_r}$ is normal. We must show that any limit value $\phi : B(0,r) \to \mathbb{C}$ coincides with $\phi_\alpha : B(0,r) \to \mathbb{C}$. Since $\phi(0) = 0$ and $\phi'(0) = 1$, it is sufficient, by uniqueness of the linearizing parametrization, to show that ϕ linearizes P_α. This is immediate by passing to the limit, as $n \to \infty$, on the equation

$$\phi_{\alpha[n]} \circ R_{\alpha[n]} = P_{\alpha[n]} \circ \phi_{\alpha[n]}.$$

■

5. Parabolic explosion for quadratic polynomials.

From now on, every time we use the notation p/q for a rational number, we mean that p and q are coprime with $q > 0$.

Let us fix a rational number p/q. Then, 0 is a parabolic fixed point of the quadratic polynomial $P_{p/q} : z \mapsto e^{2i\pi p/q} z + z^2$. It is known (see [DH1], chapter IX) that there exists a complex number $A \in \mathbb{C}^*$ such that

$$P_{p/q}^{\circ q}(z) = z + A z^{q+1} + \mathcal{O}(z^{q+2}).$$

This number shall not be mistaken for the formal invariant of the parabolic germ. Following [C], we introduce the notion of asymptotic size.

Definition 8. *If $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ such that $f(z) = z + A z^{r+1} + \mathcal{O}(z^{r+2})$, then the asymptotic size of f at 0 is the positive number*

$$L_a(f, 0) = \frac{1}{|rA|^{1/r}}.$$

Remark. This quantity measures how fast orbits attracted by 0 under iteration of f will converge to 0:

$$\forall z \text{ such that } f^{\circ n}(z) \underset{\neq}{\longrightarrow} 0, \quad |f^{\circ n}(z)| \underset{n \to \infty}{\sim} \frac{L_a(f,0)}{n^{1/r}}.$$

It follows immediately that if $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ are conjugate by a germ $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, i.e., $g = \phi \circ f \circ \phi^{-1}$ and $\phi'(0) \neq 0$, then

$$L_a(g, 0) = |\phi'(0)| L_a(f, 0).$$

Definition 9. *For each rational number p/q, let us denote by*

- *$A(p/q)$ the coefficient of z^{q+1} in the power series at 0 of $P_{p/q}^{\circ q}$ and*
- *$L_a(p/q)$ the asymptotic size of $P_{p/q}^{\circ q}$ at 0:*

$$L_a(p/q) = \frac{1}{|qA(p/q)|^{1/q}}.$$

Definition 10. *Let $\mathcal{P}_q$ be the set of parameters $\alpha \in \mathbb{C}$ such that $P_\alpha^{\circ q}$ has a parabolic fixed point with multiplier 1. For each rational number p/q, set*

$$R_{p/q} = d\left(\frac{p}{q}, \mathcal{P}_q \setminus \left\{\frac{p}{q}\right\}\right).$$

When $\alpha \neq p/q$ is a small perturbation of p/q, 0 becomes a simple fixed point of P_α and $P_\alpha^{\circ q}$ has q other fixed points close to 0. The dependence of those fixed points on α is locally holomorphic for α not in $\mathcal{P}_{p/q}$. If we add p/q, we get a holomorphic dependence in terms of the q-th root of the perturbation $\alpha - p/q$. The following proposition corresponds to proposition 2.2 in [C].

Proposition 8. *For each rational number p/q,*

(1) *there exists a function $\chi : B = B(0, R_{p/q}^{1/q}) \to \mathbb{C}$ such that*
   - $\chi(0) = 0$,
   - $\chi'(0) \neq 0$,
   - $\forall \delta \in B$, $\left\langle \chi(\delta), \chi(\zeta\delta), \ldots, \chi(\zeta^{q-1}\delta) \right\rangle$ *forms a cycle of period q of P_α with $\zeta = e^{2i\pi p/q}$ and*
   $$\alpha = \frac{p}{q} + \delta^q \ .$$
   This understands the following relation: $\forall \delta \in B$,
   $$\chi(\zeta\delta) = P_\alpha(\chi(\delta)) \ .$$

(2) *Any function satisfying these conditions is of the form $\delta \mapsto \chi(\zeta^k \delta)$ for some $k \in \{0, \ldots, q-1\}$.*

(3) *A computation yields*
$$\chi'(0)^q = -\frac{2i\pi q}{A(p/q)}$$

We prefer in this article to normalize χ differently. We will use the symbol ψ for the new function, and define it by $\psi(\delta) = \chi(\delta/\chi'(\delta))$ wherever it is defined. This amounts to replace the relation $\alpha = \frac{p}{q} + \delta^q$ by
$$\alpha = \frac{p}{q} - \frac{A(p/q)}{2i\pi q}\delta^q.$$
There are two advantages to do this. First, this function does not depend on the choice of χ among the q possibilities. Second, is to give a nice statement of proposition 12. Let
$$r_{p/q} = \left| \frac{2\pi q R_{p/q}}{A(p/q)} \right|^{1/q},$$
and let us give the version of proposition 8 that we will use here.

Proposition 9. *For each rational number p/q, there exists a unique holomorphic function $\psi : B(0, r_{p/q}) \to \mathbb{C}$ such that*

(1) $\psi(0) = 0$,
(2) $\psi'(0) = 1$,
(3) *for any $\delta \in B(0, r_{p/q})$,*
$$\left\langle \psi(\delta), \psi(\zeta\delta), \psi(\zeta^2\delta), \ldots, \psi(\zeta^{q-1}\delta) \right\rangle$$
forms a cycle of period q of P_α with $\zeta = e^{2i\pi p/q}$ and
$$\alpha = \frac{p}{q} - \frac{A(p/q)}{2i\pi q}\delta^q.$$

In particular, $\forall \delta \in B(0, r_{p/q})$,

$$\psi(\zeta\delta) = P_\alpha(\psi(\delta)).$$

Remark. Note that we consider complex perturbations of p/q.

Proof. Set $A = A(p/q)$ and $r = r_{p/q}$. Let us write

$$\alpha(\delta) = \frac{p}{q} - \frac{A}{2i\pi q}\delta^q$$

The function from $B(0,r) \times \mathbb{C}$ to $\mathbb{C}$ defined by $(\delta, z) \mapsto P^{\circ q}_{\alpha(\delta)}(z) - z$ vanishes at $(\delta, 0)$. Thus it factors into $P^{\circ q}_{\alpha(\delta)}(z) - z = z \cdot h(\delta, z)$ where $h : B(0,r) \times \mathbb{C} \to \mathbb{C}$ is analytic. One computes that $h(\delta, 0) = e^{-A\delta^q} - 1$. Let us consider the subset $\mathcal{M}$ of $B(0,r) \times \mathbb{C}$ that consists in pairs (δ, z) for which $h(\delta, z) = 0$. For all $(\delta, z) \in \mathcal{M}$, z is a periodic point of $P_{\alpha(\delta)}$ of period dividing q. Let $\pi_1 : B(0,r) \times \mathbb{C} \to B(0,r)$ be the projection to the first coordinate: $\pi_1(\delta, z) = \delta$. Let $\pi_2 : B(0,r) \times \mathbb{C} \to \mathbb{C}$ be the projection to the second coordinate: $\pi_2(\delta, z) = z$.

Lemma 7. *The restriction* $\pi_1 : \mathcal{M} \to B(0,r)$ *is proper.*

Proof. Let us take a sequence $(\delta_n, z_n) \in \mathcal{M}$ such that δ_n converges to some $\delta \in B(0,r)$. Observe that if $|z| > 1 + |e^{2i\pi\alpha}|$, then

$$|P_\alpha(z)| = |z| \cdot |e^{2i\pi\alpha} + z| > |z|,$$

and so, z cannot be a periodic point of P_α. As a consequence, $|z_n| \le 1 + |e^{2i\pi\alpha(\delta_n)}|$. Now, since $\delta_n \in B(0,r)$, we have $\alpha(\delta_n) \in B(p/q, R_{p/q})$. Moreover $R_{p/q} \le 1$, since $\mathbb{Z} \subset \mathcal{P}_q$ for every q. As a consequence, the sequence z_n is bounded (it belongs to $B(0, 1 + e^{2\pi})$) and so, extracting a subsequence if necessary, we may assume that z_n converges to some point $z \in \mathbb{C}$. Passing to the limit in the equation $P^{\circ q}_{\alpha(\delta_n)}(z_n) = z_n$, we obtain $P^{\circ q}_{\alpha(\delta)}(z) = z$. Thus $(\delta, z) \in \mathcal{M}$ and $\pi_1 : \mathcal{M} \to B(0,r)$ is proper. □

At points $(\delta, z) \in \mathcal{M} \setminus \{(0,0)\}$, the derivative of $h(\delta, z)$ with respect to z is equal to

$$\frac{m-1}{z} \neq 0,$$

where $m = (P^{\circ q}_{\alpha(\delta)})'(z)$, which in the case the periodic point z has period q is it's multiplier. Thus, by the implicit function theorem, δ is a local coordinate for $\mathcal{M}$. The point $(0,0)$ belongs to $\mathcal{M}$ and for (δ, z) close to $(0,0)$, a computation yields the following expansion

$$h(\delta, z) = A \cdot (z^q - \delta^q) + \text{higher order terms}.$$

More precisely, "higher order terms"$= \mathcal{O}(\delta^q z) + \mathcal{O}(z^{q+1})$. Thus $\mathcal{M}$ is locally the union of q smooth manifolds through $(\delta, z) = (0,0)$, for which δ is a local parameter, as can be proved by introducing the new variable $\lambda = z/\delta$. This gives a new chart (δ, λ), with $(\delta, \lambda) \in B(0,r) \times \mathbb{C}$, that extends the complex manifold "$(\delta, z) \in B(0,r) \times \mathbb{C} - \{(0,0)\}$" into some complex manifold $\mathcal{C}$. Note that the projection η from $\mathcal{C}$ back to $B(0,r) \times \mathbb{C}$ is well defined and analytic. We let $\widetilde{\pi}_1 = \pi_1 \circ \eta : \mathcal{C} \to B(0,r)$. The manifold $\mathcal{M} \setminus \{(0,0)\}$ extends in the chart (δ, λ) into a complex submanifold $\widetilde{\mathcal{M}}$ (the proper transform of $\mathcal{M}$) by adding the q points $(\delta, \lambda) = (0, \exp(i2\pi k/q))$ for $k \in \mathbb{Z}$. The map η is proper from $\widetilde{\mathcal{M}}$ to $\mathcal{M}$, and we have seen that the map π_1 is proper from $\mathcal{M}$ to $B(0,r)$, thus their composition $\widetilde{\pi}_1$

is proper from $\widetilde{\mathcal{M}}$ to $B(0,r)$. The fact that δ is always a local coordinate translates in "$\widetilde{\pi}_1|_{\widetilde{\mathcal{M}}}$ is locally invertible". Thus $\widetilde{\pi}_1|_{\widetilde{\mathcal{M}}}$ is a covering. Since $B(0,r)$ is simply connected, this is a trivial covering. This means $\widetilde{\mathcal{M}}$ is a disjoint union of graphs of functions of δ over $B(0,r)$. More rigorously, we mean that there exist $m > 0$ and, for $j = 1, \ldots, m$, analytic functions $\sigma_j : B(0,r)$ mapping into $\mathcal{C}$, such that $\forall \delta \in B(0,r)$, $\widetilde{\pi}_1 \circ \sigma_j(\delta) = \delta$, and $\widetilde{\mathcal{M}}$ is the disjoint union of the images of the σ_j. We may suppose σ_1 is the one whose image goes through $(\delta, \lambda) = (0,1)$.

For uniqueness, let us consider a candidate function ψ: by hypothesis, we have $\psi(0) = 0$, $\psi'(0) = 1$ and for every $\delta \in B(0,r)$, the point $(\delta, z) = (\delta, \psi(\delta))$ belongs to $\mathcal{M}$. Since

$$\frac{\psi(\delta)}{\delta} \underset{\delta \to 0}{\longrightarrow} 1,$$

the map $\delta \mapsto (\delta, z) = (\delta, \psi(\delta))$ lifts to a continuous function $\widetilde{\psi}$ from $B(0,r)$ to $\widetilde{\mathcal{M}}$ mapping 0 to $(\delta, \lambda) = (0,1)$. Since the union of leaves composing $\widetilde{\mathcal{M}}$ is disjoint, and $\widetilde{\psi}(0) = \sigma_1(0)$, we have by continuity $\widetilde{\psi} = \sigma_1$. This proves uniqueness.

For existence, we just take $\psi = \pi_2 \circ \eta \circ \sigma_1$. There remains to prove that $\forall \delta \in B(0,r)$, $\psi(e^{2i\pi p/q} z) = P_{\alpha(\delta)}(\psi(\delta))$. For this, let $g(\delta) = P_{\alpha(\delta)}(\psi(e^{-2i\pi p/q}\delta))$. This function satisfies conditions (1), (2) and (3) of the statement, thus by uniqueness it is equal to ψ. ■

The function ψ depends on p/q and from now on, we will denote it by $\psi_{p/q}$. Our goal is to understand how the maps $\psi_{p/q}$ behave when p/q tends to some irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In order to be able to say something, we must first understand how the numbers $R_{p/q}$ and $r_{p/q}$ behave as $q \to \infty$. This is the object of the following two propositions.

Proposition 10. *There exists a constant C such that for every rational number p/q, we have*

$$R_{p/q} \geq \frac{C}{q^3}.$$

Proof. The proof relies on Douady's landing theorem and the Yoccoz inequality (see [Hu] or [P1]).

Let us choose a rational number p/q, and assume that $\alpha \in \mathcal{P}_q \setminus \{p/q\}$. In other words, assume $\alpha \neq p/q$ and $P_\alpha^{\circ q}$ has a multiple fixed point z_0. Then, P_α has a parabolic cycle $\langle z_0, z_1, \ldots, z_{q_1 - 1}\rangle$ of period q_1 dividing q, and the immediate basin of this parabolic cycle contains the critical point $\omega_0 = -e^{2i\pi\alpha}/2$ of P_α. As a consequence, the Julia set $J(P_\alpha)$ is connected and all other periodic cycles of P_α are repelling.

It follows that the fixed point 0 is either parabolic or repelling.

In the first case, $\alpha = p'/q$ with $p' \in \mathbb{Z}$ not necessarily prime to q. Then, the distance between α and p/q is trivially bounded from below by $1/q$.

In the second case, α belongs to the lower half-plane $\{\mathrm{Im}(\alpha) < 0\}$. At this point, we need combinatorial arguments. We will refer mainly to [GM]. (The reader may also look at [Hu], [DH1], and to further work of these authors and also Branner, Kiwi, Mc Mullen, Sullivan, Thurston, and others.) Since the Julia set is connected, Douady's landing theorem asserts that there are finitely many (let's say s) external rays landing at the repelling point $z = 0$. Those s rays can be ordered cyclically with respect to their arguments. They are permuted by P_α and each ray is mapped to the one which is r further counter-clockwise for some $r < s$ and prime to s (see [GM], Lemma 2.2 and the remark that follows). Then, the Yoccoz inequality implies that α belongs to the closed disk of radius $(\log 2)/(2\pi s)$ tangent to the real axis at r/s (see for example [Hu]). A key result depending on the fact we are working with quadratic polynomials is that we necessarily have $q > s$ (see also [C], subsection 6.2 pages 54-55). Indeed, the s rays together with their common end point divide the complex plane into s connected components called *sectors*. Let us call S_0 the one that contains the critical point, and $S_1, \ldots, S_{s-1}$ the others, in the order given by the external argument of their bounding rays. According to [GM], lemma 2.5, for $i \neq 0$, f maps S_i homeomorphically to S_j where $j = i + r \bmod s$, whereas S_0 is mapped to all of $\mathbb{C}$. Moreover, the critical point is mapped in S_r (because otherwise, the preimage of S_r in S_0 would be all of S_0, and thus P would be an isomorphism from S_0 to S_r). It follows that $P^{-1}(S_r)$ has only one connected component U, contained in S_0. The critical point belongs to the immediate basin of a point z_0 in the parabolic cycle. It follows that $z_0 \in U$. Thus, the first q iterates $f^{\circ k}(z_0)$ for $k = 0, \ldots, q-1$ visit each sector, before $f^q(z_0)$ falls back into S_0. This implies it has period $\geq s$, thus $q \geq s$.

Let us now assume by contradiction that $q = s$. Then z_0 has period s and multiplier 1. Moreover, $f^{\circ s} : \overline{U} \to \overline{S_0}$ is *weakly polynomial-like of degree* 2 according to definition 3.6 in [GM]. According to lemma 3.7 in [GM] (a version of the Lefschetz Fixed Point Theorem), the sum of Lefschetz indices (which are positive integers) of the fixed points of this map is equal to 2. However, the index of z_0 is at least 2, and there is at least another fixed point which is $z = 0$. This leads to contradiction.

Thus $q > s$. Pythagoras theorem then gives

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{2\pi}\sqrt{\left(2\pi\left(\frac{r}{s} - \frac{p}{q}\right)\right)^2 + \left(\frac{\log 2}{s}\right)^2} - \frac{\log 2}{s}.$$

Since $s < q$, an elementary computation gives

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{2\pi}{q^3 \log 2\left(\sqrt{\left(\frac{2\pi}{q\log 2}\right)^2 + 1} + 1\right)}.$$

As q tends to ∞, this last term is equivalent to $\pi/(q^3 \log 2)$, which proves the proposition. ■

Definition 11. *For each irrational number α, we define*

- $r_\alpha = 0$ *if P_α is not linearizable and*
- r_α *is the conformal radius of the Siegel disk of P_α otherwise.*

Proposition 11. *Assume α is an irrational number such that P_α has a Siegel disk and p_k/q_k are the approximants to α. Then,*

$$\liminf_{k\to\infty} r_{p_k/q_k} \geq r_\alpha.$$

Proof. The following lemma appears in Jellouli's thesis [J1] (compare also with [J2] Theorem 1).

Lemma 8. *Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is chosen so that P_α has a Siegel disk Δ_α and let $p_k/q_k \to \alpha$ be the approximants to α given by the continued fraction. Then, $P_{p_k/q_k}^{\circ q_k}$ converges uniformly to the identity on every compact subset of Δ_α.*

Proof. Let $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$ be the isomorphism that fixes 0 with derivative 1 and let $r < r_\alpha$ be an arbitrary radius (in the end, we will let r tend to r_α). For each $\alpha' \in \mathbb{C}$, consider the map $g_{\alpha'} = \phi_\alpha^{-1} \circ P_{\alpha'} \circ \phi_\alpha$ which is defined on a neighborhood of 0 in $B(0,r_\alpha)$. As $\alpha' \to \alpha$, $P_{\alpha'}$ converges to P_α uniformly on every compact subset of Δ_α. It follows that $g_{\alpha'}(z)/(e^{2i\pi\alpha'}z)$ converges uniformly to 1 on every compact subset of $B^*(0,r_\alpha)$. By the maximum modulus principle, the convergence is in fact uniform on $B(0,r_\alpha)$. Let us choose $\varepsilon > 0$ sufficiently small so that

$$\forall \alpha' \in B(\alpha,\varepsilon), \quad g_{\alpha'} \text{ is defined on } B(0,r) \quad \text{and} \quad \left| \frac{g_{\alpha'}(z)}{e^{2i\pi\alpha'}z} - 1 \right| < 1.$$

Under this condition, we may define a function u analytic on $B(\alpha,\varepsilon) \times B(0,r)$ by

$$u(\alpha', Z) = \log\left(\frac{g_{\alpha'}(z)}{e^{2i\pi\alpha'}z}\right),$$

where $\log : B(1,1) \to \mathbb{C}$ is the branch of logarithm that sends 1 to 0. Note that the function $Z \mapsto u(\alpha', e^{2i\pi Z})$ is $\mathbb{Z}$-periodic.

Without loss of generality, we may assume that u is analytic on a neighborhood of $\overline{B(\alpha,\varepsilon)} \times \overline{B(0,r)}$, and thus, bounded by some constant M on $B(\alpha,\varepsilon) \times B(0,r)$.

Let us now consider the upper half-plane

$$\mathbb{H}_r = \left\{ Z \in \mathbb{C} \;\middle|\; \mathrm{Im}(Z) > \frac{1}{2\pi}\log\frac{1}{r} \right\}.$$

We then define a function $G_{\alpha'} : \mathbb{H}_r \to \mathbb{H}$ by

$$G_{\alpha'}(Z) = Z + \alpha' + u(\alpha', e^{2i\pi Z}).$$

We have

$$g_{\alpha'}(e^{2i\pi Z}) = e^{2i\pi G_{\alpha'}(Z)}.$$

Thus, we have lifted $g_{\alpha'} : B(0,r) \to B(0,r_\alpha)$ to a map $G_{\alpha'} : \mathbb{H}_r \to \mathbb{H}$. Since $P_{\alpha'} \to P_\alpha$ as $\alpha' \to \alpha$, we have $u(\alpha,z) = 0$ for every $z \in B(0,r)$. It follows from the maximum modulus principle that $\forall \alpha' \in B(\alpha,\varepsilon)$, $\forall z \in B(0,r)$,

$$\left| \frac{u(\alpha',z) - u(\alpha,z)}{\alpha' - \alpha} \right| \le \max_{\alpha' \in \partial B(\alpha,\varepsilon)} \left| \frac{u(\alpha',z) - u(\alpha,z)}{\alpha' - \alpha} \right| \le \frac{M}{\varepsilon}.$$

Thus, setting $K = M/\varepsilon$, we get

$$\forall \alpha' \in B(\alpha,\varepsilon),\ \forall z \in B(0,r), \quad |u(\alpha',z)| \le K|\alpha' - \alpha|.$$

We will now use a fundamental property of the approximants p_k/q_k of α:

$$\left| \alpha - \frac{p_k}{q_k} \right| \le \frac{1}{q_k^2}.$$

If $p_k/q_k \in B(\alpha,\varepsilon)$, we therefore have

$$\left| G_{p_k/q_k}(Z) - Z - \frac{p_k}{q_k} \right| \le K \left| \frac{p_k}{q_k} - \alpha \right| \le \frac{K}{q_k^2}.$$

It follows by induction that if $p_k/q_k \in B(\alpha,\varepsilon)$ and $\mathrm{Im}(Z) > \dfrac{1}{2\pi}\log\dfrac{1}{r} + \dfrac{K}{q_k}$, then for any $m = 0, \ldots, q_k$, we have: $G_{p_k/q_k}^{\circ m}(Z)$ is defined and

$$\left| G_{p_k/q_k}^{\circ m}(Z) - Z - m\frac{p_k}{q_k} \right| \le m\frac{K}{q_k^2}.$$

As a consequence, if $p_k/q_k \in B(\alpha,\varepsilon)$ and $\mathrm{Im}(Z) > \frac{1}{2\pi}\log\frac{1}{r} + \frac{K}{q_k}$, then

$$\left|G^{\circ q_k}_{p_k/q_k}(Z) - Z - p_k\right| \le \frac{K}{q_k},$$

and when k is sufficiently large, $g^{\circ q_k}_{p_k/q_k}$ is uniformly close to the identity on a disk centered at 0 of radius arbitrarily close to r. The lemma follows since r can be chosen arbitrarily close to r_α. □

Again, let $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$ be the linearizing map which fixes 0 with derivative 1. For each $k \ge 0$, set

$$g_k = \phi_\alpha \circ P_{p_k/q_k} \circ \phi_\alpha^{-1}.$$

Then, since $\phi'_\alpha(0) = 1$, an elementary computation gives

$$g_k^{\circ q_k} = z + A(p_k/q_k) z^{1+q_k} + \mathcal{O}(|z|^{2+q_k}).$$

The previous lemma implies that $g_k^{\circ q_k}$ converges uniformly to the identity on every compact subset of $B(0,r_\alpha)$. For any radius $r < r_\alpha$, we may find an integer N so that $g_k^{\circ q_k}$ is defined on $B(0,r)$ for $n \ge N$. Since $g_k^{\circ q_k}$ takes its values in $B(0,r_\alpha)$, we have

$$|A(p_k/q_k)| = \frac{1}{2\pi}\left|\int_{\partial B(0,r)} \frac{g_k^{\circ q_k}(z)}{z^{2+q_k}} dz\right| \le \frac{r_\alpha}{r^{1+q_k}}.$$

Therefore,

$$|A(p_k/q_k)|^{1/q_k} \le \frac{1}{r}\left(\frac{r_\alpha}{r}\right)^{1/q_k} \underset{n\to\infty}{\longrightarrow} \frac{1}{r}.$$

Finally, since $R_{p/q} > C/q^3$, we see that

$$[2\pi q R_{p/q}]^{1/q} \underset{q\to\infty}{\longrightarrow} 1,$$

and so

$$\liminf_{k\to\infty} r_{p_k/q_k} = \liminf_{k\to\infty} \frac{1}{|A(p_k/q_k)|^{1/q_k}} \ge r.$$

The result follows by letting $r \to r_\alpha$. ■

We may now study the asymptotic behavior of the functions ψ_{p_k/q_k} as $k \to \infty$.

Proposition 12. *Assume $\alpha \in \mathbb{R}$ is an irrational number such that P_α has a Siegel disk Δ_α and p_k/q_k are the approximants to α. Then,*

- $\lim_{k\to\infty} r_{p_k/q_k} = r_\alpha$ *and*
- *the sequence of functions $\psi_{p_k/q_k} : B(0, r_{p_k/q_k}) \to \mathbb{C}$ converges uniformly on every compact subset of $B(0,r_\alpha)$ to the isomorphism $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$ which fixes 0 with derivative 1.*

Proof. We have just seen that

$$\liminf_{k\to\infty} r_{p_k/q_k} \ge r_\alpha.$$

Therefore, given any radius $r < r_\alpha$, the function ψ_{p_k/q_k} is defined on the disk $B(0,r)$ for $k \ge k_r$ large enough. If $\alpha \in B(p/q, R_{p/q})$ and z is a periodic point of P_α, then, as mentioned in Lemma 7, $|z| \le 1 + e^{2\pi}$. Therefore, the functions ψ_{p_k/q_k} all take their values in the disk $B(0, 1+e^{2\pi})$. It follows that the sequence of functions

$$\left(\psi_{p_k/q_k} : B(0,r) \to B(0,1+e^{2\pi})\right)_{k\ge k_r}$$

is normal. Let $\psi : B(0,r) \to \mathbb{C}$ be a limit value. We have

$$\psi_{p_k/q_k}(e^{2i\pi p_k/q_k}\delta) = P_{p_k/q_k} \circ \psi_{p_k/q_k}(\delta).$$

Passing to the limit as $k \to \infty$, we see that for any $\delta \in B(0,r)$, we have

$$\psi(e^{2i\pi\alpha}\delta) = P_\alpha \circ \psi(\delta).$$

Therefore, ψ coincides with the linearizing parametrization of P_α that fixes 0 with derivative 1. As a consequence, the whole sequence $\delta \mapsto \psi_{p_k/q_k}(\delta)$ converges on compact subsets of $B(0,r_\alpha)$ to the isomorphism $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$ which fixes 0 with derivative 1.

Now, let r be defined by

$$r = \limsup_{k\to\infty} r_{p_k/q_k}.$$

Extracting a subsequence if necessary, we may assume that the sequence r_{p_k/q_k} converges to r. Then, the same argument as above shows that the extracted subsequence ψ_{p_k/q_k} converges on compact subsets of $B(0,r)$ to a holomorphic map $\phi : B(0,r) \to \mathbb{C}$ which fixes 0 with derivative 1 and linearizes P_α. In particular, the linearizing parametrization $\phi_\alpha : B(0,r_\alpha) \to \Delta_\alpha$ is holomorphic on the disk of radius r and so,

$$r \le r_\alpha.$$

■

Corollary 4. *Assume α is an irrational number such that P_α has a Siegel disk and p_k/q_k are the approximants to α. Then,*

$$\lim_{k\to\infty} L_a(p_k/q_k) = r_\alpha.$$

Proof. Since $C/q^3 \le R_{p/q} \le 1$ and

$$r_{p/q} = [2\pi q^2 R_{p/q}]^{1/q} L_a(p/q),$$

we see that the quantities $r_{p/q}$ and $L_a(p/q)$ are equivalent as $q \to \infty$. ■

Corollary 5. *Assume α is a Bruno number and let p_k/q_k be the approximants to α defined by the continued fraction. Assume $\alpha[n]$ is a sequence of real numbers such that*

$$\left|\alpha[n] - \frac{p_n}{q_n}\right|^{1/q_n} \underset{n\to\infty}{\longrightarrow} \lambda < 1.$$

Then, the sequence of compact sets

$$O_n = \psi_{p_n/q_n}\left\{\delta_n e^{2i\pi k/q_n} \;\middle|\; k = 1,\ldots,q_n\right\}$$

converges to $\phi_\alpha(\partial B(0,\lambda r_\alpha))$ for the Hausdorff topology on compact subsets of $\mathbb{C}$. Moreover, the conformal radii $r_{\alpha[n]}$ of the Siegel disks of the quadratic polynomials $P_{\alpha[n]}$ satisfy:

$$\limsup_{n\to\infty} r_{\alpha[n]} \le \lambda r_\alpha.$$

Proof. For each n, let δ_n be a complex number such that

$$\frac{p_n}{q_n} + i\frac{A(p_n/q_n)}{2\pi q_n}\delta_n^{q_n} = \alpha[n].$$

As $n \to \infty$,

$$|\delta_n| = \left|\alpha[n] - \frac{p_n}{q_n}\right|^{1/q_n} \cdot \left|\frac{2\pi q_n}{A(p_n/q_n)}\right|^{1/q_n} \to \lambda r_\alpha.$$

Moreover, the sequence ψ_{p_n/q_n} converges to the linearizing parametrization $\phi_\alpha : B(0, r_\alpha) \to \Delta_\alpha$. Therefore, the sequence of compact sets

$$O_n = \psi_{p_n/q_n} \left\{ \delta_n e^{2i\pi k/q_n} \;\middle|\; k = 1, \ldots, q_n \right\}$$

converges to $\phi_\alpha(\partial B(0, \lambda r_\alpha))$ for the Hausdorff topology on compact subsets of $\mathbb{C}$.

Let us assume that r is a limit value of a subsequence $r_{\alpha(n_k)}$. Then, for any $r' < r$, if k is sufficiently large, $\phi_{\alpha(n_k)}$ is defined on the disk $B(0, r')$. The maps $\phi_{\alpha(n_k)} : B(0, r') \to \mathbb{C}$ are univalent, fix 0 and have derivative 1 at the origin. Therefore, extracting a further subsequence if necessary, we may assume that the sequence $\phi_{\alpha(n_k)} : B(0, r') \to \Delta_{\alpha(n_k)}$ converges to a limit $\phi : B(0, r') \to \mathbb{C}$. The map $\phi_{\alpha(n_k)}$ take its values in the Siegel disk $\Delta_{\alpha(n_k)}$, and so, it omits the periodic cycle O_n of $P_{\alpha[n]}$. As a consequence, the limit map ϕ must omit $\phi_\alpha(\partial B(0, \lambda r_\alpha))$.

Therefore, the map $\phi_\alpha^{-1} \circ \phi$ sends $B(0, r')$ into $B(0, \lambda r_\alpha)$, it fixes 0 and has derivative 1 at 0. Thus, by Schwarz's lemma, $r' \leq \lambda r_\alpha$. Letting $r' \to r$ shows that any limit value of a subsequence $r_{\alpha(n_k)}$ is less or equal than λr_α, q.e.d. ■

6. Main lemma

In this section, $\alpha \in \mathbb{R}$ is any Bruno number, and A any real number such that $A > 1$. Let us recall the following definition: for $n \geq 1$,

$$\mathcal{T}(\alpha, A, n) = [a_0, a_1, \ldots, a_n, A_n, 1, 1, \ldots]$$

where $A_n = \lfloor A^{q_n} \rfloor$ is the integer part of A^{q_n}.

Let us emphasize two properties of the sequence $\alpha[n] = \mathcal{T}(\alpha, A, N)$:

$$\Phi(\alpha[n]) \longrightarrow \Phi(\alpha) + \log A \;, \tag{5}$$

$$\left|\alpha[n] - \frac{p_n}{q_n}\right|^{1/q_n} \longrightarrow \frac{1}{A} \;. \tag{6}$$

The first is proved in proposition 12, the second follows for instance from equation (2) page 13:

$$\left|\alpha[n] - \frac{p_n}{q_n}\right| = \frac{|\beta_n|}{q_n} \sim \frac{1}{q_n^2 A_n}$$

Proposition 13. *For each $n \geq 0$, set $\alpha[n] = \mathcal{T}(\alpha, A, n)$. Then,*

$$\liminf_{n\to\infty} r_{\alpha[n]} \geq \frac{r_\alpha}{A},$$

and the linearizing parametrizations $\phi_{\alpha[n]}$ converge uniformly to ϕ_α on every compact subset of $B(0, r_\alpha/A)$.

This is just corollary 7, which applies according to (5).

Proposition 14. *For each $n \geq 0$, set $\alpha[n] = \mathcal{T}(\alpha, A, n)$. Then, $P_{\alpha[n]}$ has a cycle O_n with the following limit for the Hausdorff topology on compact sets:*

$$\lim_{n\to\infty} O_n = \phi_\alpha\left(\partial B\left(0, \frac{r_\alpha}{A}\right)\right).$$

The result follows from corollary 5 with $\lambda = 1/A$, which applies according to (6).

Figure 4 shows the boundary of the Siegel disks for $\alpha = (\sqrt{5}+1)/2 = [1, 1, 1, \ldots]$, and $\alpha[n]$, $n = 5, \ldots, 8$ with $A_n = \lfloor 1.5^{q_n} \rfloor$. The reader should try and convince himself that as n grows, this boundary oscillates more and more between $\partial\Delta_\alpha$ and $\phi_\alpha\left(\frac{2}{3}\partial B(0, r_\alpha)\right)$ which both appear in the last frame.

7. Conclusion.

As mentioned in the introduction, Zakeri and Petersen proved that there exists quadratic Siegel disks whose boundaries are Jordan curve containing the critical point but that are not quasicircles. They even have an arithmetical condition for this to be true: when $\alpha = [a_0, a_1, a_2, \ldots]$ with a_n unbounded but $\log a_n = \mathcal{O}(\sqrt{n})$ as $n \to \infty$.

The examples of quadratic Siegel disks constructed by Herman and which do not contain the critical point in their boundaries are quasicircles. The authors do not know if one can control the regularity of this boundary with Herman's methods.

The techniques we developed in this article are very flexible. We think that we can apply them in order to prove the existence of Siegel disks whose boundaries are Jordan curves avoiding the critical point but are not quasicircles. More generally, we ask the following.

Problem 1. Given an integer $k \geq 0$, does there exist a quadratic Siegel disk whose boundary is C^k but not C^{k+1}?

We think our techniques yield a positive answer.

We can also ask about the Hausdorff dimension of the boundaries of Siegel disks. It is known that when $\alpha = [a_0, a_1, a_2, \ldots]$ with a_n bounded, the Hausdorff dimension is greater then 1 (Graczyk-Jones [GJ]) and less than 2 (because it is a quasi-circle). In the case of Siegel disks with smooth boundaries, the Hausdorff dimension is obviously equal to 1. This naturally yields the following.

Problem 2. Does there exist a quadratic Siegel disk whose boundary is a Jordan curve with Hausdorff dimension 2?

Remark. We think we can produce a quadratic Siegel disk whose boundary does not contain the critical point and has packing dimension 2 and Hausdorff dimension 1. The problem of producing a Siegel disk whose boundary has Hausdorff dimension 2 seems more tricky.

Next, the quadratic Siegel disks we produce are accumulated by small cycles. This is the way we control that the Siegel disk is not larger than expected. Pérez-Marco has produced maps which are univalent in the unit disk, have Siegel disks with smooth boundaries, but without small cycles.

Problem 3. Does there exist a quadratic polynomial having a Siegel disk whose boundary is not accumulated by small cycles?

Finally, it is known that when α satisfies an arithmetical condition $\mathcal{H}$, called the Herman condition, the critical point is on the boundary of the Siegel disk. It would be interesting to quantify the construction we give in this article.

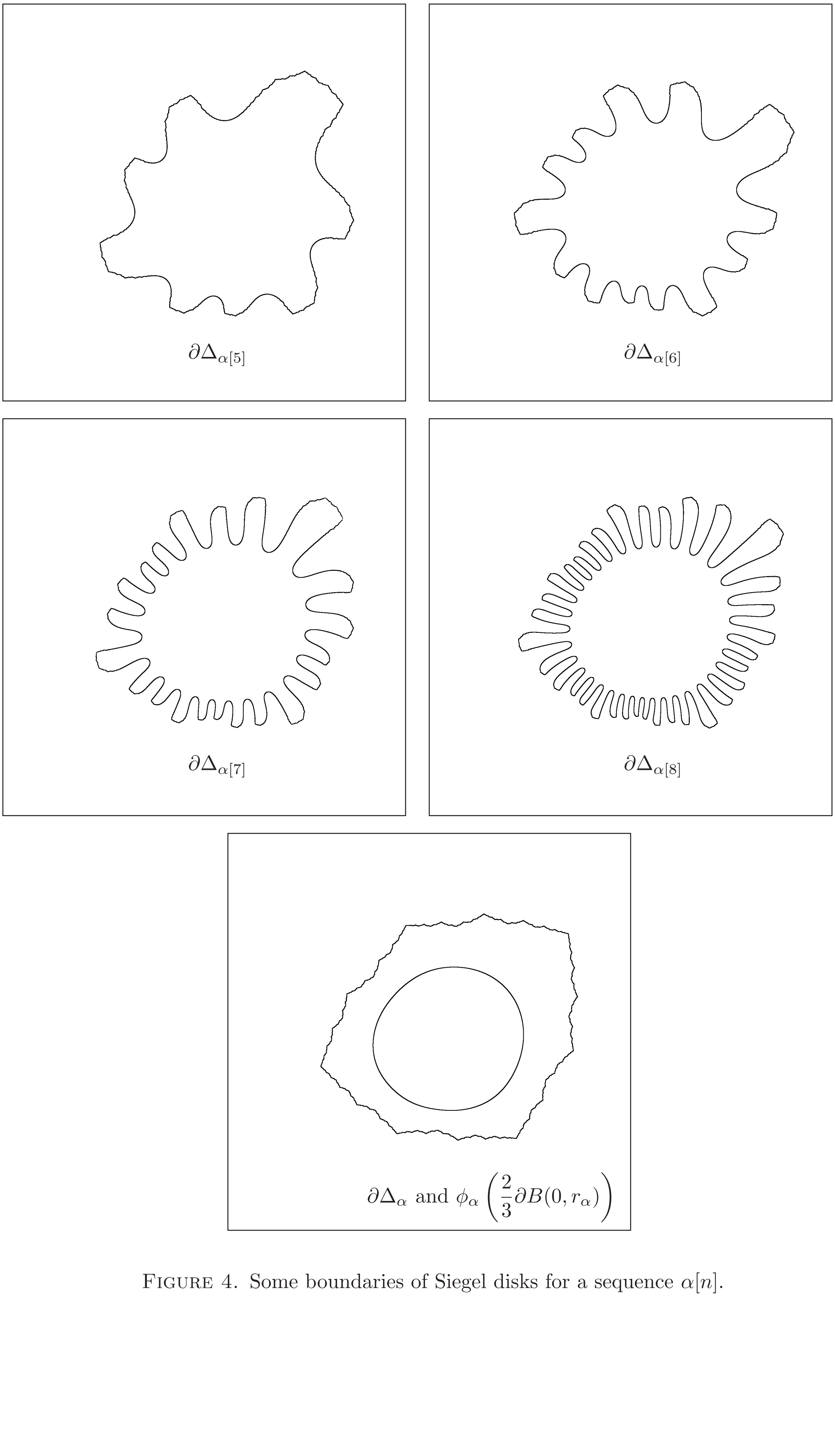

FIGURE 4. Some boundaries of Siegel disks for a sequence $\alpha[n]$.

Problem 4. Give an arithmetical condition which ensures that the critical point is not on the boundary of the Siegel disk. Or, give an arithmetical condition which ensures that the boundary of the Siegel disk is smooth.

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