A parabolic Pommerenke-Levin-Yoccoz inequality.

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ABSTRACT. In a recent preprint [B], Bergweiler relates the number of critical points contained in the immediate basin of a multiple fixed point β of a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$, the number N of attracting petals and the residue $\iota(f,\beta)$ of the 1-form $dz/(z-f(z))$ at β. In this article, we present a different approach to the same problem, which we had been developing independently at the same time. We apply our method to answer a question raised by Bergweiler. In particular, we prove that when there are only N grand orbit equivalence classes of critical points in the immediate basin, then

$$\Re\left(\frac{N+1}{2} - \iota(f,\beta)\right) > \frac{N}{\pi^2}.$$

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Contents

1 Introduction.

In this article, we will restrict our study to rational maps. The results we obtain may be generalized to finite type mappings (see [Ep1] for the appropriate definitions). In particular, the special case of finite type meromorphic functions $f : \mathbb{C} \to \mathbb{P}^1$ has been considered by Bergweiler in [B]. The techniques we use are similar to the ones described by Shishikura in [Sh2].

In the first section $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ having a parabolic fixed point at 0 with multiplier $e^{2i\pi p/q}$. In the rest of the article, $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map or a polynomial having a fixed point β with multiplier $e^{2i\pi p/q}$. It is known (see the Appendix) that there exist an integer $\nu \geq 1$ called the *parabolic multiplicity* of β, a complex number $\alpha \in \mathbb{C}$ called the *formal invariant* of f at β, and a local analytic coordinate $\varphi : (\mathbb{P}^1, \beta) \to (\mathbb{C}, 0)$ defined in a neighborhood of β, such that the expression of f in this coordinate is

$$\varphi \circ f \circ \varphi^{-1}(z) = e^{2i\pi p/q} z\Big(1 + z^{\nu q} + \alpha z^{2\nu q}\Big) + \mathcal{O}\Big(|z|^{2\nu q+2}\Big).$$

In the following, we will use the notation $N = \nu q$. When $\beta \neq \infty$ and $q = 1$, the complex number α is the residue of the 1-form $dz/(z - f(z))$ at β. In this article we will use another way of encoding the formal invariant α.

Definition 1 *We define the* résidu itératif *of f at β by*

$$\text{résit}(f, \beta) = \frac{N+1}{2} - \alpha.$$

The résidu itératif is determined by the formal invariant and vice versa. The term "résidu itératif" is due to Écalle [Éc]. For our purposes, it is a better way of encoding the formal invariant for two reasons: it will have an understandable geometric interpretation as a *Grötzsch defect* and it behaves nicely under iteration. Indeed, an easy (but computational) induction shows that for any integer $n \geq 1$, we have

$$f^{\circ n}(z) = e^{2i\pi np/q} z \left(1 + nz^N + n^2 \left(\frac{N+1}{2} - \frac{1}{n}\text{résit}(f,\beta)\right) z^{2N}\right) + \mathcal{O}\left(|z|^{2N+2}\right).$$

A germ

$$z \mapsto \lambda z(1 + az^N + bz^{2N}) + \mathcal{O}\left(|z|^{2N+2}\right)$$

is conjugate (via a change of variable $z = \rho w$ with $\rho^N = a$) to

$$w \mapsto \lambda w \left(1 + w^N + \frac{b}{a^2} w^{2N}\right) + \mathcal{O}\left(|w|^{2N+2}\right).$$

It follows immediately that for any integer $n \geq 1$, we have

$$\text{résit}(f^{\circ n}, \beta) = \frac{1}{n}\text{résit}(f, \beta).$$

The local dynamics of f at β is well understood. It has been completely classified topologically by Camacho [C] and analytically by Voronin [V] and Écalle [Éc] (see also [Ma] and [MR]). In particular, it is known that there exist N *attracting petals* $\mathcal{P}_{att,k}$ and N *repelling petals* $\mathcal{P}_{rep,k}$, ordered cyclically with counterclockwise orientation, such that the image of the attracting petal $\mathcal{P}_{att,k}$ is contained in the petal $\mathcal{P}_{att,k+\nu p}$, whereas the image of the repelling petal $\mathcal{P}_{rep,k}$ contains the petal $\mathcal{P}_{rep,k+\nu p}$. Under iteration of f, the orbit of every point contained in an attracting petal converges to the parabolic fixed point.

The global dynamics is also well understood. Since f is a rational map, it has a *Fatou set* Ω_f on which the family of iterates of f is normal. The *Julia set* $J_f = \mathbb{P}^1 \setminus \Omega_f$ is the closure of the set of repelling cycles. The parabolic fixed point β has a *basin of attraction*

$$W_\beta = \{z \in \mathbb{C} \mid f^{\circ n}(z) \underset{n \to +\infty}{\xrightarrow{\neq}} \beta\}$$

which is a union of connected components of the Fatou set. The attracting petals $\mathcal{P}_{att,k}$ are contained in W_β. We denote by Ω_k the connected component of W_β which contains $\mathcal{P}_{att,k}$ and we refer to the union of these components as the *immediate basin* of β. Figure 1 shows the basin (dark and light grey) and the immediate basin (dark grey) of 0 for the polynomial $z \mapsto z(1+z^2)$. It also shows attracting and repelling petals.

Figure 1: The parabolic basin of 0 for the polynomial $P(z) = z + z^3$ is colored grey. The parabolic multiplicity is 2. There are two attracting petals and two repelling petals. The connected components of the immediate basin are dark grey. The sepals are light grey.

Fatou [F1] proved that the immediate basin of β contains at least ν critical points of f. In [B], Bergweiler shows that the number of critical points contained in the immediate basin of β and the formal invariant are related. He implicitly proves the following proposition.

Theorem A. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree d having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Denote by Ω_k, $k \in \mathbb{Z}/N\mathbb{Z}$, the connected components of the immediate basin of β. Assume those connected components are simply connected and choose uniformizing maps $\varphi_k : \Omega_k \to \mathbb{D}$. Then, the mapping $F_k = \varphi_k \circ f^{\circ q} \circ \varphi_k^{-1}$ is a Blaschke product having a parabolic fixed point β_k, the invariant* $\text{résit}(F_k, \beta_k)$ *is real, and*

$$\Re\Big(\text{résit}(f^{\circ q}, \beta)\Big) \geq \frac{1}{2} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \text{résit}(F_k, \beta_k).$$

Bergweiler then studies the case where the immediate basin of β contains exactly ν distinct critical points of f, one for each component. The following corollary follows easily from Theorem A. We were aware of the application to rational maps having simple critical points but we never thought about the applications to mappings having multiple critical points.

Corollary. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. If the immediate basin of β contains exactly ν simple critical points of f, then,*

$$\Re(\text{résit}(f^{\circ q}, \beta)) \geq \frac{N}{4}.$$

If the immediate basin of β contains exactly ν critical points ω_j with multiplicities m_j, then,

$$\Re(\text{résit}(f^{\circ q}, \beta)) \geq \frac{3N}{20} + \frac{3q}{10} \sum_{j=1}^{\nu} \frac{1}{m_j(m_j+2)} > \frac{3N}{20}.$$

Remark. Equalities are possible if and only if $N = 1$ or $N = 2$.

In particular, Bergweiler deduces that when $\Re(\text{résit}(f^{\circ q}, \beta)) < N/4$, the immediate basin of β contains at least $\nu + 1$ critical points of f counted with multiplicity (so that there are at least 2, counting multiplicity, in some component of the immediate basin) and that when $\Re(\text{résit}(f^{\circ q}, \beta)) \leq 3N/20$, the immediate basin of β contains at least $\nu + 1$ (possibly multiple) distinct critical points of f. In this article, we will present a different approach to those results. The proof is inspired by the so-called Pommerenke-Levin-Yoccoz inequality (see for example [Po], [L], [Pe] or [H]) We will then show that Bergweiler's estimates can be improved in the case where f is a polynomial. However, the inequality we obtain is probably not optimal.

Theorem B. (Pommerenke-Levin-Yoccoz inequality for parabolic fixed points). *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Denote by K_β the connected component of K_P that contains β and by Ω_k, $k \in \mathbb{Z}/N\mathbb{Z}$, the connected components of the immediate basin of β. Choose uniformizing maps $\varphi_k : \Omega_k \to \mathbb{D}$, and let β_k be the unique parabolic fixed point of the Blaschke product $F_k = \varphi_k \circ P^{\circ q} \circ \varphi_k^{-1}$. Then, we have the inequality*

$$\Re(\text{résit}(P^{\circ q}, \beta)) > \frac{m}{2q \log d} + \frac{1}{2} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \text{résit}(F_k, \beta_k),$$

where $m \geq N$ is the number of accesses to β in $\mathbb{C} \setminus K_\beta$.

Corollary. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Denote by K_β the connected component of K_P that contains β and by $m \geq N$ the number of accesses to β in $\mathbb{C} \setminus K_\beta$. If the immediate basin of β contains exactly ν simple critical points of P, then,*

$$\Re(\text{résit}(P^{\circ q}, \beta)) \geq \frac{m}{2q \log d} + \frac{N}{4}.$$

If the immediate basin of β contains exactly ν critical points ω_j with multiplicities m_j, then,

$$\Re(\text{résit}(P^{\circ q}, \beta)) \geq \frac{m}{2q \log d} + \frac{3N}{20} + \frac{3q}{10} \sum_{j=1}^{\nu} \frac{1}{m_j(m_j+2)} > \frac{m}{2q \log d} + \frac{3N}{20}.$$

Example. In the case of the cubic polynomial $P(z) = z + z^3$, the filled-in Julia set is connected, and there are exactly 2 accesses to the parabolic fixed point 0 in $\mathbb{C} \setminus K_P$. Each connected component of the immediate basin contains exactly one simple critical point (those critical points are denoted by ω_0 and ω_1 on Figure 1). Thus, the parabolic Pommerenke-Levin-Yoccoz inequality gives

$$\Re(\text{résit}(P, 0)) = \frac{3}{2} > \frac{1}{\log 3} + \frac{1}{2} \sim 1.410239226.$$

Finally, we will answer a question raised by Bergweiler, by finding a universal positive lower bound to the résidu itératif of f at β when the number of grand orbit equivalence classes of critical points in the immediate basin of β is minimal.

Definition 2 *Define two critical points to be* grand orbit equivalent *if their orbits intersect.*

We do not know whether the following inequality is optimal or not.

Theorem C. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Assume that there are exactly ν grand orbit equivalence classes of critical points in the immediate basin of β. Then,*

$$\Re(\text{résit}(f^{\circ q}, \beta)) > \frac{N}{\pi^2}.$$

2 The Fatou flower and the Voronin-Écalle invariants.

In this section, we start by recalling classical results from the analytic classification of parabolic germs $f:(\mathbb{C},0)\to(\mathbb{C},0)$. In particular, we recall the definition of attracting and repelling petals $\mathcal{P}_{att,k}$ and $\mathcal{P}_{rep,k}$ and of attracting and repelling Fatou coordinates $\varphi_{att,k}:\mathcal{P}_{att,k}\to\mathbb{C}$ and $\varphi_{rep,k}:\mathcal{P}_{rep,k}\to\mathbb{C}$. With Fatou coordinates, we then define horn maps (also frequently called first return maps or Écalle maps) which are germs $h_k^+:(\mathbb{C},0)\to(\mathbb{C},0)$ and $h_k^-:(\mathbb{P}^1,\infty)\to(\mathbb{P}^1,\infty)$. We finally make explicit a crucial relation between the product of the multipliers of the horn maps $h_k^\pm$ at 0 and ∞ and the résidu itératif.

2.1 The Fatou flower.

Let $f:(\mathbb{C},0)\to(\mathbb{C},0)$ be a germ of the form

$$f(z)=e^{2i\pi p/q}z\Big(1+z^N+\alpha z^{2N}\Big)+\mathcal{O}\Big(|z|^{2N+2}\Big),$$

with $N=\nu q$.

Definition 3 (ATTRACTING AND REPELLING PETALS) *For any real number $\rho>0$, denote by $\widehat{\mathcal{P}}_{att}$ and $\widehat{\mathcal{P}}_{rep}$ the sectors*

$$\widehat{\mathcal{P}}_{att}=\Big\{Z\in\mathbb{C}\;\Big|\;\rho-\Re(Z)<|\Im(Z)|\Big\},\quad\text{and}\quad\widehat{\mathcal{P}}_{rep}=\Big\{Z\in\mathbb{C}\;\Big|\;\rho+\Re(Z)<|\Im(Z)|\Big\}.$$

Then, for any $k\in\mathbb{Z}/N\mathbb{Z}$, denote by $\mathcal{P}_{att,k}$ and $\mathcal{P}_{rep,k}$ the sets

$$\begin{aligned}\mathcal{P}_{att,k}&=\left\{z\in\mathbb{C}^*\;\middle|\;\frac{(2k-2)\pi}{N}<\mathrm{Arg}(z)<\frac{2k\pi}{N}\text{ and }-\frac{1}{Nqz^N}\in\widehat{\mathcal{P}}_{att}\right\}\quad\text{and}\\ \mathcal{P}_{rep,k}&=\left\{z\in\mathbb{C}^*\;\middle|\;\frac{(2k-1)\pi}{N}<\mathrm{Arg}(z)<\frac{(2k+1)\pi}{N}\text{ and }-\frac{1}{Nqz^N}\in\widehat{\mathcal{P}}_{rep}\right\}.\end{aligned}$$

If ρ is sufficiently large, then for each $k\in\mathbb{Z}/N\mathbb{Z}$, the change of variable $Z=-1/(Nqz^N)$ conjugates $f^{\circ q}:\mathcal{P}_{att,k}\to\mathcal{P}_{att,k}$ to a mapping $F_{att,k}:\widehat{\mathcal{P}}_{att}\to\widehat{\mathcal{P}}_{att}$ satisfying the asymptotic development

$$F_{att,k}(Z)=Z+1+\frac{A}{Z}+\mathcal{O}\left(\frac{1}{|Z|^2}\right),\quad\text{with}\quad A=\frac{1}{N}\text{résit}(f^{\circ q},0).$$

Similarly, it conjugates $f^{\circ q} : \mathcal{P}_{rep,k} \to f^{\circ q}(\mathcal{P}_{rep,k})$ to a mapping $F_{rep,k} : \widehat{\mathcal{P}}_{rep} \to F_{rep,k}(\widehat{\mathcal{P}}_{rep})$ satisfying the asymptotic development

$$F_{rep,k}(Z) = Z + 1 + \frac{A}{Z} + \mathcal{O}\left(\frac{1}{|Z|^2}\right), \quad \text{with} \quad A = \frac{1}{N}\text{résit}(f^{\circ q}, 0).$$

We fix once and for all such a sufficiently large real ρ.

Definition 4 (Sepals) *If $N = 1$, we denote by U^+ the connected component of $\mathcal{P}_{att} \cap \mathcal{P}_{rep}$ which is contained in the upper half-plane and by U^- the one contained in the lower half-plane. Otherwise, we denote by U_k^+, $k \in \mathbb{Z}/N\mathbb{Z}$, the intersection $\mathcal{P}_{rep,k} \cap \mathcal{P}_{att,k+1}$ and by U_k^- the intersection $\mathcal{P}_{rep,k} \cap \mathcal{P}_{att,k}$. We define the sepals $\mathcal{S}_k^\pm$ by*

$$\mathcal{S}_k^\pm = \bigcup_{n \in \mathbb{Z}} f^{\circ nq}(U_k^\pm).$$

Each sepal $\mathcal{S}_k^-$ is a Jordan domain which intersects the two petals $\mathcal{P}_{att,k}$ and $\mathcal{P}_{rep,k}$, while each sepal $\mathcal{S}_k^+$ is a Jordan domain which intersects the two petals $\mathcal{P}_{rep,k}$ and $\mathcal{P}_{att,k+1}$. The forward $f^{\circ q}$-orbit of any point $z \in \mathcal{S}_k^- \cup \mathcal{S}_{k-1}^+$ eventually enters the attracting petal $\mathcal{P}_{att,k}$ and the backward $f^{\circ q}$-orbit of any point $z \in \mathcal{S}_k^- \cup \mathcal{S}_k^+$ eventually enters the repelling petal $\mathcal{P}_{rep,k}$. Figure 3 shows an example of sets U^+ and U^- (dark grey) and Figure 1 shows an example of sepals (light grey).

2.2 The Fatou coordinates.

It is well known that there exist attracting (respectively repelling) Fatou coordinates $\Phi_{att,k} : \widehat{\mathcal{P}}_{att} \to \mathbb{C}$ (respectively $\Phi_{rep,k} : \widehat{\mathcal{P}}_{rep} \to \mathbb{C}$) conjugating $F_{att,k}$ (respectively $F_{rep,k}$) to the translation $Z \mapsto Z + 1$. These Fatou coordinates are uniquely defined up to addition of a complex constant. Moreover, the Fatou coordinates admit the asymptotic developments

$$(1) \qquad \begin{cases} \Phi_{att,k}(Z) &= Z - A\log_{att}(Z) + C_{att,k} + \mathcal{O}\left(\frac{1}{|Z|}\right), \quad \text{and} \\ \Phi_{rep,k}(Z) &= Z - A\log_{rep}(Z) + C_{rep,k} + \mathcal{O}\left(\frac{1}{|Z|}\right), \end{cases}$$

where $A = \dfrac{1}{N}\text{résit}(f^{\circ q}, 0)$, $C_{att,k}$ and $C_{rep,k}$ are constants, and $\log_{att}$ and $\log_{rep}$ are branches of logarithms defined respectively in $\mathbb{C} \setminus \mathbb{R}^-$ and $\mathbb{C} \setminus \mathbb{R}^+$.

One may then define attracting Fatou coordinates $\varphi_{att,k} : \mathcal{P}_{att,k} \to \mathbb{C}$ and repelling Fatou coordinates $\varphi_{rep,k} : \mathcal{P}_{rep,k} \to \mathbb{C}$ by

$$\varphi_{att,k}(z) = \Phi_{att,k}\left(-\frac{1}{Nqz^N}\right) \quad \text{and} \quad \varphi_{rep,k}(z) = \Phi_{rep,k}\left(-\frac{1}{Nqz^N}\right).$$

Those Fatou coordinates conjugate $f^{\circ q}$ to the translation $Z \mapsto Z+1$.

The attracting Fatou coordinates $\varphi_{att,k}$ extend analytically to $\mathcal{P}_{att,k} \cup \mathcal{S}_k^- \cup \mathcal{S}_{k-1}^+$ via the formula

$$\varphi_{att,k}(z) = \varphi_{att,k}(f^{\circ nq}(z)) - n,$$

where n is chosen large enough so that $f^{\circ nq}(z) \in \mathcal{P}_{att,k}$. Similarly, the repelling Fatou coordinates $\varphi_{rep,k}$ extend analytically to $\mathcal{P}_{rep,k} \cup \mathcal{S}_k^- \cup \mathcal{S}_k^+$ via the formula

$$\varphi_{rep,k}(z) = \varphi_{rep,k}\left([f^{-1}]^{\circ nq}(z)\right) + n,$$

where n is chosen large enough so that $[f^{-1}]^{\circ nq}(z) \in \mathcal{P}_{rep,k}$.

2.3 The horn maps.

Definition 5 (Lifted horn maps) *Let us define* $V_k^- = \varphi_{rep,k}(\mathcal{S}_k^-)$, $V_k^+ = \varphi_{rep,k}(\mathcal{S}_k^+)$, $W_k^- = \varphi_{att,k}(\mathcal{S}_k^-)$ *and* $W_k^+ = \varphi_{att,k+1}(\mathcal{S}_k^+)$. *Then, denote by* $H_k^- : V_k^- \to W_k^-$ *the restriction of* $\varphi_{att,k} \circ (\varphi_{rep,k})^{-1}$ *to* V_k^- *and by* $H_k^+ : V_k^+ \to W_k^+$ *the restriction of* $\varphi_{att,k+1} \circ (\varphi_{rep,k})^{-1}$ *to* V_k^+. *We refer to* $H_k^\pm$ *as* lifted horn maps *for* f.

Remark. Since Fatou coordinates are uniquely defined up to an additive constant, lifted horn maps are uniquely defined up to pre- and post-composition with a translation.

The regions $V_k^\pm$ and $W_k^\pm$ are invariant by translation by 1. Moreover, the asymptotic development of the Fatou coordinates implies that the regions V_k^+ and W_k^+ contain an upper half-plane, whereas the regions V_k^- and W_k^- contain a lower half-plane. Consequently, under the projection $\pi : Z \mapsto \zeta = e^{2i\pi Z}$, the regions V_k^+ and W_k^+ project to punctured neighborhoods $\mathcal{V}_k^+$ and $\mathcal{W}_k^+$ of 0, whereas V_k^- and W_k^- project to punctured neighborhoods $\mathcal{V}_k^-$ and $\mathcal{W}_k^-$ of ∞.

The lifted horn maps $H_k^\pm$ satisfy $H_k^\pm(Z+1) = H_k^\pm(Z) + 1$ on $V_k^\pm$. Thus, they project to mappings $h_k^\pm : \mathcal{V}_k^\pm \to \mathcal{W}_k^\pm$ such that the following diagram commutes:

$$\begin{tikzcd} V_k^\pm \arrow[r, "H_k^\pm"] \arrow[d, "\pi"'] & W_k^\pm \arrow[d, "\pi"] \\ \mathcal{V}_k^\pm \arrow[r, "h_k^\pm"'] & \mathcal{W}_k^\pm. \end{tikzcd}$$

Definition 6 (Horn maps) *The maps* $h_k^\pm$ *are called* horn maps *for* f*.*

Remark. The horn maps (or their inverse) are the Voronin-Écalle invariants which play a crucial role in the classification, up to analytic conjugacy, of parabolic germs.

Since the lifted horn maps $H_k^\pm$ are uniquely defined up to pre- and post-composition with a translation, the horn maps $h_k^\pm$ are uniquely defined up to pre- and post-multiplication by constants. More precisely, if $h_k^\pm$ and $\widetilde{h}_k^\pm$ are horn maps for f, then there exist constants $\alpha_k \in \mathbb{C}^*$ and $\beta_k \in \mathbb{C}^*$ such that

$$\widetilde{h}_k^-(\zeta) = \alpha_k h_k^-\left(\frac{\zeta}{\beta_k}\right) \quad \text{and} \quad \widetilde{h}_k^+(\zeta) = \alpha_{k+1} h_k^-\left(\frac{\zeta}{\beta_k}\right).$$

Indeed, there exist constants $a_k \in \mathbb{C}$ and $b_k \in \mathbb{C}$ such that for all $k \in \mathbb{Z}/N\mathbb{Z}$, we have

$$\widetilde{\varphi}_{att,k} = \varphi_{att,k} + a_k \quad \text{and} \quad \widetilde{\varphi}_{rep,k} = \varphi_{rep,k} + b_k.$$

Then, for all $k \in \mathbb{Z}/N\mathbb{Z}$, we get

$$\widetilde{H}_k^-(Z) = H_k^-(Z - b_k) + a_k \quad \text{and} \quad \widetilde{H}_k^+(Z) = H_k^+(Z - b_k) + a_{k+1}.$$

Projecting via π, and using the notation $\alpha_k = e^{2i\pi a_k}$ and $\beta_k = e^{2i\pi b_k}$, we easily derive that

$$\widetilde{h}_k^-(\zeta) = \alpha_k h_k^-\left(\frac{\zeta}{\beta_k}\right) \quad \text{and} \quad \widetilde{h}_k^+(\zeta) = \alpha_{k+1} h_k^-\left(\frac{\zeta}{\beta_k}\right).$$

2.4 Multipliers of the horn maps and the résidu itératif.

On the sector $\{Z \in \mathbb{C} \mid \rho + |\Re(Z)| < \Im(Z)\}$, we have

$$\Phi_{att,k+1}(Z) = Z - A\log_{att}(Z) + \mathcal{O}(1) \quad \text{and} \quad \Phi_{rep,k}(Z) = Z - A\log_{rep}(Z) + \mathcal{O}(1),$$

where $A = \text{résit}(f^{\circ q}, 0)/(N)$. Since

$$H_k^+ = \Phi_{att,k+1} \circ (\Phi_{rep,k})^{-1},$$

we see that $H_k^+(Z) = Z + \mathcal{O}(1)$. This proves that $h_k^+(\zeta) \to 0$ as $\zeta \to 0$. Thus, the horn maps h_k^+ extend analytically to 0 by $h_k^+(0) = 0$. One shows similarly that the horn maps h_k^- extend analytically to ∞ by $h_k^-(\infty) = \infty$. In the following, the horn maps h_k^- will be considered as germs $h_k^- : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$ and the horn maps h_k^+ will be considered as germs $h_k^+ : (\mathbb{P}^1, 0) \to (\mathbb{P}^1, 0)$.

We now come to the key result (equation (2)) relating the one formal invariant of f to the horn maps. We will see below that this result has an understandable geometric interpretation as a Grötzsch defect (equations (3) and (5)).

Proposition 1 *Assume $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ having a parabolic fixed point at 0 with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Let $h_k^- : (\mathbb{P}^1, \infty) \to (\mathbb{P}^1, \infty)$ and $h_k^+ : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $k \in \mathbb{Z}/N\mathbb{Z}$, be horn maps for f. Let λ_k^+ be the multiplier of 0 as a fixed point of the horn map h_k^+. Similarly, let λ_k^- be the multiplier of ∞ as a fixed point of the horn map h_k^-. Then,*

$$\prod_{k \in \mathbb{Z}/N\mathbb{Z}} \left(\lambda_k^- \lambda_k^+ \right) = e^{4\pi^2 \text{résit}(f^{\circ q}, 0)}. \tag{2}$$

Remark. Since the horn maps are only defined up to pre- and post-multiplication by constants, the multipliers are not canonically defined. However, this proposition shows that their product does not depend on the various choices made.

Proof. Let us first show that the product of the multipliers does not depend on the choice of Fatou coordinates. Let $h_k^\pm$ and $\widetilde{h}_k^\pm$ be two systems of horn maps for f. We know that there exist constants $\alpha_k \in \mathbb{C}^*$ and $\beta_k \in \mathbb{C}^*$ such that

$$\widetilde{h}_k^-(\zeta) = \alpha_k h_k^- \left(\frac{\zeta}{\beta_k} \right) \quad \text{and} \quad \widetilde{h}_k^+(\zeta) = \alpha_{k+1} h_k^- \left(\frac{\zeta}{\beta_k} \right).$$

This in turn yields

$$\widetilde{\lambda}_k^- = \frac{\beta_k}{\alpha_k} \lambda_k^- \quad \text{and} \quad \widetilde{\lambda}_k^+ = \frac{\alpha_{k+1}}{\beta_k} \lambda_k^+.$$

Thus, we get

$$\begin{aligned}\prod_{k\in\mathbb{Z}/N\mathbb{Z}}\left(\widetilde{\lambda}_k^-\widetilde{\lambda}_k^+\right) &= \left(\frac{\beta_0}{\alpha_0}\frac{\alpha_1}{\beta_0}\right)\left(\frac{\beta_1}{\alpha_1}\frac{\alpha_2}{\beta_1}\right)\cdots\left(\frac{\beta_{N-1}}{\alpha_{N-1}}\frac{\alpha_0}{\beta_{N-1}}\right)\prod_{k\in\mathbb{Z}/N\mathbb{Z}}\left(\lambda_k^-\lambda_k^+\right)\\ &= \prod_{k\in\mathbb{Z}/N\mathbb{Z}}\left(\lambda_k^-\lambda_k^+\right).\end{aligned}$$

To compute this product, observe that

$$\lambda_k^- = \lim_{\zeta\to\infty}\frac{\zeta}{h_k^-(\zeta)} \quad\text{and}\quad \lambda_k^+ = \lim_{\zeta\to 0}\frac{h_k^+(\zeta)}{\zeta}.$$

Since,

$$h_k^-\left(e^{2i\pi\Phi_{rep,k}(Z)}\right) = e^{2i\pi\Phi_{att,k}(Z)} \quad\text{and}\quad h_k^+\left(e^{2i\pi\Phi_{rep,k}(Z)}\right) = e^{2i\pi\Phi_{att,k+1}(Z)},$$

we have

$$\lambda_k^- = \lim_{\Im(Z)\to-\infty} e^{2i\pi[\Phi_{rep,k}(Z)-\Phi_{att,k}(Z)]}$$

and

$$\lambda_k^+ = \lim_{\Im(Z)\to+\infty} e^{2i\pi[\Phi_{att,k+1}(Z)-\Phi_{rep,k}(Z)]}.$$

We are allowed to normalize the Fatou coordinates the way we want. Since Fatou coordinates are defined up to addition of a complex constant and in view of the asymptotic developments (1), we may arrange that, for all $k\in\mathbb{Z}/N\mathbb{Z}$,

$$\lim_{\Im(Z)\to-\infty}\Phi_{att,k}(Z)-\Phi_{rep,k}(Z) = -2i\pi A$$

and

$$\lim_{\Im(Z)\to+\infty}\Phi_{att,k+1}(Z)-\Phi_{rep,k}(Z) = 0.$$

Indeed, in (1) we may take $\log_{att}$ to be the branch of logarithm defined on $\mathbb{C}\setminus\mathbb{R}^-$ which takes the value 0 at 1, take $\log_{rep}$ to be the branch of logarithm defined on $\mathbb{C}\setminus\mathbb{R}^+$ which takes the value $i\pi$ at -1 and take all the constants $C_{att,k}$ and $C_{rep,k}$ to be 0. With these normalizations, the multipliers λ_k^- are equal to $e^{4\pi^2 A}$ and the multipliers λ_k^+ are all equal to 1. Hence, the product of multipliers is equal to $e^{4\pi^2 NA}$. This proves the proposition since $NA = \text{résit}(f^{\circ q},0)$. ■

2.5 The case of a rational map.

Let us now consider the case of a rational map f having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$. As mentioned above, there exists a local change of coordinates $\varphi : (\mathbb{P}^1, \beta) \to (\mathbb{C}, 0)$, in which the expression of f is

$$\varphi \circ f \circ \varphi^{-1}(z) = e^{2i\pi p/q} z \Big(1 + z^N + \alpha z^{2N}\Big) + \mathcal{O}\Big(|z|^{2N+2}\Big).$$

The preceding analysis applies, proving the existence of Fatou coordinates $\varphi_{att,k} : \mathcal{P}_{att,k} \to \mathbb{C}$ and $\varphi_{rep,k} : \mathcal{P}_{rep,k} \to \mathbb{C}$ and of horn maps $h_k^\pm$. Our goal is to prove that those horn maps are defined on natural maximal domains and are ramified coverings above $\mathbb{C}^*$ in an appropriate sense. This result will only be used in the last section, in order to handle the case of multiply connected immediate basins.

Recall that each attracting petal $\mathcal{P}_{att,k}$ is contained in a connected component Ω_k of the immediate basin of β. The attracting Fatou coordinate $\varphi_{att,k} : \mathcal{P}_{att,k} \to \mathbb{C}$ extends to a holomorphic function $\varphi_{att,k} : \Omega_k \to \mathbb{C}$ via the formula

$$\varphi_{att,k}(z) = \varphi_k(f^{\circ nq}(z)) - n,$$

where n is chosen large enough so that $f^{\circ nq}(z)$ belongs to the attracting petal $\mathcal{P}_{att,k}$. Similarly, the inverse $\psi_{rep,k}$ of the repelling Fatou coordinate $\varphi_{rep,k} : \mathcal{P}_{rep,k} \to \mathbb{C}$ extends to a meromorphic function $\psi_{rep,k} : \mathbb{C} \to \mathbb{P}^1$ via the formula

$$\psi_{rep,k}(z) = f^{\circ nq}(\psi_{rep,k}(z - n)).$$

Definition 7 (see Figures 2 and 4) *For each $k \in \mathbb{Z}/N\mathbb{Z}$, we define D_k^+ (respectively D_k^-) to be the connected component of $\psi_{rep,k}^{-1}(\Omega_{k+1})$ (respectively of $\psi_{rep,k}^{-1}(\Omega_k)$ which contains an upper half-plane (respectively a lower half-plane). In addition, we define $\mathcal{D}_k^\pm$ to be the projection of $D_k^\pm$ via the projection $\pi : Z \mapsto e^{2i\pi Z}$.*

Then, for each $k \in \mathbb{Z}/N\mathbb{Z}$, the lifted horn maps $H_k^\pm$ are defined on the domains $D_k^\pm$ by the formulae

$$H_k^- = \varphi_{att,k} \circ \psi_{rep,k} \quad \text{and} \quad H_k^+ = \varphi_{att,k+1} \circ \psi_{rep,k}.$$

They project to horn maps $h_k^\pm : \mathcal{D}_k^\pm \to \mathbb{C}^*$. We will show that those horn maps are ramified coverings in an appropriate sense and locate their critical values.

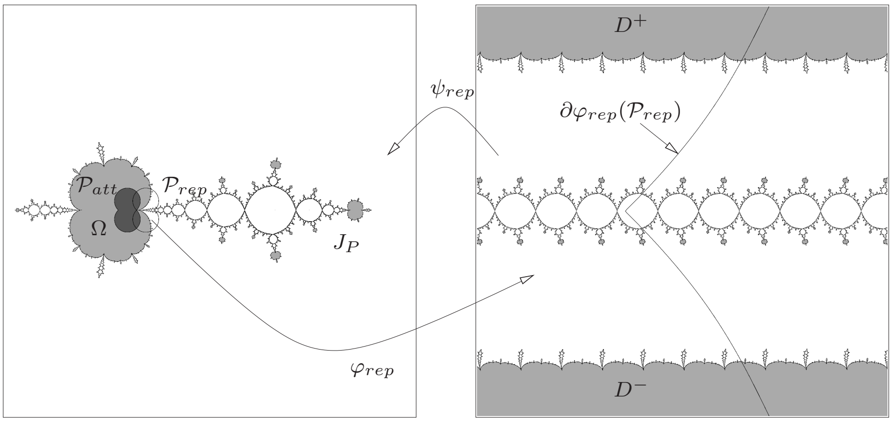

Figure 2: The polynomial $P(z) = z + z^2 - .46z^3$ has a parabolic fixed point at 0. Left: the Julia set of the polynomial P. The parabolic basin of 0 is colored grey. Right: the preimage of J_P by the meromorphic function $\psi_{rep} : \mathbb{C} \to \mathbb{P}^1$.

Definition 8 *An analytic mapping $h : X \to Y$, where X and Y are Riemann surfaces, is said to be a* ramified covering *if and only if for each point $y \in Y$, there exists a neighborhood V of y such that every connected component of $f^{-1}(V)$ is proper over V. A point $x \in X$ is said to be critical if the local degree of f at x is greater than* 1. *The image of a critical point is called a critical value.*

Remark. It follows from the definition of ramified covering that the local degree of f at x is well defined and positive for every $x \in X$.

The composition of ramified coverings is a ramified covering. Thus, in order to prove that the horn maps are ramified coverings, it is sufficient to prove that $\varphi_{att,k} : \Omega_k \to \mathbb{C}$, $\psi_{rep,k} : D_k^- \to \Omega_k$, and $\psi_{rep,k} : D_k^+ \to \Omega_{k+1}$, are ramified coverings.

Proposition 2 *The extended attracting Fatou coordinate $\varphi_{att,k} : \Omega_k \to \mathbb{C}$ is a ramified covering. The critical points of $\varphi_{att,k}$ are exactly the pre-critical points of $f^{\circ q}$ which are in Ω_k.*

Proof. Given any bounded disk $V \subset \mathbb{C}$ and any connected component U of $\varphi_{att,k}^{-1}(V)$, observe that the restriction $\varphi_{att,k} : U \to V$ may be decomposed as

$$U \xrightarrow{f^{\circ nq}} f^{\circ nq}(U) \xrightarrow{\varphi_{att,k} - n} V.$$

where n is chosen large enough so that $n + V \subset \varphi_{att,k}(\mathcal{P}_{att,k})$. It follows that this restriction is a proper mapping (because f is proper), so the extended Fatou coordinate $\varphi_{att,k} : \Omega_k \to \mathbb{C}$ is a ramified covering.

Differentiating the formula $\varphi_{att,k}(f^{\circ q}(z)) = \varphi_{att,k}(z) + 1$, one easily shows that the critical points of $\varphi_{att,k}$ are exactly the pre-critical points of $f^{\circ q}$ contained in Ω_k. ■

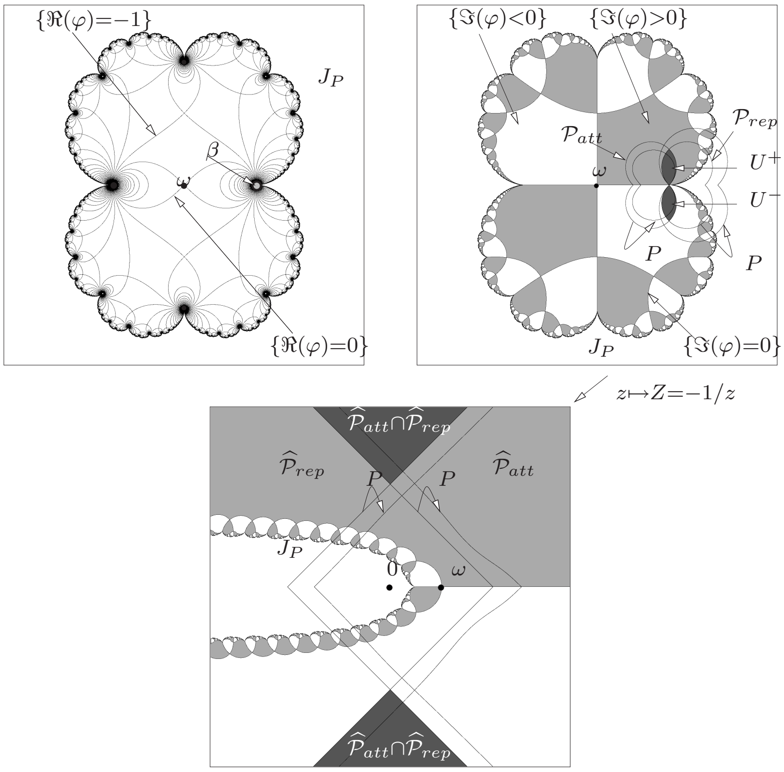

Figure 3: The polynomial $P(z) = z + z^2$ has a parabolic fixed point at 0 with immediate basin Ω. Any attracting Fatou coordinate extends to a ramified covering $\varphi_{att} : \Omega \to \mathbb{C}$. We can normalize φ_{att} so that $\varphi_{att}(\omega) = 0$, where $\omega = -1/2$ is the unique critical point of P. Left: the curves $\{z \in \Omega \mid \Re(\varphi_{att}(z)) \in \mathbb{Z}\}$. Right: the basin Ω is tiled by the regions $\{z \in \Omega \mid \Im(\varphi_{att}(z)) < 0\}$ and $\{z \in \Omega \mid \Im(\varphi_{att}(z)) > 0\}$. Down: the same picture in the coordinate $Z = -1/z$.

Figure 3 illustrates the covering property of φ_{att} in the case of the polynomial $z \mapsto z + z^2$.

The treatment of the repelling Fatou coordinate is more subtle. The meromorphic function $\psi_{rep,k} : \mathbb{C} \to \mathbb{C}$ is not a ramified covering.

Definition 9 *Let us define $\mathcal{C}_f$ to be the set of critical points of f, $\mathcal{P}_f$ to be the post-critical set of f:*

$$\mathcal{P}_f = \bigcup_{\omega\in\mathcal{C}_f} \bigcup_{n\geq 1} f^{\circ n}(\omega).$$

Moreover, define $\mathcal{A}_f$ to be the accumulation set of all critical orbits:

$$\mathcal{A}_f = \bigcap_{n\in\mathbb{N}} f^{\circ n}\left(\overline{\mathcal{P}_f}\right).$$

Proposition 3 *The meromorphic function $\psi_{rep,k} : \mathbb{C} \to \mathbb{C}$ restricts to a ramified covering*

$$\psi_{rep,k} : \psi_{rep,k}^{-1}(\mathbb{P}^1 \setminus \mathcal{A}_f) \to \mathbb{P}^1 \setminus \mathcal{A}_f.$$

A point $Z \in \mathbb{C}$ is a critical point of $\psi_{rep,k}$ if and only if there exists an integer $n \geq 1$ such that $\psi_{rep,k}(Z-n)$ is a critical point of $f^{\circ q}$.

Remark. This proposition shows that $\psi_{rep,k} : \psi_{rep,k}^{-1}(\mathbb{P}^1 \setminus \overline{\mathcal{P}_f}) \to \mathbb{P}^1 \setminus \overline{\mathcal{P}_f}$ is a covering map, but we will need a slightly stronger result.
Proof. We have to prove that given any point $y \in \mathbb{P}^1 \setminus \mathcal{A}_f$, there exists a neighborhood V of y such that any connected component of $\psi_{rep,k}^{-1}(V)$ is proper over V.

Let V be any neighborhood of y and let U be a connected component of $\psi_{rep,k}^{-1}(V)$. For any $n \in \mathbb{N}$, set $U_n = \{z \in \mathbb{C} \mid z+n \in U\}$ and $V_n = \psi_{rep,k}(U_n)$. For any $z \in \mathbb{C}$ and any $n \in \mathbb{N}$, we have

$$\psi_{rep,k}(z) = f^{\circ nq}(\psi_{rep,k}(z-n)).$$

Consequently, V_n is a connected component of $f^{-nq}(V)$. In particular, $f^{\circ nq} : V_n \to V$ is proper, and the proposition will follow if we prove that for a good choice of V and for n large enough, $\psi_{rep,k} : U_n \to V_n$ is proper. In fact, we will show that we may choose V carefully enough so that for all sufficiently large n, V_n is contained in the repelling petal $\mathcal{P}_{rep,k}$. In that case, $\psi_{rep,k} : U_n \to V_n$ is an isomorphism, and the proof is completed.

Since, by assumption, y is not accumulated by the critical orbits, we may assume that V is relatively compact in $\{y\}\cup(\mathbb{P}^1\setminus\overline{\mathcal{P}_f})$. Observe that $\beta\in\mathcal{A}_f$, and thus, there exists an integer n_1 such that

$$V\cap f^{\circ n_1 q}(\mathcal{P}_{att,k}\cup\mathcal{P}_{att,k+1})=\emptyset.$$

Since $f^{\circ q}(\mathcal{P}_{att,k}\cup\mathcal{P}_{att,k+1})\subset\mathcal{P}_{att,k}\cup\mathcal{P}_{att,k+1}$, it follows that for any $n\in\mathbb{N}$,

$$V_n\cap f^{\circ n_1 q}(\mathcal{P}_{att,k}\cup\mathcal{P}_{att,k+1})=\emptyset.$$

Moreover, there are only finitely many n's (possibly none) such that y is a critical value of $f^{\circ nq}$. Thus, we may choose $n_2\geq 0$ such that $f^{-n_2 q}(V)$ is relatively compact in $\mathbb{P}^1\setminus\overline{\mathcal{P}_f}$. For any connected component V' of $f^{-n_2 q}(V)$, we see that the restriction $f^{\circ n_2 q}:V'\to V$ is proper (because f is proper) and has at most one critical value at y (by the choice of V). Thus, V_{n_2} is simply connected and relatively compact in $\mathbb{P}^1\setminus\overline{\mathcal{P}_f}$. It follows that for any integer $n\geq n_2$, the sequence of inverse branches $g_n:V_{n_2}\to V_n$ of $f^{\circ(n-n_2)q}$ forms a normal family for uniform convergence on V_{n_2} (and not just on compact subsets of V_{n_2}).

Given any $M>0$, there exists an integer n_M such that for any $n\geq n_M$, the set U_n intersects the left half-plane $\{Z\in\mathbb{C}\mid\Re(Z)<-M\}$. Since M can be arbitrarily large, it follows that for any $\varepsilon>0$ and any n sufficiently large, the set V_n intersects the disk $\mathbb{D}_\varepsilon$ of radius ε centered at β. However, we observed that

$$V_n\cap f^{\circ n_1 q}(\mathcal{P}_{att,k}\cup\mathcal{P}_{att,k+1})=\emptyset.$$

It follows from Hurwitz's theorem that the sequence $g_n:V_{n_2}\to V_n$ converges uniformly on V_{n_2} to the mapping which is constantly equal to β. Consequently, for any $\varepsilon>0$, we may choose n large enough so that V_n intersects $\mathbb{D}_\varepsilon$, does not intersect $f^{\circ n_1 q}(\mathcal{P}_{att,k}\cup\mathcal{P}_{att,k+1})$, and has diameter less than ε. By choosing ε small enough, we see that V_n is contained in the repelling petal $\mathcal{P}_{rep,k}$ as required.

We finally see that $\psi_{rep,k}:U\to V$ is an isomorphism if and only if $f^{\circ nq}:V_n\to V$ is unramified. This proves the statement for the critical points of $\psi_{rep,k}$. ■

Observe that the grand-orbit of any point $z\in\Omega_k$ is mapped by $\varphi_{att,k}$ to a $\mathbb{Z}$-orbit. Such an orbit projects to exactly one point via the projection $\pi:Z\mapsto e^{2i\pi Z}$. This is in particular true if z is a critical point of $f^{\circ q}$ contained in Ω_k.

Definition 10 *For any $k \in \mathbb{Z}/N\mathbb{Z}$, we define $\mathcal{C}_k$ to be the set of critical points of $f^{\circ q}$ contained in Ω_k and we set $\mathcal{V}_k = \pi \circ \varphi_{att,k}(\mathcal{C}_k)$.*

Proposition 4 *For any $k \in \mathbb{Z}/N\mathbb{Z}$, the horn maps $h_k^{\pm} : \mathcal{D}_k^{\pm} \to \mathbb{C}^*$ are ramified coverings. The set of critical values of $h_k^- : \mathcal{D}_k^- \to \mathbb{C}^*$ is exactly the set $\mathcal{V}_k$, and the set of critical values of $h_k^+ : \mathcal{D}_k^+ \to \mathbb{C}^*$ is exactly the set $\mathcal{V}_{k+1}$.*

Remark. In particular, we see that the horn map h_k^- restricts to a covering map above $\mathbb{C}^* \setminus \mathcal{V}_k$ and the horn map h_k^+ restricts to a covering above $\mathbb{C}^* \setminus \mathcal{V}_{k+1}$.
PROOF. Every orbit is discrete in the immediate basin of β. Thus, $\Omega_k \cap \mathcal{A}_f = \emptyset$. Moreover $H_k^- : D_k^- \to \mathbb{C}$ is the composition of $\psi_{rep,k} : D_k^- \to \Omega_k$ and $\varphi_{att,k} : \Omega_k \to \mathbb{C}$. This shows that for any $k \in \mathbb{Z}/N\mathbb{Z}$, the lifted horn map $H_k^- : D_k^- \to \mathbb{C}$ is a ramified covering. Similarly, one can show that $H_k^+ : D_k^+ \to \mathbb{C}$ is a ramified covering. It follows immediately that $h_k^{\pm} : \mathcal{D}_k^{\pm} \to \mathbb{C}^*$ is a ramified covering.

We will now show that the set of critical values of $h_k^- : \mathcal{D}_k^- \to \Omega_k$ is $\mathcal{V}_k$. A similar proof shows that the set of critical values of $h_k^+ : \mathcal{D}_k^+ \to \Omega_{k+1}$ is $\mathcal{V}_{k+1}$. The critical values of h_k^- are exactly the images of the critical values of H_k^- by π. A critical value of H_k^- is either a critical value of $\varphi_{att,k}$ or the image by $\varphi_{att,k}$ of a critical value of $\psi_{rep,k}$. We know that the critical values of $\varphi_{att,k} : \Omega_k \to \mathbb{C}$ are the pre-critical points of $f^{\circ q}$ contained in Ω_k, i.e., the points in the backward orbit of $\mathcal{C}_k$. Moreover, if $Z \in D_k^-$ is a critical point of $\psi_{rep,k}$, then there exists an integer n such that $Z - n \in D_k^-$ is mapped by $\psi_{rep,k}$ to a critical point ω of $f^{\circ q}$. This critical point is contained in Ω_k and $f^{\circ nq}(\omega) = \psi_{rep,k}(Z)$. Thus, the critical values of $\psi_{rep,k} : D_k^- \to \Omega_k$ are the points of the forward orbit of $\mathcal{C}_k$. This shows that the critical values of H_k^- are exactly the images by $\varphi_{att,k}$ of the points which belong to the grand-orbit of $\mathcal{C}_k$. ■

Example. Figure 4 gives an idea of how lifted horn maps and horn maps behave in the case of the polynomial $f(z) = z + z^2$. This polynomial has a unique critical point $\omega \in \mathbb{C}$. We can normalize the attracting Fatou coordinate φ_{att} so that it sends ω to 0. Figure 4 shows the sets $D^{\pm}$ and $\mathcal{D}^+$. The domains D^+ and D^- are tiled by the preimages of the upper and lower half-planes by H^+ and H^-. More precisely, the lifted horn maps $H^{\pm}$ are isomorphisms between each grey tile and the upper half-plane, whereas they are isomorphisms between each white tile and the lower half-plane. In the case of $\mathcal{D}^+$, the horn map h^+ is an isomorphism between the punctured grey

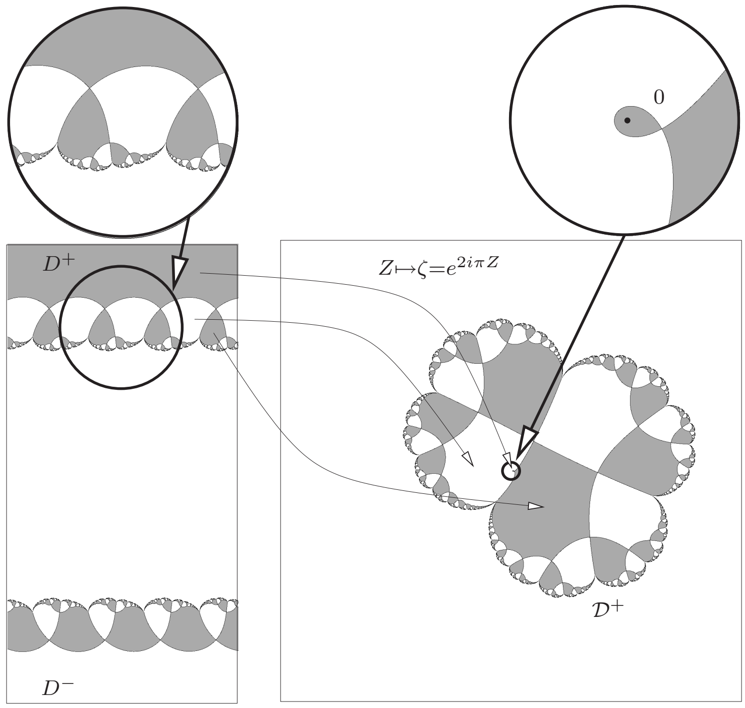

Figure 4: Left: The tiled domains D^+ and D^-. Right: the tiled domain $\mathcal{D}^+$.

tile containing 0 in its boundary and the punctured disk $\mathbb{D}^*$. It is a universal covering from the other grey tiles to the punctured disk $\mathbb{D}^*$, and it is a universal covering from each white tile to $\mathbb{C} \setminus \overline{\mathbb{D}}$.

3 The Grötzsch defect and applications.

In this section, we will prove Theorem A, its corollary and Theorem B.

3.1 Definition of the Grötzsch defect.

Let us first give a formal definition of Grötzsch defect. The notion of Grötzsch defect and the relevance to the Pommerenke-Levin-Yoccoz inequality are due to Carsten Petersen. One formulation is as follows.

Definition 11 *We say that a compact $E \subset \mathbb{C}^*$ is* equatorial *if and only if $\mathbb{P}^1 \setminus E$ consists of two simply connected components D^+ containing* 0*, and*

D^- containing ∞. Given any equatorial compact set E, let $\varphi^+ : D^+ \to \mathbb{D}$ (respectively $\varphi^- : D^- \to \mathbb{P}^1 \setminus \overline{\mathbb{D}}$) be a Riemann map fixing 0 (respectively ∞) with multiplier ρ^+ (respectively ρ^-). We define the Grötzsch defect *of E to be*

$$\mathrm{defect}(E) = \frac{1}{2\pi} \log \left|\rho^- \rho^+\right|.$$

For example, the Grötzsch defect of a round cylinder $E = \{z \in \mathbb{C}^* \mid r \leq |z| \leq R\}$ is the modulus of E, i.e., $\mathrm{defect}(E) = \frac{1}{2\pi}(\log(R) - \log(r))$.

Lemma 1 *(a) The Grötzsch defect of any equatorial compact set $E \subset \mathbb{C}^*$ is nonnegative. It vanishes if and only if E is a circle centered at 0.*

(b) If E contains disjoint equatorial annuli A_i of moduli M_i, then $\mathrm{defect}(E) \geq \sum M_i$*. Equality holds if and only if the annuli A_i are all round annuli centered at 0 and $E = \bigcup \overline{A_i}$.*

Proof. The proof is based on a classical length-area argument. It is a variant of Grötzsch's inequality. Statement (a) is in fact a degenerate case of statement (b). We leave the proof of (a) to the reader and we prove (b).

Consider the flat metric $|dz|^2/|z|^2$ on $\mathbb{C}^*$. We claim that the area of an equatorial annulus A is at least equal to its modulus, with equality if and only if A is a round annulus centered at 0. Indeed, let M be this modulus and denote by $\mathcal{A}$ the annulus

$$\mathcal{A} = \{z = x + iy \in \mathbb{C} \mid 0 < y < M\}/\mathbb{Z}.$$

Moreover, let $\varphi : \mathcal{A} \to A$ be a conformal parameterization of A. Then,

$$\mathrm{area}(A) = \int_{y=0}^{M}\int_{x=0}^{1} \frac{|\varphi'(x+iy)|^2}{|\varphi(x+iy)|^2} dx dy \geq \int_{y=0}^{M} \left(\int_{x=0}^{1} \frac{|\varphi'(x+iy)|}{|\varphi(x+iy)|} dx\right)^2 dy \geq M.$$

In the first inequality, we used Schwarz's inequality, and in the second one we used the observation that the length of any equatorial curve is at least 1. Equality in Schwarz's inequality can only be achieved if φ'/φ constant, and the length of an equatorial curve is 1 only if it is a round circle centered at 0. This precisely implies that A is a round annulus centered at 0. One easily checks that the area of such an annulus is equal to its modulus.

Now, as in the definition of the Grötzsch defect, let D^+ and D^- be the two simply connected components of $\mathbb{P}^1 \setminus E$. Let $\varphi^+ : D^+ \to \mathbb{D}$ (respectively

$\varphi^- : D^- \to \mathbb{P}^1 \setminus \overline{\mathbb{D}})$ be a Riemann map fixing 0 (respectively ∞) with multiplier ρ^+ (respectively ρ^-). Denote by $A(M)$ the annulus

$$A(M) = \left\{ z \in D^+ \ : \ e^{-2\pi M} < |\varphi^+(z)| \right\} \cup E \cup \left\{ z \in D^- \ : \ e^{2\pi M} > |\varphi^-(z)| \right\}.$$

The boundary of this annulus consists of two Jordan curves $\gamma^+(M)$ and $\gamma^-(M)$; $\gamma^+(M)$ is contained in between two circles of radii $|\rho^+|e^{-2\pi M} + o(1)$ and $\gamma^-(M)$ is contained in between two circles of radii $|\frac{1}{\rho^-}|e^{2\pi M} + o(1)$. Thus, using the definition of Grötzsch defect, we have

$$\mathrm{area}(A) = 2M + \mathrm{defect}(E) + o(1).$$

Moreover, the annulus A contains the equatorial annuli A_i as well as the annuli

$$\{z \in D^+ \mid e^{-2\pi M} < |\varphi^+(z)| < 1\}, \quad \text{and} \quad \{z \in D^- \mid 1 < |\varphi^-(z)| < e^{2\pi M}\}.$$

Those annuli are all disjoint and thus,

$$2M + \mathrm{defect}(E) + o(1) \geq M + M + \sum M_i,$$

which implies the first part of (b).

To get the second part of (b), observe that when $M' > M$, the annulus $A(M')$ contains $A(M)$ and $A(M') \setminus A(M)$ is the disjoint union of two annuli, each one of modulus $M' - M$. As previously, Schwarz's inequality implies that

$$\mathrm{area}(A(M') \setminus A(M)) \geq 2(M' - M).$$

In particular, the function $\mathrm{area}(A(M)) - 2M$ is increasing with M. Therefore, for all $M > 0$, we have

$$\mathrm{defect}(E) = \lim_{M \to +\infty} \mathrm{area}(A(M)) - 2M \geq \mathrm{area}(A(M)) - 2M \geq \sum M_i.$$

As a consequence, we see that when $\mathrm{defect}(E) = \sum M_i$, we must have

$$\mathrm{area}(A(M)) = 2M + \sum M_i$$

for all $M > 0$. This means that we have equality in Schwarz's inequality, which can only be achieved when the annuli A_i are all round annuli centered at 0 and $E = \bigcup \overline{A_i}$. □

3.2 Rational maps with simply connected parabolic basins.

The following result has been proved independently by W. Bergweiler [B]. Even though his proof and ours are based on similar ideas such as asymptotics of Fatou coordinates and conjugation to Blaschke products, there are various differences between the arguments.

Theorem A. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree d having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Denote by Ω_k, $k \in \mathbb{Z}/N\mathbb{Z}$, the connected components of the immediate basin of β. Assume that these connected components are simply connected and choose uniformizing maps $\varphi_k : \Omega_k \to \mathbb{D}$. Then, the mapping $F_k = \varphi_k \circ f^{\circ q} \circ \varphi_k^{-1}$ is a Blaschke product having a parabolic fixed point β_k, the invariant* $\text{résit}(F_k, \beta_k)$ *is real, and*

$$\Re\Big(\text{résit}(f^{\circ q}, \beta)\Big) \geq \frac{1}{2} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \text{résit}(F_k, \beta_k).$$

Proof. For each $k \in \mathbb{Z}/N\mathbb{Z}$, the map $f^{\circ q} : \Omega_k \to \Omega_k$ is a proper mapping. Hence, the mapping $F_k = \varphi_k \circ f^{\circ q} \circ \varphi_k^{-1}$ is a proper mapping from the unit disk to itself. The proper holomorphic mappings from $\mathbb{D}$ to $\mathbb{D}$ are exactly the Blaschke products.

Lemma 2 (Fatou [F3]) *The Blaschke product F_k has a unique nonrepelling fixed point β_k. The Julia set of F_k is the unit circle S^1. The fixed point β_k is parabolic with multiplier* 1 *and parabolic multiplicity* 2*. Its résidu itératif* $\text{résit}(F_k, \beta_k)$ *is real.*

Proof. It is known that the Julia set of a Blaschke product is contained in S^1 and is either a Cantor set or S^1 (because S^1 is totally invariant).

If J_{F_k} is a Cantor set, then the Fatou set has exactly one connected component Ω. This Fatou component is fixed. It is an attracting or parabolic domain and the corresponding attracting or parabolic fixed point β_k belongs to S^1. We will show by contradiction that this case is not possible. Recall that each quotient $\Omega_k/f^{\circ q}$ is a Riemann surface isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$. Since $f^{\circ q} : \Omega_k \to \Omega_k$ is conjugate to $F_k : \mathbb{D} \to \mathbb{D}$, we see that $\mathbb{D}/F_k$ is also a Riemann surface isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$. However, if β_k were an attracting fixed point with multiplier λ, the Riemann surface Ω/F_k

would be isomorphic to the torus $\mathbb{C}^*/\lambda$ and $\mathbb{D}/F_k$ would be isomorphic to the annulus $\mathbb{H}/\lambda$. Such an annulus is never isomorphic to $\mathbb{C}/\mathbb{Z}$, which gives a contradiction. If β_k were a parabolic fixed point, the Riemann surface Ω/F_k would be isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ and $\mathbb{D}/F_k$ would be isomorphic to the cylinder $\mathbb{H}/\mathbb{Z}$. Again, this cylinder is not isomorphic to $\mathbb{C}/\mathbb{Z}$, which gives a contradiction.

The above discussion shows that $J_{F_k} = S^1$. Hence $\mathbb{D}$ is a fixed Fatou component. Since $F_k : \mathbb{D} \to \mathbb{D}$ is conjugate to $f^{\circ q} : \Omega_k \to \Omega_k$, the orbit of every point in $\mathbb{D}$ leaves every compact subset of $\mathbb{D}$. Thus, $\mathbb{D}$ is a parabolic domain for F_k and the orbit of every point in $\mathbb{D}$ converges to a parabolic fixed point $\beta_k \in S^1$ with multiplier 1. Using the symmetry $z \mapsto 1/\overline{z}$, we see that the basin of attraction of this parabolic fixed point has two connected components: $\mathbb{D}$ and $\mathbb{P}^1 \setminus \overline{\mathbb{D}}$. Hence, the parabolic multiplicity of β_k is equal to 2. Since the anti-holomorphic map $z \mapsto 1/\overline{z}$ conjugates F_k to itself, the formal invariant of F_k at β_k is necessarily real, whence résit(F_k, β_k) is also real. $\square$

For each $k \in \mathbb{Z}/N\mathbb{Z}$, the Blaschke product F_k has a parabolic fixed point β_k with parabolic multiplicity 2. The immediate basin of β_k has two connected components: $\Omega_{k,0} = \mathbb{D}$ and $\Omega_{k,1} = \mathbb{P}^1 \setminus \overline{\mathbb{D}}$ (see Figure 5). There are two attracting petals $P_{att,k,0} \subset \Omega_{k,0}$ and $\mathcal{P}_{att,k,1} \subset \Omega_{k,1}$. There are two extended attracting Fatou coordinates $\varphi_{att,k,j} : \Omega_{k,j} \to \mathbb{C}$, $j \in \mathbb{Z}/2\mathbb{Z}$. Similarly, there are two repelling petals $\mathcal{P}_{rep,k,j}$ and two repelling Fatou coordinates $\varphi_{rep,k,j} : \mathcal{P}_{rep,k,j} \to \mathbb{C}$. Their inverses extend to meromorphic mappings $\psi_{rep,k,j} : \mathbb{C} \to \mathbb{P}^1$ so that we may define the domains

$$D^-_{k,j} = \psi^{-1}_{rep,k,j}(\Omega_{k,j}) \quad \text{and} \quad D^+_{k,j} = \psi^{-1}_{rep,k,j}(\Omega_{k,j+1}).$$

The lifted horn maps $H^{\pm}_{k,j} : D^{\pm}_{k,j} \to \mathbb{C}$ are then defined by

$$H^-_{k,j} = \varphi_{att,k,j} \circ \psi_{rep,k,j} \quad \text{and} \quad H^+_{k,j} = \varphi_{att,k,j+1} \circ \psi_{rep,k,j}.$$

The lifted horn maps $H^{\pm}_{k,j}$ descend to horn maps $h^{\pm}_{k,j} : \mathcal{D}^{\pm}_{k,j} \to \mathbb{C}$ via the projection $\pi : Z \mapsto e^{2i\pi Z}$. We claim that we may normalize the repelling Fatou coordinates so that for each $j \in \mathbb{Z}/2\mathbb{Z}$, we have $\psi^{-1}_{rep,k,j}(S^1) = \mathbb{R}$. Indeed, since the repelling Fatou coordinates are only defined up to translation, we can normalize them so that $\varphi_{rep,k,j}(S^1 \cap \mathcal{P}_{rep,k,j})$ intersects $\mathbb{R}$. As F_k commutes with the reflection $z \mapsto 1/\overline{z}$, each repelling Fatou coordinate conjugates the mapping $z \mapsto 1/\overline{z}$ to an anti-holomorphic mapping which is defined in a left

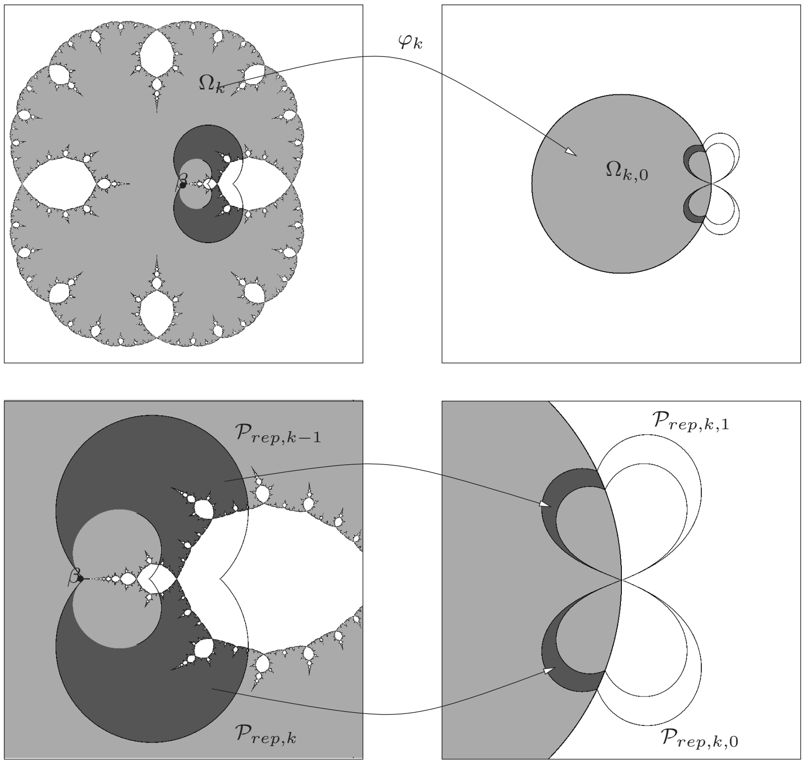

Figure 5: The conformal representation $\varphi_k : \Omega_k \to \mathbb{D}$ induces conformal representations $\varphi_k^- : \mathcal{D}_k^- \to \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\varphi_k^+ : \mathcal{D}_k^+ \to \mathbb{D}^*$.

half-plane and which commutes with the unit translation. Such a mapping is necessarily a symmetry with respect to an horizontal line. This shows that $\varphi_{rep,k,j}(S^1 \cap \mathcal{P}_{rep,k,j}) \subset \mathbb{R}$. Since $F_k \circ \psi_{rep,k,j}(Z) = \psi_{rep,k,j}(Z+1)$, and since $F_k^{-1}(S^1) = S^1$, we conclude that $\psi_{rep,k,j}^{-1}(S^1) = \mathbb{R}$. It follows immediately that for each $k \in \mathbb{Z}/N\mathbb{Z}$ and each $j \in \mathbb{Z}/2\mathbb{Z}$, we have

$$D_{k,j}^+ = \mathbb{H}^+, \quad D_{k,j}^- = \mathbb{H}^-, \quad \mathcal{D}_{k,j}^+ = \mathbb{D} \quad \text{and} \quad \mathcal{D}_{k,j}^- = \mathbb{P}^1 \setminus \overline{\mathbb{D}}.$$

Let us now come back to the proof of the inequality

$$\Re\Big(\text{résit}(f^{\circ q}, \beta)\Big) \geq \frac{1}{2} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \text{résit}(F_k, \beta_k).$$

First, recall that φ_k conjugates $f^{\circ q} : \Omega_k \to \Omega_k$ to $F_k : \Omega_{k,0} \to \Omega_{k,0}$ and thereby induces isomorphisms $\varphi_k^- : \mathcal{D}_k^- \to \mathcal{D}_{k,0}^-$ and $\varphi_k^+ : \mathcal{D}_{k-1}^+ \to \mathcal{D}_{k,1}^+$. Let

ρ_k^+ be the multiplier of φ_k^+ at 0 and ρ_k^- be the multiplier of φ_k^- at ∞. Since $\mathcal{D}_{k,0}^- = \mathbb{P}^1 \setminus \overline{\mathbb{D}}$ and $\mathcal{D}_{k,1}^+ = \mathbb{D}$, we see that the Grötzsch defect of the equatorial compact set $E_k = \mathbb{C}^* \setminus (\mathcal{D}_k^+ \cup \mathcal{D}_k^-)$ is

$$\mathrm{defect}(E_k) = \rho_k^- \rho_{k-1}^+.$$

Observe that for any $k \in \mathbb{Z}/N\mathbb{Z}$, we may normalize the attracting Fatou coordinates $\varphi_{att,k,0}$ and $\varphi_{att,k,1}$ so that

$$\varphi_{att,k,0}(z) = \varphi_{att,k} \circ \varphi_k^{-1}(z) \quad \text{and} \quad \varphi_{att,k,1}(z) = \overline{\varphi_{att,k} \circ \varphi_k^{-1}(\overline{z})},$$

whence

$$h_{k,0}^+(z) = \overline{h_{k,0}^-(1/\overline{z})} \quad \text{and} \quad h_{k,1}^-(z) = \overline{h_{k,1}^+(1/\overline{z})}.$$

Moreover, the following two diagrams commute:

$$\begin{tikzcd}
 & \mathbb{C}^* & \\
 & \mathbb{C} \arrow[u, "\pi"] & \\
\Omega_k \arrow[ur, "\varphi_{att,k}"] \arrow[rr, "\varphi_k"] & & \Omega_{k,0} = \mathbb{D} \arrow[ul, "\varphi_{att,k,0}"'] \\
D_k^- \arrow[u, "\psi_{rep,k}"] \arrow[d, "\pi"'] & & D_{k,0}^- = \mathbb{H}^- \arrow[u, "\psi_{rep,k,0}"'] \arrow[d, "\pi"] \\
\mathcal{D}_k^- \arrow[rr, "\varphi_k^-"'] & & \mathcal{D}_{k,0}^- = \mathbb{C} \setminus \overline{\mathbb{D}}
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathbb{C}^* & \\
 & \mathbb{C} \arrow[u, "\pi"] & \\
\Omega_k \arrow[ur, "\varphi_{att,k}"] \arrow[rr, "\varphi_k"] & & \Omega_{k,0} = \mathbb{D} \arrow[ul, "\varphi_{att,k,0}"'] \\
D_k^+ \arrow[u, "\psi_{rep,k-1}"] \arrow[d, "\pi"'] & & D_{k,1}^+ = \mathbb{H}^+ \arrow[u, "\psi_{rep,k,1}"'] \arrow[d, "\pi"] \\
\mathcal{D}_k^+ \arrow[rr, "\varphi_k^-"'] & & \mathcal{D}_{k,1}^+ = \mathbb{D}
\end{tikzcd}.$$

Consequently, we have $h_k^- = h_{k,0}^- \circ \varphi_k^-$ and $h_{k-1}^+ = h_{k,1}^+ \circ \varphi_k^+$. In particular, we see that for any $k \in \mathbb{Z}/N\mathbb{Z}$, we have

$$\lambda_{k,0}^+ = \overline{\lambda_{k,0}^-}, \quad \lambda_{k,1}^- = \overline{\lambda_{k,1}^+}, \quad \lambda_k^- = \lambda_{k,0}^- \rho_k^- \quad \text{and} \quad \lambda_{k-1}^+ = \lambda_{k,1}^+ \rho_k^+,$$

where $\lambda_{k,j}^\pm$ is the multiplier of the horn map $h_{k,j}^\pm$.

Applying Proposition 1, we deduce that

$$e^{4\pi^2 \text{résit}(f^{\circ q},\beta)} = \prod_{k\in\mathbb{Z}/N\mathbb{Z}} \left(\lambda_k^- \lambda_k^+\right) = \prod_{k\in\mathbb{Z}/N\mathbb{Z}} \left(\lambda_{k,0}^- \lambda_{k,1}^+\right) \prod_{k\in\mathbb{Z}/N\mathbb{Z}} \left(\rho_k^- \rho_k^+\right).$$

Taking the logarithm of the modulus of each member gives

$$\begin{aligned}
4\pi^2 \Re(\text{résit}(f^{\circ q},\beta)) &= \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \log\left|\lambda_{k,0}^- \lambda_{k,1}^+\right| + \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \log\left|\rho_k^- \rho_k^+\right| \\
&= \frac{1}{2} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \log\left|\lambda_{k,0}^- \lambda_{k,1}^+\right|^2 + \sum_{k\in\mathbb{Z}/N\mathbb{Z}} 2\pi\, \text{defect}(E_k).
\end{aligned}$$

On the other hand, it similarly follows from Proposition 1 that

$$e^{4\pi^2 \text{résit}(F_k,\beta_k)} = \prod_{j\in\mathbb{Z}/2\mathbb{Z}} \left(\lambda_{k,j}^- \lambda_{k,j}^+\right) = \left|\lambda_{k,0}^- \lambda_{k,1}^+\right|^2.$$

Combining these observations, we obtain the following equality which will be employed again in the proof of Theorem B.

$$\Re(\text{résit}(f^{\circ q},\beta)) = \frac{1}{2} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \text{résit}(F_k,\beta_k) + \frac{1}{2\pi} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \text{defect}(E_k). \tag{3}$$

This concludes the proof of Theorem A since we proved that the Grötzsch defect of an equatorial compact set is always nonnegative. ■

Corollary. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. If the immediate basin of β contains exactly ν simple critical points of f, then,*

$$\Re(\text{résit}(f^{\circ q}, \beta)) \geq \frac{N}{4}.$$

If the immediate basin of β contains exactly ν critical points ω_j with multiplicities m_j, then,

$$\Re(\text{résit}(f^{\circ q}, \beta)) \geq \frac{3N}{20} + \frac{3q}{10}\sum_{j=1}^{\nu} \frac{1}{m_j(m_j+2)} > \frac{3N}{20}.$$

Proof. Let us call Ω_k, $k \in \mathbb{Z}/N\mathbb{Z}$, the N connected components of the immediate basin of β. We know that each connected component Ω_k contains at least, and thus exactly, one critical point Ω_k of $f^{\circ q}$ (otherwise, the extended Fatou coordinate $\varphi_{att,k} : \Omega_k \to \mathbb{C}$ would be an isomorphism). Let us first show that Ω_k is necessarily simply connected.

Lemma 3 (Fatou [F2]) *Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map and Ω be a fixed Fatou component containing exactly one critical point of g, possibly multiple of multiplicity m. Then Ω is simply connected and $g : \Omega \to \Omega$ is of degree $m+1$.*

Proof. Since Ω contains a critical point of g, it cannot be a Siegel disk or a Herman ring. Hence, it is necessarily the immediate basin of attraction of an attracting fixed point, a superattracting fixed point or (as in the case at hand) a parabolic fixed point with multiplier 1. In each case, there exist simply connected domains U_1 and U_0 such that $U_0 \subset U_1$, the restriction $g : U_1 \to U_0$ is a (possibly ramified) covering, and the orbit of every point in Ω eventually enters U_0. Then we may inductively define U_n to be the connected component of $g^{-1}(U_{n-1})$ which contains U_{n-1}. Note that $\chi(U_0) = 1$, where χ denotes the euler characteristic. We will show that $\chi(U_n) = 1$, whence U_n is simply connected, for evey positive integer n. Note thta as each U_n is a propoer subset of the Riemann sphere, we necessarily have $\chi(U_n) \leq 1$; we will establish the converse inequality $\chi(U_n) \geq 1$ by induction on n.

Observe that the mapping $g : U_0 \to U_1$ is proper, so by the Riemann-Hurwitz formula, we have

$$\chi(U_n) = d_n \chi(U_{n-1}) - k_n,$$

where d_n is the degree of $g : U_n \to U_{n-1}$ and where k_n is the number of critical points of g in U_n. By assumption, we have $k_n = 0$ or $k_n = m$. Moreover, we claim that $d_n - k_n \geq 1$ for all $n \geq 1$. Indeed, if $k_n = 0$ then $d_n - k_n = d_n \geq 1$; on the other hand, if $k_n = m$ the the critical value is contained in U_{n-1} and has at least $m+1$ preimages (counting multiplicity) in U_n, whence $d_n \geq m+1$ and therefore $d_n - k_n \geq (m+1) - m = 1$. Finally, if $\chi(U_{n-1}) \geq 1$ then we have

$$\chi(U_n) = d_n \chi(U_{n-1}) - k_n = d_n - k_n \geq 1.$$

Thus, each region U_n is simply connected. It follows that $\Omega = \bigcup U_n$ is simply connected and furthermore $g : \Omega \to \Omega$ has degree $m+1$ by the Riemann-Hurwitz formula. □

To summarize, we have shown that $f^{\circ q} : \Omega_k \to \Omega_k$ is a proper mapping of degree 2 and that Ω_k is simply connected. Now choose a uniformizing map $\varphi_k : \Omega_k \to \mathbb{D}$. We have seen in Theorem A that $F_k = \varphi_k \circ f^{\circ q} \circ \varphi_k^{-1}$ has a parabolic fixed point $\beta_k \in S^1$. Counting multiplicity, there are $\deg F_k + 1 = 3$ fixed points of F_k, and as β_k is a multiple fixed point with parabolic multiplicity 2, it follows that β_k is the only fixed point of F_k. In view of the Holomorphic Index Formula (see [M2] chapter 12) asserts that $\iota(F_k, \beta_k) = 1$ and consequently

$$\text{résit}(F_k, \beta_k) = \frac{2+1}{2} - 1 = \frac{1}{2}.$$

This completes the proof of the corollary in the case of simple critical points:

$$\Re(\text{résit}(f^{\circ q}, \beta)) \geq \frac{1}{2} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \text{résit}(F_k, \beta_k) = \frac{N}{4}.$$

Our treatment of multiple critical points is patterned after Bergweiler's discussion in [B]. His crucial observation is that if the immediate basin contains exactly ν (possibly multiple) critical points, then one may still control real part of the residu itératif. Indeed, for each $j = 1, \ldots, \nu$, the cycle $\langle \Omega_j, \Omega_{j+\nu}, \ldots, \Omega_{j+\nu(q-1)} \rangle$ contains at least, and thus exactly, one critical point ω_j of f. Let us denote by m_j the multiplicity of this critical point. Lemma 3 asserts that for any $j = 1, \ldots, \nu$ and for any $l \in \mathbb{Z}/q\mathbb{Z}$, the connected component $\Omega_{j+\nu l}$ is simply connected and the restriction $f^{\circ q} : \Omega_{j+\nu l} \to \Omega_{j+\nu l}$ is a proper mapping of degree $m_j + 1$. This restriction is conjugate to the unique Blaschke product of degree $m_j + 1$ with critical points of multiplicity m_j at 0 and ∞ and a parabolic fixed point at 1, that is to

$$F_j(z) = \frac{z^{m_j+1} + a}{1 + az^{m_j+1}}, \quad \text{with} \quad a = \frac{m_j}{m_j + 2}.$$

As

$$\text{résit}(F_j, 1) = \frac{3}{10} + \frac{3}{5m_j(m_j + 2)} > \frac{3}{10},$$

it follows that

$$\begin{aligned} \Re(\text{résit}(f^{\circ q}, \beta)) &\geq \frac{1}{2} \sum_{l \in \mathbb{Z}/q\mathbb{Z}} \sum_{j=1}^{\nu} \left(\frac{3}{10} + \frac{3}{5m_j(m_j + 2)} \right) \\ &= \frac{3N}{20} + \frac{3q}{10} \sum_{j=1}^{\nu} \frac{1}{m_j(m_j + 2)} > \frac{3N}{20}. \end{aligned}$$

■

Example. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a quadratic rational map with a parabolic fixed point β of multiplier 1 and parabolic multiplicity $\nu = 1$. By conjugating f with a Möbius transformation, we may assume that the multiple fixed point is ∞, and that f is tangent to $Z \mapsto Z + 1$ near ∞. We may express such a map as

$$f_A(Z) = Z + 1 + \frac{A}{Z}, \quad A \in \mathbb{C}^*.$$

Figure 6 shows the locus $\mathcal{L}$ of parameters $A \in \mathbb{C}^*$ for which the Julia set J_{f_A} is connected.

Claim. *The Julia set of f_A is connected if and only if the basin of attraction Ω_A of the parabolic fixed point contains only one critical point.*

Indeed, since the basin Ω_A contains at least one critical point of f_A, the restriction $f_A : \Omega_A \to \Omega_A$ is a proper mapping of degree 2 – note that in this caseΩ_A is totally invariant and that the Julia set of f_A is equal to the boundary of Ω_A. In view of Lemma 3 if Ω_A contains only one critical point then Ω_A is simply connected, whence the Julia set J_{f_A} is connected. On the other hand, if both critical points lie in Ω_A then $\chi(\Omega_A) = 2\chi(\Omega_A) - 2$ by the Riemann-Hurwitz formula, so that $\chi(\Omega_A)$ is either 2 or $-\infty$. As Ω_A is a

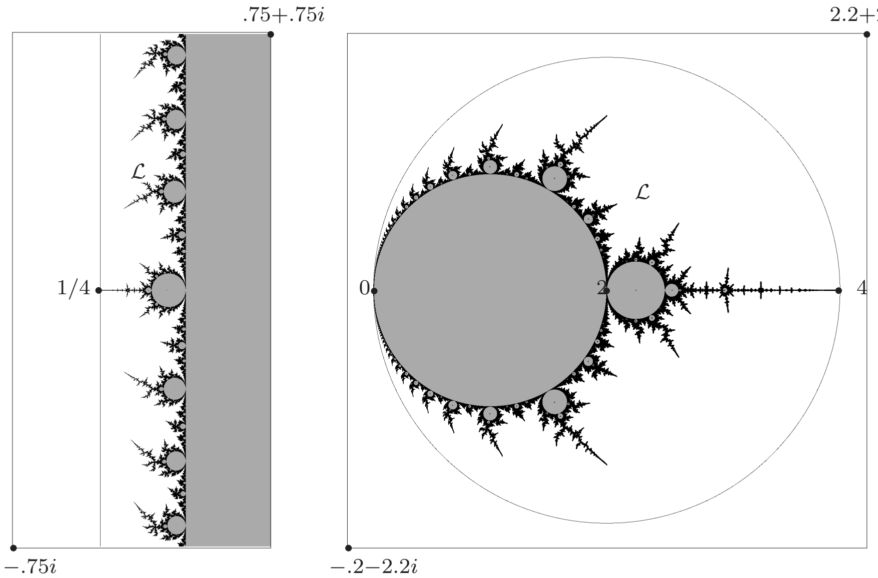

Figure 6: The connectedness locus $\mathcal{L}$ of the family $\{Z \mapsto Z + 1 + A/Z\}$, $A \in \mathbb{C}^*$. Left: in the coordinates A. Right: in the coordinates $1/A$.

proper subset of the Riemann spheer, we have $\chi(\Omega_A) \leq 1$, whence $\chi(\Omega_A) = -\infty$; in particular, J_f is disconnected (and therefore totally disconnected – see [M1]).

As an application of the above corollary, we deduce that the connectedness locus $\mathcal{L}$ is contained in the half-plane $\{A \in \mathbb{C}^* \mid \Re(A) > 1/4\}$. Indeed, the résidu itératif of f at ∞ is A. Figure 6 also shows the set $\mathcal{L}$ in the coordinate $1/A$. The image of the half-plane $\{A \in \mathbb{C} \mid \Re(A) > 1/4\}$ by the mapping $A \mapsto 1/A$ is the disk of radius 2 centered at 2.

3.3 Polynomials.

In this subsection, we assume that P is a polynomial of degree d. The following result is proved in [M2] (see Theorems 18.11, 18.12 and 18.13) and deals with polynomials having a connected Julia set.

Proposition 5 *Let P be a polynomial having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Assume that the filled-in Julia set K_P is connected. Then, each repelling petal $\mathcal{P}_{rep,k}$, contains at least one external ray which lands at β. Moreover, β is the landing point of only finitely many external rays, all of which are necessarily periodic with exact period q.*

Let us generalize this result for disconnected Julia sets.

Proposition 6 *Let P be a polynomial having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Denote by K_β the connected component of K_P that contains β. Then, there are finitely many accesses to β in $\mathbb{C} \setminus K_\beta$. They are all periodic with period q.*

PROOF. Denote by $G_P : \mathbb{C} \to [0, +\infty[$ the Green function defined by

$$G_P(z) = \lim_{n \to +\infty} \frac{1}{d^n} \max\{\log |P^{\circ n}(z)|, 0\}.$$

Let η be the minimum, over all the escaping critical points ω, of $G_P(\omega)$ and let U (respectively V) be the connected component of $\mathbb{C} \setminus G_P^{-1}(\eta)$ (respectively $\mathbb{C} \setminus G_P^{-1}(d\eta)$ containing K_β. Then, $P : U \to V$ is a polynomial-like mapping whose filled-in Julia set is K_β. The proposition now follows immediately from the straightening theorem for polynomial-like mappings [DH] and from proposition 5. ∎

Since the connected components Ω_k, $k \in \mathbb{Z}/N\mathbb{Z}$, of the immediate basin of β are Fatou components, and since P is a polynomial, we see that the components Ω_k are simply connected. As in Theorem A, we can choose unifomizing maps $\varphi_k : \Omega_k \to \mathbb{D}$. Then, the proper mapping $P^{\circ q} : \Omega_k \to \Omega_k$ is conjugate to a Blaschke product $F_k = \varphi_k \circ P^{\circ q} \circ \varphi_k^{-1} : \mathbb{D} \to \mathbb{D}$ having a parabolic fixed point $\beta_k \in S^1$.

Theorem B. (POMMERENKE-LEVIN-YOCCOZ INEQUALITY FOR PARABOLIC FIXED POINTS). *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Denote by K_β the connected component of K_P that contains β and by Ω_k, $k \in \mathbb{Z}/N\mathbb{Z}$, the connected components of the immediate basin of β. Choose uniformizing maps $\varphi_k : \Omega_k \to \mathbb{D}$, and let β_k be the unique parabolic fixed point of the Blaschke product $F_k = \varphi_k \circ P^{\circ q} \circ \varphi_k^{-1}$. Then, we have the inequality*

$$\Re(\text{résit}(P^{\circ q}, \beta)) > \frac{m}{2q \log d} + \frac{1}{2} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \text{résit}(F_k, \beta_k),$$

where $m \geq N$ is the number of accesses to β in $\mathbb{C} \setminus K_\beta$.

PROOF. The proof follows from equality (3). We will show that the m accesses to β in $\mathbb{C} \setminus K_\beta$ give rise to m disjoint equatorial annuli contained in the disjoint union of the equatorial compact sets E_k, $k \in \mathbb{Z}/N\mathbb{Z}$. We will also show that the moduli of those annuli are at least $\pi/(q \log d)$. Hence,

$$\sum_{k \in \mathbb{Z}/N\mathbb{Z}} \text{defect}(E_k) \geq \frac{m\pi}{q \log d}. \tag{4}$$

This almost gives the required result. We will have to prove that the inequality is strict.

Let us first show that each access to β in $\mathbb{C} \setminus K_\beta$ gives rise to an equatorial annulus of modulus at least $\pi/(q \log d)$ contained in one of the equatorial compact sets E_k, $k \in \mathbb{Z}/N\mathbb{Z}$. By considering $P^{\circ q}$ instead of P, we may assume that the multiplier at β is 1 and each access is fixed. Thus, without loss of generality, we may assume that $q = 1$. When K_P is connected, it is known that each access to β in $\mathbb{C} \setminus K_\beta$ gives rise to an equatorial annulus of modulus exactly equal to $\pi/\log d$ (combine [M2] theorem 18.13 with [H] proposition 3.5). The proof essentially proceeds as follows.

(a) The set of accesses to β in $\mathbb{C} \setminus K_P$ is in one-to-one correspondence with the set of connected components of $\bigcup_{k \in \mathbb{Z}/N\mathbb{Z}} \mathcal{P}_{rep,k} \setminus K_P$ modulo P. Each such component is an equatorial annulus.

(b) The modulus of this annulus can be computed in the following way. Consider the conformal representation $\varphi : \mathbb{C} \setminus K_P \to \mathbb{C} \setminus \overline{\mathbb{D}}$. When correctly normalized, this conformal representation conjugates $P : \mathbb{C} \setminus K_P \to \mathbb{C} \setminus K_P$ to the mapping $f(z) = z^d$. An access to β in $\mathbb{C} \setminus K_P$ corresponds to a fixed point α of f in S^1. The modulus of the corresponding equatorial annulus is $\pi/\log \lambda$, where $\lambda = d$ is the multiplier of f at α.

We have to show that the proof can be generalized when K_P is disconnected. Since P restricts to a polynomial-like mapping whose filled-in Julia set K_β is connected, (a) generalizes immediately by replacing K_P by K_β.

The difficulty consists in generalizing (b), i.e., estimating the modulus of the corresponding equatorial annulus. Observe that since the immediate basin of β is contained in K_β, this component is not reduced to a point. Then, define $V = \mathbb{P}^1 \setminus K_\beta$ and $U = \mathbb{P}^1 \setminus P^{-1}(K_\beta)$. Since K_β is forward invariant, we see that $U \subset V$ and $P : U \to V$ is a proper mapping. Choose $\varphi : V \to V' = \mathbb{P}^1 \setminus \overline{\mathbb{D}}$ to be an isomorphism fixing ∞. This conformal representation conjugates $P : U \to V$ to a proper mapping $f : U' = \varphi(U) \to$

V'. By Schwarz's reflection principle, this mapping extends analytically in a neighborhood of S^1. Each access to β in V corresponds to a fixed point α of f in S^1. The modulus of the equatorial annulus corresponding to this access is $\pi/\log\lambda$, where λ is the multiplier of f at α. In order to prove that the modulus is at least $\pi/\log d$, we must show that $\lambda \leq d$. We will show that for any $z \in U'$, we have $|f(z)| \leq |z|^d$. This shows that f is less expanding that the mapping $z \mapsto z^d$ on S^1, and consequently the multiplier at any of its fixed point $\alpha \in S^1$ is less than d. In other words, for $\epsilon > 0$, we have

$$|f(\alpha+\varepsilon\alpha))| = 1 + f'(\alpha)\varepsilon + o(\varepsilon) \leq 1 + d\varepsilon + o(\varepsilon) = |\alpha+\varepsilon\alpha|^d,$$

and thus, $f'(\alpha) \leq d$.

Let us now prove that for any $z \in U'$, we have $|f(z)| \leq |z|^d$. First observe that since P has no poles in $U \setminus \{\infty\}$, f is holomorphic in $U' \setminus \{\infty\}$. Besides, since P has a superattracting fixed point at ∞ with local degree d, so does f. In particular, the mapping $f(z)/z^d$ is holomorphic in a neighborhood of ∞, and thus, it is holomorphic through the whole set U'. Since $f : U' \to V'$ is proper, we see that $|f(z)|$ tends to 1 as z tends to the boundary of U'. Thus, when z tends to the boundary of U', we see that $|f(z)/z^d|$ can not accumulate values greater than 1. By the maximum modulus principle, we see that for any $z \in U'$, we have $|f(z)/z^d| \leq 1$.

We finally have to prove that in (4), equality cannot be achieved. Let us assume that equality is achieved. Then, Lemma 1 asserts that the m equatorial annuli corresponding to the accesses to β in $\mathbb{C} \setminus K_\beta$ are round annuli. As a consequence, the boundaries of those annuli are union of circles. It follows the image of ∂K_β by the meromorphic function $\psi_{rep,k}$ is equal to a finite union of horizontal lines (with at least two distinct lines). Since $\psi_{rep,k} : \mathcal{P}_{rep,k} \to \mathbb{C}$ is univalent, we see that the intersection $\partial K_\beta \cap \mathcal{P}_{rep,k}$ consists in finitely many $\mathbb{R}$-analytic arcs and the asymptotics of $\psi_{rep,k}$ near β show that the union of those arcs form a cusp at β. One of these analytic arcs, γ say, contains some point $z_0 \in \partial K_\beta$ and there exists a simply connected domain U such that $\partial K_\beta \cap U = \gamma \cap U$. Since K_β is the filled-in Julia set of a polynomial-like restriction of P, for m sufficiently large, $\partial K_\beta = P^{\circ m}(\gamma \cap U)$ and thus, ∂K_β is an analytic curve. As in [St] Theorem 3 page 140, we can conclude that ∂K_β must be an analytic Jordan curve or Jordan arc. This is not possible since β belongs to this curve and there is a cusp at β. ■

The above argument can be used to generalize the classical Pommerenke-Levin-Yoccoz inequality which relates the combinatorial rotation number of the repelling fixed point β of a polynomial P to its multiplier. The standard hypothesis is that K_P is connected. However, Jin [Ji] has shown that one need only assume that the connected component of K_P that contains β is not reduced to a point and obtain the same relation between the combinatorial rotation number and the multiplier.

Example. We may apply those results to the family of cubic polynomials $P_\alpha(z) = z + z^2 + \alpha z^3$. We denote by $\mathcal{L}$ the *connectedness locus* of this family: the set of parameters $\alpha \in \mathbb{C}$ for which the Julia set is connected. Figure 7 shows the locus $\mathcal{L}$ in the coordinate α (left) and in the coordinate $1/\alpha$ (right). The polynomial $P_\alpha : \mathbb{C} \to \mathbb{C}$ has a parabolic fixed point at 0

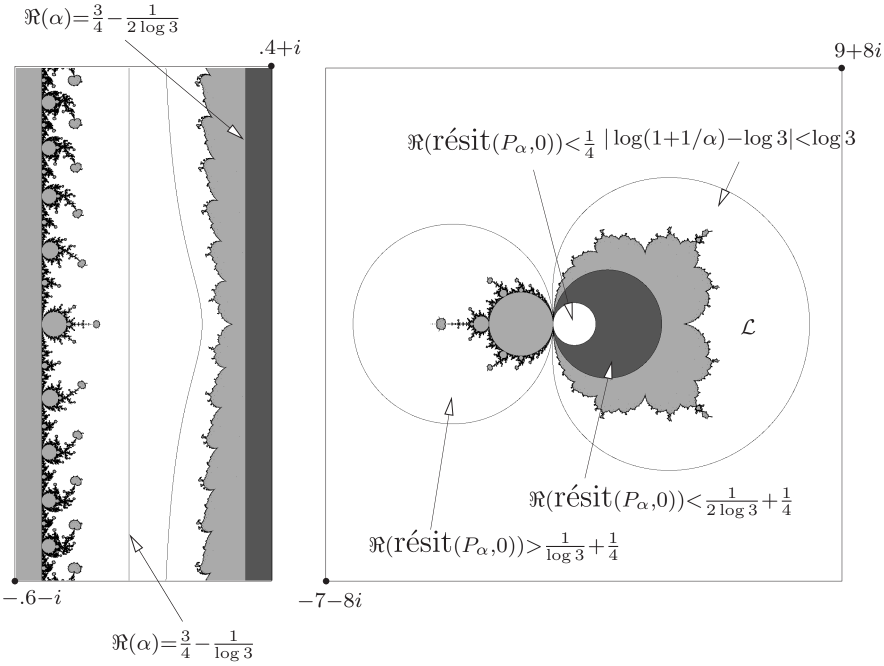

Figure 7: The connectedness locus of the family $P_\alpha(z) = z + z^2 + \alpha z^3$. Left: in the coordinate α. Right: in the coordinate $1/\alpha$.

with multiplier 1, parabolic multiplicity 1 and formal invariant α. Hence, $\text{résit}(P_\alpha, 0) = 1 - \alpha$. Moreover, the polynomial P_α has two (finite) critical points. The immediate basin of 0 may contain one or both critical points. Theorem A shows that when $\Re(\text{résit}(P_\alpha, 0) < 1/4$, i.e., when $\Re(\alpha) > 3/4$, the immediate basin contains two critical points. This inequality corresponds to the white disk centered at 2/3 with radius 2/3 in the coordinate $1/\alpha$ – it is known that the set of parameters α for which both critical points lie in the immediate basin of 0 is the connected component of the interior of $\mathcal{L}$ which contains this disk.

Theorem B shows that when the immediate basin contains only one critical point, we have

$$\Re(\text{résit}(P_\alpha, 0)) = 1 - \Re(\alpha) > \frac{1}{2\log 3} + \frac{1}{4}.$$

Thus, in the closed disk $\{\alpha \in \mathbb{C} \mid \Re(\alpha) \geq 3/4 - 1/(2\log 3)\}$, the immediate basin of 0 contains both critical points of P_α and the Julia set is connected. This improves the previous result. In the coordinate $1/\alpha$ this region is the disk of radius R_1 centered at R_1, with

$$R_1 = \frac{2\log 3}{3\log 3 - 2} \sim 1.695602768.$$

As P_α is a cubic polynomial, there are two fixed external rays (for monic polynomials of degree 3 these rays would have argument 0 and $1/2$). It is known that when the Julia set is connected, each such ray lands at a fixed point which is either repelling or parabolic with multiplier 1. We know that at least one of them lands at 0. When both land at 0, Theorem B shows that

$$\Re(\text{résit}(P_\alpha, 0)) = 1 - \Re(\alpha) \geq \frac{1}{\log 3} + \frac{1}{4}.$$

In the coordinate $1/\alpha$, this region is the disk of radius R_2 centered at $-R_2$, with

$$R_2 = \frac{2\log 3}{4 - 3\log 3} \sim 3.120334586.$$

Finally, when one of the two fixed rays do not land at 0, then it lands instead at a repelling fixed point with combinatorial rotation number 0. There is only one fixed point other than 0, namely $-1/\alpha$, and its multiplier is $1 + 1/\alpha$. The classical Pommerenke-Levin-Yoccoz inequality asserts that one of the branches of $\log(1+1/\alpha)$ belongs to the closed disk with radius $\log 3$ centered at $\log 3$.

4 The number of grand orbit equivalence classes of critical points.

In this section, we again assume that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map, but we no longer require that the components Ω_k be simply connected: consequently, we may no longer uniformize to obtain finite Blaschke products. Nevertheless, our techniques still yield interesting results. Here we show that the résidu itératif and the number of grand orbit equivalence classes of critical points in the immediate basin of β are related.

Theorem C. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map having a parabolic fixed point β with multiplier $e^{2i\pi p/q}$ and parabolic multiplicity ν. Assume that there are exactly ν grand orbit equivalence classes of critical points in the immediate basin of β. Then*

$$\Re(\text{résit}(f^{\circ q}, \beta)) > \frac{N}{\pi^2}.$$

Proof. Observe that by assumption, there may be several critical points of $f^{\circ q}$ in a given component Ω_k, but their grand-orbits by $f^{\circ q}$ must coincide. Thus, the set $\mathcal{V}_k$ (see definition 10) contains only one point. We may normalize the attracting Fatou coordinates $\varphi_{att,k}$ so that $\mathcal{V}_k = \{1\}$. The horn maps $h_k^\pm : \mathcal{D}_k^\pm \to \mathbb{C}^*$ are then ramified only above 1 and the lifted horn maps $H_k^\pm : D_k^\pm \to \mathbb{C}$ are ramified only above $\mathbb{Z}$.

Step 1. Let us define $V = (\mathbb{C} \setminus \mathbb{R}) \cup]0, 1[$. Since V is a simply connected domain which omits the critical values of H_k^+, there exists an inverse branch G_k^+ of H_k^+ defined on V which maps points with large positive imaginary part to points with large positive imaginary part. Similarly, there exists an inverse branch G_k^- of H_k^- defined on V which maps points with large negative imaginary part to points with large negative imaginary part. We set

$$U_k^\pm = \bigcup_{n\in\mathbb{Z}} \Big(n + G_k^\pm(V)\Big) \quad \text{and} \quad \mathcal{U}_k^\pm = \pi(U_k^\pm).$$

Figure 8 shows the preimages of the upper and lower half-planes by the lifted horn map of the rational map $f(Z) = Z + 1 + \frac{1}{9Z}$. This rational map has a parabolic fixed point at ∞ with multiplier 1 and parabolic multiplicity 1 (whence $N = 1$) and résidu itératif is 1/9. The critical points are $\pm 1/3$: these are grand orbit equivalent since $f(-1/3) = 1/3$. Figure 8 shows the sets $U^\pm$ and $\mathcal{U}^\pm$.

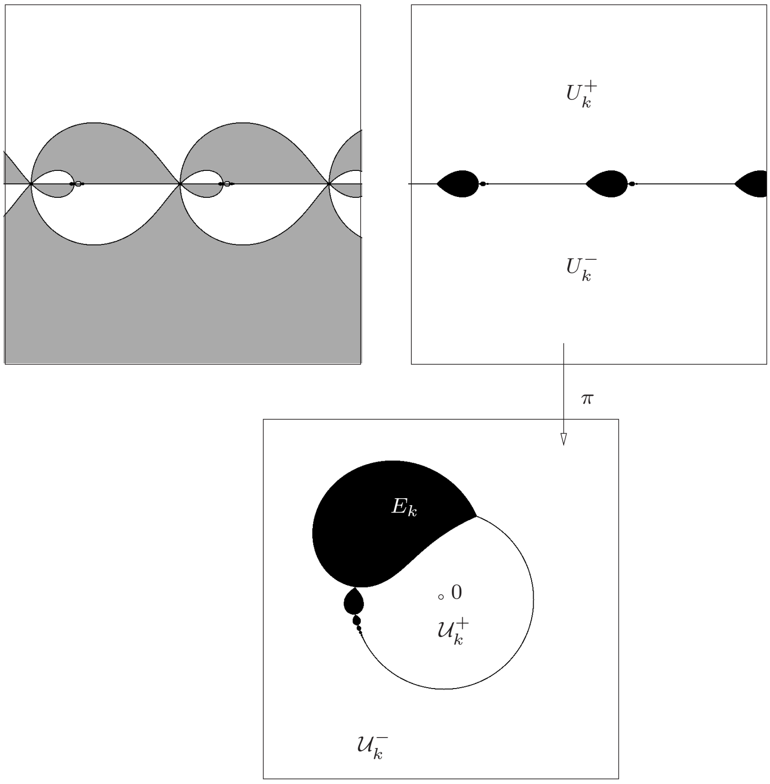

Figure 8: The sets $U_k^\pm$, $\mathcal{U}_k^\pm$ and E_k.

LEMMA 4 *For any $k \in \mathbb{Z}/N\mathbb{Z}$:*

(a) the sets $\mathcal{U}_k^+$ and $\mathcal{U}_k^-$ are disjoint;

(b) the sets $G_k^+(\mathbb{H}^-)$ and $G_k^-(\mathbb{H}^+)$ are disjoint from all their translates by an integer.

PROOF. Let us first prove (a). If $N > 1$, the immediate basin has several disjoint connected components Ω_k, $k \in \mathbb{Z}/N\mathbb{Z}$. The domains D_k^- and D_k^+ are connected components of $\psi_{rep,k}^{-1}(\Omega_k)$ and $\psi_{rep,k}^{-1}(\Omega_{k+1})$ respectively, so the domains of definition of H_k^- and H_k^+ are disjoint. It follows that the sets $U_k^- \subset D_k^-$ and $U_k^+ \subset D_k^+$ are also disjoint.

Let us now assume that $N = 1$ and that the domains D^- and D^+ intersect. As both are connected components of $\psi_{rep}^{-1}(\Omega)$, we have $D^- = D^+$. Moreover, since both lifted horn maps are defined by $H^\pm = \varphi_{att} \circ \psi_{rep}$, we have $H^- = H^+$. We denote by $H : D \to \mathbb{C}$ this lifted horn map. Now, G^+ and G^- are both inverse branches of H, and if the sets $G^+(V)$ and $G^-(V)$ were intersecting, the two inverse branches G^- and G^+ would be equal. We will show that this is not possible.

Assume, by contradiction, that this is the case and denote by $G : V \to \mathbb{C}$ this inverse branch. We know that $H(Z+1) = H(Z)+1$ and $H(Z) = Z + \mathcal{O}(1)$ when $|\Im(Z)| \to +\infty$. Thus, when $|\Im(Z)|$ is sufficiently large, the only preimage of $H(Z)+1$ with large imaginary part is $Z+1$. Since $G : V \to \mathbb{C}$ is an inverse branch of H that sends points with large imaginary part to points with large imaginary part, we see that $G(Z+1) = G(Z)+1$ when $|\Im(Z)|$ is sufficiently large. By the identity theorem, this remains true on the connected components of $V \cap (V-1)$ which contains points with large imaginary part, i.e., on $\mathbb{H}^+ \cup \mathbb{H}^-$. It follows that we can extend G analytically to $\mathbb{C} \setminus \mathbb{Z} = \bigcup_{n\in\mathbb{Z}}(n+V)$ using the formula $G(Z+1) = G(Z)+1$. Since the imaginary part of $G(Z)$ is bounded when $\Im(Z)$ is sufficiently small, the singularities of $G : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ are removable. Hence, G extends to an entire mapping $G : \mathbb{C} \to \mathbb{C}$. Since $H \circ G = \mathrm{Id}$, it follows that $H : \mathbb{C} \to \mathbb{C}$ is an isomorphism. This gives the required contradiction since H is ramified above $\mathbb{Z}$.

Let us now prove (b). Assume by contradiction that there exists an integer $n \in \mathbb{Z}$ and a point $Z_0 \in G_k^+(\mathbb{H}^-)$ such that $Z_0 + n \in G_k^+(\mathbb{H}^-)$. Then, the two mappings $Z \mapsto G_k^+(Z)$ and $Z \mapsto G_k^+(Z-n)+n$ are inverse branches of H_k^+ defined on the lower half-plane $\mathbb{H}^-$. They both send $H_k^+(Z_0+n) = H_k^+(Z_0)+n$ to Z_0+n. Hence, those two branches are equal and $G_k^+(Z-n) = G_k^+(Z)-n$ on $\mathbb{H}^-$. This shows that $G_k^+(\mathbb{H}^-)$ is invariant by the translation $Z \mapsto Z+n$. The Riemann surface $G_k^+(\mathbb{H}^-)/n\mathbb{Z} \subset \mathbb{C}/n\mathbb{Z}$ is isomorphic to the cylinder $\mathbb{H}^-/n\mathbb{Z}$ and has infinite modulus. Thus, the imaginary of Z cannot be bounded on $G_k^+(\mathbb{H}^-)$. Since $G_k^+(\mathbb{H}^-)$ avoids $G_k^+(\mathbb{H}^+)$ which contains an upper half-plane, we see that $G_k^+(\mathbb{H}^-)$ contains points with arbitrary large negative imaginary part. It follows that $G_k^+(Z+1) = G_k^+(Z)+1$ on $\mathbb{H}^+ \cup \mathbb{H}^-$. This yields to the same contradiction as in (a). $\square$

It follows that $U_k^\pm$ is simply connected and $\pi : U_k^\pm \to \mathcal{U}_k^\pm$ is a universal covering. Thus, the sets $\mathcal{U}_k^\pm$ are doubly connected. Moreover, $\mathcal{U}_k^+$ is a punctured neighborhood of 0 and $\mathcal{U}_k^-$ is a punctured neighborhood of ∞. Hence, $\mathcal{U}_k^+ \cup \{0\}$ and $\mathcal{U}_k^- \cup \{\infty\}$ are simply connected. Consequently,

$$E_k = \mathbb{C}^* \setminus (\mathcal{U}_k^+ \cup \mathcal{U}_k^-)$$

is an equatorial compact set. Our goal is now to prove the following equality:

$$\Re(\text{résit}(f^{\circ q},\beta)) = \frac{N}{\pi^2} + \frac{1}{2\pi}\sum_{k\in\mathbb{Z}/N\mathbb{Z}} \text{defect}(E_k). \tag{5}$$

This will complete the proof of the theorem since the Grötzsch defect of E_k is positive.

Step 2. Let us introduce the ramified coverings $\Gamma : \mathbb{C}\setminus(\mathbb{Z}+1/2)\to\mathbb{C}$ and $\gamma : \mathbb{C}^*\setminus\{-1\}\to\mathbb{C}^*$ defined by

$$\Gamma(Z) = Z - \frac{1}{\pi}\tan(\pi Z) \quad \text{and} \quad \gamma(\zeta) = \zeta e^{2(1-\zeta)/(1+\zeta)}.$$

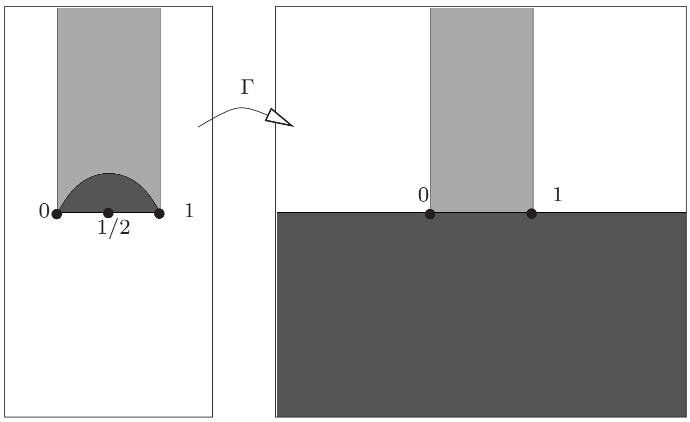

Figure 9: The isomorphism $\Gamma : \{Z\in\mathbb{H}^+ \mid 0<\Re(Z)<1\}\to\mathbb{H}^-\cup\{Z\in\mathbb{C}\mid 0<\Re(Z)<1\}$.

Those ramified coverings are related by the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}\setminus(\mathbb{Z}+1/2) \arrow[r, "\Gamma"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi"] \\
\mathbb{C}^*\setminus\{-1\} \arrow[r, "\gamma"'] & \mathbb{C}^*
\end{tikzcd}.$$

The multipliers of γ at 0 and ∞ are both equal to e^2. The critical points and critical values of K are the integers, whereas γ has only one critical point and one critical value: 1.

Lemma 5 (see figure 9) *The mapping* Γ *restricts to an isomorphism*

$$\Gamma : \{Z \in \mathbb{H}^+ \mid 0 < \Re(Z) < 1\} \to \mathbb{H}^- \cup \{Z \in \mathbb{C} \mid 0 < \Re(Z) < 1\}.$$

Proof. This follows from the fact that Γ has no pole in $\{Z \in \mathbb{H}^+ \mid 0 < \Re(Z) < 1\}$, combined with the fact that when Z turns once around $\{Z \in \mathbb{H}^+ \mid 0 < \Re(Z) < 1\}$ with counterclockwise orientation, $\Gamma(Z)$ turns once around $\{Z \in \mathbb{C} \mid 0 < \Re(Z) < 1\}$ with counterclockwise orientation. We leave the details to the reader. $\square$

Let us define $\chi^+ : V \to \mathbb{C}$ to be the inverse branch of Γ which maps points with large positive imaginary part to points with large positive imaginary part. Define $\chi^- : V \to \mathbb{C}$ to be the inverse branch of Γ which maps points with large negative imaginary part to points with large negative imaginary part. Then, set

$$W^\pm = \bigcup_{n \in \mathbb{Z}} \Big(n + \chi^\pm(V)\Big) \quad \text{and} \quad \mathcal{W}^\pm = \pi(W^\pm).$$

Lemma 5 implies that $W^+ = \mathbb{H}^+$ and $\mathcal{W}^+ = \mathbb{D}^*$. Similarly, one can prove that $W^- = \mathbb{H}^-$ and $\mathcal{W}^- = \mathbb{C} \setminus \overline{\mathbb{D}}$.

Step 3. Finally, let $\Phi_k^\pm : G_k^\pm(V) \to \chi^\pm(V)$ be the isomorphism defined by $\Phi_k^\pm = \chi^\pm \circ H_k^\pm$. Since $H_k^\pm$ and $\chi^\pm$ commute with translation by 1, so does $\Phi_k^\pm$. Thus, we can extend $\Phi_k^\pm$ to an isomorphism $\Phi_k^\pm : U_k^\pm \to W^\pm$ using the formula $\Phi_k^\pm(Z+1) = \Phi_k^\pm(Z) + 1$. Those isomorphisms project via π to isomorphisms $\varphi_k^+ : \mathcal{U}_k^+ \to \mathbb{D}^*$ and $\varphi_k^- : \mathcal{U}_k^- \to \mathbb{C} \setminus \overline{\mathbb{D}}$. It follows that

$$\text{defect}(E_k) = \frac{1}{2\pi} \log \left|\rho_k^- \rho_k^-\right|,$$

where $\rho_k^\pm$ is the multiplier of the map $\varphi_k^\pm$.

Moreover, the relation $H_k^\pm = \Gamma \circ \Phi_k^\pm$ projects to $h_k^\pm = \gamma \circ \varphi_k^\pm$ (see figure 10). The multipliers of γ at 0 and at ∞ are both equal to e^2. Thus, we have

$$\lambda_k^\pm = e^2 \cdot \rho_k^\pm$$

and

$$\begin{aligned} 4\pi^2 \Re(\text{résit}(f^{\circ q}, \beta)) &= \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \log \left|\lambda_k^- \lambda_k^+\right| \\ &= \sum_{k \in \mathbb{Z}/N\mathbb{Z}} 4 + \log \left|\rho_k^- \rho_k^+\right| = 4N + \sum_{k \in \mathbb{Z}/N\mathbb{Z}} 2\pi \,\text{defect}(E_k). \end{aligned}$$

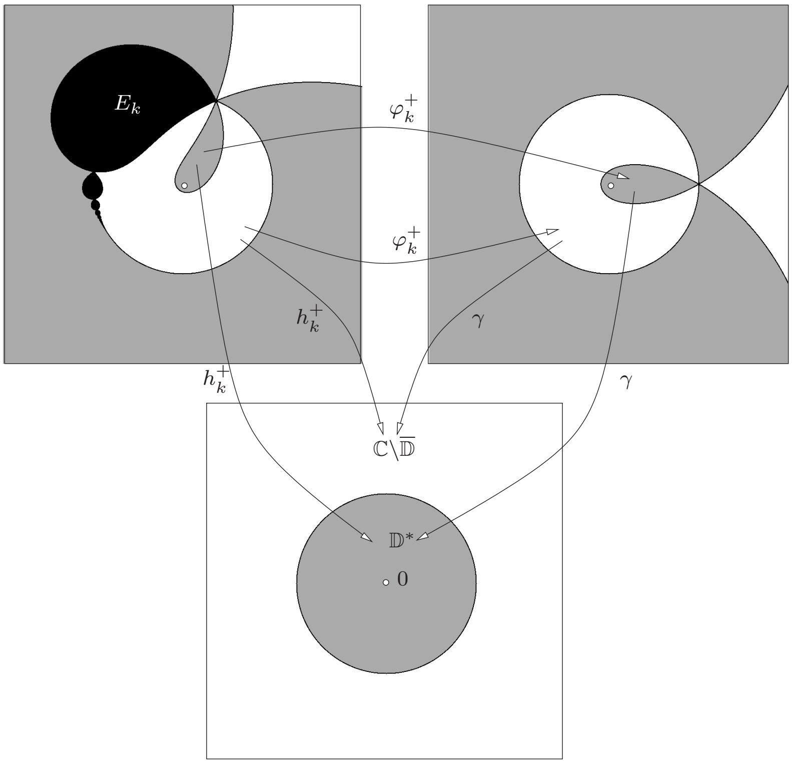

Figure 10: The relation $h_k^\pm = \gamma \circ \varphi_k^\pm$.

Dividing on both sides by $4\pi^2$ gives the required equality (5). ■

Example. Figure 11 illustrates this inequality in the case of quadratic rational maps $Z \mapsto Z+1+A/Z$. The parameters for which the two critical points are grand orbit equivalent are exactly the corners of the tiles. It seems that the left-most one is $A = 1/9 > 1/\pi^2$. We see that the estimate we obtain is good, but we do not know whether it is optimal.

5 More results.

The techniques introduced in this article can be used in order to get other inequalities relating the résidu itératif at a parabolic fixed point to the behaviour of the critical orbits contained in the basin of this fixed point. For example, one may as well prove that when the sets $\mathcal{V}_k$ (see definition 10) are

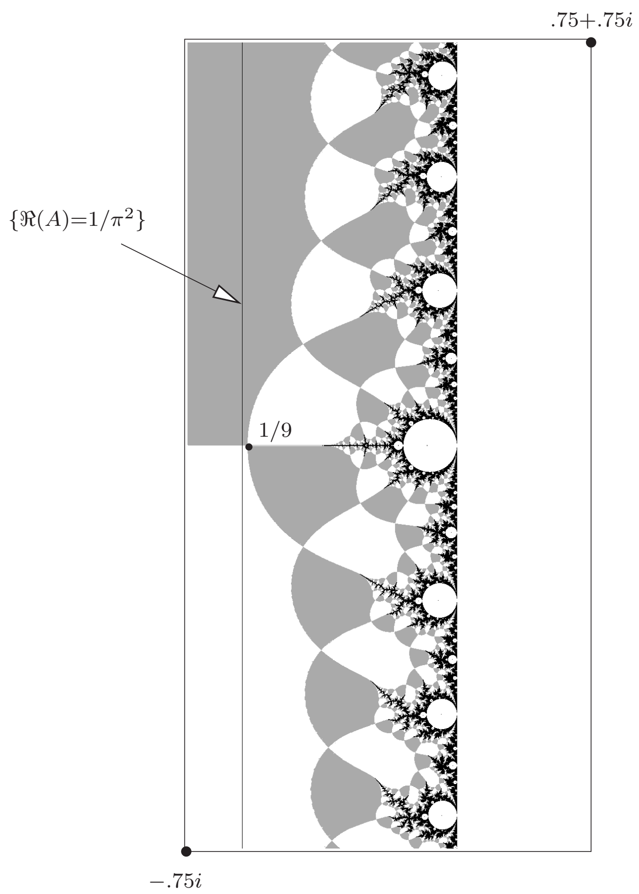

Figure 11: Critical orbit relations for quadratic rational maps with a multiple fixed point: $Z \mapsto Z + 1 + A/Z$.

contained in equatorial compact sets E_k, then

$$\Re(\text{résit}(f^{\circ q}, \beta)) \geq \frac{1}{2\pi} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \text{defect}(E_k).$$

In particular, if each set $\mathcal{V}_k$ is contained in a circle centered at 0, then we have $\Re(\text{résit}(f, \beta)) \geq 0$. This had already been observed in [Ep2] where Epstein writes: "considerations of the return maps on Écalle cylinders shows in fact that there are at least $\nu + 1$ critical values with infinite forward orbit in the immediate basin of a parabolic-attracting or parabolic-indifferent cycle of degeneracy ν". This is the case, for example, when f is the n-th iterate of some rational map g satisfying the assumptions of Theorem C (observe that $\text{résit}(f,\beta) = \dfrac{1}{n}\text{résit}(g,\beta) \underset{n\to+\infty}{\longrightarrow} 0$).

In [Je] (see Proposition 2), Jellouli proves among other things that for the quadratic polynomial $P(z) = e^{2i\pi p/q}z + z^2$, we have

$$\Re(\iota(P^{\circ q},0)) \le \frac{q+1}{2}.$$

This is equivalent to $\Re(\text{résit}(P^{\circ q},0)) \ge 0$. Our results show that Jellouli's inequality can be improved to

$$\Re(\iota(P^{\circ q},0)) \le \frac{q+1}{2} - \left(\frac{1}{2\log 2} + \frac{q}{4}\right).$$

Finally, the original motivation of our study was to try and see whether Epstein's generalization of the Fatou-Shishikura inequality (see [Ep3]) could be obtained using Shishikura's arguments (see [Sh1]). The answer is yes, and we get an improved inequality.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map and let M be a real number. Associate to each non-repelling cycle $c = \{\alpha_0, \dots, \alpha_{n-1}\}$ of f a number $\gamma_M(c)$ defined to be

$$\gamma_M(c) = \begin{cases} 0 & \text{if } c \text{ is superattracting,} \\ 1 & \text{if } c \text{ is attracting, not superattracting,} \\ 1 & \text{if } c \text{ is irrationally indifferent,} \\ \nu & \text{if } c \text{ is parabolic, and } \Re(\text{résit}(f^{\circ(nq)},\alpha_0)) > \nu q M \\ & (\nu \text{ is the parabolic multiplicity and} \\ & \text{the multiplier of the cycle is } e^{2i\pi p/q}), \\ \nu+1 & \text{if } c \text{ is parabolic, and } \Re(\text{résit}(f^{\circ(nq)},\alpha_0)) \le \nu q M. \end{cases}$$

We count the number $\gamma_M(f)$ of non-repelling cycles of f counted with this multiplicity:

$$\gamma_M(f) = \sum \gamma_M(c),$$

where the sum is taken over all the non-repelling cycles of f.

Then, define the number $\delta(f)$ of infinite tails of critical orbits. An infinite tail of critical orbit is a non-(pre)periodic sequence $(\omega, f(\omega), f^{\circ 2}(\omega), \dots)$, where ω is a critical point of f. Two infinite tails of critical orbit $(\omega_1, f(\omega_1), f^{\circ 2}(\omega_1), \dots)$, and $(\omega_2, f(\omega_2), f^{\circ 2}(\omega_2), \dots)$ are equivalent if $f^{\circ i}(\omega_1) = f^{\circ j}(\omega_2)$ for some

$i, j \geq 0$. The number $\delta(f)$ of infinite tails of critical orbits is the number of equivalence classes.

In [Ep3], Epstein proves that

$$\delta(f) \geq \gamma_0(f).$$

Using Shishikura's perturbative arguments, combined with our Theorem C, one may improve this result:

$$\delta(f) \geq \gamma_{1/\pi^2}(f).$$

This result is probably not optimal and one may have $\delta(f) \geq \gamma_{1/9}(f)$.

Appendix: The parabolic multiplicity and the formal invariant.

In this appendix, we recall the definitions of the parabolic multiplicity ν and the formal invariant α. We then prove that any parabolic germ of the form $z \mapsto e^{2i\pi p/q} z + \mathcal{O}(|z|^2)$ is analytically conjugate to a germ of the form $z \mapsto e^{2i\pi p/q} z(1 + z^{\nu q} + \alpha z^{2\nu q}) + \mathcal{O}(|z|^{2\nu q+2})$. This appendix is extracted from [BH].

Definition 12 *Let $\beta \in \mathbb{P}^1$ be a fixed point of an analytic mapping f defined in a neighborhood of β. If $\beta \neq \infty$, we define the* multiplicity *of β as a fixed point of f to be the residue*

$$\frac{1}{2i\pi} \int_\gamma \frac{1 - f'(z)}{z - f(z)} dz,$$

where γ is a small loop turning once around β with counter-clockwise orientation. If $\beta = \infty$, we define the multiplicity of β to be the multiplicity of 0 as a fixed point of $1/f(1/z)$.

If $\beta \neq \infty$, the multiplicity of β as a fixed point of f is equal to the multiplicity of β as a of the equation $f(z) - z = 0$. In particular, the multiplicity of a fixed point is equal to 1 when the multiplier λ is not 1, and it is greater otherwise.

The following proposition asserts that the multiplicity of a fixed point β is an analytic invariant. Hence, the definition of the multiplicity at infinity makes sense.

Proposition 7 *The multiplicity of a fixed point β is an analytic invariant. More precisely, if $g = \varphi \circ f \circ \varphi^{-1}$, where φ is a local analytic isomorphism, then the multiplicity of β as a fixed point of f is equal to the multiplicity of $\varphi(\beta)$ as a fixed point of g.*

PROOF. The multiplicity is equal to 1 if and only if the multiplier differs from 1. Since the multiplier at a fixed point is an analytic invariant, we see that being a simple fixed point does not depend on the choice of coordinates.

Now, to prove the proposition for a multiple fixed point (i.e., when the multiplier is equal to 1), consider the family of mappings

$$f_\varepsilon(z) = f(z) + \varepsilon, \quad \varepsilon \in \mathbb{C}.$$

Call m the multiplicity of β as a fixed point of f. When $\varepsilon \neq 0$ is small, a multiple fixed point β of f will split up into a cluster of m nearby simple fixed points. Since the integral of $(1 - f'(z))/(z - f(z))$ on γ varies continuously with ε, and since this integral is a sum of residues which are analytic invariants when $\varepsilon \neq 0$, it follows that the multiplicity of β is also an analytic invariant. ■

Proposition 8 *Let $f(z) = e^{2i\pi p/q} z + \mathcal{O}(|z|^2)$ be an analytic mapping defined in a neighborhood of 0 and having a parabolic fixed point at 0.*

a) The multiplicity of 0 as a fixed point of $f^{\circ q}$ is of the form $\nu q + 1$ for some integer $\nu \geq 1$, called the parabolic multiplicity *of the fixed point.*

b) The mapping f can be analytically conjugated to a mapping of the form

$$g(z) = e^{2i\pi p/q} z\Big(1 + z^{\nu q}\Big) + \mathcal{O}\Big(|z|^{\nu q + 2}\Big).$$

Remark. The parabolic multiplicity of a parabolic fixed point is not the same as the multiplicity of the fixed point.

PROOF. Set $\lambda = e^{2i\pi p/q}$. If $f(z) = \lambda z + a z^k + \mathcal{O}\left(z^{k+1}\right)$, then setting $\varphi(z) = z + b z^k$, we find that

$$\varphi \circ f \circ \varphi^{-1}(z) = \lambda\Big(z + (a + b(\lambda^k - \lambda)) z^k\Big) + \mathcal{O}(z^{k+1}).$$

If k is not congruent to 1 modulo q, we can set

$$b = \frac{a}{\lambda - \lambda^k}$$

so that $\varphi \circ f \circ \varphi^{-1}(z) = \lambda z + \mathcal{O}\left(z^{k+1}\right)$. This proves that by successive conjugations with mappings of the form $\varphi(z) = z + bz^k$, we can eliminate terms with powers that are not congruent to 1 modulo q, so the first term we cannot eliminate this way will have a power of the form $\nu q + 1$ for some ν.

We will now show that $\nu q + 1$ is the multiplicity of 0 as a fixed point of $f^{\circ q}$. Let $g(z) = \lambda(z + az^{\nu q+1}) + \mathcal{O}(|z|^{\nu q+2})$ be this conjugate of f. One can easily check by induction that for any $n \geq 1$, we have

$$g^{\circ n}(z) = \lambda^n\Big(z + naz^{\nu q+1}\Big) + \mathcal{O}\Big(|z|^{\nu q+2}\Big).$$

In particular, for $n = q$, we have $\lambda^n = 1$, and thus,

$$g^{\circ q}(z) = z + qaz^{\nu q+1} + \mathcal{O}\Big(|z|^{\nu q+2}\Big).$$

This shows that the multiplicity of 0 as a fixed point of $g^{\circ q}$ is equal to $\nu q+1$. Since the maps $f^{\circ q}$ and $g^{\circ q}$ are analytically conjugate, this proves part (a) of the proposition.

The only thing left to show is that the coefficient a can be taken to be 1; it is easy to show that a linear change of variables $w = \rho z$ will accomplish this if we take $\rho^{\nu q} = a$. ∎

The next proposition shows that f has only one formal invariant α.

Proposition 9 *Let $f(z) = e^{2i\pi p/q}z + \mathcal{O}(|z|^2)$ be an analytic mapping defined in a neighborhood of 0 and having a parabolic fixed point at 0.*

a) There exists a number α such that for any integer $k > 2\nu q + 1$, there is a polynomial $\varphi_k(z)$ with $\varphi_k(0) = 0$, $\varphi_k'(0) = 1$ and

$$\varphi_k \circ f \circ \varphi_k^{-1}(z) = e^{2i\pi p/q} z\Big(1 + z^{\nu q} + \alpha z^{2\nu q}\Big) + \mathcal{O}\left(|z|^k\right).$$

b) There exists a formal power series $\varphi(z) = z + \mathcal{O}(|z|^2)$ such that

$$\varphi \circ f \circ \varphi^{-1}(z) = e^{2i\pi p/q} z\Big(1 + z^{\nu q} + \alpha z^{2\nu q}\Big).$$

Remark. This series usually diverges.

Proof. We will show that we can eliminate terms degree by degree in the expansion of f. We have seen in the proof of Proposition 8 that we can easily eliminate any term of degree k, if k is not congruent to 1 modulo q.

Again, we set $\lambda = e^{2i\pi p/q}$ and we suppose that

$$f(z) = \lambda\Big(z + z^{\nu q+1} + az^{mq+1}\Big) + \mathcal{O}\Big(|z|^{mq+2}\Big)$$

with $\nu < m < 2\nu$. Then setting

$$\varphi(z) = z(1 + bz^{(m-\nu)q}) \quad \text{with} \quad b = \frac{a}{(2\nu - m)q},$$

we will show that

$$\varphi \circ f \circ \varphi^{-1}(z) = \lambda\Big(z + z^{\nu q+1}\Big) + \mathcal{O}\Big(|z|^{mq+2}\Big).$$

Define $g = \varphi \circ f \circ \varphi^{-1}$ and assume $g(z) = \lambda z(1 + z^{\nu q} + \mu z^k) + \mathcal{O}(|z|^{k+2})$ for some integer $k \le mq$. We want to show that $\mu = 0$. We will do it by comparing the power series of $\varphi \circ f$ and $g \circ \varphi$:

$$\begin{aligned} \varphi \circ f(z) &= \lambda z\Big(1 + bz^{(m-\nu)q} + z^{\nu q} + (a + b + (m-\nu)q)z^{mq}\Big) + \mathcal{O}\Big(|z|^{mq+2}\Big) \\ g \circ \varphi(z) &= \lambda z\Big(1 + bz^{(m-\nu)q} + z^{\nu q} + (b + b\nu q)z^{mq} + \mu z^k\Big) + \mathcal{O}\Big(|z|^{k+2}\Big). \end{aligned}$$

By choice of b, we have $a + b + (m-\nu)q = b + b\nu q$. Hence, the equality of power series of $\varphi \circ f$ and $g \circ \varphi$ yields $\mu = 0$.

Next suppose that

$$f(z) = \lambda z + z^{\nu q+1} + \alpha z^{2\nu q+1} + az^{mq+1} + \mathcal{O}(z^{mq+2})$$

with $m > 2\nu$. Then setting

$$\varphi(z) = z(1 + bz^{(m-\nu)q}) \quad \text{with} \quad b = \frac{a}{(2\nu - m)q},$$

the same computation shows that

$$(\varphi \circ f \circ \varphi^{-1})(z) = \lambda z + z^{\nu q+1} + \alpha z^{2\nu q+1} + \mathcal{O}(z^{mq+2}).$$

This proves (a). Moreover, since all the polynomials φ we have used were of the form $z + b_k z^k$, and that there was at most one of any given degree, the composition of all these polynomials is a well defined formal power series. This proves (b). ■

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