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Quadratic Julia Sets with Positive Area

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Abstract

We recently proved the existence of quadratic polynomials having a Julia set with positive Lebesgue measure. We present the ideas of the proof and the techniques involved.

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1. Introduction

We study the dynamics of polynomials $P : \mathbb{C} \to \mathbb{C}$, i.e. the sequences defined by induction:

$$z_0 \in \mathbb{C}, \quad z_{n+1} = P(z_n).$$

The sequence (z_n) is called the *orbit* of z_0.

Definition 1. The filled-in Julia set $K(P)$ is the set of points $z_0 \in \mathbb{C}$ with bounded orbits. The Julia set $J(P)$ is the boundary of $K(P)$.

The filled-in Julia set $K(P)$ is a compact subset of $\mathbb{C}$ and so, its boundary $J(P)$ has empty interior. Points outside $K(P)$ have an orbit tending to ∞.

This subject has its roots in complex analysis, strongly linked to Montel's theorem on normal families. In particular, the family of iterates $(P^{\circ n})_{n\geq 0}$ is normal on the complement of $J(P)$ (called the Fatou set of P) and on any open set intersecting the Julia set $J(P)$, the sequence of iterates is not normal, since such an open set contains points with bounded orbit and points whose orbit tends to ∞. Thus, the Julia set $J(P)$ may be viewed as the *chaotic set* for the dynamics of P.

Periodic points play an important role from a dynamical point of view. A *periodic point* of P of period p is a point z such that $P^{\circ p}(z) = z$ for some

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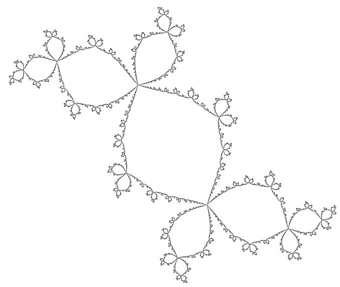

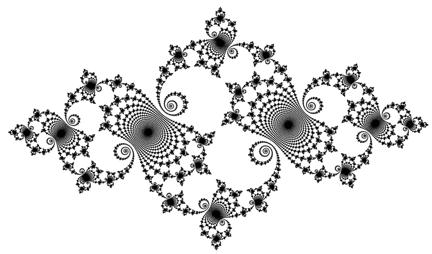

Figure 1. Left: the Julia set of a quadratic polynomial for which the critical point is periodic of period 3. It is known as the Douady Rabbit. Right: the Julia set of a quadratic polynomial with an unbounded critical orbit. The Julia set is a Cantor set.

smallest integer $p \geq 1$. The set $\{z, P(z), \ldots, P^{\circ(p-1)}(z)\}$ is a *periodic cycle*. The periodic point is repelling (respectively attracting, superattracting, indifferent) if its *multiplier* $\lambda = (P^{\circ p})'(z)$ satisfies $|\lambda| > 1$ (respectively $0 < |\lambda| < 1$, $\lambda = 0$, $|\lambda| = 1$). The Julia set $J(P)$ may equivalently be defined as the closure of the set of repelling periodic points of P.

Fatou observed that the dynamics of a polynomial P is intimately related to the behavior of the orbit of the critical points of P. A *critical point* of P is a point $\omega \in \mathbb{C}$ for which $P'(\omega) = 0$. In particular, Fatou proved that $K(P)$ is connected if and only if all the critical points of P are in $K(P)$. Further, when all the critical points of P are in the complement of $K(P)$, then $K(P) = J(P)$ is a Cantor set.

Fatou suggested that one should apply to $J(P)$ the methods of Borel-Lebesgue for the measure of sets. This naturally yields the following question.

Question. *What can we say about the Lebesgue measure of the Julia set of a polynomial?*

Until recently, the common belief was that Julia sets of polynomials always had area (Lebesgue measure) zero. It is known that the area of $J(P)$ is zero in several cases, in particular when $J(P)$ does not contain critical points of P or when the orbit of any critical point of P contained in $J(P)$ is finite ([DH] or [L1]).

In the rest of the article, we will mainly focus on the case of quadratic polynomials

$$Q_\lambda(z) = \lambda z + z^2 \quad \text{with} \quad \lambda \in \mathbb{C}.$$

Such a polynomial has a fixed point at 0 with multiplier λ and a unique critical point $\omega_\lambda = -\lambda/2$. So, we have the following dichotomy: $K(Q_\lambda)$ is connected if the orbit of ω_λ is bounded and is a Cantor set otherwise. We shall denote by $\mathcal{M}$ the set of parameters $\lambda \in \mathbb{C}$ for which $K(Q_\lambda)$ is connected (see Figure 2).

The area of $J(Q_\lambda)$ is zero:

- when λ is outside the connectivity locus $\mathcal{M}$;

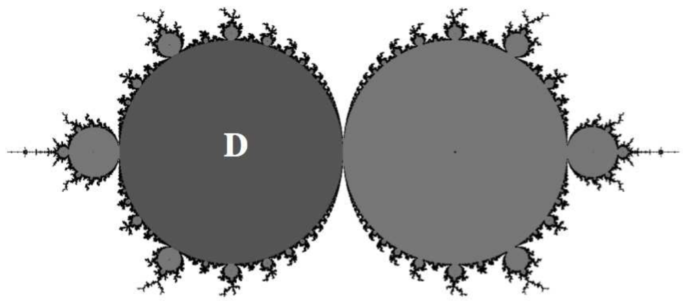

Figure 2. The set $\mathcal{M}$ of parameters $\lambda \in \mathbb{C}$ for which $J(Q_\lambda)$ is connected. It contains the unit disk $\mathbb{D}$ for which Q_λ has an attracting fixed point at 0.

- when Q_λ has a (super)attracting cycle (conjecturally, this is true for all λ in the interior of $\mathcal{M}$, and according to [MSS], if there were a parameter λ in the interior of $\mathcal{M}$ for which Q_λ does not have an attracting cycle, it is known that $J(Q_\lambda)$ would necessarily have positive area);
- for a generic (in the sense of Baire) λ in the boundary of $\mathcal{M}$ ([L1] or [L2]),
- if Q_λ is not infinitely renormalizable ([L3] or [Sh]), a condition that we will not define here;
- if $\lambda = e^{2i\pi\alpha}$ with $\alpha = \mathrm{a}_0 + \cfrac{1}{\mathrm{a}_1 + \cfrac{1}{\mathrm{a}_2 + \ddots}}$ and $\log \mathrm{a}_n = \mathcal{O}(\sqrt{n})$ ([PZ]); this condition on α holds for almost every $\alpha \in \mathbb{R}/\mathbb{Z}$.

In the 1990's, Douady proposed a program to show that there exist complex numbers λ of modulus 1 so that the area of $J(Q_\lambda)$ is positive. After a major breakthrough by the second author [C1], we finally brought Douady's program to completion in 2005. For a presentation of Douady's initial program, the reader is invited to consult [C2].

Theorem 1.1 ([BC2]). *There exist λ of modulus 1 such that $J(Q_\lambda)$ has positive area.*

We will present the ideas of the proof and the techniques involved.

2. Quadratic Polynomials with an Indifferent Fixed Point

We may classify the quadratic polynomials Q_λ with $|\lambda| = 1$ in three categories as follows. First, let us note $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R}/\mathbb{Z}$ and set

$$P_\alpha(z) = e^{i2\pi\alpha} z + z^2, \quad K_\alpha = K(P_\alpha) \quad \text{and} \quad J_\alpha = J(P_\alpha).$$

If $\alpha \in \mathbb{Q}/\mathbb{Z}$, we say that 0 is a parabolic fixed point of P_α. In that case, K_α has interior and $0 \in J_\alpha$. The orbit of a point in the interior of K_α converges to 0. The Julia set J_α has area zero.

If $\alpha \in (\mathbb{R} - \mathbb{Q})/\mathbb{Z}$, the dynamical behavior of P_α near 0 depends subtly on the arithmetical properties of α. We have the following dichotomy.

- If α is sufficiently Liouville, then $J_\alpha = K_\alpha$. Any neighborhood of 0 contains points with bounded orbit and points whose orbit tends to ∞. Cremer proved that the set of such angles α is G_δ dense in $\mathbb{R}/\mathbb{Z}$. We say that P_α has a Cremer fixed point at 0.
- If α is badly approximated by rational numbers, then 0 is in the interior of K_α. In that case, we denote by Δ_α the component of the interior of K_α that contains 0. Then P_α is holomorphically conjugate to the aperiodic rotation $R_\alpha : z \mapsto e^{2\pi i\alpha} z$: there is an analytic isomorphism ϕ between the unit disk $\mathbb{D}$ and Δ_α such that $\phi(0) = 0$ and $\phi \circ R_\alpha = P_\alpha \circ \phi$. One says that the polynomial P_α is *linearizable* and the component Δ_α is called a *Siegel disk*. Siegel [Si] proved that this property holds when α is Diophantine, in particular for a set of full measure in $\mathbb{R}/\mathbb{Z}$ (α is Diophantine if there are constants $c > 0$ and $\tau \geq 2$ such that $|\alpha - p/q| > c/q^\tau$ for all rational numbers p/q).

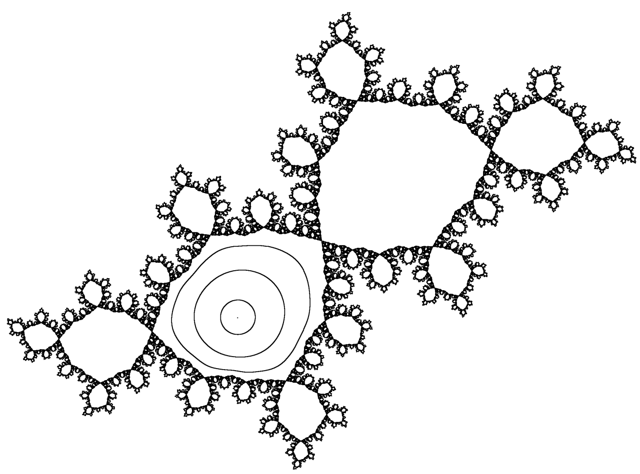

Figure 3. The Julia set of P_α for $\alpha = (\sqrt{5} - 1)/2$. We have drawn the orbits of some points in the Siegel disk. Each orbit accumulates on a $\mathbb{R}$-analytic circle.

In fact, there is a complete arithmetic characterization of the two previous sets of angles. Let $(p_n/q_n)_{n\geq 0}$ be the approximants to α given by the continued fraction algorithm. Brjuno [Brj] proved that when

$$B(\alpha) = \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n} < +\infty,$$

the polynomial P_α is linearizable. Yoccoz [Y] proved that when $B(\alpha) = +\infty$, the polynomial P_α has a Cremer fixed point at 0. In addition, any neighborhood of 0 contains a cycle which is not reduced to $\{0\}$.

We have the following refined versions of our theorem.

Theorem 2.1. *There exist angles $\alpha \in (\mathbb{R} - \mathbb{Q})/\mathbb{Z}$ for which P_α has a Cremer fixed point at* 0 *and* $\mathrm{area}(J_\alpha) > 0$.

Theorem 2.2. *There exist angles $\alpha \in (\mathbb{R} - \mathbb{Q})/\mathbb{Z}$ for which P_α has a Siegel disk and* $\mathrm{area}(J_\alpha) > 0$.

We will now sketch the proof of the first theorem. The proof of the second theorem relies on similar ideas.

3. Strategy of the Proof

Proposition 3.1. *The function $\alpha \mapsto \mathrm{area}(K_\alpha) \in [0, +\infty[$ is upper semi-continuous.*

In other words, if $\alpha_n \to \alpha$, then

$$\limsup_{n\to\infty} \mathrm{area}(K_{\alpha_n}) \le \mathrm{area}(K_\alpha).$$

Proof. Every open set containing K_α contains $K_{\alpha'}$ for α' close enough to α. □

We shall see that the existence of Julia sets with positive area is an immediate consequence of the following key proposition which is illustrated by Figure 4.

Proposition 3.2. *There exists a non empty set $\mathcal{S}$ of Diophantine numbers such that: for all $\alpha \in \mathcal{S}$ and all $\varepsilon > 0$, there exists $\alpha' \in \mathcal{S}$ with*

- $|\alpha' - \alpha| < \varepsilon$,
- $P_{\alpha'}$ *has a cycle in* $D(0, \varepsilon) \setminus \{0\}$ *and*
- $\mathrm{area}(K_{\alpha'}) \ge (1 - \varepsilon)\mathrm{area}(K_\alpha)$.

With this proposition, one concludes as follows. First, we choose ε_n in $(0, 1)$ so that $\prod(1 - \varepsilon_n) > 0$. Then, we construct $(\theta_n \in \mathcal{S})$ so that:

- (θ_n) is a Cauchy sequence.
- $\mathrm{area}(K_{\theta_n}) \ge (1 - \varepsilon_n)\mathrm{area}(K_{\theta_{n-1}})$.
- For $\theta = \lim \theta_n$, the polynomial P_θ has small cycles.

Since θ_n is Diophantine, $K(P_{\theta_n})$ has non empty interior and so, its area is positive. Since P_θ has small cycles, it is not linearizable, and so $J_\theta = K_\theta$. By upper semi-continuity of the function $\alpha \mapsto \mathrm{area}(K_\alpha)$, we have

$$\mathrm{area}(J_\theta) = \mathrm{area}(K_\theta) \ge \limsup_{n\to+\infty} \mathrm{area}(K_{\theta_n}) \ge \mathrm{area}(K_{\theta_0}) \cdot \prod_{n\ge1}(1 - \varepsilon_n) > 0.$$

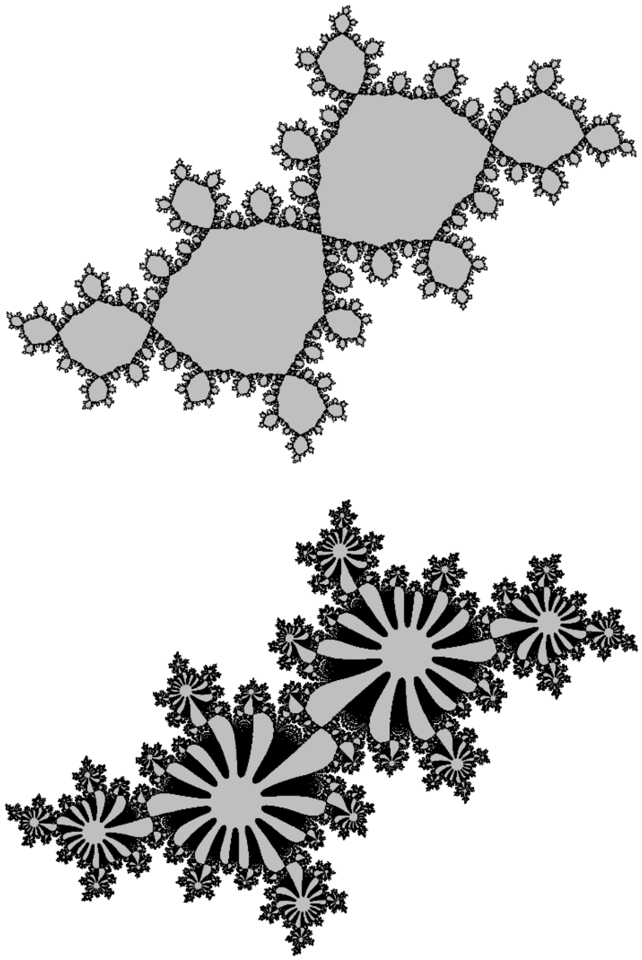

Figure 4. Two filled-in Julia sets K_α (top) and $K_{\alpha'}$ (bottom), with α' a well-chosen perturbation of α. If α and α' are chosen carefully enough the loss of measure from K_α to $K_{\alpha'}$ is small.

4. The Set $\mathcal{S}$

For $\alpha \in \mathbb{R} - \mathbb{Q}$, let us use the continued fraction notation

$$[\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots] = \mathrm{a}_0 + \cfrac{1}{\mathrm{a}_1 + \cfrac{1}{\mathrm{a}_2 + \ddots}}.$$

Recall that an irrational number $\alpha = [\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots]$ is of *bounded type* if the sequence (a_n) is bounded.

Definition 2. For $N \geq 1$, denote by $\mathcal{S}_N$ be the set of irrational numbers $\alpha = [\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots]$ of bounded type such that $\mathrm{a}_n \geq N$ for all $n \geq 1$.

Proposition 4.1. *If $N \geq 1$ is a sufficiently large integer, then Proposition 3.2 holds with $\mathcal{S} = \mathcal{S}_N$.*

5. McMullen's Results on Siegel Disks of Bounded Type

As we shall see below, the proof of Proposition 4.1 reduces to the following result that is illustrated on Figure 5.

Lemma 5.1. *If $N \geq 1$ is a sufficiently large integer, then for all $\alpha \in \mathcal{S}_N$, there is a sequence $\alpha_n \in \mathcal{S}_N$ converging to α such that*

- *P_{α_n} has a cycle converging to 0 as $n \to \infty$,*
- *for all open set $U \subset \Delta_\alpha$, we have*
$$\liminf_{n\to\infty} \mathrm{area}(U \cap \Delta_{\alpha_n}) \geq \frac{1}{2}\mathrm{area}(U \cap \Delta_\alpha) \quad \textit{and}$$
- *$\overline{\Delta}_{\alpha_n} \to \overline{\Delta}_\alpha$ for the Hausdorff topology on compact subsets of $\mathbb{C}$.*

The second assertion says that asymptotically, the Siegel disks Δ_{α_n} are at least 1/2-dense in Δ_α.

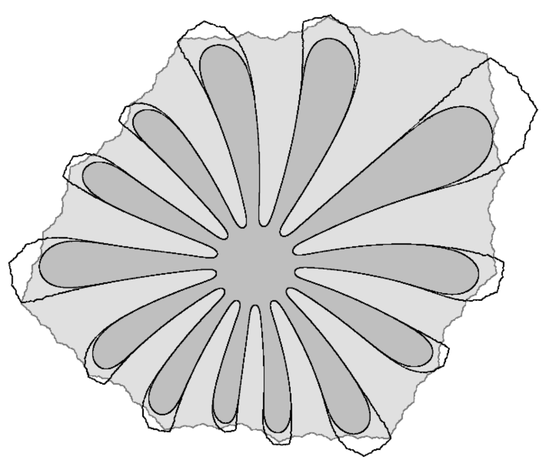

Figure 5. Illustration of lemma 5.1 for $N = 1$, $\alpha = [0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \ldots]$ and $\alpha' = [0, 1, 1, 1, 1, 1, 1, 1, 10^{10}, 1, 1, \ldots]$. The Siegel disk Δ_α is colored light grey. The boundary of the Siegel disk $\Delta_{\alpha'}$ is drawn. The darker set is the set of points in $\Delta_{\alpha'}$ whose orbit under iteration of $P_{\alpha'}$ remains in Δ_α.

We then use an argument of *toll belts* inspired by work of McMullen [McM] to promote the loss of $1/2$ for the area of Siegel disks to an arbitrarily small loss for the area of the filled-in Julia sets. For the argument of toll belts to work, we need that α is of bounded type and $\overline{\Delta}_{\alpha_n} \to \overline{\Delta}_\alpha$ as $n \to \infty$. More precisely, we use the following result of McMullen.

Theorem 5.2 (McMullen). *Assume α is a bounded type irrational and $\delta > 0$. Then, every point $z \in \partial\Delta_\alpha$ is a Lebesgue density point of the set $K(\delta)$ of points whose orbit under iteration of P_α remains at distance less than δ from Δ_α and eventually intersect Δ_α.*

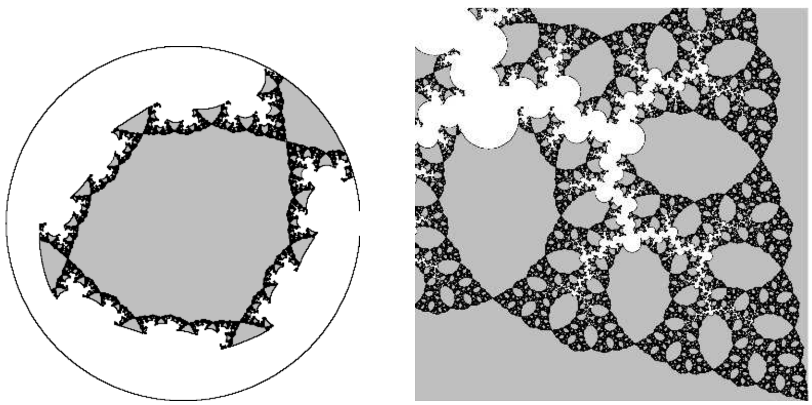

Figure 6. If $\alpha = (\sqrt{5}-1)/2$, the critical point of P_α is a Lebesgue density point of the set of points whose orbit remain in $D(0,1)$. Left: the set of points whose orbit remains in $D(0,1)$. Right: a zoom near the critical point.

Proof of Proposition 4.1 assuming lemma 5.1. Assume $\alpha \in \mathcal{S}_N$ and let $(\alpha_n)_{n\geq 0}$ be a sequence of $\mathcal{S}_N$ given by lemma 5.1. Denote by K (resp. K_n) the filled-in Julia set of P_α (resp. P_{α_n}) and by Δ (resp. Δ_n) its Siegel disk. We know that asymptotically, the Siegel disks Δ_n are at least $1/2$-dense in the Siegel disk Δ. We want to show that $\mathrm{area}(K_n) \to \mathrm{area}(K)$, which amounts to proving that the density of K_n in Δ tends to 1 as $n \to \infty$.

For all S, one can find a finite nested sequence of *toll belts* $(W_s)_{1\leq s\leq S}$

$$W_s = \left\{z \in \mathbb{C}\ ;\ 2\delta_s < d(z,\Delta) < 8\delta_s\right\} \quad \text{with} \quad 8\delta_{s+1} < \delta_s,$$

surrounding the Siegel disk Δ such that for n large enough the following holds.

- The orbit under iteration of P_{α_n} of any point in $\Delta \setminus K_n$ must pass through all the toll belts.
- Thanks to Lemma 5.1, the toll belts surround the Siegel disk Δ_n.

- Thanks to Theorem 5.2 and Lemma 5.1, under the iterates of P_{α_n}, at least $1/2-\varepsilon$ of points in the toll belt W_{s+1} will be captured by the Siegel disk Δ_n without being able to enter the toll belt W_s.
- Since the toll belts surround the Siegel disk Δ_n, they are free of the postcritical set of P_{α_n}. This gives us Koebe control of points passing through the belt, implying that at most $1/2+\varepsilon$ of points in Δ that manage to reach W_{s+1} under iteration of P_{α_n} will manage to reach W_s.

As a consequence, at most $(1/2+\varepsilon)^S$ points in Δ can have an orbit under iteration of P_{α_n} that passes through all the belts and we are done by choosing S large enough. □

6. The Sequence (α_n)

We claim that if N is a large enough integer and if $\alpha = [\mathrm{a}_0, \mathrm{a}_1, \ldots] \in \mathcal{S}_N$, then Lemma 5.1 holds for the sequence (α_n) defined by

$$\alpha_n = [\mathrm{a}_0, \mathrm{a}_1, \ldots, \mathrm{a}_n, A_n, N, N, N, \ldots] \tag{1}$$

where the sequence (A_n) is chosen so that

$$A_n \geq N, \quad \sqrt[q_n]{A_n} \underset{n\to+\infty}{\longrightarrow} +\infty \quad \text{and} \quad \sqrt[q_n]{\log A_n} \underset{n\to+\infty}{\longrightarrow} 1.$$

Lemma 5.1 has three parts which can be treated one at a time: the existence of a cycle of P_{α_n} close to 0, the density of the perturbed Siegel disk Δ_{α_n} within Δ_α and the Hausdorff convergence of $\overline{\Delta}_{\alpha_n}$ to $\overline{\Delta}_\alpha$.

7. The Control of the Cycle

In order to prove the existence of a cycle of P_{α_n} close to 0, we use a result of the second author [C1].

Proposition 7.1. *Assume P_α has a Siegel disk. Let (p_n/q_n) be the approximant to α given by the continued fraction algorithm. Let $\chi : \mathbb{D} \to \Delta_\alpha$ be an isomorphism which sends 0 to 0. There is a sequence (r_n) converging to 1 and a sequence of univalent maps $(\chi_n : D(0, r_n) \to \Delta_\alpha)$ converging locally uniformly to $\chi : \mathbb{D} \to \Delta_\alpha$ such that the following holds: if (α_n) is a sequence converging to α with $\limsup \sqrt[q_n]{|\alpha_n - p_n/q_n|} < 1$ and if C_n is the set of q_n-th roots of $\alpha_n - p_n/q_n$, then for n large enough, $\chi_n(C_n)$ is a cycle of period q_n of P_{α_n}.*

The functions $\chi_n : D(0, r_n) \to \Delta_\alpha$ are called *explosion functions*. They control the explosion, as α goes away from p_n/q_n, of the cycle of period q_n of P_α which coalesces at 0 when $\alpha = p_n/q_n$.

Now, observe that the sequence (α_n) defined by Equation (1) satisfies

$$\alpha_n - \frac{p_n}{q_n} \underset{n\to\infty}{\sim} \frac{(-1)^n}{q_n^2 A_n}.$$

Since $\sqrt[q_n]{A_n} \to +\infty$, we have that $\sqrt[q_n]{|\alpha_n - p_n/q_n|} \to 0$.

Thus, for n sufficiently large, the set C_n of q_n-th roots of $\alpha_n - p_n/q_n$ is contained in an arbitrarily small neighborhood of 0. The sequence (χ_n) converges locally uniformly to χ. So, for n large enough, the set $\chi_n(C_n)$, which is a cycle of P_{α_n}, is contained in an arbitrarily small neighborhood of 0

8. The Density of Perturbed Siegel Disks

We still assume that (α_n) is defined by Equation (1).

In order to control the density of the Siegel disks Δ_{α_n} within the Siegel disk Δ_α, we may work in the coordinates given by the explosion functions χ_n. In other words, we set

$$f_n = \chi_n^{-1} \circ P_{\alpha_n} \circ \chi_n.$$

As $n \to \infty$, the domain of f_n eventually contains any compact subset of $\mathbb{D}$. The sequence (f_n) converges locally uniformly to the rotation R_α. The map f_n fixes 0 with derivative $e^{2\pi i\alpha_n}$ and has a Siegel disk Δ_n whose image by χ_n is contained in the Siegel disk Δ_{α_n} of P_{α_n}.

We want to prove that asymptotically as $n \to \infty$, the Siegel disks Δ_n are 1/2-dense in the unit disk. For this purpose, it is not enough to compare the dynamics of f_n with the dynamics of a rotation. Instead, we will compare it with the (real) dynamics of an appropriate polynomial vector field ξ_n.

Note that by property of the explosion functions χ_n, the set C_n of q_n-th roots of $\varepsilon_n = \alpha_n - p_n/q_n$ is a periodic cycle of f_n of period q_n. Let ξ_n be the polynomial vector field which has simple roots exactly at 0 and the points of C_n and which has derivative $2\pi i q_n \varepsilon_n$ at 0. Then, the time-1 map of ξ_n fixes 0 and the points of C_n (which are also fixed points of $f_n^{\circ q_n}$) with multiplier $e^{2\pi i q_n\varepsilon_n}$ at 0 (which is also the multiplier of $f_n^{\circ q_n}$ at 0). Thanks to those properties, there is a good hope that the time-1 map of ξ_n very well approximates $f_n^{\circ q_n}$. This vector field is

$$\xi_n = \xi_n(z)\frac{\mathrm{d}}{\mathrm{d}z} = 2\pi i q_n z(\varepsilon_n - z^{q_n})\frac{\mathrm{d}}{\mathrm{d}z}.$$

We have an explicit description of the vector field ξ_n which is invariant under the rotation $z \mapsto e^{2\pi i/q_n}z$. For all $\rho < 1$ and all n sufficiently large, the set $X_n(\rho)$ defined below is invariant under the real flow of the vector field ξ_n:

$$X_n(\rho) = \left\{ z \in \mathbb{C} \; ; \; \frac{z^{q_n}}{z^{q_n} - \varepsilon_n} \in D(0, s_n) \right\} \quad \text{with} \quad s_n = \frac{\rho^{q_n}}{\rho^{q_n} + |\varepsilon_n|}.$$

This set looks like an amoeba with q_n arms. Asymptotically, the density of $X_n(\rho)$ in $D(0,\rho)$ is at least 1/2.

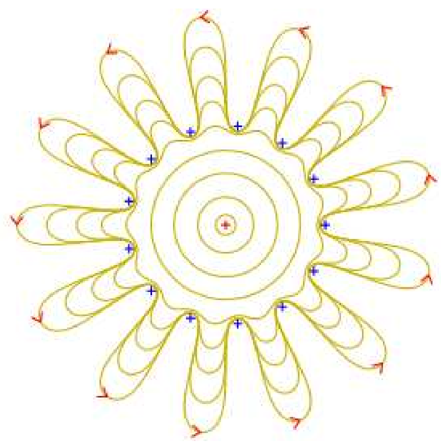

Figure 7. Some real trajectories for the vector field ξ_n; zeroes of the vector field are shown.

Using very careful estimates on how close $f_n^{\circ q_n}$ is to the time-1 map of the vector field ξ_n and using Yoccoz renormalization techniques [Y], we obtain the following result which implies the required control on the asymptotic density of Δ_n within $\mathbb{D}$.

Proposition 8.1. *For all $\rho < 1$, if n is large enough, the set $X_n(\rho)$ is contained in the Siegel disk Δ_n of f_n.*

9. Hausdorff Convergence of Perturbed Siegel Disks

In order to prove the Hausdorff convergence of $\overline{\Delta}_{\alpha_n}$ to $\overline{\Delta}_\alpha$, we use techniques of *near parabolic renormalization* introduced recently by Inou and Shishikura [IS]. Those techniques are far too elaborate for us to present them here.

Let us however insist on the fact that it is to apply those techniques that we have to assume that the entries in the continued fraction expansion of α are large enough ($\mathrm{a}_n \geq N$ for all n).

10. Further Questions

Our proof of existence of quadratic polynomials having a Julia set of positive area is *a priori* not constructive. It would be interesting to have informations regarding the set of $\alpha \in \mathbb{R}$ for which the Julia set J_α has positive area.

Theorem 10.1 (Petersen, Zakeri). *For almost every $\alpha \in \mathbb{R}$, we have* $\mathrm{area}(J_\alpha) = 0$.

Question. *Is the set of parameters* $\alpha \in \mathbb{R}$ *for which* $\mathrm{area}(J_\alpha) > 0$ *a* G_δ*-dense set?*

Now that we have proved the existence of $\alpha \in \mathbb{R} - \mathbb{Z}$ for which $J_\alpha = K_\alpha$ has positive area, we can change the question. Indeed, we do not know of a single example of a non linearizable quadratic polynomial P_α with $\alpha \in \mathbb{R} - \mathbb{Q}$ for which the Julia set has area zero. It may well be that all such Julia set have positive area.

Question. *Is there* $\alpha \in \mathbb{R}$ *such that* $J_\alpha = K_\alpha$ *and* $\mathrm{area}(J_\alpha) = 0$*?*

A key point in our proof was the observation that the function $\alpha \mapsto \mathrm{area}(K_\alpha)$ is upper semicontinuous. It would be interesting to have additional informations regarding its continuity properties.

Theorem 10.2 (Douady). *The function* $\alpha \mapsto \mathrm{area}(K_\alpha)$ *is discontinuous at rational numbers.*

Question. *Is the function* $\alpha \mapsto \mathrm{area}(K_\alpha)$ *continuous at irrational numbers?*

The techniques we have been developing for studying the area of Julia sets already had fruitful applications, in particular for the study of Siegel disks. Answering a question of Herman, we proved the following result in collaboration with A. Avila.

Theorem 10.3 ([ABC]). *There exist* $\alpha \in \mathbb{R}$ *such that* P_α *has a Siegel disk whose boundary is a smooth (*C^∞*) Jordan curve.*

In that case, the boundary of the Siegel disk P_α cannot contain the critical point of P_α. This is in contrast to the following result of Petersen and Zakeri.

Theorem 10.4 (Petersen-Zakeri). *For almost every* $\alpha \in \mathbb{R}$*,* P_α *has a Siegel disk whose boundary is a Jordan curve passing through the critical point* $\omega_\lambda = -\lambda/2$.

This raises naturally the following questions.

Question. *If* P_α *has a Siegel disk, is the boundary of* Δ_α *always a Jordan curve?*

Question. *For which values of* α *does* P_α *have a Siegel disk whose boundary contains the critical point?*

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