Farey Curves

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We study the analytic function η defined on the unit disk by $\eta(\lambda) = \lim_{n\to+\infty} P_\lambda^{\circ n}(-\lambda/2)/\lambda^n$, where P_λ is the quadratic polynomial $z \mapsto \lambda z + z^2$.

INTRODUCTION

Consider the family of quadratic polynomials $P_\lambda : \mathbb{C} \to \mathbb{C}$, where λ is a complex number with $|\lambda| \le 1$, defined by

$$P_\lambda(z) = \lambda z + z^2.$$

We will examine this polynomial from a dynamical point of view; that is, we will be interested in its behaviour under iteration.

The first observation is that the polynomial P_λ has a fixed point at zero: $P_\lambda(0) = 0$. Schröder [1871] proved that when $0 < \lambda < 1$, the polynomial P_λ behaves near 0 like its linear part: $\zeta \mapsto \lambda\zeta$. More precisely, he proved that there exists a unique analytic germ $\varphi_\lambda : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$\varphi_\lambda(z) = z + \mathcal{O}(|z|^2),$$

such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{C},0) & \xrightarrow{P_\lambda} & (\mathbb{C},0) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ (\mathbb{C},0) & \xrightarrow{\zeta \mapsto \lambda\zeta} & (\mathbb{C},0). \end{array}$$

We should think of $\zeta = \varphi_\lambda(z)$ as a new variable which is defined on a neighborhood of 0. Under the change of coordinates, the expression of P_λ simply becomes the multiplication by λ. This germ can be extended to the basin of attraction

$$U_\lambda = \{z \in \mathbb{C} \mid \lim_{n\to\infty} P_\lambda^{\circ n}(z) = 0\},$$

using the formula

$$\varphi_\lambda(z) = \lim_{n\to\infty} \frac{1}{\lambda^n} P_\lambda^{\circ n}(z).$$

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Fatou [1906] proved that the basin of attraction U_λ always contains the critical point $\omega_\lambda = -\lambda/2$, and this result arguably opened the entire field of holomorphic dynamics.

What remains of those results when $|\lambda| = 1$? If $\lambda = e^{2i\pi p/q}$ is a root of unity, then P is not linearizable. Indeed, if it were linearizable, then $P^{\circ q}$ would be equal to the identity. Cremer [1932] proved that if the Schröder equation

$$\varphi_\lambda(P_\lambda(z)) = \lambda\varphi_\lambda(z)$$

has a solution for $|\lambda| = 1$, then the mapping φ_λ will extend to a simply connected region $U_\lambda \subset \mathbb{C}$, called a Siegel disk. Cremer probably believed that this case could not occur for $|\lambda| = 1$. The first existence result for such disks was found by Siegel [1942], in one of the landmark papers of the twentieth century.

In the late 1980's Yoccoz found an amazingly simple proof that for almost every $\lambda \in S^1$, the polynomial P_λ has a Siegel disk. His proof is based on the fact that a bounded nonconstant analytic function $\eta : \mathbb{D} \to \mathbb{C}$ (where $\mathbb{D}$ is the open unit disc) has radial limits almost everywhere, and that the limit superior of the modulus of η vanishes only at a set of measure 0 as λ tends to the boundary.

1. THE FUNCTION η

Let λ be in $\mathbb{D}$, and let

$$\varphi_\lambda(z) = \lim_{n\to\infty} \frac{1}{\lambda^n} P_\lambda^{\circ n}(z)$$

be the linearizing map for $P_\lambda(z) = \lambda z + z^2$ defined in the basin U_λ. The critical point $\omega_\lambda = -\lambda/2$ belongs to this basin, so we can define

$$\eta(\lambda) = \varphi_\lambda(\omega_\lambda) = \lim_{n\to\infty} \frac{1}{\lambda^n} P_\lambda^{\circ n}(-\lambda/2).$$

Proposition 1.1. *The function $\eta(\lambda)$ is analytic and bounded in the unit disk.*

Proof. The critical values of φ_λ are the points $\eta(\lambda)/\lambda^n$, for $n \geq 0$. Hence, the disk $\mathbb{D}(0, |\eta(\lambda)|)$ does not contain any critical value of φ_λ. Since it is simply connected, there exists an inverse branch

$$\psi_\lambda = \varphi_\lambda^{-1} : \mathbb{D}(0, |\eta(\lambda)|) \to U_\lambda$$

defined on $\mathbb{D}(0, |\eta(\lambda)|)$ and such that $\psi_\lambda(0) = 0$.

Then, the function $\chi_\lambda(\xi) = \psi_\lambda(\xi\eta(\lambda))$ is an injective analytic function on the unit disk, which vanishes at 0. Moreover, $\chi_\lambda(\xi)$ belongs to the basin of attraction U_λ, hence it is in the filled Julia set $K(P_\lambda)$, which is known to be inside the disk $\mathbb{D}(0, 4)$. In particular,

$$|\chi_\lambda(\xi)| < 4. \quad (1\text{–}1)$$

Since $\varphi'_\lambda(0) = 1$, we obtain $\chi'_\lambda(0) = \eta(\lambda)$. Now we have seen that χ_λ is a univalent mapping from the unit disk to the disk $\mathbb{D}(0, 4)$. Since $\chi_\lambda(0) = 0$, Schwarz's Lemma shows that

$$|\eta(\lambda)| = |\chi'_\lambda(0)| \leq 4. \quad (1\text{–}2)$$

To see that η is analytic, notice that

$$\frac{1}{\lambda^n} P_\lambda^{\circ n}(-\lambda/2)$$

is a rational map of λ, analytic on $\mathbb{D}^*$. It is not difficult to see that the convergence is uniform on compact subsets of $\mathbb{D}^*$. Hence, η is analytic in $\mathbb{D}^*$. Since by equation (1–2) it is bounded by 4, the removability theorem shows that it is analytic in all $\mathbb{D}$. □

Since $\eta : \mathbb{D} \to \mathbb{C}$ is a bounded analytic function that does not vanish identically, it has radial limits almost everywhere, and the set of $\mu \in S^1$ such that

$$\limsup_{\lambda\to\mu} |\eta(\lambda)| = 0$$

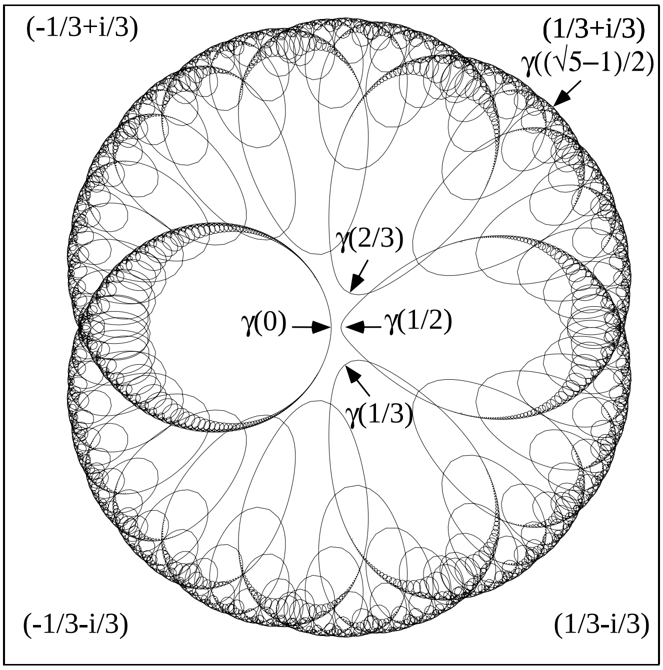

FIGURE 1. The image of the circle $|\lambda| = .9999$ by the map $\eta(\lambda)$, where $\gamma(t) = \eta(.9999\, e^{2\pi i t})$.

is of measure 0. This function $\eta(\lambda) : \mathbb{D} \to \mathbb{C}$ is quite surprising. We will see that if $\eta(\lambda)$ does not tend to 0 when $\lambda \to \mu \in S^1$, then the polynomial $P(z) = \mu z + z^2$ has a Siegel disk. In particular, the limit must be 0 when $\mu \in S^1$ is a root of unity. On the other hand, the radial limit must be positive on a set of measure 1. In order to understand the boundary behaviour of the function η, we have drawn the curve $\gamma(t) = \eta(.9999\, e^{2\pi i t})$ on Figure 1.

2. YOCCOZ'S PROOF

Theorem 2.1. *There exists a set $S \subset S^1$ of measure 1 such that if $\lambda \in S$, the quadratic polynomial $P_\lambda(z) = \lambda z + z^2$ is linearizable in a neighborhood of 0.*

Proof. We have seen (equation (1–1)) that the family $\{\chi_\lambda\}_{\lambda\in\mathbb{D}}$ is uniformly bounded by 4. By Montel's theorem, the family $\{\chi_\lambda\}_{\lambda\in\mathbb{D}}$ is normal. So any sequence χ_{λ_n} with $\lambda_n \to \mu \in S^1$ has a convergent subsequence, and the limit χ is either identically 0 or injective on the unit disk. Notice that $\chi_\lambda(\lambda z) = P_\lambda \circ \chi_\lambda(z)$. Hence, if the limit function χ is injective, it satisfies $\chi(\mu z) = P_\mu \circ \chi(z)$, i.e., χ^{-1} linearizes P_μ near 0.

The function

$$R(\mu) = \limsup_{\lambda\to\mu} |\eta(\lambda)|$$

on the unit circle vanishes only on a set of measure 0. Since $\chi_\lambda'(0) = \eta(\lambda)$, if $\mu = e^{2\pi i\theta}$ is in the complement S of this set, then a sequence λ_n converging to μ can be chosen such that the sequence χ_{λ_n} converges to χ, and

$$|\chi'(0)| = \left|\lim_{n\to\infty} \chi'_{\lambda_n}(0)\right| = R(\mu) > 0,$$

so that the limit is not constant. □

The proof above gives little insight into the nature of angles θ for which the polynomial

$$P(z) = e^{2\pi i\theta} z + z^2$$

is or is not linearizable. We will now give a result which should bring out the arithmetic nature of such angles. The first such result was obtained by Siegel in 1942. Brjuno [1971] and Yoccoz [1995] have since completely solved the problem.

Theorem 2.2. *Set $\lambda = e^{2\pi i\theta}$. The polynomial P_λ is linearizable if and only if $\theta \in \mathbb{R}\setminus\mathbb{Q}$ and*

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty, \qquad (2–1)$$

where p_n/q_n is the n-th convergent of the continued fraction of θ.

An angle θ satisfying the condition (2–1) above is called a Brjuno number.

Even though the Brjuno numbers have full measure on the circle, the complement is a dense uncountable set (fat in the sense of Baire). Yoccoz's proof shows that if θ is not a Brjuno number, then $\eta(\lambda)$ tends to 0 when $\lambda \in \mathbb{D}$ tends to $e^{2i\pi\theta}$. This explains the weird boundary behaviour of the function $\eta(\lambda)$ which appears on Figure 1.

3. PICTURES

Figure 2 shows the images of the circles $|\lambda| = .5$, $|\lambda| = .75$, $|\lambda| = .9$, $|\lambda| = .99$, $|\lambda| = .999$, and $|\lambda| = .9999$ under the map $\eta(\lambda)$.

We must justify how those pictures are drawn. Recall that

$$\eta(\lambda) \sim \lim_{n\to\infty} \frac{1}{\lambda^n} P_\lambda^{\circ n}\Big(-\frac{\lambda}{2}\Big).$$

On the other hand, we know that

$$\eta(\lambda) = \varphi_\lambda\Big(-\frac{\lambda}{2}\Big) = \frac{1}{\lambda^n}\varphi_\lambda\left(P_\lambda^{\circ n}\Big(-\frac{\lambda}{2}\Big)\right).$$

Hence, we just need to control how close φ_λ is to the identity near 0. This can be done using classical distortion estimates for univalent functions in the unit disk.

Lemma 3.1. *The map φ_λ is univalent on the disk $\mathbb{D}_R$ centered at 0 of radius*

$$R = \inf\{|\lambda|/2, 1 - |\lambda|\}.$$

Proof. It is not difficult to show that $P_\lambda(\mathbb{D}_R) \subset \mathbb{D}_R$ for all $R < 1 - |\lambda|$. This implies that $\mathbb{D}_R \subset U_\lambda$, and so φ_λ is analytic on $\mathbb{D}_R$. The polynomial P_λ is univalent on the half-plane $\mathrm{Re}(z/\lambda) > -\frac{1}{2}$ and in particular it is univalent on $\mathbb{D}_R$ for $R \le |\lambda|/2$. So, letting $R = \inf\{|\lambda|/2,\ 1-|\lambda|\}$, every iterate $P_\lambda^{\circ n}$ for $n \ge 0$ is univalent on $\mathbb{D}_R$. Finally, the linearizing map φ_λ is locally univalent near the origin and the functional equation $\varphi_\lambda \circ P_\lambda = \lambda\cdot\varphi_\lambda$ yields the desired univalence of φ_λ. □

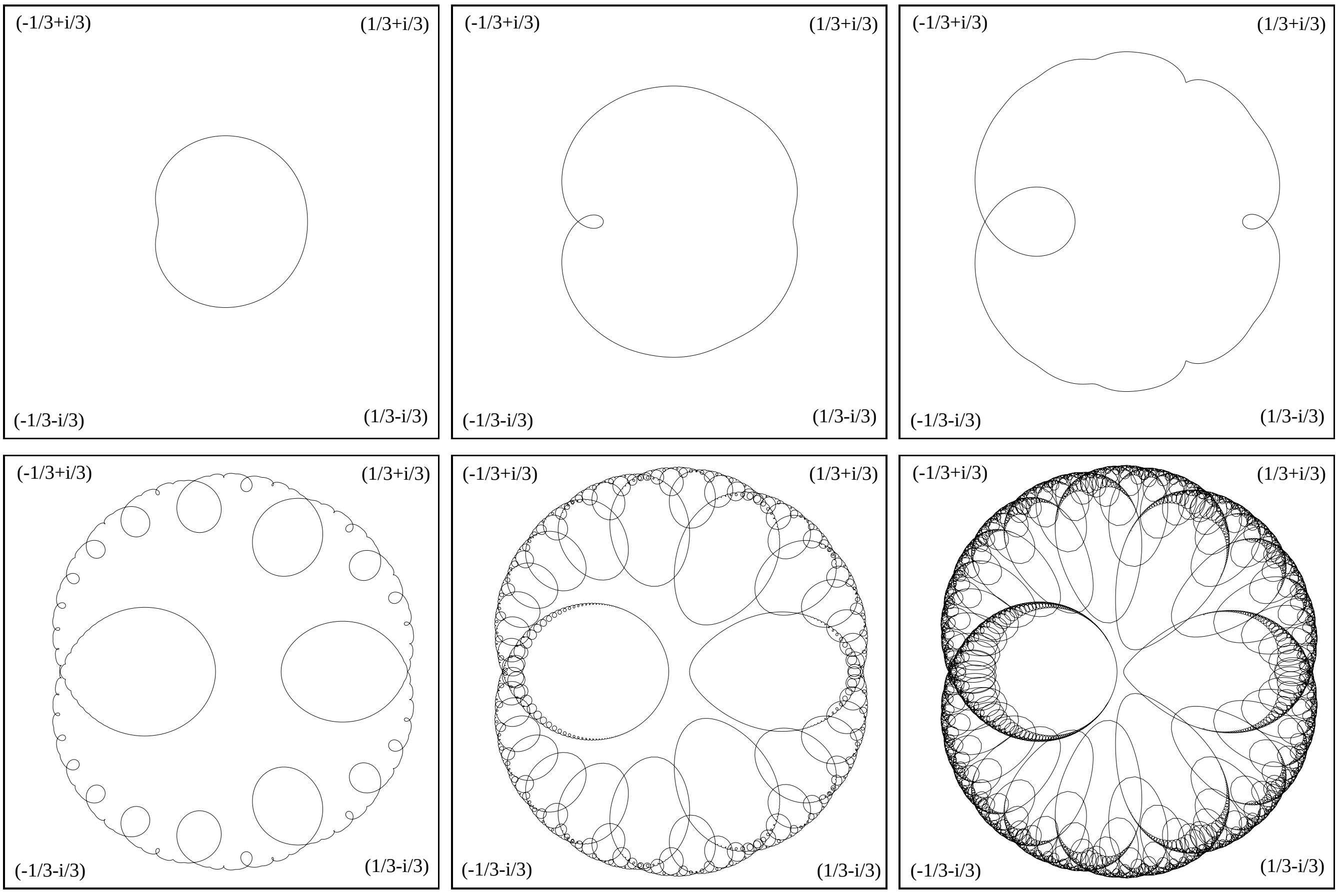

FIGURE 2. Images of the circles $|\lambda| = .5$, $|\lambda| = .75$, $|\lambda| = .9$, $|\lambda| = .99$, $|\lambda| = .999$, and $|\lambda| = .9999$ under $\eta(\lambda)$.

Lemma 3.2. *If $|z| < R/2$, where R is the lesser of $|\lambda|/2$ and $1 - |\lambda|$, then*

$$|\varphi_\lambda(z) - z| < \frac{6}{R}|z|^2.$$

Proof. First notice that the function

$$f(w) = \frac{1}{R}\varphi_\lambda(Rw),$$

is univalent in $\mathbb{D}$, $f(0) = 0$ and $f'(0) = 1$. The result is then a consequence of the distortion theorem applied to f. Indeed, it is possible to prove that when $|w| \le r < 1$,

$$|f(w) - w| \le \frac{2-r}{(1-r)^2}|w|^2.$$

For more details, see [Duren 1983]. In particular, if $|w| \le 1/2$, then $|f(w) - w| \le 6|w|^2$, and replacing w by z/R, we get

$$|\varphi_\lambda(z) - z| < \frac{6}{R}|z|^2. \qquad \square$$

The lemma gives rise to a method of computing arbitrarily good approximations of $\eta(\lambda)$. Since

$$P_\lambda^{\circ n}(-\lambda/2) \to 0$$

and

$$\eta(\lambda) = \frac{1}{\lambda^n}\varphi_\lambda\left(P_\lambda^{\circ n}\left(-\frac{\lambda}{2}\right)\right),$$

we can choose $n = n(\lambda)$ such that $|P_\lambda^{\circ n}(-\lambda/2)| < \inf\{R/2, R\varepsilon/6\}$. If we set

$$\tilde{\eta}(\lambda) = \frac{P_\lambda^{\circ n}(-\lambda/2)}{\lambda^n}$$

then

$$|\eta(\lambda) - \tilde{\eta}(\lambda)| \le \frac{1}{|\lambda|^n}\frac{6}{R}|P_\lambda^{\circ n}(-\lambda/2)|^2 \le \varepsilon|\tilde{\eta}(\lambda)|.$$

Hence $\tilde{\eta}$ approximates η with a relative error that is bounded by ε. Using this we have drawn the approximations of $\eta(\{|\lambda| = \rho\})$, for increasing ρ shown in Figure 2, with a precision $\varepsilon = 10^{-3}$.

4. OBSERVATIONS AND QUESTIONS

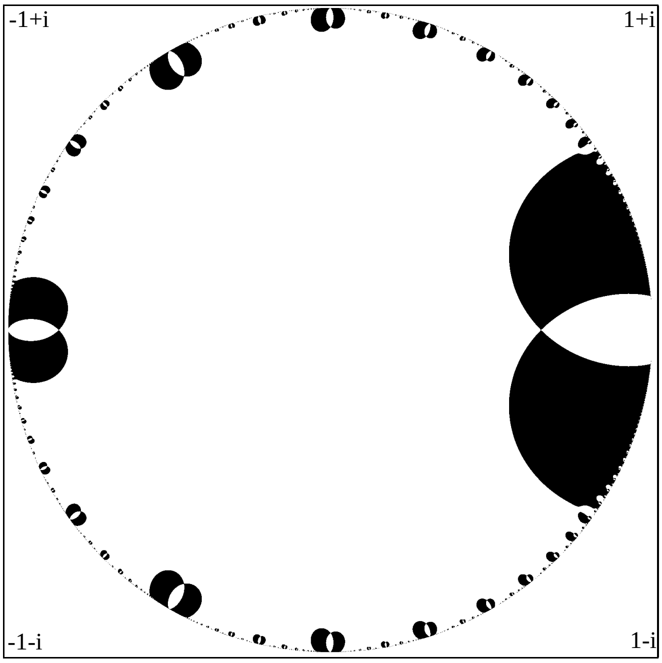

FIGURE 3. The critical points of η are the corner points.

We have proved that the function η has a nonvanishing radial limit almost everywhere. Besides, if θ is not a Brjuno number, we have seen that

$$\lim_{\lambda\to e^{2i\pi\theta}} \eta(\lambda) = 0.$$

In fact, Yoccoz [1995] obtained a much stronger result. He proved that the radial limit

$$\lim_{r\to 1} \eta\left(re^{2i\pi\theta}\right)$$

exists everywhere and is equal to the conformal radius of the Siegel disk of $P_{e^{2i\pi\theta}}$. In particular, the radial limit is positive when θ is a Brjuno number; it is equal to 0 when $\theta = p/q$ is rational.

Moreover, in Figure 2, we observe that as ρ increases, "bubbles" are formed and seem to reach towards the origin. It is very natural to try to see what those bubbles correspond to. We have seen that if $\theta \in \mathbb{Q}$ is rational, then

$$\lim_{\lambda\to e^{2i\pi\theta}} \eta(\lambda) = 0.$$

One problem is to understand what relation there is between the bubbles and rational angles θ.

A bubble is formed each time a curve of the family $\eta(\{|\lambda| = \rho\})$, $0 < \rho < 1$, passes through a critical point of η. Hence, understanding the relation between the bubbles and the rational angles amounts to understanding the relation of the critical points of η and those rational angles. To study the critical points of η, we must study η'. The following limit is uniform on every compact subset of $\mathbb{D}$:

$$\eta'(\lambda) = \lim \frac{d\left(P_\lambda^{\circ n}(-\lambda/2)/\lambda^n\right)}{d\lambda}.$$

So by choosing a large enough n for each λ, we can estimate η'. In Figure 3 we color pixels black if the corresponding λ satisfies $\mathrm{Re}(\eta'(\lambda)^2) > 0$ and otherwise we color the pixels white. A point λ is a critical point of η of degree n if and only if $2n$ regions of black and white meet in a sufficiently small neighborhood of λ. That is, the critical points are the corner points in the figure; all of them appear to be simple critical points.

The position of the critical points seems to reflect the structure of the Farey tree (see Figure 4). The Farey tree is defined by induction:

- The rational numbers of depth 0 are $0/1$ and $1/1$.
- The only rational number of depth 1 is $1/2$, and there is an edge between $1/2$ and each number of depth 0.
- Now if p/q is a rational number of depth n, we let p_1/q_1 and p_2/q_2 be the two rational numbers of depth less than n closest to p/q, such that $p_1/q_1 < p/q < p_2/q_2$. Then,
$$\frac{p_1+p}{q_1+q} \quad \text{and} \quad \frac{p_2+p}{q_2+q}$$
are rational numbers of depth $n+1$ and there is an edge between p/q and each of them.

The Farey tree contains all rational numbers in $[0, 1]$, and it is illustrated, up to depth 14, in Figure 4. A rational number p/q of depth n is drawn in the disk with argument $2\pi p/q$ and radius $r^{1/n}$. We chose $r = .65$ to show the similarity between the position of the critical points and the structure of the Farey tree.

Figure 4 suggests the following conjecture: for each rational number p/q, there is a gradient curve of $|\eta|$ joining $e^{2\pi ip/q}$ to a critical point of η; these segments define a bijective correspondance between $\mathbb{Q}/\mathbb{Z}$ and the critical points of η.

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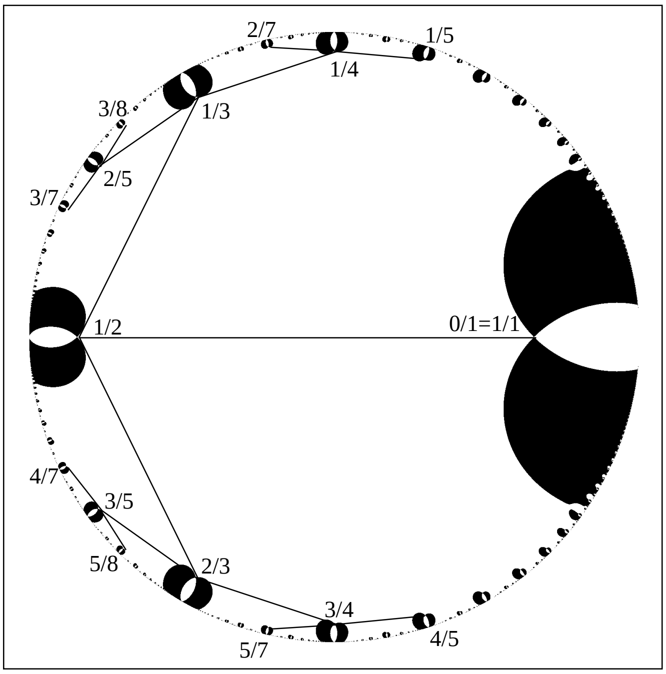

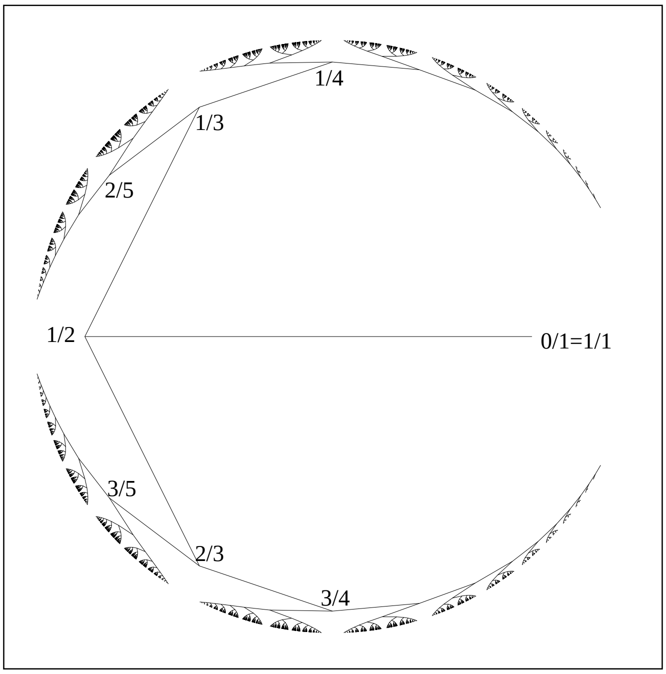

FIGURE 4. The critical points of η and the Farey tree up to depth 14.

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