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The Brjuno function continuously estimates the size of quadratic Siegel disks

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Abstract

If α is an irrational number, Yoccoz defined the Brjuno function Φ by

$$\Phi(\alpha) = \sum_{n\geq 0} \alpha_0\alpha_1\cdots\alpha_{n-1}\log\frac{1}{\alpha_n},$$

where α_0 is the fractional part of α and α_{n+1} is the fractional part of $1/\alpha_n$. The numbers α such that $\Phi(\alpha) < +\infty$ are called the Brjuno numbers.

The quadratic polynomial $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$ has an indifferent fixed point at the origin. If P_α is linearizable, we let $r(\alpha)$ be the conformal radius of the Siegel disk and we set $r(\alpha) = 0$ otherwise.

Yoccoz [Y] proved that $\Phi(\alpha) = +\infty$ if and only if $r(\alpha) = 0$ and that the restriction of $\alpha \mapsto \Phi(\alpha) + \log r(\alpha)$ to the set of Brjuno numbers is bounded from below by a universal constant. In [BC2], we proved that it is also bounded from above by a universal constant. In fact, Marmi, Moussa and Yoccoz [MMY] conjecture that this function extends to $\mathbb{R}$ as a Hölder function of exponent 1/2. In this article, we prove that there is a continuous extension to $\mathbb{R}$.

Contents

1. Introduction

2. Statement of results
   2.1. The value of Υ at rational numbers
   2.2. The value of Υ at Cremer numbers
   2.3. Strategy of the proof

3. Parabolic explosion
   3.1. Outline
   3.2. Definitions
   3.3. A preliminary lemma: Getting some room for holomorphic motions
   3.4. The loss of conformal radius when one removes the exploding cycle
   3.5. A short remark: Denominators of convergents and Fibonacci numbers
   3.6. The key inequality for the upper bound
   3.7. Application to the proof of Theorem 2: Υ at Cremer numbers

4. Proof of inequality (4) (the upper bound)
  4.1. Irrational numbers
  4.2. Rational numbers: Outline
  4.3. Rational numbers
  4.4. Proof of Lemma 3: Removing external rays for α close to p/q

5. Yoccoz's renormalization techniques
  5.1. Outline
  5.2. Renormalization principle
  5.3. Proof of Propostion 3: The uniformization L is close to a linear map
  5.4. Controlling the height of renormalization

6. Proof of inequality (5) (the lower bound) in most cases
  6.1. Renormalizing a map close to a translation
  6.2. Brjuno numbers
  6.3. Rational numbers
  6.4. Cremer numbers whose Pérez-Marco sum converges

7. Proof of inequality (5) when the Pérez-Marco sum diverges
  7.1. Parabolic explosion
  7.2. Renormalization

Appendix A. Extracts from [BC2]

Acknowledgements

References

1. Introduction.

For any irrational number $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, we denote by $(p_n/q_n)_{n\geq 0}$ the approximants to α given by its continued fraction expansion (by convention, $p_0 = \lfloor\alpha\rfloor$ is the integer part of α and $q_0 = 1$).

Remark. Every time we use the notation p/q for a rational number, we mean that $q > 0$ and p and q are coprime.

We denote by $\lfloor\alpha\rfloor \in \mathbb{Z}$ the integer part of α, i.e., the largest integer $n \leq \alpha$, by $\{\alpha\} = \alpha - \lfloor\alpha\rfloor$ the fractional part of α, and we define $(\alpha_n)_{n\geq 0}$ recursively by setting $\alpha_0 = \{\alpha\}$ and $\alpha_{n+1} = \{1/\alpha_n\}$. We then define $\beta_{-1} = 1$ and $\beta_n = \alpha_0\alpha_1\cdots\alpha_n$.

Definition 1 (Yoccoz's Brjuno function). If α is an irrational number, we define

$$\Phi(\alpha) = \sum_{n=0}^{+\infty} \beta_{n-1} \log \frac{1}{\alpha_n}.$$

If α is a rational number we define $\Phi(\alpha) = +\infty$. Irrational numbers for which $\Phi(\alpha) < \infty$ are called Brjuno numbers. Other irrational numbers are called Cremer numbers.

Remark. In terms of α_n, the definition reads

$$\Phi(\alpha) = \log\frac{1}{\alpha_0} + \alpha_0 \log\frac{1}{\alpha_1} + \alpha_0\alpha_1 \log\frac{1}{\alpha_2} + \cdots$$

Remark. The set $\mathcal{B}$ of Brjuno numbers has full measure in $\mathbb{R}$. It contains the set of all Diophantine numbers, i.e., numbers for which $\log q_{n+1} = \mathcal{O}(\log q_n)$.

We study the quadratic polynomials

$$P_\alpha : z \mapsto e^{2i\pi\alpha} z + z^2$$

for $\alpha \in \mathbb{R}$. It is known that such P_α is linearizable — and so, has a Siegel disk — if and only if α is a Brjuno number.

Definition 2. If $U \subsetneq \mathbb{C}$ is a simply connected domain containing 0, we denote by $\mathrm{rad}(U)$ the conformal radius of U at 0, i.e., $\mathrm{rad}(U) = |\phi'(0)|$ where $\phi : (\mathbb{D}, 0) \to (U, 0)$ is any conformal representation.

Definition 3. For any Brjuno number $\alpha \in \mathcal{B}$, we denote by $r(\alpha)$ the conformal radius at 0 of the Siegel disk of the quadratic polynomial P_α. If $\alpha \in \mathbb{R} \setminus \mathcal{B}$, we define $r(\alpha) = 0$.

Remark. The functions $\alpha \mapsto \Phi(\alpha)$ and $\alpha \mapsto \log r(\alpha)$, defined on $\mathcal{B}$, are highly discontinuous: for instance they respectively tend to $+\infty$ and $-\infty$ at every rational number.

It is known that there exists a constant C_0 such that for any Brjuno number $\alpha \in \mathcal{B}$ and any univalent map $f : \mathbb{D} \to \mathbb{C}$ which fixes 0 with derivative $e^{2i\pi\alpha}$, f has a Siegel disk Δ_f which contains $B(0, r)$ with $\Phi(\alpha) + \log r \geq -C_0$. In particular, for all $\alpha \in \mathcal{B}$, we have

$$\Phi(\alpha) + \log r(\alpha) \geq -C_0 - \log 2. \tag{1}$$

Indeed, P_α is injective on $B(0, 1/2)$.

Remark. The existence of Δ_f is due to Brjuno [Brj]. The lower bound (1) is due to Yoccoz [Y].

In [BC2], we proved that there exists a universal constant C_1 such that for all $\alpha \in \mathcal{B}$, we have

$$\Phi(\alpha) + \log r(\alpha) \leq C_1. \tag{2}$$

Inequalities (1) and (2) imply that $\Phi(\alpha)+\log r(\alpha)$ is uniformly bounded on $\mathcal{B}$:

$$(\exists C\in\mathbb{R}),\ (\forall\alpha\in\mathcal{B}),\quad |\Phi(\alpha)+\log r(\alpha)|\le C. \tag{3}$$

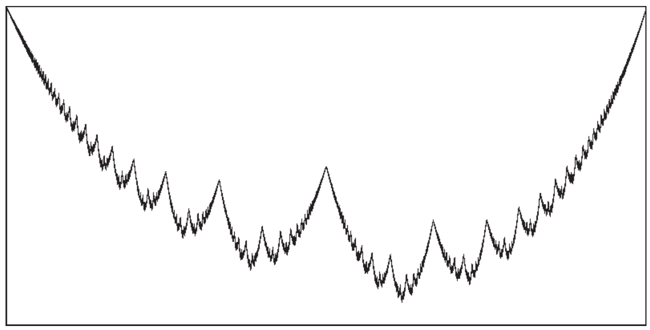

Figure 1: The graph of the function $\alpha\mapsto\Phi(\alpha)+\log r(\alpha)$ with $\alpha\in[0,1]$. The range is $[0,\log(2\pi)]$.

In this article we prove the following result which was conjectured by Marmi [Ma].

Theorem 1 (Main Theorem). *The function $\alpha\mapsto\Phi(\alpha)+\log r(\alpha)$ extends to $\mathbb{R}$ as a continuous function.*

In fact, Marmi, Moussa and Yoccoz made the following stronger conjecture ([MMY] and [Ca]).

Conjecture 1. *The function $\alpha\mapsto\Phi(\alpha)+\log r(\alpha)$ —which is well-defined on $\mathcal{B}$— is Hölder of exponent $1/2$.*

Remark. Since $\mathcal{B}$ is dense in $\mathbb{R}$, being $1/2$-Hölder on $\mathcal{B}$ and having a $1/2$-Hölder extension to $\mathbb{R}$ are equivalent, and the extension is unique.

Remark. In [Y], Yoccoz uses a modified version of continued fractions. He defines a sequence $\tilde\alpha_n$ defined by $\tilde\alpha_0=d(\alpha,\mathbb{Z})$ and $\tilde\alpha_{n+1}=d(1/\tilde\alpha_n,\mathbb{Z})$. The corresponding function $\widetilde\Phi$ defined by

$$\widetilde\Phi(\alpha)=\sum_{n\ge0}\tilde\alpha_0\cdots\tilde\alpha_{n-1}\log\frac{1}{\tilde\alpha_n}$$

has the additional property that $\widetilde\Phi(1-\alpha)=\widetilde\Phi(\alpha)$. Figure 2 shows the graph of the function $\alpha\mapsto\widetilde\Phi(\alpha)+\log r(\alpha)$. Theorem 4.6 in [MMY] asserts that the restriction of $\Phi-\widetilde\Phi$ to $\mathcal{B}$ extends to $\mathbb{R}$ as a $1/2$-Hölder continuous periodic function with period one. It has two consequences: first, the Marmi-Moussa-Yoccoz conjecture is equivalent with Φ replaced by $\widetilde\Phi$. Second, with Theorem 1

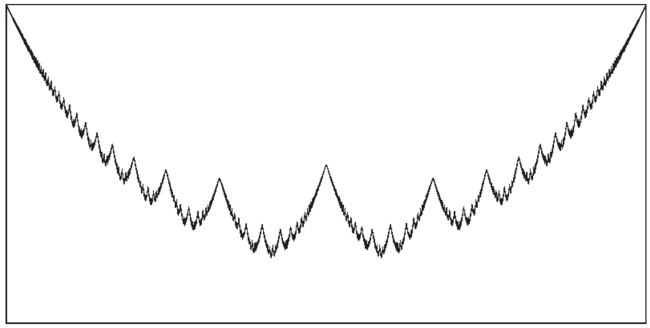

Figure 2: The graph of the function $\alpha \mapsto \widetilde{\Phi}(\alpha) + \log r(\alpha)$ with $\alpha \in [0,1]$. The range is $[0, \log(2\pi)]$.

it implies that the function $\alpha \mapsto \widetilde{\Phi}(\alpha) + \log r(\alpha)$ extends to $\mathbb{R}$ as a continuous function.

2. Statement of results

The function $\Phi(\alpha) + \log r(\alpha)$ is defined on the set of Brjuno numbers $\mathcal{B}$. In this section, we will define an extension $\Upsilon : \mathbb{R} \to \mathbb{R}$ and in the rest of the article, we will show that for all $\alpha \in \mathbb{R}$,

$$\lim_{\alpha' \to \alpha,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') = \Upsilon(\alpha).$$

It is an easy exercise to prove that Υ is then continuous.

Remark. For $\alpha \in \mathbb{Q}$, we will give an explicit formula for $\Upsilon(\alpha)$.

Definition 4. For $\alpha \in \mathcal{B}$, we set

$$\Upsilon(\alpha) = \Phi(\alpha) + \log r(\alpha).$$

2.1. *The value of* Υ *at rational numbers.* A rational number $\alpha = p/q \in \mathbb{Q}$ has two finite continued fraction expansions, corresponding to two sequences of approximants p_n/q_n, two sequences α_n, and two sequences β_n. One of the sequences α_n is provided by the usual algorithm: $\alpha_0 = \{\alpha\}$ and $\alpha_{n+1} = \{1/\alpha_n\}$, which eventually gives $\alpha_m = 0$ for some $m \in \mathbb{N}$, after which the sequence is not defined any more. The other has the same α_k for $k < m$, its $\alpha_m = 1$, and has one more term, $\alpha_{m+1} = 0$.[1]

In both cases, the sequence β is defined by $\beta_{-1} = 1$ and $\beta_n = \alpha_0 \cdots \alpha_n$. Let $n_0 = m$ or $m+1$ be the last index of the sequence α_n of p/q that we chose.

[1]A number α' tending to p/q has its α'_k that tends to the α_k of p/q for all $k < m$. According to whether α' tends to p/q from the left or the right, α'_m tends to one of the two values defined above, that is 0 or 1, the correspondence depending on the parity of m. Moreover, if it is 1, then α'_{m+1} tends to 0. This motivates the two definitions we made.

We have $\alpha_{n_0} = 0$. We can form the finite sum

$$\Phi_{\text{trunc}}(p/q) = \sum_{n=0}^{n_0-1} \beta_{n-1} \log \frac{1}{\alpha_n}$$

(with the convention that a sum $\sum_{n=0}^{n=-1} \cdots$ is equal to 0). It turns out to be independent of the choice between the two values of n_0, as can easily be checked.

Examples.
$$\begin{aligned}
\Phi_{\text{trunc}}(0/1) &= 0 \\
\Phi_{\text{trunc}}(1/2) &= \log 2 \\
\Phi_{\text{trunc}}(1/3) &= \log 3 \\
\Phi_{\text{trunc}}(2/3) &= \log \tfrac{3}{2} + \tfrac{2}{3} \log 2
\end{aligned}$$

The following two definitions and their relations with the conformal radii of Siegel disks appear in [Ch].

Definition 5. Assume $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ having a multiple fixed point at the origin whose Taylor expansion is

$$f(z) = z + Az^{k+1} + \mathcal{O}(z^{k+2}), \quad \text{with} \quad A \in \mathbb{C}^*.$$

The asymptotic size of f at 0 is defined by

$$L_a(f, 0) = \left| \frac{1}{kA} \right|^{1/k}.$$

The map $P_{p/q}$ fixes 0 with derivative $e^{2i\pi p/q}$. Therefore, its q-th iterate is tangent to the identity, and we make the following definition.

Definition 6. Assume $p/q \in \mathbb{Q}$ is a rational number. Then, we define

$$L_a(p/q) = L_a(P_{p/q}^{\circ q}, 0).$$

For $P_{p/q}$, it turns out that $k = q$ (see [DH, Ch. IX]).

Definition 7. For all rational number p/q, we define

$$\Upsilon\left(\frac{p}{q}\right) = \Phi_{\text{trunc}}\left(\frac{p}{q}\right) + \log L_a\left(\frac{p}{q}\right) + \frac{\log 2\pi}{q}.$$

Examples (approximate values rounded to the nearest decimal).

$$
\begin{array}{lllllll}
L_a(0/1) & = & 1 & \Upsilon(0/1) & = & \log 2\pi & = 1.8379\ldots \\
L_a(1/2) & = & \frac{1}{2} & \Upsilon(1/2) & = & \frac{\log 2\pi}{2} & = 0.9189\ldots \\
L_a(1/3) & = & \frac{1}{3^{\frac12}7^{\frac16}} & \Upsilon(1/3) & = & \frac{\log 3}{2} - \frac{\log 7}{6} + \frac{\log 2\pi}{3} & = 0.8376\ldots \\
L_a(2/3) & = & \frac{1}{3^{\frac12}7^{\frac16}} & \Upsilon(2/3) & = & \frac{\log 3}{2} - \frac{\log 7}{6} + \frac{\log \pi}{3} & = 0.6066\ldots.
\end{array}
$$

2.2. *The value of* Υ *at Cremer numbers.*

Definition 8. For all irrational number α and all integer $n \geq 0$, we define

$$\Phi_n(\alpha) = \sum_{k=0}^{n} \beta_{k-1} \log \frac{1}{\alpha_k}.$$

We recall that a domain $U \subset \mathbb{C}$ is hyperbolic if and only if its universal cover is isomorphic to $\mathbb{D}$ as a Riemann surface. We also recall that it is equivalent to $\mathbb{C} \setminus U$ containing at least two points.

Definition 9. If $U \subset \mathbb{C}$ is a hyperbolic connected domain containing 0, we denote by $\mathrm{rad}(U)$ the conformal radius of U at 0, i.e., $\mathrm{rad}(U) = |\pi'(0)|$ where $\pi : (\mathbb{D}, 0) \to (U, 0)$ is any universal covering.

Remark. This definition of conformal radius coincides with the one given in the introduction in the case of simply connected domains.

Definition 10. For all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and all integer $n \geq 0$, we define

$$X_n(\alpha) = \{z \in \mathbb{C}^* \mid z \text{ is a periodic point of } P_\alpha \text{ of period } \leq q_n\}$$

where p_n/q_n are the approximants to α,

$$r_n(\alpha) = \mathrm{rad}(\mathbb{C} \setminus X_n(\alpha)) \quad \text{and} \quad d_n(\alpha) = d(0, X_n(\alpha)).$$

Remark. If $n \geq 2$, then $q_n \geq 2$, $X_n(\alpha)$ contains at least two points and $r_n(\alpha) \in]0, +\infty[$. Moreover, for $n \geq 2$, the function $\alpha \mapsto \log r_n(\alpha)$ is well-defined and continuous in a neighborhood of every point $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

For all irrational number α, the sequence $(r_n(\alpha))_{n \geq 0}$ is decreasing and converges to $r(\alpha)$ as $n \to \infty$. Indeed, if 0 is not linearizable, it is accumulated by periodic points of P_α.[2] If 0 is linearizable, the Siegel disk Δ_α is contained in $\mathbb{C} \setminus X_n(\alpha)$ for all $n \geq 0$ and the boundary of Δ_α is accumulated by periodic

[2] In fact, Yoccoz proved that 0 is accumulated by whole cycles.

points of P_α.[3] Since P_α is tangent the rotation of angle α and α is irrational, if 0 is not linearizable, then

$$r_n(\alpha) \underset{n\to+\infty}{\sim} d_n(\alpha).$$

If α is a Brjuno number, then

$$\lim_{n\to\infty} \Phi_n(\alpha) + \log r_n(\alpha) = \Upsilon(\alpha).$$

In Section 3, we will prove the following theorem.

THEOREM 2. *For all Cremer numbers α, the sequence*

$$\Phi_n(\alpha) + \log r_n(\alpha)$$

has a finite limit when $n \longrightarrow +\infty$.

Definition 11. For all Cremer numbers α, we define

$$\Upsilon(\alpha) = \lim_{n\to+\infty} \Phi_n(\alpha) + \log r_n(\alpha)$$

Remark. This definition is equivalent to

$$\Upsilon(\alpha) = \lim_{n\to+\infty} \Phi_n(\alpha) + \log d_n(\alpha).$$

2.3. *Strategy of the proof.* Our goal is to prove that for all $\alpha \in \mathbb{R}$, the value of $\Upsilon(\alpha)$ defined previously (see Definitions 4, 7 and 11) is the limit of $\Phi(\alpha') + \log r(\alpha')$ as $\alpha' \in \mathcal{B}$ tends to α. The strategy consists in bounding $\Phi(\alpha') + \log r(\alpha')$ from above and from below as $\alpha' \in \mathcal{B}$ tends to α.

The upper bound follows from techniques of parabolic explosion developed in [Ch] and [BC2]. We present them in Section 3, and in Section 4 we show that for all $\alpha \in \mathbb{R}$,

$$\limsup_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \leq \Upsilon(\alpha). \tag{4}$$

The lower bound essentially follows from techniques of renormalization introduced by Yoccoz in [Y]. He uses estimates which are valid for all maps which are univalent in $\mathbb{D}$ and fix 0 with derivative of modulus 1. In our case, we will need to improve those estimates for maps which are close to rotations and maps which have at most one fixed point in $\mathbb{D}^*$ (see §5). In Sections 6 and 7 we show that for all $\alpha \in \mathbb{R}$,

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Upsilon(\alpha). \tag{5}$$

Let us mention that inequality (4) without inequality (5) (respectively inequality (5) without inequality (4)) is not sufficient to conclude that Υ is upper semi-continuous (respectively lower semi-continuous) since we only consider approximating α by sequences of Brjuno numbers.

[3] It is not known whether $\partial\Delta_\alpha$ is always accumulated by whole cycles.

3. Parabolic explosion

In this section, we first present the techniques of parabolic explosion. We then apply those techniques in order to prove Theorem 2.

3.1. *Outline.* Here, we informally describe what will be done in Section 3. Let α be irrational. Recall that $r_{n-1}(\alpha)$ is the conformal radius at 0 of the complement of $X_{n-1}(\alpha)$, the set of non zero periodic points of period $\leq q_{n-1}$. When we increment $n-1$ to n, $X_{n-1}(\alpha)$ contains more periodic points, hence $r_{n-1}(\alpha)$ decreases. Among the points removed from $\mathbb{C} \setminus X_{n-1}(\alpha)$, we single out a particular cycle $\mathcal{C}$. We will prove that this cycle induces a decrease in conformal radius, of at least $\beta_{n-1} \log \frac{1}{\alpha_n}$, up to a tame error term.

What is this cycle $\mathcal{C}$? The approximant p_n/q_n is close to α. Therefore P_α is a perturbation of P_{p_n/q_n}. The latter has a parabolic fixed point at 0. The perturbations of P_{p_n/q_n} have a cycle $\mathcal{C}$ of period q_n close to 0.

Why a decrease of $\beta_{n-1} \log \frac{1}{\alpha_n}$? The points in the cycle turn out to depend analytically on the q_n-th root of the perturbation. It follows from a version of Schwarz's lemma that the cycle cannot go significantly farther than $|\alpha - p_n/q_n|^{1/q_n}$ times the conformal radius of the region where the explosion takes place. We will see that the cycle cannot collide with the points of $X_{n-1}(\alpha)$. In terms of logarithms of conformal radii, this implies that there must be a decrease of $\frac{-1}{q_n} \log |\alpha - p_n/q_n|$. The theory of continued fractions approximates this value by $\beta_{n-1} \log \frac{1}{\alpha_n}$.

Unfortunately there are several technical difficulties. They will induce error terms of order $\frac{1}{q_n} \log q_n$. Among them:

- One needs p_n/q_n to be a good enough approximant to α. When it is not, the claimed decrease may not be true, but it is then small enough to be swallowed by the error term.
- The set $X_{n-1}(\alpha)$ depends on α and thus, during the explosion, the cycle avoids a set which moves with α. We have to show that this motion is small (by proving that there is a holomorphic motion defined on a domain in the parameter space much bigger than the domain on which the explosion is defined). And we have to prove that this small motion induces a small error term.

Other technical difficulties are addressed in this section.

3.2. *Definitions.* Assume $p/q \in \mathbb{Q}$ is a rational number. The origin is a parabolic fixed point for the quadratic polynomial $P_{p/q}$. It is known (see [DH, Ch. IX]) that there exists a complex number $A \in \mathbb{C}^*$ such that

$$P_{p/q}^{\circ q}(z) = z + Az^{q+1} + \mathcal{O}(z^{q+2}).$$

Thus, $P_{p/q}^{\circ q}$ has a fixed point of multiplicity $q+1$ at the origin. By Rouché's theorem, when α is close to p/q, the polynomial $P_\alpha^{\circ q}$ has $q+1$ fixed points close to 0. One coincides with 0. The others form a cycle of period q for P_α. More precisely, we have the following proposition (see [Ch] or [BC2, Prop. 1] for a proof).

Proposition 1. *Let p/q be a rational number, and $\zeta = e^{2i\pi p/q}$. There exists an analytic function $\chi : B(0, 1/q^{3/q}) \to \mathbb{C}$ such that $\chi(0) = 0$ and for any $\delta \in B(0, 1/q^{3/q}) \setminus \{0\}$, $\chi(\delta) \neq 0$ and the set*

$$\Big\langle \chi(\delta), \chi(\zeta\delta), \chi(\zeta^2\delta), \dots, \chi(\zeta^{q-1}\delta) \Big\rangle$$

forms a cycle of period q of $P_{p/q+\delta^q}$. We will note $\chi = \chi_{p/q}$, since it depends on p/q.

In other words, the points of the cycles depend analytically, not on the perturbation $\alpha - p/q$ but on its q-th root δ. Moreover, these q points are given by a single analytic function χ, applied to the q values of the q-th root. The proposition also gives a lower bound on the size of the disk on which this holds.

Remark. Observe that $\delta \in B(0, 1/q^{3/q})$ if and only if $\alpha = p/q + \delta^q \in B(p/q, 1/q^3)$.

In the following definition, note that α is a *complex* number.

Definition 12. For all $p/q \in \mathbb{Q}$ and all $\alpha \in B(p/q, 1/q^3)$, we define

$$\mathcal{C}_{p/q}(\alpha) = \chi_{p/q} \left\{ \sqrt[q]{\alpha - p/q} \right\},$$

where $\sqrt[q]{z}$ denotes the *set* of complex q-th roots of z.

The set $\mathcal{C}_{p/q}(\alpha)$ is a cycle of period q for P_α, except when $\alpha = p/q$, in which case it is reduced to $\{0\}$. In particular, if α is irrational, $p/q = p_n/q_n$ is an approximant to α and $|\alpha - p_n/q_n| < 1/q_n^3$, then $\mathcal{C}_{p_n/q_n}(\alpha) \subset X_n(\alpha)$. Note that when $|\alpha_0 - p/q| < 1/2q^3$, the cycle $\mathcal{C}_{p/q}(\alpha)$ is defined for all $\alpha \in B(\alpha_0, 1/2q^3)$, and not reduced to $\{0\}$.

3.3. *A preliminary lemma: Getting some room for holomorphic motions.* Recall the following classical fact: a periodic point of P_α can be locally followed holomorphically in terms of α as long as its multiplier is different from 1 (as can be proved using the Implicit Function Theorem). The following lemma gives us room to do that.

Lemma 1. *Assume $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let p_n/q_n be an approximant to α_0 with $q_n \geq 2$. Assume $\alpha \in \mathbb{C}$, $\alpha \neq p_n/q_n$, $q \leq q_n$ and $P_\alpha^{\circ q}$ has a multiple fixed point. Then,*

$$|\alpha_0 - \alpha| \geq \frac{1}{2q_n^3}.$$

Proof. Either $\alpha = p/q$ for some integer p. Within the disk $B(\alpha_0, 1/2q_n^3)$, the only possibility is $p/q = p_n/q_n$. Or α belongs to a Yoccoz disk of radius $\log 2/(2\pi q') < 1/8q'$ tangent to the real axis at p'/q' for some rational number p'/q' with $q' < q \le q_n$ (see [Ch, Part I, §6.2], or [BC1, Lemma 1], or [BC2, Lemma 1]). By a well-known property of approximants, we have

$$|q'\alpha_0 - p'| \ge |q_{n-1}\alpha_0 - p_{n-1}| \ge \frac{1}{q_n + q_{n-1}} \ge \frac{1}{2q_n}.$$

Moreover, by Pythagoras' theorem,

$$\begin{aligned}|\alpha - \alpha_0| &\ge \frac{1}{q'}\left(\sqrt{(q'\alpha_0 - p')^2 + (1/8)^2} - 1/8\right)\\ &\ge \frac{1}{q_n}\left(\sqrt{1/\left(2q_n\right)^2 + 1/8^2} - 1/8\right)\\ &= \frac{1/(2q_n)^2}{q_n\left(\sqrt{1/\left(2q_n\right)^2 + 1/8^2} + 1/8\right)}\\ &\ge \frac{1}{2q_n^3}\cdot\frac{1}{2\left(\sqrt{1/4^2 + 1/8^2} + 1/8\right)} \quad \ge \frac{1}{2q_n^3}.\end{aligned}$$

□

COROLLARY 1. *Assume $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$ and let p_n/q_n be an approximant to α_0 with $q_n \ge 2$. The set*

$$X(\alpha) = \{z \in \mathbb{C}^* \mid z \text{ is a periodic point of } P_\alpha \text{ of period } \le q_n\}$$

moves holomorphically with respect to $\alpha \in B(\alpha_0, 1/2q_{n+1}^3)$.

Proof. If the set $X(\alpha)$ fails to move holomorphically at a point $\alpha \in \mathbb{C}$, then, for some integer $q \le q_n$, $P_\alpha^{\circ q}$ has a multiple fixed point. Either $\alpha = p_n/q_n$, and (according to a property of approximants) $|\alpha - \alpha_0| \ge 1/(2q_n q_{n+1}) > 1/2q_{n+1}^3$. Or $\alpha \ne p_n/q_n$, and by the previous lemma $|\alpha - \alpha_0| \ge 1/2q_n^3 > 1/2q_{n+1}^3$. □

3.4. *The loss of conformal radius when one removes the exploding cycle.* In the next lemma we investigate the loss of conformal radius of a domain when we remove the cycle $\mathcal{C}_{p/q}(\alpha_0)$ from it. It mainly concerns the case when p/q is a good enough approximant of α_0 but for convenience with respect to the next chapters, we made a statement valid for all p/q.

LEMMA 2. *There exists $C \in \mathbb{R}$ such that for all $\alpha_0 \in \mathbb{R}\setminus\mathbb{Q}$ and all $p/q \in \mathbb{Q}$ with $q \ge 2$, the following holds. Assume $V(\alpha) \ni 0$ is an open set that moves holomorphically with respect to $\alpha \in B(\alpha_0, 1/2q^3)$.*

- *If $|\alpha_0 - p/q| \ge 1/2q^3$, set $V'(\alpha_0) = V(\alpha_0)$.*
- *If $|\alpha_0 - p/q| < 1/2q^3$, assume $\mathcal{C}_{p/q}(\alpha) \subset V(\alpha)$ for all $\alpha \in B(\alpha_0, 1/2q^3)$ and set $V'(\alpha_0) = V(\alpha_0) \setminus \mathcal{C}_{p/q}(\alpha_0)$.*

Then,

$$\log \frac{\mathrm{rad}(V'(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} \leq \frac{\log|\alpha_0 - p/q|}{q} + C\frac{\log q}{q}.$$

Remark. The first case will turn out to be trivial. For the second case, before giving the proof, let us informally explain what happens. The explosion of the multiple fixed point coming from $\alpha = p/q$ is analytic with respect to the q-th roots δ of $\alpha - p/q$, and is defined on a disk of radius almost 1 (up to a tame error term). When $\alpha = \alpha_0$, the q parameters δ have modulus $|\alpha_0 - p/q|^{1/q}$. Now the explosion takes place in $V(\alpha)$. When q is big, there are many values of δ, tightly packed on the circle of radius $|\alpha_0 - p/q|^{1/q}$. If $V(\alpha)$ did not depend on α, if it were simply connected, if the parameters δ covered all the circle, and if the explosion were defined for all $\delta \in \mathbb{D}$, Schwarz's lemma would imply that removing the cycle from $V(\alpha_0)$ decreases its conformal conformal radius of at least a factor $|\alpha_0 - p/q|^{1/q}$, which in terms of logarithms means $\log(\mathrm{rad}(V'(\alpha_0)) \leq \log(\mathrm{rad}(V'(\alpha_0)) + \frac{1}{q}\log|\alpha_0 - p/q|$ (the last term is negative). None of these 4 assumptions are true, but in each case, we can prove that the error we make is of order $\frac{1}{q}\log q$ (this is done in [BC2], and we copied here in the appendix the statements of the relevant theorems).

Proof of Lemma 2. Let us first assume that $|\alpha_0 - p/q| \geq 1/2q^4 \geq 1/q^5$ (this comprises the case $V'(\alpha_0) = V(\alpha_0)$). Then,

$$\log|\alpha_0 - p/q| + 5\log q \geq 0$$

and the lemma follows trivially with $C = 5$ since

$$\log \frac{\mathrm{rad}(V'(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} \leq 0.$$

So, let us assume that $|\alpha_0 - p/q| < 1/2q^4$. Then,

$$B \stackrel{\mathrm{def}}{=} B(p/q, 1/2q^4) \subset B(\alpha_0, 1/q^4) \subset B(\alpha_0, 1/2q^3).$$

We set

$$U = \{\delta \in \mathbb{C} \mid p/q + \delta^q \in B\} \quad \text{and} \quad S = \{\delta \in U \mid p/q + \delta^q = \alpha_0\}.$$

Note that $\chi_{p/q}(S) = \mathcal{C}_{p/q}(\alpha_0)$.

The radius of the disk U is $1/(2q^4)^{1/q}$ and the set S consists in q points equidistributed on a circle of radius $|\alpha_0 - p/q|^{1/q}$. So, according to Proposition 11 (see the appendix), we have

$$\log \frac{\mathrm{rad}(U \setminus S)}{\mathrm{rad}(U)} < \log \frac{|\alpha_0 - p/q|^{1/q}}{1/(2q^4)^{1/q}} + \frac{C}{q}$$

for some universal constant C.

According to Proposition 12 (see the appendix), there exists for $\alpha \in B(\alpha_0, 1/2q^3)$ an analytic family of universal coverings $\pi_\alpha : \widetilde{V}(\alpha) \to V(\alpha)$, where $\widetilde{V}(\alpha)$ are open subsets of $B(0,4)$, and $\widetilde{V}(\alpha_0) = \mathbb{D}$. The set $V(\alpha)$ moves holomorphically with $\alpha \in B(\alpha_0, 1/2q^3)$ and when $\delta \in U$, $\alpha(\delta) = p/q + \delta^q$ belongs to $B \subset B(\alpha_0, 1/q^4)$. For $\alpha \in B$, the sets $\widetilde{V}(\alpha)$ are all contained in some ball $B(0,\rho)$ with

$$\log \rho = \frac{2\log 4}{1 + \dfrac{1/2q^3}{1/q^4}} = \frac{\log 16}{1 + q/2}.$$

The map $\chi_{p/q}$ "lifts" to a map $\phi : U \to B(0,\rho)$ such that $\phi(\delta) \in \widetilde{V}(\alpha(\delta))$. It follows from the definitions that,

$$\begin{aligned} \log \frac{\mathrm{rad}(V'(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} &= \log \frac{\mathrm{rad}(V(\alpha_0) \setminus \mathcal{C}_{p/q}(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} \\ &= \log \frac{\mathrm{rad}\left(\widetilde{V}(\alpha_0) \setminus \pi_{\alpha_0}^{-1}\left(\chi_{p/q}(S)\right)\right)}{\mathrm{rad}(\widetilde{V}(\alpha_0))}. \end{aligned}$$

Now $\widetilde{V}(\alpha_0) = \mathbb{D}$ and $\phi(S) \subset \pi_{\alpha_0}^{-1}(\chi_{p/q}(S))$, thus

$$\log \frac{\mathrm{rad}(V'(\alpha_0))}{\mathrm{rad}(V(\alpha_0))} \le \log \mathrm{rad}(\mathbb{D} \setminus \phi(S)) \le \log \mathrm{rad}(B(0,\rho) \setminus \phi(S)).$$

The range of the function ϕ needs not to be a subset of $\mathbb{D}$, but we know from Proposition 10 (see the appendix), that

$$\begin{aligned} \log \mathrm{rad}(B(0,\rho) \setminus \phi(S)) &\le \log \frac{\mathrm{rad}(U \setminus S)}{\mathrm{rad}(U)} + \log \rho \\ &\le \frac{\log|\alpha_0 - p/q|}{q} + 4\frac{\log q}{q} + \frac{\log 2}{q} + \frac{C}{q} + \frac{\log 16}{1 + q/2} \\ &\le \frac{\log|\alpha_0 - p/q|}{q} + C' \frac{\log q}{q} \end{aligned}$$

for some universal constant C'. □

3.5. *A short remark*: *Denominators of convergents and Fibonacci numbers.* Let F_n be the smallest possible value of q_n over all irrationals α, where p_n/q_n is the n-th approximant to α. Then F_n is the Fibonacci sequence defined by

$$F_{-1} = 0,\ F_0 = 1,\ F_{n+1} = F_n + F_{n-1}.$$

The first terms are

$$F_{-1} = 0,\ F_0 = 1,\ F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5,\ \ldots .$$

The function $x \mapsto \log(x)/x$ is decreasing on $[e, +\infty[$, thus

$$\text{for all } n \ge 3,\ \frac{\log q_n}{q_n} \le \frac{\log F_n}{F_n}.$$

For $n = 1$ and 2, the biggest possible value of $\log(q_n)/q_n$ is $\log(3)/3$.

3.6. *The key inequality for the upper bound.* The next proposition tells us that for all irrational α, the sequence $\Phi_n(\alpha)+\log r_n(\alpha)$ is essentially decreasing, in the sense that it cannot increase too fast.

PROPOSITION 2. *There exists a constant $C \in \mathbb{R}$ such that for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and all $n \geq 1$ such that $q_n \geq 2$ (with p_n/q_n the approximants to α), we have*

$$\Big(\Phi_{n+1}(\alpha) + \log r_{n+1}(\alpha)\Big) - \Big(\Phi_n(\alpha) + \log r_n(\alpha)\Big) \leq C\frac{\log q_{n+1}}{q_{n+1}}.$$

Proof. Let us fix $\alpha_0 \in \mathbb{R} \setminus \mathbb{Q}$ and choose n so that $q_n \geq 2$. We want to apply Lemma 2 with $p/q = p_{n+1}/q_{n+1}$ and

$$V(\alpha) = \mathbb{C} \setminus \{z \in \mathbb{C}^* \mid z \text{ is a periodic point of } P_\alpha \text{ of period } \leq q_n\}.$$

By definition, $0 \in V(\alpha)$ and by Corollary 1, the set $V(\alpha)$ moves holomorphically with respect to $\alpha \in B(\alpha_0, 1/2q_{n+1}^3)$. Also, $V(\alpha)$ contains the periodic cycles of P_α of period q_{n+1} and so, if $|\alpha_0 - p/q| < 1/2q^3$, then $\mathcal{C}_{p/q}(\alpha) \subset V(\alpha)$ for all $\alpha \in B(\alpha_0, 1/2q^3)$. As in Lemma 2, if $|\alpha_0 - p/q| \geq 1/2q^3$, we set $V'(\alpha_0) = V(\alpha_0)$ and otherwise, we set $V'(\alpha_0) = V(\alpha_0) \setminus \mathcal{C}_{p/q}(\alpha_0)$. Then,

$$r_n(\alpha_0) = \mathrm{rad}(V(\alpha_0)) \quad \text{and} \quad r_{n+1}(\alpha_0) \leq \mathrm{rad}(V'(\alpha_0)).$$

So, Lemma 2 implies that

$$\begin{aligned}\log r_{n+1}(\alpha_0) - \log r_n(\alpha_0) &\leq \frac{\log|\alpha_0 - p_{n+1}/q_{n+1}|}{q_{n+1}} + C\frac{\log q_{n+1}}{q_{n+1}}\\ &= \frac{\log \beta_{n+1}}{q_{n+1}} + (C-1)\frac{\log q_{n+1}}{q_{n+1}}.\end{aligned}$$

Since $\beta_{n+1} \leq \alpha_{n+1}$ and $1/q_{n+1} \geq \beta_n$:

$$\begin{aligned}\log r_{n+1}(\alpha_0) - \log r_n(\alpha_0) &\leq -\beta_n \log\frac{1}{\alpha_{n+1}} + (C-1)\frac{\log q_{n+1}}{q_{n+1}}\\ &= -\Phi_{n+1}(\alpha_0) + \Phi_n(\alpha_0) + (C-1)\frac{\log q_{n+1}}{q_{n+1}}\end{aligned}$$

for some universal constant C. □

The bound we gave depends on α, but for each n, the supremum over all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is exponentially decreasing with respect to n (according to §3.5).

3.7. *Application to the proof of Theorem 2: Υ at Cremer numbers.* Yoccoz's work [Y] implies that there exists a constant C_0' such that for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and all $n \geq 0$,

$$\Phi_n(\alpha) + \log r_n(\alpha) \geq C_0',$$

(compare with inequality (1)). Now assume α is a Cremer number, and define $u_n = \Phi_n(\alpha) + \log r_n(\alpha)$. Then u_n is bounded from below.

The sequence u_n is not decreasing, but it is “essentially decreasing”, in the sense that Proposition 2 gives us

$$u_{n+1} - u_n \leq C\frac{\log q_{n+1}}{q_{n+1}}$$

and $(\log q_{n+1})/q_{n+1}$ decreases exponentially fast. Therefore the sequence

$$v_n = u_n - \sum_{k=0}^{n} C\frac{\log q_k}{q_k}$$

is decreasing and bounded from below, thus convergent. It follows that u_n converges.

4. Proof of inequality (4) (the upper bound)

4.1. *Irrational numbers.* We will now show that for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$,

$$\limsup_{\alpha' \to \alpha,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') \leq \Upsilon(\alpha).$$

Let us fix $\varepsilon > 0$. We must show that for $\alpha' \in \mathcal{B}$ sufficiently close to α, $\Phi(\alpha') + \log r(\alpha') \leq \Upsilon(\alpha) + \varepsilon$. Remember that as $n \to \infty$, $\Phi_n(\alpha) + \log r_n(\alpha) \to \Upsilon(\alpha)$. So, let us choose n_0 large enough so that

$$\Phi_{n_0}(\alpha) + \log r_{n_0}(\alpha) \leq \Upsilon(\alpha) + \varepsilon/3.$$

Increasing n_0 if necessary, we may also assume that $n_0 \geq 2$ and

$$\sum_{n \geq n_0} C\frac{\log F_{n+1}}{F_{n+1}} \leq \varepsilon/3,$$

where C is the constant in Proposition 2. In a neighborhood of α, the functions Φ_{n_0} and $\log r_{n_0}$ are continuous. So, if α' is sufficiently close to α,

$$\Phi_{n_0}(\alpha') + \log r_{n_0}(\alpha') \leq \Phi_{n_0}(\alpha) + \log r_{n_0}(\alpha) + \varepsilon/3$$

and summing the inequality of Proposition 2 from $n = n_0$ to $n = +\infty$ yields

$$\Phi(\alpha') + \log r(\alpha') \leq \Upsilon(\alpha) + \varepsilon. \qquad \square$$

4.2. *Rational numbers*: *Outline.* We will show that

$$\limsup_{\alpha' \to p/q,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') \leq \Upsilon(p/q).$$

Suppose $\alpha' \to p/q$ from one side (either left or right). Then for α' close enough to p/q the continued fraction expansion of α' starts with $[\mathrm{a}_0, \ldots, \mathrm{a}_{n_0}, \ldots]$. Here $[\mathrm{a}_0, \ldots, \mathrm{a}_{n_0}]$ is one of the two finite continued fraction expansions of the rational number p/q (see §2.1). The other expansion is produced by α' converging to p/q from the other side. The cycle $\mathcal{C}_{p_n/q_n}(\alpha')$ tends to 0, and according to Section 3, its distance to 0 is roughly

$d = L_a(p/q)|2\pi q^2(\alpha' - p/q)|^{1/q}$. This cycle is approximately on a regular polygon centered at 0. Therefore, the logarithm of $r_{n_0}(\alpha')$, the conformal radius of $\mathbb{C} \setminus X_{n_0}(\alpha')$, is essentially bounded from above by $\log d = \log L_a(p/q) + \frac{1}{q}\log(q^2\varepsilon) + \frac{\log 2\pi}{q}$, where $\varepsilon = |\alpha' - p/q|$. Now, in the sum defining the Brjuno function, the partial sum of the terms from rank 0 up to $n_0 - 1$ (that we denoted $\Phi_{n_0-1}(\alpha')$) tends to $\Phi_{\text{trunc}}(p/q)$. The term of rank n_0 has expansion $-\frac{1}{q}\log(q^2\varepsilon) + o(1)$ as $\varepsilon \longrightarrow 0$. Thus,

$$\Phi_{n_0}(\alpha') + \log r_{n_0}(\alpha') \leq \Phi_{\text{trunc}}(p/q) + \frac{\log 2\pi}{q} + \log L_a(p/q) + o(1).$$

Then, we add the inequalities of Proposition 2, for n from n_0 to $+\infty$ and obtain

$$(\Phi(\alpha') + \log r(\alpha')) - (\Phi_{n_0}(\alpha') + \log r_{n_0}(\alpha')) \leq C' \frac{\log q_{n_0+1}}{q_{n_0+1}}$$

where C' is a universal constant. Now, remark that $q_{n_0+1} \longrightarrow +\infty$ when $\alpha' \longrightarrow p/q$. This yields the announced upper bound $\Upsilon(p/q)$.

In the simplified explanation above, we cheated when we claimed that the logarithm of the conformal radius of $\mathbb{C} \setminus X_{n_0}(\alpha')$ is less than $\log d + o(1)$. In reality, for each q, it is less than $\log d + C_q + o(1)$, with $C_q > 0$. So, we add to $X_{n_0}(\alpha')$ the external rays landing at the cycle $\mathcal{C}_{p_n/q_n}(\alpha')$. We then prove that the logarithm of the conformal radius of the complement of the rays is less than $\log d + o(1)$.

Remark. We do not use the theory of parabolic enrichment (geometric limits, Lavaurs maps, Ecalle maps, horn maps and Fatou coordinates).

4.3. *Rational numbers.* In the whole section, we will use the notation

$$\varepsilon = \alpha' - p/q.$$

For $\alpha' \in \mathbb{C}$ and $\theta \in \mathbb{R}$, we will also denote by $R_{\alpha'}(\theta)$ the external ray of argument θ of $P_{\alpha'}$. The external rays for the Mandelbrot set will be denoted by $R_M(\theta)$.

The polynomial P_α is conjugate to the quadratic polynomial $z \mapsto z^2 + c$ with $c = e^{2i\pi\alpha}/2 - e^{4i\pi\alpha}/4$. When $\mathrm{Im}(\alpha) \longrightarrow -\infty$ and $\mathrm{Re}(\alpha) \longrightarrow \widetilde{\theta}$, then $|c| \longrightarrow +\infty$ and $\arg c \longrightarrow 2\widetilde{\theta} + \frac{1}{2}$ mod 1. Given $\widetilde{\theta} \in \mathbb{R}$, we will denote by $\mathcal{R}(\widetilde{\theta})$ the connected component of the preimage of $R_M(2\widetilde{\theta} + 1/2)$ by $\alpha \mapsto c$, whose real part tends to $\widetilde{\theta}$.

When α is real, the parameter c is on the boundary of the main cardioid of the Mandelbrot set. If $\alpha = p/q \notin \mathbb{Z}$, then $c \neq 1/4$ and there are two external rays of M landing at c. We denote by $\theta^- < \theta^+$ their arguments in $]0, 1[$. The arguments θ^+ and θ^- are periodic of period q under multiplication by 2 modulo 1. They belong to the same orbit Θ. In the dynamical plane of $P_{p/q}$, the rays $R_{p/q}(\theta)$, $\theta \in \Theta$, form a periodic cycle of rays which land at 0.

If $p/q \in \mathbb{Z}$, the dynamical ray of argument 0 is fixed and lands at 0. We set $\theta^- = \theta^+ = 0$ and $\Theta = \{0\}$.

Let us recall the following rule: the ray $R_{\alpha'}(\theta)$ moves holomorphically with α' as long as c does not belong to the closure of the union of the $R_M(2^k\theta)$ for $k \in \mathbb{N}^*$.

Definition 13. When $\alpha' \in \mathbb{R}$ is close to p/q, the rays $R_{\alpha'}(\theta)$, $\theta \in \Theta$, form a cycle of rays which land on the cycle $\mathcal{C}_{p/q}(\alpha')$. We denote by $Y(\alpha')$ the union of $\mathcal{C}_{p/q}(\alpha')$ and this cycle of rays.

Figure 3 shows the rays of argument 1/7, 2/7 and 4/7 and the boundary of the Siegel disk for the polynomial $P_{(1/3)+\varepsilon}$ for $\varepsilon = \sqrt{2}/1000$ and $\varepsilon = \sqrt{2}/10000$.

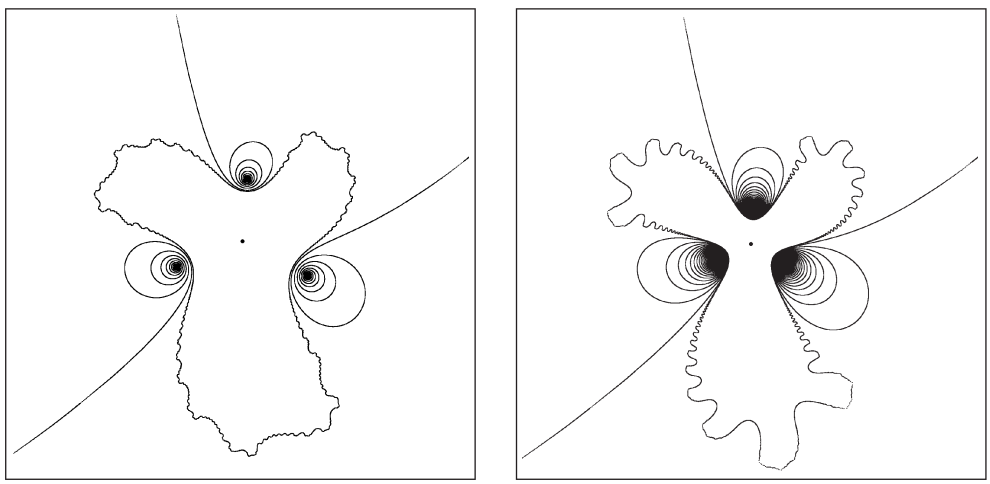

Figure 3: The rays of argument 1/7, 2/7 and 4/7 and the boundary of the Siegel disk for the polynomial $P_{(1/3)+\varepsilon}$: left for $\varepsilon = \sqrt{2}/1000$ and right for $\varepsilon = \sqrt{2}/10000$.

If ε is irrational and is close enough to 0, then p/q is an approximant p'_{n_0}/q'_{n_0} to α', and its index n_0 is the same number as in Section 2.1 and depends on the sign of ε. As $\alpha' \to p/q$, $\log \operatorname{rad}(\mathbb{C} \setminus Y(\alpha')) \to -\infty$ and $\beta'_{n_0-1} \log(1/\alpha'_{n_0}) \to -\infty$. We postpone the proof of the following lemma to Section 4.4.

Lemma 3. *We have*

$$\limsup_{\alpha' \to p/q,\ \alpha' \in \mathbb{R}\setminus\mathbb{Q}} \log \operatorname{rad}(\mathbb{C} \setminus Y(\alpha')) + \beta'_{n_0-1} \log \frac{1}{\alpha'_{n_0}} \le \log L_a\left(\frac{p}{q}\right) + \frac{\log 2\pi}{q}.$$

When α' is close to p/q but not necessarily real, the dynamical rays of argument $\theta \in \Theta$ may bifurcate. In a neighborhood of p/q, this precisely occurs when $c' = e^{2i\pi\alpha'}/2 - e^{4i\pi\alpha'}/4$ belongs to $\mathcal{R}_M(\theta^+)$ or $\mathcal{R}_M(\theta^-)$.

LEMMA 4. *There exists a constant* $c \in]0,1]$, *which depends on* p/q, *such that the following holds. Assume* $\alpha' \in \mathbb{R}\setminus\mathbb{Q}$ *and* p/q *is an approximant to* α'. *Let* n_0 *be its index. Let* p'_{n_0+1}/q'_{n_0+1} *be* α'*'s next approximant. Then, for all* $\alpha'' \in B(\alpha', c/(q'_{n_0+1})^2)$, *the dynamical rays of argument* $\theta \in \Theta$ *do not bifurcate. In particular,* $Y(\alpha'')$ *moves holomorphically with respect to* $\alpha'' \in B(\alpha', c/(q'_{n_0+1})^2)$.

Proof. There is exactly one pair $\widetilde{\theta}^- < \widetilde{\theta}^+$, with with $2\widetilde{\theta}^+ + 1/2 = \theta^+$ (mod 1) and $2\widetilde{\theta}^- + 1/2 = \theta^-$ (mod 1) such that $\mathcal{R}(\widetilde{\theta}^+)$ and $\mathcal{R}(\widetilde{\theta}^-)$ land on p/q. The rays $\mathcal{R}(\widetilde{\theta}^+)$ and $\mathcal{R}(\widetilde{\theta}^-)$ are separated from the upper half plane (that corresponds to the cardioid by $\alpha \mapsto c$), by a smooth curve having a contact of order 2 with the real line, at p/q. Also, the other external rays $R_M(\theta')$ for $\theta' \in \Theta\setminus\{\theta^+,\theta^-\}$ do not land on the cardioid. Therefore, there exists a constant $c' > 0$ such that the dynamical rays of argument $\theta \in \Theta$ do not bifurcate when $\alpha'' \in B(\alpha', c'|\alpha' - p/q|^2)$. The result follows since

$$\left|\alpha' - \frac{p}{q}\right|^2 \geq \left(\frac{1}{2q'_{n_0}q'_{n_0+1}}\right)^2 = \frac{1}{4q^2(q'_{n_0+1})^2}. \qquad \square$$

Let us choose c as in Lemma 4 and $\alpha' \in \mathcal{B}$ sufficiently close to p/q so that $q'_{n_0+1} > 1/2c$ (we denote by p'_n/q'_n the approximants to α'). Then, the set $Y(\alpha'')$ moves holomorphically with respect to $\alpha'' \in B(\alpha', 1/2(q'_{n_0+1})^3)$. Let us also assume that $q'_{n_0+1} \geq 2$

LEMMA 5. *Under the assumptions above, we have*

$$\Phi(\alpha') + \log r(\alpha') \leq \Phi_{n_0}(\alpha') + \log \operatorname{rad}(\mathbb{C}\setminus Y(\alpha')) + (C-1)\sum_{n\geq n_0+1}\frac{\log q'_n}{q'_n},$$

where C *is the constant provided by Lemma* 2.

Proof. For $\alpha'' \in B(\alpha', 1/2(q'_{n_0+1})^3)$, let us define $V_{n_0}(\alpha'') = \mathbb{C}\setminus Y(\alpha'')$ and by induction, for $n \geq n_0+1$ and $\alpha'' \in B(\alpha', 1/2(q'_{n+1})^3)$, let us define

- $V_n(\alpha'') = V_{n-1}(\alpha'') \setminus \mathcal{C}_{p'_n/q'_n}(\alpha'')$ if $|\alpha' - p'_n/q'_n| < 1/2(q'_n)^3$ and
- $V_n(\alpha'') = V_{n-1}(\alpha'')$ otherwise.

Then, the hypotheses of Lemma 2 are satisfied and (as in Proposition 2), we have

$$\begin{aligned}\log \operatorname{rad}(V_n(\alpha')) - \log \operatorname{rad}(V_{n-1}(\alpha')) &\leq \frac{\log|\alpha' - p'_n/q'_n|}{q'_n} + C\frac{\log q'_n}{q'_n}\\ &\leq -\Phi_n(\alpha') + \Phi_{n-1}(\alpha') + (C-1)\frac{\log q'_n}{q'_n},\end{aligned}$$

where C is the constant provided by Lemma 2. The Siegel disk $\Delta_{\alpha'}$ is contained in the intersection of the sets $V_n(\alpha')$, and so,

$$\log r(\alpha') - \log \operatorname{rad}(V_{n_0}(\alpha')) \leq -\Phi(\alpha') + \Phi_{n_0}(\alpha') + (C-1) \sum_{n \geq n_0+1} \frac{\log q_n'}{q_n'}.$$

□

As α' tends to p/q, each q'_{n_0+k} (for $k \geq 1$) tends to ∞, thus the n_0+k-th summand tends to 0. Since the sum is dominated by a summable sequence $(\log(F_n)/F_n)$, this yields

$$\sum_{n \geq n_0+1} \frac{\log q_n'}{q_n'} \to 0.$$

Moreover, $\Phi_{n_0-1}(\alpha')$ converges to $\Phi_{\text{trunc}}(p/q)$ and by Lemma 3,

$$\limsup_{\alpha' \to p/q,\ \alpha' \in \mathbb{R}\setminus\mathbb{Q}} \Phi_{n_0}(\alpha') + \log \operatorname{rad}(\mathbb{C} \setminus Y(\alpha')) \leq \Upsilon(p/q).$$

This completes the proof of inequality (4).

4.4. *Proof of Lemma* 3: *Removing external rays for α close to p/q.* We recall that $\alpha' = p/q + \varepsilon$ is real, and that n_0 depends on the sign of ε.

Lemma 6. *For $\varepsilon \in \mathbb{R}^*$ small enough, let z_ε be a periodic point of $P_{\alpha'}$ in the cycle $\mathcal{C}_{p/q}(\alpha')$. Then,*

$$\log |z_\varepsilon| + \beta'_{n_0-1} \log \frac{1}{\alpha'_{n_0}} = \log L_a\left(\frac{p}{q}\right) + \frac{\log 2\pi}{q} + \mathcal{O}(\varepsilon^{1/q}).$$

Proof. By definition of the asymptotic size, we have

$$L_a(p/q) = \left|\frac{1}{qA}\right|^{1/q} \quad \text{with} \quad P_{p/q}^{\circ q}(z) = z + Az^{q+1} + \mathcal{O}(z^{q+2}).$$

Moreover, $P_{p/q+\varepsilon}^{\circ q}(0) = 0$ and $(P_{p/q+\varepsilon}^{\circ q})'(0) = e^{2i\pi q\varepsilon}$. So

$$P_{p/q+\varepsilon}^{\circ q}(z) = e^{2i\pi q\varepsilon} z + Az^{q+1} + \mathcal{O}(\varepsilon z^2).$$

We know that $z_\varepsilon \longrightarrow 0$ and that $P_{p/q+\varepsilon}^{\circ q}(z_\varepsilon) = z_\varepsilon$. Therefore, we have

$$z_\varepsilon^q = \frac{1 - e^{2i\pi q\varepsilon}}{A} \left(1 + \mathcal{O}(z_\varepsilon)\right) = \frac{-2i\pi q\varepsilon}{A} \left(1 + \mathcal{O}(z_\varepsilon) + \mathcal{O}(\varepsilon)\right).$$

Thus, $z_\varepsilon = \mathcal{O}(\varepsilon^{1/q})$ and

$$\log |z_\varepsilon| = \frac{1}{q} \log \left|\frac{2\pi q\varepsilon}{A}\right| + \mathcal{O}(\varepsilon^{1/q}).$$

Observe that

$$\frac{1}{q} \log \left|\frac{2\pi q\varepsilon}{A}\right| = \log L_a\left(\frac{p}{q}\right) + \frac{\log 2\pi}{q} + \frac{1}{q} \log q^2 |\varepsilon|.$$

Now, if α' is sufficiently close to p/q, then the n_0-th approximant p'_{n_0}/q'_{n_0} to α' is p/q, and therefore when ε's sign is fixed, n_0 is fixed, and the numbers q'_{n_0} and q'_{n_0-1} are constants. We have

$$\begin{aligned}\beta'_{n_0-1} &= |q'_{n_0-1}\alpha' - p'_{n_0-1}| = \left|q'_{n_0-1}\Big(\frac{p_{n_0}}{q_{n_0}} + \varepsilon\Big) - p'_{n_0-1}\right| \\ &= \left|\frac{1}{q'_{n_0}} \pm q'_{n_0-1}\varepsilon\right| = \frac{1}{q'_{n_0}} + \mathcal{O}(\varepsilon), \quad \text{and} \\ \beta'_{n_0} &= q'_{n_0}|\varepsilon|, \quad \text{thus} \\ \alpha'_{n_0} &= \frac{\beta'_{n_0}}{\beta'_{n_0-1}} = (q'_{n_0})^2|\varepsilon|(1+\mathcal{O}(\varepsilon)).\end{aligned}$$

Thus, we have

$$\beta'_{n_0-1}\log|\alpha'_{n_0}| = \Big(\frac{1}{q} + \mathcal{O}(\varepsilon)\Big)\log\big(q^2|\varepsilon|(1+\mathcal{O}(\varepsilon))\big) = \frac{1}{q}\log q^2|\varepsilon| + \mathcal{O}(\varepsilon\log|\varepsilon|).$$

□

Let us now study the dynamical behaviour of $P_{p/q+\varepsilon}$ at the scale of z_ε. For this purpose, we rescale the dynamical plane. More precisely, we introduce the conjugate polynomial

$$Q_\varepsilon : w \mapsto \frac{1}{z_\varepsilon}P_{p/q+\varepsilon}(z_\varepsilon w).$$

This polynomial is conjugate to $P_{p/q+\varepsilon}$. It fixes 0 with derivative $e^{2i\pi(p/q+\varepsilon)}$ and has a cycle of period q containing 1.

As $\varepsilon \to 0$, Q_ε converges uniformly on every compact subset of $\mathbb{C}$ to the rotation $w \mapsto e^{2i\pi p/q}w$. Hence, $Q_\varepsilon^{\circ q}$ converges uniformly on every compact subset of $\mathbb{C}$ to the identity. However, the limit of the dynamics of Q_ε is richer than the dynamics of the identity. In some sense, it contains the real flow of the vector field $2i\pi q w(1-w^q)\frac{\partial}{\partial w}$.

Lemma 7. *We have*

$$Q_\varepsilon^{\circ q}(w) = w + 2i\pi q\varepsilon w(1-w^q) + \varepsilon R_\varepsilon(w),$$

with $R_\varepsilon \to 0$ uniformly on every compact subset of $\mathbb{C}$ as $\varepsilon \to 0$.

Proof. Since

$$P_{p/q+\varepsilon}^{\circ q}(z) = e^{2i\pi q\varepsilon}z + Az^{q+1} + \mathcal{O}(\varepsilon z^2),$$

we have

$$\begin{aligned}\frac{1}{z_\varepsilon}P_{p/q+\varepsilon}^{\circ q}(z_\varepsilon w) &= e^{2i\pi q\varepsilon}w + Az_\varepsilon^q w^{q+1} + \mathcal{O}(\varepsilon z_\varepsilon w^2) \\ &= w + 2i\pi q\varepsilon(w - w^{q+1}) + \mathcal{O}(\varepsilon^{1+1/q}w^2) + \mathcal{O}(\varepsilon^2 w).\end{aligned}$$

□

Figure 4 shows some trajectories of the real flow of the vector field $2i\pi qw(1-w^q)\frac{\partial}{\partial w}$ for $q=3$. The origin is a center and its basin Ω is colored light grey.

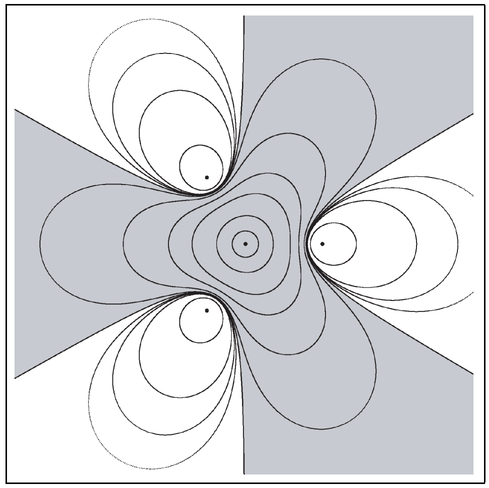

Figure 4: Some trajectories of the real flow of the vector field $2i\pi qw(1-w^q)\frac{\partial}{\partial w}$ for $q=3$.

Let us now define

$$Y_\varepsilon = \frac{1}{z_\varepsilon} Y\left(\frac{p}{q}+\varepsilon\right).$$

The set Y_ε contains 1 and we have

$$\log \operatorname{rad}(\mathbb{C}\setminus Y(p/q+\varepsilon)) = \log \operatorname{rad}(Y_\varepsilon) + \log |z_\varepsilon|.$$

Thus, we must show that

$$\limsup_{\varepsilon\to 0,\ \varepsilon\in\mathbb{R}} \log \operatorname{rad}(\mathbb{C}\setminus Y_\varepsilon) \le 0.$$

Set $\overline{Y_\varepsilon} = Y_\varepsilon \cup \{\infty\}$. This set is compact in $\mathbb{P}^1$. Without loss of generality, extracting a subsequence if necessary, we may assume that it converges for the Hausdorff topology on compact subsets of $\mathbb{P}^1$ to some limit $\overline{Y_0}$ as $\varepsilon \to 0$. We define $Y_0 = \overline{Y}_0 \setminus \{\infty\}$. Each $\overline{Y_\varepsilon}$ is connected and contains 1 and ∞. Passing to the limit, we see that $\overline{Y_0}$ is also connected and contains 1 and ∞. Moreover, Q_ε converges uniformly on compact subsets of $\mathbb{C}$ to the rotation $w\mapsto e^{2i\pi p/q}w$. Since $Q_\varepsilon(Y_\varepsilon)=Y_\varepsilon$, we see that Y_0 is invariant under this rotation. Note that $Q_\varepsilon^{\circ q}(Y_\varepsilon)\subset Y_\varepsilon$ and

$$Q_\varepsilon^{\circ q}(w) = w + 2i\pi q\varepsilon w(1-w^q) + \varepsilon R_\varepsilon(w)$$

with $R_\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $\varepsilon\to 0$. It follows that Y_0 is forward invariant under the real flow of the vector field $2i\pi qw(1-w^q)\frac{\partial}{\partial w}$.

Consider the map $\phi : w \mapsto \zeta = w^q/(w^q - 1)$. It is the composition of $w \mapsto w^q$, (which identifies the quotient of $\mathbb{P}^1$ under the rotation of angle $1/q$ with $\mathbb{P}^1$), with a Moebius transformation fixing 0, sending 1 to ∞, and ∞ to 1. It sends the above vector field to the circular vector field $(2\pi q^2)i\zeta\frac{\partial}{\partial\zeta}$. It follows that Y_0 contains the set $\phi^{-1}(\mathbb{C} \setminus \mathbb{D})$. Thus, we have

$$\limsup_{\varepsilon\to 0,\ \varepsilon\in\mathbb{R}} \log \operatorname{rad}(\mathbb{C} \setminus Y_\varepsilon) \leq \log \operatorname{rad}(\phi^{-1}(\mathbb{D})) = 0.$$

The proof of Lemma 3 is completed.

5. Yoccoz's renormalization techniques

In this section, we present the techniques of renormalization developed by Yoccoz [Y].

5.1. *Outline.* This outline is somewhat informal, rigorous treatment is made in the other subsections of Section 5. The proof of the lower bound being technical, we think it is useful to present some of the ideas in a lighter way.

5.1.1. *The renormalization.* Assume $\alpha_0 \in]0, 1[$ and let $f_0 : \mathbb{D} \to \mathbb{C}$ be a univalent holomorphic map fixing 0 with derivative $e^{2i\pi\alpha_0}$. We would like to make the following construction: take a sector $\mathcal{U}_0$ between the segment $[0, 1]$ and its image by f_0 (the one with angle α_0 at the vertex 0). The Riemann surface $\mathcal{V}_0$ obtained as the quotient of $\mathcal{U}_0$ with $[0, 1]$ identified with its image by f_0 is a punctured disk. The first-return map to $\mathcal{U}_0$ associated to f_0 induces a holomorphic map $g : \mathcal{V}_0' \to \mathcal{V}_0$ with $\mathcal{V}_0' \subset \mathcal{V}_0$. We can identify $\mathcal{V}_0$ with $B(0, S_0) \setminus \{0\}$ where S_0 is chosen so that $\mathbb{D}^* \subset \mathcal{V}_0'$. Then, g is univalent and extends at the origin by $g(0) = 0$ and $g'(0) = e^{-2i\pi\alpha_1}$ with $\alpha_1 = \{1/\alpha_0\}$. The renormalized map f_1 is defined as the restriction to $\mathbb{D}$ of $\overline{g(\overline{z})}$, which has derivative $e^{2i\pi\alpha_1}$ at the origin.

One problem that may happen is that the curve $f([0, 1])$ may cross its image, preventing the Riemann surface to be well defined. For the renormalization to be well defined, we need to assume that f is close enough to the rotation R_α. Or we can make the construction with a sector of smaller radius. Therefore, we introduce a radius $\rho_0 < 1$, and consider only the sector $\mathcal{U}_0$ between the segment $[0, \rho_0]$ and its image by f_0. In this theory, the control on ρ_0 is central. We will not try to associate a canonical value of ρ_0 to a given map f_0. In fact the choice will depend on the setting.

If the map $f_0 : B(0, \rho_0) \to \mathbb{C}$ were the rotation of angle α_0, we could choose $S_0 = 1$ and the canonical map from $\mathcal{U}_0$ to $\mathcal{V}_0$ would be $z \mapsto (z/\rho_0)^{1/\alpha_0}$. We will always choose ρ_0 such that S_0 can be taken close to 1 and that the canonical map from $\mathcal{V}_0$ to $\mathcal{U}_0$ is close to $z \mapsto (z/\rho_0)^{1/\alpha_0}$.

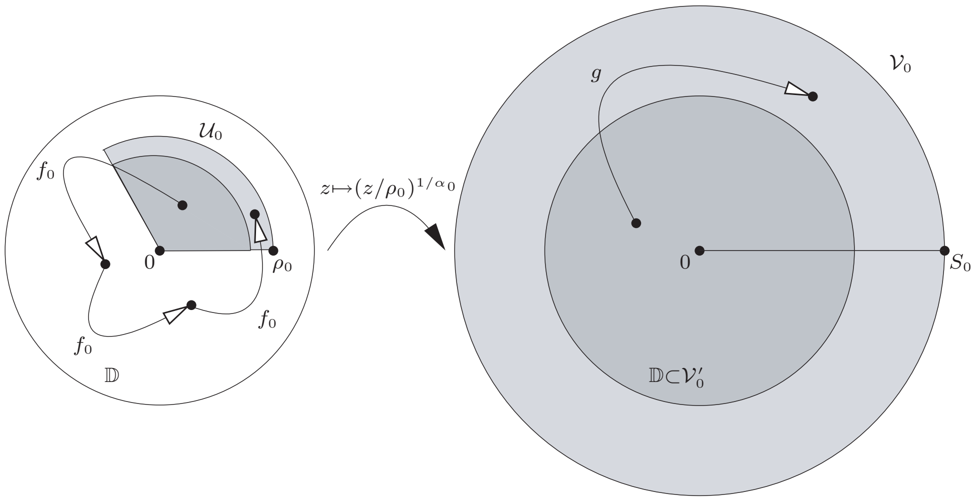

Figure 5: The construction of the renormalized map.

Given a fixed $\alpha_0 \in]0,1[$, if $f : \mathbb{D} \to \mathbb{C}$ is a map fixing 0 (its multiplier may be $\neq e^{2i\pi\alpha_0}$), if $f \longrightarrow R_{\alpha_0}$, then we can take $\rho_0 \longrightarrow 1$. Moreover, its renormalization tends to R_{α_1}.

5.1.2. *The size of Siegel disks.* We can repeat inductively the renormalization construction: given a univalent map $f_n : \mathbb{D} \to \mathbb{C}$ which fixes 0 with derivative $e^{2i\pi\alpha_n}$, we choose ρ_n and we let f_{n+1} be the renormalization of f_n.

The crux of the matter is that essentially, f_0 can be iterated infinitely many times on the disk $B(0, \sigma_0)$ with

$$\sigma_0 = \rho_0 \cdot \rho_1^{\alpha_0} \cdot \rho_2^{\alpha_0\alpha_1} \cdots$$

(it follows easily that $B(0, \sigma_0)$ is contained in the Siegel disk of f_0). Indeed, since f_0 is close to a rotation on the disk $B(0, \rho_0)$, if $|z_0| < \sigma_0 < \rho_0$, its forward orbit under iteration of f_0 intersects $\mathcal{U}_0$ at a point z_0'. The image of z_0' in $\mathcal{V}_0$ is a point z_1 of modulus close to

$$|z_0/\rho_0|^{1/\alpha_0} < \sigma_1 = \rho_1 \cdot \rho_2^{\alpha_1} \cdot \rho_3^{\alpha_1\alpha_2} \cdots .$$

Then, the forward orbit of z_1 under iteration of f_1 intersects $\mathcal{U}_1$ at a point z_1' and the image of z_1' in $\mathcal{V}_2$ is a point z_2 with modulus close to

$$|z_1/\rho_1|^{1/\alpha_1} < \sigma_2 = \rho_2 \cdot \rho_3^{\alpha_2} \cdot \rho_4^{\alpha_2\alpha_3} \cdots ,$$

and so on … Since f_n is a n-th renormalization of f_0, being able to iterate f_n at z_n means that we can iterate f_0 at z_0 many times, and since n is arbitrarily large, we can iterate f_0 at z_0 infinitely many times.

5.1.3. *Yoccoz's lower bound.* In order to bound the conformal radius of a Siegel disk from below, we must find a good enough lower bound for ρ_n. The set of univalent maps $f:\mathbb{D}\to\mathbb{C}$ such that $f(0)=0$ and $|f'(0)|=1$ is compact. It follows rather easily than one can always take $\rho_n = c\alpha_n$ for some universal constant c. This gives

$$\begin{aligned}\log\sigma_0 &= \log(c\alpha_0)+\alpha_0\log(c\alpha_1)+\alpha_0\alpha_1\log(c\alpha_2)+\dots\\ &= -\Phi(\alpha_0)+(1+\alpha_0+\alpha_0\alpha_1+\dots)\log c\\ &\geq -\Phi(\alpha_0)+4\log c.\end{aligned}$$

This is essentially how Yoccoz proves inequality (1): the Siegel disk of a univalent map $f_0:\mathbb{D}\to\mathbb{C}$ fixing 0 with derivative $e^{2i\pi\alpha_0}$ contains a disk $B(0,\sigma_0)$ with

$$\log\sigma_0 \geq -\Phi(\alpha_0)-C_0$$

for some universal constant C_0.

5.1.4. *Perturbing a Siegel disk.* Assume α is a Brjuno number. Let $\phi:\mathbb{D}\to\Delta(\alpha)$ be the linearization. Conjugating by ϕ^{-1}, the family $P_{\alpha'}$ becomes a family $g_{\alpha'}$ of maps tending to R_α, uniformly on every compact subset of $\mathbb{D}$, as $\alpha'\longrightarrow\alpha$. We will give a lower bound on the size the Siegel disks $\Delta(g_{\alpha'})$. Conjugating back by ϕ multiplies conformal radii by $r(\alpha)$, and the Siegel disk of $P_{\alpha'}$ must contain $\phi(\Delta(g_{\alpha'}))$.

Consider the sequence of renormalized maps $(f_n)_{n\geq 0}$ with $f_0=g_{\alpha'}$. As $\alpha'\to\alpha$, $f_0\to R_{\alpha_0}$ uniformly on every compact subset of $\mathbb{D}$. Thus, we can choose ρ_0 close to 1 and as $\alpha'\longrightarrow\alpha$, the renormalized map f_1 converges to the rotation R_{α_1} uniformly on every compact subset of $\mathbb{D}$.

Given $n>0$, if α' is sufficiently close to α, we can repeat this argument n times: we can take $\rho_0=\rho_1=\rho_{n-1}\underset{\alpha'\to\alpha}{=}1-o(1)$. Afterwards, we can take $\rho_n=c\alpha'_n$, $\rho_{n+1}=c\alpha'_{n+1},\dots$, for some universal constant c, as in Section 5.1.3.

The Siegel disk of $g_{\alpha'}$ contains $B(0,\sigma)$ with

$$\log\sigma = -(1+\beta'_0+\dots+\beta'_{n-2})o(1)+\sum_{k=n}^{+\infty}\beta'_{k-1}\log\alpha'_k+(\beta'_{n-1}+\beta'_n+\dots)\log c.$$

It follows that

$$\log r(\alpha')\geq\log r(\alpha)-\sum_{k=n}^{+\infty}\beta'_{k-1}\log\frac{1}{\alpha'_k}-\beta'_{n-1}C_0+o(1)$$

with a universal constant C_0 as in Section 5.1.3. Adding $\Phi(\alpha')$ on both sides yields

$$\Upsilon(\alpha')\geq\log r(\alpha)+\sum_{k=0}^{n-1}\beta'_{k-1}\log\frac{1}{\alpha'_k}-\beta'_{n-1}C_0+o(1).$$

For all $k < n$, the term $\beta'_k \log \frac{1}{\alpha'_k}$ tends to $\beta_k \log \frac{1}{\alpha_k}$ as $\alpha' \longrightarrow \alpha$. Thus,

$$\liminf_{\alpha' \in \mathcal{B} \to \alpha} \Upsilon(\alpha') \geq \log r(\alpha) + \sum_{k=0}^{n-1} \beta_{k-1} \log \frac{1}{\alpha_k} - \beta_{n-1} C_0.$$

Recall that α is a Brjuno number, thus passing to the limit $n \longrightarrow +\infty$ (whence $\beta_{n-1} \longrightarrow 0$):

$$\liminf_{\alpha' \in \mathcal{B} \to \alpha} \Upsilon(\alpha') \geq \log r(\alpha) + \sum_{k=0}^{+\infty} \beta_{k-1} \log \frac{1}{\alpha_k} = \Upsilon(\alpha).$$

5.1.5. *Perturbation of a parabolic point.* Assume $\alpha' \longrightarrow p/q$ on one side. This corresponds to one of the two continued fractions of p/q (see Section 2.1): $[\mathrm{a}_0, \dots, \mathrm{a}_{n_0}]$. Recall that we defined

$$\Upsilon(p/q) = \log L_a(p/q) + \Phi_{\mathrm{trunc}}(p/q) + \frac{\log 2\pi}{q}.$$

The cycle $\mathcal{C}_{p/q}(\alpha')$ (see Definition 12) tends to 0. This cycle is approximately on a regular polygon centered at 0, and of radius d, where (according to Section 3):

$$\log d = \log L_a(p/q) + \beta'_{n_0-1} \log \alpha'_{n_0} + \frac{\log 2\pi}{q} + o(1).$$

When we rescale by a factor d, we conjugate $P_{\alpha'}$ to a polynomial $Q_{\alpha'}$ such that $Q_{\alpha'}^{\circ q}$ tends to the identity along an explicit and fixed vector field (which depends only on q). This vector field has a center at 0. The maximal domain of linearization turns out to have conformal radius 1. Consider the change of variable which sends this domain to the unit disk.

In this new coordinate, the polynomials $P_{\alpha'}$ are conjugate to maps $g_{\alpha'}$ which converge to the rotation R_α, uniformly on every compact subset of $\mathbb{D}$. As in Section 5.1.4, we can construct the sequence of renormalizations $(f_n)_{n \geq 0}$ with $f_0 = g_{\alpha'}$, taking ρ_0, ρ_1, … ρ_{n_0-1} close to 1. This time, $\alpha'_{n_0} \to \alpha_{n_0} = 0$ and as $\alpha' \to \alpha$. The n_0-th renormalization f_{n_0} tends to the identity uniformly on every compact subset of $\mathbb{D}$. It tends to the identity along a vector field of rotation, which allows us to take ρ_{n_0} close to 1. Then, we take $\rho_n = c\alpha'_n$ for $n > n_0$.

As in Section 5.1.4 it follows that as $\alpha' \longrightarrow p/q$,

$$\log r(\alpha') \geq \log d - \sum_{k=n_0+1}^{+\infty} \beta'_{k-1} \log \frac{1}{\alpha'_k} - \beta'_{n_0} C_0 + o(1).$$

Adding $\Phi(\alpha')$ on both sides, using the expansion of $\log d$, and using $\beta'_{n_0} \to \beta_{n_0} = 0$ yields

$$\Upsilon(\alpha') \geq \log L_a(p/q) + \sum_{k=0}^{n_0-1} \beta'_{k-1} \log \frac{1}{\alpha'_k} + \frac{\log 2\pi}{q} + o(1) = \Upsilon(p/q) + o(1).$$

5.1.6. *Perturbing a Cremer point under the Pérez-Marco condition.* In this case, we will construct the sequence of renormalized maps $(f_n)_{n\geq 0}$ without changing coordinates, i.e., with $f_0 = P_{\alpha'}$.

We first consider the case $\alpha' \longrightarrow \alpha$ with

$$\sum \beta_{k-1} \log\log \frac{e}{\alpha_k} < +\infty.$$

Remember that $r_n(\alpha)$ stands for the conformal radius of $\mathbb{C} \setminus X_n(\alpha)$, where $X_n(\alpha)$ is the set of non zero periodic points of period $\leq q_n$. We have $r_n(\alpha) \sim d_n(\alpha)$ where $d_n(\alpha)$ is the distance of 0 to $X_n(\alpha)$.

We will choose $\rho_0 = d_{n_1}(\alpha)$ for some large n_1. As $n_1 \longrightarrow \infty$, $d_{n_1}(\alpha) \longrightarrow 0$. Thus, given n_0, if n_1 is large enough and α' is sufficiently close to α, the renormalized maps $f_1, f_2, \ldots, f_{n_0}$ will be close to rotations on $\mathbb{D}$, and we can take $\rho_1, \rho_2, \ldots, \rho_{n_0}$ close to 1. Since the map f_0 does not have periodic cycles of period $\leq q_n$ on $B(0, \rho_0)$, it turns out that the maps f_{n_0+1}, f_{n_0+2}, $\ldots$ f_{n_1} do not have fixed points in $\mathbb{D}^*$. In that case, Pérez-Marco proved that we can take $\rho_n = c/\log(e/\alpha_n)$ for $n_0 + 1 \leq n \leq n_1$ and for some universal constant c. As usual, for $n > n_1$, we can take $\rho_n = c\alpha_n$.

It follows that

$$\log r(\alpha') \geq \log d_{n_1}(\alpha) + o(1) - \sum_{k=n_0+1}^{n_1} \beta'_{k-1} \log\log \frac{e}{\alpha'_k} - \sum_{k=n_1+1}^{+\infty} \beta'_{k-1} \log \frac{1}{\alpha'_k} - \beta'_{n_0} C_0$$

with $o(1) \longrightarrow 0$ as $\alpha' \longrightarrow \alpha$. Adding $\Phi(\alpha')$ on both sides and letting $\alpha' \longrightarrow \alpha$ yields

$$\liminf_{\alpha' \in \mathcal{B} \to \alpha} \Upsilon(\alpha') \geq \log d_{n_1}(\alpha) + \Phi_{n_1}(\alpha) - \sum_{k=n_0+1}^{n_1} \beta_{k-1} \log\log \frac{e}{\alpha_k} - \beta_{n_0} C_0.$$

Since the series $\sum \beta_{k-1} \log\log \frac{e}{\alpha_k}$ is convergent, letting first $n_1 \longrightarrow \infty$ and then $n_0 \longrightarrow \infty$ gives

$$\liminf_{\alpha' \in \mathcal{B} \to \alpha} \Upsilon(\alpha') \geq \Upsilon(\alpha) - \lim_{n_0 \to \infty} \left(\sum_{k=n_0+1}^{+\infty} \beta_{k-1} \log\log \frac{e}{\alpha_k} + \beta_{n_0} C_0 \right) = \Upsilon(\alpha).$$

5.1.7. *Perturbation of a Cremer point with good approximants.* The last case is the most difficult. In all the cases which have not been covered yet, we have

$$\sup \beta_{n-1} \log \frac{1}{\alpha_n} = +\infty.$$

When $\beta_{n-1} \log \frac{1}{\alpha_n}$ is large, we say that p_n/q_n is a good approximant.

Consider n_0 such that p_{n_0}/q_{n_0} is a good approximant. For $n \neq n_0$ we will take $\rho_n = c\alpha_n$. We will now explain how we choose ρ_{n_0}. On the one hand, it follows from our techniques of parabolic explosion that the distance $d_{n_0}(\alpha')$

between 0 and $\mathcal{C}_{p_{n_0}/q_{n_0}}(\alpha')$ satisfies $\log d_{n_0}(\alpha') \le -\Phi_{n_0}(\alpha') + C_0$. On the other hand, it follows from Yoccoz's lower bound on the size of Siegel disks and from parabolic explosion that the other cycles of period $\le q_{n_0}$ lie outside a disk $B(0, \sigma_{n_0}(\alpha'))$ with $\log \sigma_{n_0}(\alpha') = -\Phi_{n_0-1}(\alpha') - C_0$. Note that

$$\log \sigma_{n_0}(\alpha') - \log d_{n_0}(\alpha') \ge \beta_{n_0-1} \log \frac{1}{\alpha_{n_0}} - 2C_0.$$

Thus, if p_{n_0}/q_{n_0} is a good approximant, the cycle $\mathcal{C}_{p_{n_0}/q_{n_0}}(\alpha')$ is very close to 0 compared to the other cycles of period $\le q_{n_0}$.

We will see that the (n_0-1)-th renormalization of $f_0 = P_{\alpha'}$ is a univalent map $f_{n_0-1} : \mathbb{D} \to \mathbb{C}$ having only two fixed points in $\mathbb{D}$: 0 and a point ζ_{n_0}, the (n_0-1)-th renormalization of the cycle $\mathcal{C}_{p_{n_0}/q_{n_0}}(\alpha')$. Since for $n < n_0$ the canonical map from $\mathcal{U}_n$ to $\mathcal{V}_n$ is close to $z \mapsto (z/\rho_n)^{1/\alpha'_n}$, we have

$$\begin{aligned} d_{n_0}(\alpha') &\simeq \rho_0 \left(\rho_1 \ldots \left(\rho_{n_0-2} \left(\rho_{n_0-1} \, |\zeta_{n_0}|^{\alpha'_{n_0-1}} \right)^{\alpha'_{n_0-2}} \right)^{\cdots} \right)^{\alpha'_0} \\ &= \rho_0 \rho_1^{\beta'_0} \rho_2^{\beta'_1} \ldots \rho_{n_0-1}^{\beta'_{n_0-2}} |\zeta_{n_0}|^{\beta'_{n_0-1}}. \end{aligned}$$

We will show that instead of taking $\rho_{n_0} = c\alpha'_{n_0}$, we can take ρ_{n_0} close to $|\zeta_{n_0}|$ so that $\log \rho_{n_0} \simeq \log |\zeta_{n_0}|$. It will follow that

$$\log \rho_0 + \beta'_0 \log \rho_1 + \ldots + \beta'_{n_0-1} \log \rho_{n_0} \simeq \log d_{n_0}(\alpha').$$

As a consequence

$$\liminf_{\alpha' \in \mathcal{B} \to \alpha} \Upsilon(\alpha') \ge \log d_{n_0}(\alpha) + \Phi_{n_0}(\alpha) - \beta_{n_0} C_0.$$

Again, we conclude that

$$\liminf_{\alpha' \in \mathcal{B} \to \alpha} \Upsilon(\alpha') \ge \Upsilon(\alpha)$$

by letting $n_0 \longrightarrow +\infty$ with $\beta_{n_0-1} \log \frac{1}{\alpha_{n_0}} \to +\infty$.

5.2. *Renormalization principle.* Here, we recall what Pérez-Marco writes in [PM, §III], adapting it to the setting of maps which are close to translations.

Remark. There will be many constants in the discussion. Their sharp value is not important for the application we will make here, so we did not try to optimize them. Moreover, in many estimates where $C\delta$ appears, it can be weakened to $\varepsilon(\delta)$, where $\varepsilon(x) \underset{x \to 0}{\longrightarrow} 0$, while still applying to our proof.

We denote by T the translation $Z \mapsto Z+1$, by $S(\alpha)$ the space of univalent mappings $F : \mathbb{H} \to \mathbb{C}$ such that $F \circ T = T \circ F$ and such that $F(Z) - Z \to \alpha$ as $\mathrm{Im}(Z) \to +\infty$. This space is compact for the topology of uniform convergence on compact subsets of $\mathbb{H}$.

Given $\delta > 0$, we denote by $S_\delta(\alpha)$ the space of maps $F \in S(\alpha)$ such that

$$(\forall Z \in \mathbb{H}) \qquad |F(Z) - Z - \alpha| \le \delta\alpha \quad \text{and} \quad |F'(Z) - 1| \le \delta. \tag{6}$$

Such a function F extends continuously to $\mathbb{H} \cup \mathbb{R}$.

Step 1. Assume $F \in S_\delta(\alpha)$ and define $l = i\mathbb{R}^+$ and $l' = [0, F(0)]$. If δ is sufficiently small (for example $\delta < 1/10$), $l \cup l' \cup F(l)$ bounds an open strip $\mathcal{U}$ in $\mathbb{C}$. Gluing the curves l and $F(l)$ in the boundary of $\overline{\mathcal{U}}$ via F, we obtain a surface $\mathcal{V}$, whose remaining boundary corresponds to the segment l'. Its interior is a Riemann surface for the complex structure inherited from $\overline{\mathcal{U}}$ (the gluing is analytic). It is biholomorphic to the punctured disk $\mathbb{D}^*$. Lifting via $Z \mapsto z = e^{2i\pi Z}$, we get an injective holomorphic map $L : \mathcal{U} \to \mathbb{H}$ which extends continuously to $\overline{\mathcal{U}}$ and such that

$$(\forall Z \in l) \qquad L(F(Z)) = L(Z) + 1.$$

We normalize L by requiring $L(0) = 0$.

Proposition 3. *For all $\delta \in]0, 1/10[$, all $\alpha \in]0, 1[$, all $F \in S_\delta(\alpha)$, and all $Z \in \overline{\mathcal{U}}$,*

$$\mathrm{Im}(Z) - 2\delta < \alpha \mathrm{Im}(L(Z)) < \mathrm{Im}(Z) + 2\delta. \tag{7}$$

Proposition 3 will be proved in section 5.3.

Proposition 4. *Under the same assumptions, the map L extends to a univalent map on*

$$\mathcal{W} = \overline{\mathcal{U}} \cup \{Z \in \mathbb{C} \ ; \ -1 \le \mathrm{Re}(Z) \le 0 \text{ and } \mathrm{Im}(Z) \ge 4\delta\}$$

and for all $Z \in \mathcal{W}$,

$$\mathrm{Im}(Z) - 5\delta < \alpha \mathrm{Im}(L(Z)) < \mathrm{Im}(Z) + 5\delta. \tag{8}$$

From now on, L will refer to this extension. The definition of $\mathcal{W}$ is so that any point $Z \in W$ is eventually mapped to $\mathcal{U}$ under iteration of F: $F^k(Z) = Z' \in \mathcal{U}$ for some $k \in \mathbb{N}$. Then, one defines $L(Z) = L(Z') - k$. In particular, L conjugates F to the translation T (see Figure 6).

Step 2. Given $\delta \in]0, 1/10[$ and $F \in S(\alpha)$, we can define inductively a sequence of univalent maps $(F_n)_{n \ge 0}$ such that $F_n \in S(\alpha_n)$. The construction depends on the choice at each step of some real number $t_n > 0$. We start with $F_0 = F - \mathrm{a}_0$ (where $\mathrm{a}_0 = \lfloor \alpha \rfloor$) and we assume that F_n is constructed. We choose t_n such that the fundamental estimates (6) hold for $\mathrm{Im}(Z) \ge t_n$ (which is always possible). It follows that $G_n : Z \mapsto F_n(Z + it_n) - it_n$ belongs to $S_\delta(\alpha_n)$. For G_n, we construct $\mathcal{U}_n$, $\mathcal{W}_n$ and L_n as above. Let H_n be defined on $L_n\{Z \in \overline{\mathcal{U}} \ ; \ \mathrm{Im}(Z) > 4\delta\}$ by $H_n(Z) = L_n \circ T^{-1} \circ L_n^{-1}$. Note that, by

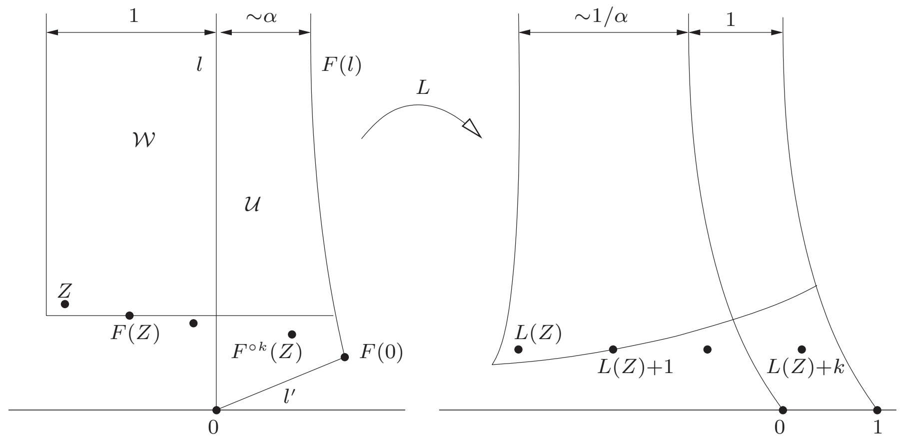

Figure 6: Construction of the map $L : \mathcal{W} \to \mathbb{H}$.

Proposition 3, for all $z \in \mathbb{H}$, if $\mathrm{Im}(Z) > 6\delta/\alpha_n$, then there exists an integer k such that $Z - k$ belongs to D, the domain of definition of H_n.

Then, $D+\mathbb{Z}$ contains the half plane "$\mathrm{Im}(Z) > 6\delta/\alpha_n$". Moreover, the map H_n commutes with the translation T on the set of points in $L_n(i[0, +\infty[\,)$ whose imaginary part is $> 6\delta/\alpha_n$. This set being analytically removable, this implies H_n extends univalently to the upper half-plane $\{Z \in \mathbb{C} \mid \mathrm{Im}(Z) > 6\delta/\alpha_n\}$. Moreover, as $\mathrm{Im}(Z) \to +\infty$, $H_n(Z) - Z \to -1/\alpha_n = -\mathrm{a}_{n+1} - \alpha_{n+1}$.

We set

$$\mathcal{W}'_n = \mathcal{W}_n + it_n$$

and we define $K_n : \mathcal{W}'_n \to \mathbb{C}$ by

$$K_n(Z) = s \circ L_n(Z - it_n) - i\frac{6\delta}{\alpha_n}$$

where $s(x + iy) = -x + iy$, and $F_{n+1} \in S(\alpha_{n+1})$ defined on $\mathbb{H}$ by

$$F_{n+1} = K_n \circ T^{-1} \circ K_n^{-1} - \mathrm{a}_{n+1}.$$

Note that on $\mathcal{W}'_n \cap F_n^{-1}(\mathcal{W}'_n)$, K_n conjugates F_n to T^{-1}. The construction of F_{n+1} is summarized on Figure 7.

Step 3. Next, to a point $Z \in \mathbb{H}$, we associate a sequence $(Z_n)_{n\geq 0}$ as follows. We define $Z_0 = Z$. If $d_n = \mathrm{Im}(Z_n) \geq 4\delta + t_n$, we choose Z'_n such that $Z_n - Z'_n \in \mathbb{Z}$ and $-1 \leq \mathrm{Re}(Z'_n) < 0$, and we define

$$Z_{n+1} = K_n(Z'_n).$$

The sequence $(Z_n)_{n\geq 0}$ may be finite or infinite. The estimates of Proposition 3 imply that for $n \geq 0$ such that Z_{n+1} is defined,

$$\mathrm{Im}(Z_n) - t_n - 11\delta \leq \alpha_n \mathrm{Im}(Z_{n+1}) \leq \mathrm{Im}(Z_n) - t_n - \delta.$$

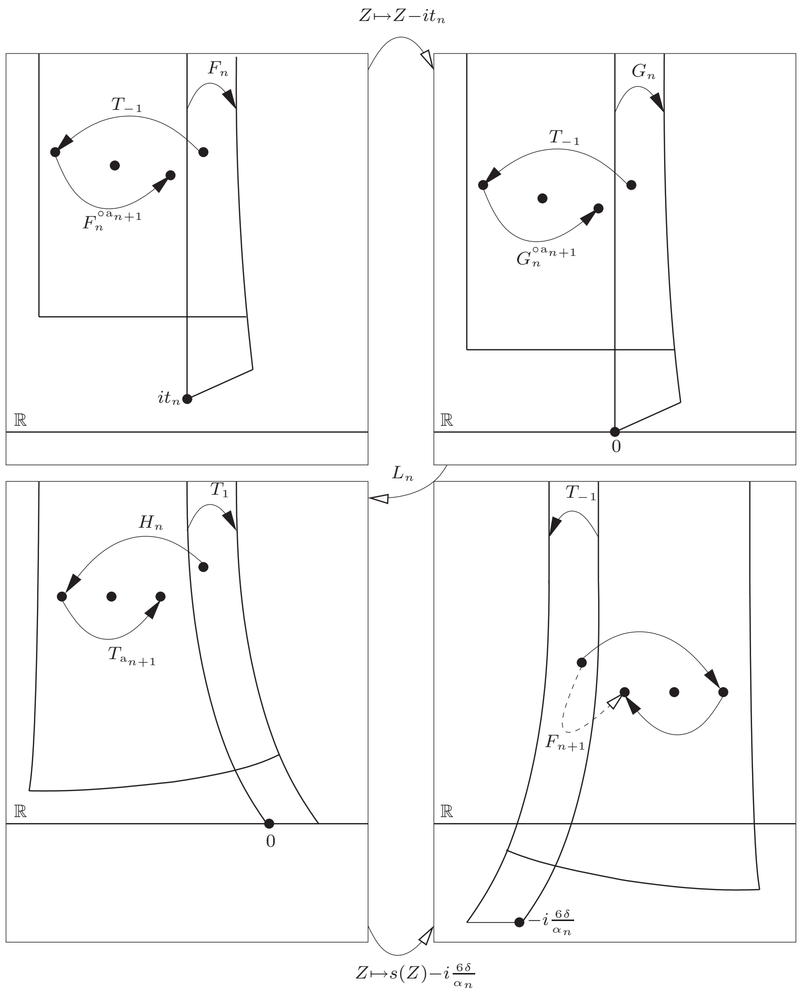

Figure 7: The construction of F_{n+1}.

For $n_0 \geq 0$:

$$\sum_{n=0}^{n_0-1} \beta_{n-1}(t_n + \delta) \leq d_0 - \beta_{n_0-1} d_{n_0} \leq \sum_{n=0}^{n_0-1} \beta_{n-1}(t_n + 11\delta), \tag{9}$$

which implies

$$\sum_{n=0}^{n_0-1} \beta_{n-1} t_n \leq d_0 - \beta_{n_0-1} d_{n_0} \leq 44\delta + \sum_{n=0}^{n_0-1} \beta_{n-1} t_n \tag{10}$$

Indeed, $1 + \beta_0 + \cdots + \beta_{n-2} \leq 4$ since $\beta_{-1} = 1$, $\beta_0 = \alpha_0 \leq 1$ and, $\beta_{n+2} \leq \beta_n/2$.

Proposition 5. *If $Z \in \mathbb{H}$ and if there exists $m \geq 0$ such that $F^{\circ m}(Z) \notin \mathbb{H}$, then the sequence $(Z_n)_{n\geq 0}$ is finite.*

Proof. Let H_n be the half plane defined by "Im$Z > t_n$". If Z_n is defined, let $1 + k_n$ (with $k_n \geq 0$) be the rank of the first iterate of Z_n under $F_n : \mathbb{H} \to \mathbb{C}$ that leaves H_n. Note that if $k_n = 0$, then Z_{n+1} is not defined. Now, if Z_{n+1} is defined and $k_{n+1} > 0$, this means that $Z_n - k_{n+1}$ is eventually mapped back to $\mathcal{U}_n$ by iteration of F_n, without leaving H_n. Therefore (since $|F_n(Z) - (Z + \alpha_n)| < \alpha_n/10$ on H_n),

$$k_{n+1} \leq \frac{11}{10}\alpha_n k_n.$$

Since $\alpha_n\alpha_{n+1} \leq 1/2$ this implies $k_{n+2} \leq \frac{121}{200}k_n$ whenever defined, from which the proposition follows. □

We can now reformulate Theorem III.1.1 in [PM] as follows.

Proposition 6. *Assume we can choose the sequence $(t_n)_{n\geq 0}$ so that the n-th renormalization F_n satisfies the fundamental estimates* (6) *when* $\mathrm{Im}(Z) > t_n$ *and so that*

$$\Phi = \sum_{n=0}^{+\infty} \beta_{n-1} t_n < +\infty.$$

Then F is linearizable and its Siegel disk contains the following upper half-plane:

$$\{Z \in \mathbb{C} \mid \mathrm{Im}(Z) > \Phi + 44\delta\}.$$

Proof. It is enough to prove that all point Z in the half plane has infinite orbit. By Proposition 5, this follows from the sequence Z_n being infinite. Indeed, assume Z_n is defined. According to the previous computations,

$$\begin{aligned}\beta_{n-1}d_n &\geq d_0 - \sum_{k=0}^{n-1}\beta_{k-1}t_k - (1 + \cdots + \beta_{n-2})11\delta \\ &= (d_0 - \Phi - 44\delta) + \beta_{n-1}t_n + \sum_{k=n+1}^{+\infty}\beta_{k-1}t_k + \\ &\quad \big(4 - (1 + \cdots + \beta_{n-1})\big)11\delta + \beta_{n-1}11\delta \\ &\geq \beta_{n-1}(t_n + 11\delta).\end{aligned}$$

Therefore, $d_n \geq t_n + 11\delta$. Since $11 > 4$, this implies Z_{n+1} is defined. □

Also, there is a correspondence between periodic orbits for F and for F_n. Given a map $F : \mathbb{H} \to \mathbb{C}$ that commutes with T, we will say that $Z \in \mathbb{C}$ is periodic with rotation number p/q when $F^q(Z) = Z + p$. In this case, p and q need not to be coprime.

Proposition 7. *Let $n_0 \geq 0$. If F_{n_0} has a fixed point with rotation number $0/1$ and imaginary part h_{n_0}, then F has a periodic orbit with rotation number p_{n_0}/q_{n_0} contained in the strip*

$$\{Z \in \mathbb{C};\ H \leq \mathrm{Im}(Z) \leq H + 44\delta\} \quad \textit{with} \quad H = \beta_{n_0-1}h_{n_0} + \sum_{n=0}^{n_0-1}\beta_{n-1}t_n.$$

Reciprocally, if F has a periodic orbit with rotation number p_{n_0}/q_{n_0} whose imaginary part h_0 satisfies $h_0 > \sum_{n=0}^{n_0-1}\beta_{n-1}t_n + 44\delta$, then F_{n_0} has a fixed point of rotation number $0/1$, and height h_{n_0} satisfying

$$h_0 - 44\delta \leq \beta_{n_0-1}h_{n_0} + \sum_{n=0}^{n_0-1}\beta_{n-1}t_n \leq h_0.$$

Proof. Same as in [PM, annex 2.e]. □

In the previous proposition, the reader should be aware that $F_{n_0}(Z) = Z + k$ with $k \in \mathbb{Z}^*$ is not considered as a fixed point with rotation number $0/1$.

5.3. *Proof of Proposition* 3: *The uniformization L is close to a linear map.* Since $F(Z) - Z - \alpha$ is periodic of period 1, we have

$$|F(Z) - Z - \alpha| \leq \delta\alpha e^{-2\pi\mathrm{Im}(Z)} \quad \text{and} \quad |F'(Z) - 1| \leq \delta e^{-2\pi\mathrm{Im}(Z)}.$$

Let B be the half-band $\{Z \in H \mid 0 < \mathrm{Re}(Z) < 1\}$. Let $H : \overline{B} \to \overline{\mathcal{U}}$ be the map defined by

$$H(X + iY) = i\alpha Y + X\big[F(i\alpha Y) - i\alpha Y\big]. \tag{11}$$

An elementary computation shows that $\|\overline{\partial}H/\partial H\|_\infty < 1$ and if we set

$$K_H = \frac{1 + |\overline{\partial}H/\partial H|}{1 - |\overline{\partial}H/\partial H|},$$

One computes that

$$\begin{aligned} \big|\partial H - \alpha\big| &\leq \alpha\delta e^{-2\pi\alpha Y} \\ \big|\overline{\partial}H\big| &\leq \alpha\delta e^{-2\pi\alpha Y} \end{aligned}$$

And therefore[4]

$$K_H(X + iY) \leq \frac{1}{1 - 2\delta e^{-2\pi\alpha Y}}.$$

Then, using $\delta < 1/10$, we have the inequality

$$K_H(X + iY) \leq 1 + \frac{5}{2}\delta e^{-2\pi\alpha Y}.$$

[4] A quick majoration yields a 4, having a 2 requires more care.

In particular, H is a $(1+\frac{5}{2}\delta)$-quasiconformal homeomorphism. Moreover, by definition

$$\mathrm{Im}(H(Z)) - \alpha\delta \le \alpha \mathrm{Im}(Z) \le \mathrm{Im}(H(Z)) + \alpha\delta,$$

and thus for all $Z \in \overline{\mathcal{U}}$, since $\alpha < 1$:

$$\mathrm{Im}(Z) - \delta \le \alpha \mathrm{Im}(H^{-1}(Z)) \le \mathrm{Im}(Z) + \delta.$$

Since L is conformal, the map $G = L \circ H$ is quasiconformal with the same dilatation as H. Moreover, $G(iY+1) = G(iY)+1$ and so, since the imaginary axis is quasiconformally removable, G extends to a quasiconformal homeomorphism $\mathbb{H} \to \mathbb{H}$. We will show that for all $Z \in \mathbb{H}$,

$$\alpha \mathrm{Im}(Z) - \delta \le \alpha \mathrm{Im}(G(Z)) \le \alpha \mathrm{Im}(Z) + \delta.$$

It follows that

$$\mathrm{Im}(Z) - 2\delta \le \alpha \mathrm{Im}(H^{-1}(Z)) - \delta \le \alpha \mathrm{Im}(L(Z)) \le \alpha \mathrm{Im}(H^{-1}(Z)) + \delta \le \mathrm{Im}(Z) + 2\delta.$$

Lemma 8. *Assume $\psi : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ is a K-quasiconformal homeomorphism. Then, for all $z \in \mathbb{D}$,*

$$4^{1-K}|z|^K \le |\psi(z)| \le 4^{1-1/K}|z|^{1/K}.$$

Proof. To prove the upper bound, note that ψ sends the annulus $\mathbb{D} \setminus [0, z]$ to an annulus separating 0 and $\psi(z)$ from S^1. The modulus is divided by at most K. So,

$$|\psi(z)| \le \mu^{-1}\left(\frac{\mu(|z|)}{K}\right),$$

where, for $r \in]0, 1[$, $\mu(r)$ is the modulus of the annulus $\mathbb{D} \setminus [0, r]$ (it is a decreasing function). The estimate

$$\mu^{-1}\left(\frac{\mu(r)}{K}\right) \le 4^{1-1/K} r^{1/K}$$

can be found in [AVV, Cor. 5.44].

The lower bound is obtained by applying the upper bound to ψ^{-1} which is K-quasiconformal. □

Lemma 9. *If $\Psi : \mathbb{H} \to \mathbb{H}$ is a K-quasiconformal homeomorphism such that $\Psi \circ T = T \circ \Psi$, then*

$$\frac{1}{K}\mathrm{Im}(Z) - \frac{K-1}{2\pi K}\log 4 \le \mathrm{Im}(\Psi(Z)) \le K\mathrm{Im}(Z) + \frac{K-1}{2\pi}\log 4.$$

Proof. Ψ is the lift, via $Z \mapsto z = e^{2i\pi Z}$, of a K-quasiconformal homeomorphism $\psi : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ as in the previous lemma. □

We now come to the control of the quasiconformal homeomorphism G.

LEMMA 10. *Let ε and η be any two positive real numbers. Assume $G : \mathbb{H} \to \mathbb{H}$ is a $(1+\varepsilon)$-quasiconformal homeomorphism such that $G \circ T = T \circ G$ and*

$$K_G(X+iY) \le 1 + \varepsilon e^{-\eta Y}.$$

Then,

$$\operatorname{Im}(Z) - \frac{\varepsilon}{\eta} - \frac{\varepsilon}{2\pi(1+\varepsilon)} \log 4 \le \operatorname{Im}(G(Z)) \le \operatorname{Im}(Z) + \frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi} \log 4,$$

which yields

$$\bigl|\operatorname{Im}(G(Z)) - \operatorname{Im}(Z)\bigr| \le \frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi} \log 4.$$

Proof. We can write $G = G_2 \circ G_1$ with

$$G_1(X+iY) = X + i\frac{1}{1+\varepsilon}\left(Y - \frac{\varepsilon}{\eta} e^{-\eta Y} + \frac{\varepsilon}{\eta}\right).$$

An elementary computation shows that

$$K_{G_1}(X+iY) = \frac{1+\varepsilon}{1+\varepsilon e^{-\eta Y}} \quad \text{and} \quad \operatorname{Im}(G_1(Z)) \le \frac{1}{1+\varepsilon}\left(\operatorname{Im}(Z) + \frac{\varepsilon}{\eta}\right).$$

So, we can apply the previous lemma to G_2 with $K = 1+\varepsilon$, which yields the upper bound for $\operatorname{Im}(G(Z))$.

To get the lower bound, we use the same argument, writing $G = G_4 \circ G_3$ with

$$G_3(X+iY) = X + i(1+\varepsilon)\left(Y + \frac{1}{\eta} \log \frac{1+\varepsilon e^{-\eta Y}}{1+\varepsilon}\right).$$

We have

$$K_{G_3}(X+iY) = \frac{1+\varepsilon}{1+\varepsilon e^{-\eta Y}} \quad \text{and} \quad \operatorname{Im}(G_3(Z)) \ge (1+\varepsilon)\left(\operatorname{Im}(Z) - \frac{\varepsilon}{\eta}\right).$$

□

To conclude the proof of the proposition, we apply the previous lemma to $\varepsilon = \frac{5}{2}\delta$ and $\eta = 2\pi\alpha$. Using $\alpha < 1$, we have

$$\frac{\varepsilon}{\eta} + \frac{\varepsilon}{2\pi} \log 4 = \frac{5\delta}{4\pi\alpha}\bigl(1 + \alpha \log 4\bigr) \le \frac{\delta}{\alpha}.$$

5.4. *Controlling the height of renormalization.* In this section, we determine an upper bound for the height t above which the fundamental estimates (6) are satisfied. The first result is due to Yoccoz (it easily follows from the compactness of $S(0)$, but the interested reader can find sharper bounds in [Y], in the lemma of §3.2, p. 26).

PROPOSITION 8. *For all* $\delta \in]0, 1/10[$, *there exists a constant* C_δ *such that for all* $F \in S(\alpha)$,

$$\mathrm{Im}(Z) \geq C_\delta \quad \Longrightarrow \quad |F'(Z) - 1| \leq \delta$$

and

$$\mathrm{Im}(Z) \geq \frac{1}{2\pi} \log \frac{1}{\alpha} + C_\delta \quad \Longrightarrow \quad |F(Z) - Z - \alpha| \leq \delta\alpha.$$

(Of course, $C_\delta \longrightarrow +\infty$ when $\delta \longrightarrow 0$.)

Remark. In particular, F can not have fixed points above $\frac{1}{2\pi}\log\frac{1}{\alpha}$ plus some universal constant.

The next result is a slight generalization of a result of Pérez-Marco.

PROPOSITION 9. *For all* $\delta \in]0, 1/10[$, *there exists a constant* C_δ *such that the following holds. Assume* $\mathrm{Im}(Z_0) \in \mathbb{H}$, $\alpha \in]0, 1[$ *and* $F \in S(\alpha)$ *has no fixed point except possibly* Z_0 *and its translates by an integer. If*

$$\mathrm{Im}(Z) \geq \mathrm{Im}(Z_0) + \frac{1}{2\pi}\Big(\log\log\frac{e}{\alpha} - \log(1 + 2\pi \mathrm{Im}(Z_0))\Big) + C_\delta$$

then

$$|F(Z) - Z - \alpha| \leq \delta\alpha.$$

One can rewrite

$$\log\log\frac{e}{\alpha} - \log(1 + 2\pi\mathrm{Im}(Z_0)) = \log\frac{1 + \log(\alpha^{-1})}{1 + 2\pi\mathrm{Im}(Z_0)}.$$

Thus for $\mathrm{Im}(Z_0) < \log(\alpha^{-1})/2\pi$, this number is positive. From this, and the remark following Proposition 8, it follows that we can take the same constants C_δ in propositions 8 and 9.

Remark. It follows that if F has no fixed point, the fundamental estimates (6) are satisfied as soon as

$$\mathrm{Im}(Z) \geq \frac{1}{2\pi}\log\log\frac{e}{\alpha} + C_\delta.$$

This result is due to Pérez-Marco [PM]. This is the form we will use in Section 6.

Remark. If $\mathrm{Im}(Z_0) \geq \dfrac{1}{2}\cdot\dfrac{1}{2\pi}\log\dfrac{1}{\alpha}$, it follows from the two propositions and an elementary computation that the fundamental estimates (6) are satisfied as soon as

$$\mathrm{Im}(Z) \geq \mathrm{Im}(Z_0) + 1 + C_\delta.$$

This is the form[5] we will use in Section 7.

[5]The assumption $\mathrm{Im}(Z_0) \geq \frac{1}{2}\cdot\frac{1}{2\pi}\log\frac{1}{\alpha}$ can be replaced by $\mathrm{Im}(Z_0) \geq \mu\cdot\frac{1}{2\pi}\log\frac{1}{\alpha}$ with $\mu \in]0, 1[$, giving the condition $\mathrm{Im}(Z) \geq \mathrm{Im}(Z_0) + \log(\mu^{-1})/2\pi + C_\delta$.

Proof of Proposition 9. Without loss of generality, we may assume that

$$\mathrm{Im}(Z_0) < \frac{1}{2\pi}\log\frac{1}{\alpha}$$

since otherwise, the result follows from Proposition 8. Let us set $r = e^{-2\pi \mathrm{Im}(Z_0)}$ if F has a fixed point at Z_0 and $r = 1$ if F has no fixed point. Then, $\alpha < r$.

Let us now define $u(Z) = F(Z) - Z$. Since u is $\mathbb{Z}$-periodic, there exists a function $g : \mathbb{D}^* \to \mathbb{C}$ such that $u(Z) = g(e^{2i\pi Z})$. The map g extends holomorphically at 0 by $g(0) = \alpha$. We need now to find an upper bound on $|z|$ which ensures that $|g(z) - \alpha| < \alpha\delta$. By compactness of $S(0)$, we can find a (universal) radius $r_0 < 1$ such that on $B(0, r_0)$, g takes its values in $B(0, e)$. Moreover, if F has a fixed point at Z_0, we define $\zeta_0 = e^{2i\pi Z_0}$. Then $g(\zeta_0) = 0$ and g does not vanish in $\mathbb{D} \setminus \{\zeta_0\}$. If F has no fixed point, g does not vanish in $\mathbb{D}$. In both cases, the map $g : B(0, r_0) \setminus \{\zeta_0\} \to B(0, e) \setminus \{0\}$ is contracting for the hyperbolic metrics.

The coefficient of the hyperbolic metrics of $B(0, e) \setminus \{0\}$ at the point α is equal to $1/(\alpha \log(e/\alpha))$, so at first approximation, points at hyperbolic distance of order $\delta/\log(e/\alpha)$ should be at Euclidean distance of order $\delta\alpha$. The lemma below makes a rigorous statement.

Lemma 11. $(\forall\delta \in]0, 1/10[)$, $(\forall\alpha \in]0, 1[)$,

$$d_{B(0,e)\setminus\{0\}}(\alpha, z) \le \frac{\delta}{2\log e/\alpha} \quad\Longrightarrow\quad |z - \alpha| \le \delta\alpha.$$

Proof. For $x < \alpha$, let $\rho(x)$ be the infimum of the coefficient of the hyperbolic metric on the Euclidean circle of center α and radius x. If $|z - \alpha| > \delta\alpha$, then the hyperbolic geodesic in $B(0, e) \setminus \{0\}$ from α to z is longer than

$$\int_0^{\delta\alpha} \rho(x)dx.$$

Let us introduce the function

$$\begin{cases} f(x) = \frac{1}{x\log e/x} & 0 < x \le 1 \\ f(x) = 1 & 1 \le x < e \end{cases}$$

Then f is decreasing, and $\rho(x) = f(x+\alpha)$. Moreover, f is C^1 and convex, and therefore above its tangents. Therefore

$$\begin{aligned} d_{B(0,e)\setminus\{0\}}(\alpha, z) &\ge \int_\alpha^{\alpha+\delta\alpha} f(x)dx \\ &\ge \int_\alpha^{\alpha+\delta\alpha} \big(f(\alpha) + (x-\alpha)f'(\alpha)\big)dx \\ &= \frac{\delta}{\log e/\alpha}\left(1 - \frac{\delta}{2}\Big(1 - \frac{1}{\log(e/\alpha)}\Big)\right) \ge c\frac{\delta}{\log e/a} \end{aligned}$$

with $c = 19/20 > 1/2$. □

The next lemma is also motivated by a hyperbolic metrics coefficient computation.

LEMMA 12. $(\forall r_0 < 1)$, $(\exists \gamma > 0)$, $(\forall \delta \in]0, 1/10[\,)$, *if* $0 < \alpha < r \le 1$, *then*

$$|z| \le \gamma\delta r \frac{\log e/r}{\log e/\alpha} \quad \Longrightarrow \quad d_{B(0,r_0)\setminus\{r\}}(0,z) \le \frac{\delta}{2\log e/\alpha}.$$

Proof. First case: $r \ge r_0/2$. When $|z| \le \delta r_0$, then

$$d_{B(0,r_0)\setminus\{r\}}(0,z) \le d_{B(0,r_0/2)}(0,z) = \log \frac{1 + 2|z|/r_0}{1 - 2|z|/r_0} \le \frac{5|z|}{r_0}.$$

Thus, when $r \ge r_0/2$, we can take any γ such that

$$\gamma \le \min_{r \in [r_0/2, 1]} \frac{r_0}{10 r \log e/r} = \frac{1}{10 \log e/r_0}.$$

Second case: $r < r_0/2$. We first solve the problem when $r_0 = 1$. Let $\rho(z)|dz|$ be the element of hyperbolic metric on $\mathbb{D} \setminus \{r\}$. A computation gives

$$\rho(z) = \frac{1 - r^2}{|1 - rz| \cdot |z - r| \cdot \log\left(\frac{|1 - rz|}{|z - r|}\right)}.$$

A majoration gives, for $|z| < r/10$, $\rho(z) < 10/(9r \log |s|^{-1})$ with $s = (z - r)/(1 - rz)$. Then, $|s| < 11r/(10 + r^2) < 11r/10$. Thus

$$\forall r \in]0, 1/2[\,,\ \forall z \text{ with } |z| \le \frac{r}{10},\ \rho(z) \le \frac{12}{r \log e/r}.$$

Therefore, for $r_0 = 1$, we can take $\gamma = \gamma_1$, with

$$\gamma_1 = 12.$$

For $r_0 \in]0, 1[$, we rescale the problem by the factor $1/r_0$, and according to what we did above, a sufficient condition on z is that

$$\left|\frac{z}{r_0}\right| < \gamma_1 \delta \frac{r}{r_0} \frac{\log e r_0/r}{\log e/\alpha},$$

Then, using $r < r_0/2$, we can take

$$\gamma \le \gamma_1 \frac{\log 2e}{\log 2e + \log r_0^{-1}}. \qquad \square$$

The two previous lemmas show that there exists $\gamma > 0$ such that for all $\delta \in]0, 1/10[$,

$$|z| \le \gamma\delta r \frac{\log e/r}{\log e/\alpha} \quad \Longrightarrow \quad |g(z) - \alpha| \le \delta\alpha.$$

As a consequence,

$$\mathrm{Im}(Z) \geq \frac{1}{2\pi}\left(\log\frac{1}{\gamma\delta} + \log\frac{1}{r} + \log\frac{\log e/\alpha}{\log e/r}\right) \quad\Longrightarrow\quad |F(Z) - Z - \alpha| \leq \delta\alpha.$$

□

6. Proof of inequality (5) (the lower bound) in most cases

6.1. *Renormalizing a map close to a translation.* Let us recall this inequality:

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Upsilon(\alpha).$$

Let us also recall a few notations from Section 5.2. Let $F_{\alpha'} \in S(\alpha')$. Assume we are given $\delta \in]0, 1/10[$ and $t > 0$ such that the map

$$G_{\alpha'} : Z \mapsto F_{\alpha'}(Z + it) - it$$

belongs to $S_\delta(\alpha')$. Then, for $G_{\alpha'}$ we can construct $\mathcal{U}_{\alpha'}$, $\mathcal{W}_{\alpha'}$ and $L_{\alpha'}$ as in Section 5.2. We then define $\mathcal{W}'_{\alpha'} = \mathcal{W}_{\alpha'} + it$, $K_{\alpha'} : \mathcal{W}'_{\alpha'} \to \mathbb{C}$ by

$$K_{\alpha'}(Z) = s \circ L_{\alpha'}(Z - it) - i\frac{6\delta}{\alpha'}$$

where $s(Z) = -\overline{Z}$ and $F_{\alpha',1} \in S(\alpha'_1)$ by

$$F_{\alpha',1} = K_{\alpha'} \circ T^{-1} \circ K_{\alpha'}^{-1} - \left\lfloor \frac{1}{\alpha'} \right\rfloor.$$

We will use the following fact several times:

Lemma 13. *Assume $\alpha' \in]0, 1[$ tends to $\alpha \in]0, 1[$ and $F_{\alpha'} \in S(\alpha')$ tends to the translation $T_\alpha : Z \mapsto Z + \alpha$ uniformly on every compact subset of $\mathbb{H}$. Then,*

(1) *Given $\delta \in]0, 1/10[$ and $t > 0$, if $F_{\alpha'}$ is sufficiently close to T_α, the map*

$$G_{\alpha'} : Z \mapsto F_{\alpha'}(Z + it) - it$$

belongs to $S_\delta(\alpha')$ (it is important that $\alpha \neq 0$), and thus $K_{\alpha'}$ and $F_{\alpha',1}$ are defined.

(2) *The map $K_{\alpha'}$ tends to $Z \mapsto (s(Z) - it - i6\delta)/\alpha$ uniformly on every compact subset of $\mathcal{W}'_{\alpha'}$ and $F_{\alpha',1}$ tends to the translation $Z \mapsto Z + \alpha_1$ uniformly on every compact subset of $\mathbb{H}$.*

Proof. The convergence of $F_{\alpha'}$ to $Z + \alpha$ is uniform on every upper half-plane of the form "$\mathrm{Im}(Z) \geq t > 0$", and $F'_{\alpha'} \longrightarrow 1$ uniformly on these half-planes, whence the first claim. As $F_{\alpha'}$ tends to T_α, $L_{\alpha'}$ tends to $Z \mapsto Z/\alpha$

uniformly on every compact subset of $\mathcal{W}$. Indeed, as in Section 5.3, we can write $L = G \circ H^{-1}$ where H is defined by equation (11) page 32. Then, H converges to $Z \mapsto \alpha Z$ uniformly on $\overline{B}$ and $G : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ is a K-quasiconformal homeomorphism such that $G(0) = 0$ and $G \circ T = T \circ G$. Moreover, $K \longrightarrow 1$ as $F_{\alpha'} \longrightarrow T_\alpha$. Thus G converges to the identity uniformly on every compact subset of $\overline{\mathbb{H}}$. □

6.2. *Brjuno numbers.* Assume $\alpha \in]0,1[$ is a Brjuno number and let $\phi_\alpha : \mathbb{D} \to \Delta_\alpha$ be a linearizing parameterization. Note that $|\phi'_\alpha(0)| = r(\alpha)$. For α' close to α, let us define

$$f_{\alpha'} = \phi_\alpha^{-1} \circ P_{\alpha'} \circ \phi_\alpha$$

on $\phi_\alpha^{-1}(\Delta_\alpha \cap P_{\alpha'}^{-1}(\Delta_\alpha))$. Since $P_\alpha(\Delta_\alpha) = \Delta_\alpha$ and $P_{\alpha'} \longrightarrow P_\alpha$ as $\alpha' \longrightarrow \alpha$, we see that when $\alpha' \longrightarrow \alpha$, f_α converges uniformly on every compact subset of $\mathbb{D}$ to the rotation of angle α. Note that when α' is a Brjuno number, $f_{\alpha'}$ has a Siegel disk of radius $\rho(\alpha') \leq r(\alpha')/r(\alpha)$. Indeed, the image of this Siegel disk by ϕ_α is contained in the Siegel disk of $P_{\alpha'}$. Finally, let $F_{\alpha'}$ be the lift of $f_{\alpha'}$ via $Z \mapsto e^{2i\pi Z}$ which satisfies $|F(Z) - Z - \alpha'| \longrightarrow 0$ when $\mathrm{Im}(Z) \longrightarrow +\infty$.

Let us now fix $\eta > 0$, $\delta \in]0, 1/10[$ and $n_0 \geq 1$. For $n \geq 0$, we will define a sequence of heights t'_n and a sequence of maps $F_{\alpha',n+1} \in S(\alpha'_{n+1})$ as in Section 5.2.

According to the fact mentioned at the beginning of Section 6, and using induction on n_0, we know that provided $\alpha' \in \mathbb{R} \setminus \mathbb{Q}$ is sufficiently close to α, we can take

$$t'_0 = \ldots = t'_{n_0} = \eta/(n_0 + 1).$$

By Proposition 8, for $n \geq n_0 + 1$, we can take

$$t'_n = \frac{1}{2\pi} \log \frac{1}{\alpha'_n} + C_\delta$$

for some constant C_δ which only depends on δ.

It follows from Proposition 6 that if $\alpha' \in \mathcal{B}$ is sufficiently close to α, we have

$$\begin{aligned} \log \frac{r(\alpha)}{r(\alpha')} \quad \leq \quad \log \frac{1}{\rho(\alpha')} &\leq 2\pi \left(\sum_{n=0}^{\infty} \beta'_{n-1} t'_n + 44\delta \right) \\ &\leq \Phi(\alpha') - \Phi_{n_0}(\alpha') + 2\pi(\eta + 4\beta'_{n_0} C_\delta + 44\delta) \end{aligned}$$

(we used $\beta'_{n_0} + \beta'_{n_0+1} + \ldots \leq 4\beta'_{n_0}$ which follows from $\beta'_{k+1} \leq \beta'_k$ and $\beta'_{k+2} \leq \beta'_k/2$). Let us rewrite it

$$\Phi(\alpha') + \log r(\alpha') \geq \Phi_{n_0}(\alpha') + \log r(\alpha) - 2\pi(\eta + 4\beta'_{n_0} C_\delta + 44\delta).$$

Letting $\alpha' \longrightarrow \alpha$ and using $\Phi_{n_0}(\alpha') \longrightarrow \Phi_{n_0}(\alpha)$ and $\beta'_{n_0} \longrightarrow \beta_{n_0}$,

$$\liminf_{\alpha' \to \alpha,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Phi_{n_0}(\alpha) + \log r(\alpha) - 2\pi(\eta + 4\beta_{n_0} C_\delta + 44\delta).$$

Now, as $n_0 \longrightarrow +\infty$, $\Phi_{n_0}(\alpha) \longrightarrow \Phi(\alpha)$ and $\beta_{n_0} \longrightarrow 0$. Thus

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Phi(\alpha) + \log r(\alpha) - 2\pi(\eta + 44\delta).$$

Since this is valid for all $\eta > 0$ and $\delta \in]0, 1/10[$, it implies

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Phi(\alpha) + \log r(\alpha) = \Upsilon(\alpha).$$

6.3. *Rational numbers.* We consider a rational number $\alpha = p/q$ and a Brjuno number α' close to p/q. Let us note α'_n and β'_n the sequences associated to α'. According to the sign of $\varepsilon = \alpha' - p/q$, we associated in Section 2.1 to $\alpha = p/q$ an integer $n_0 \in \mathbb{N}$, and finite sequences $\alpha_0, \alpha_1, \ldots, \alpha_{n_0} = 0$, and $p_0/q_0, p_1/q_1, \ldots, p_{n_0}/q_{n_0} = p/q$ such that for all $k \leq n_0$, $\alpha'_k \longrightarrow \alpha_k$, $p'_k \longrightarrow p_k$ and $q'_k \longrightarrow q_k$ when $\alpha' \longrightarrow \alpha$ on one side.

We will use the notation of Section 4.3. Let z_ε be a point of the cycle $\mathcal{C}_{p/q}(\alpha')$. To study the dynamics of $P_{p/q+\varepsilon}$ at the scale of z_ε, we defined

$$Q_\varepsilon : w \mapsto \frac{1}{z_\varepsilon} P_{p/q+\varepsilon}(z_\varepsilon w).$$

Lemma 7 asserts that

$$Q_\varepsilon^{\circ q}(w) = w + 2i\pi q\varepsilon w(1 - w^q) + \varepsilon R_\varepsilon(w), \tag{12}$$

with $R_\varepsilon \longrightarrow 0$ uniformly on every compact subset of $\mathbb{C}$ as $\varepsilon \longrightarrow 0$.

Set $\phi(w) = \omega^q/(1-\omega^q)$ and $\Omega = \phi^{-1}(\mathbb{D})$. It is the preimage by $w \mapsto w^q$ of the half plane "Re$(z) < 1/2$" and is illustrated as a gray set for $q = 3$ in Figure 4, p. 21. Let $\psi : \Omega \to \mathbb{D}$ be a holomorphic map satisfying $\psi(w)^q = \phi(w)$. Then, $\psi(0) = 0$, $|\psi'(0)| = 1$ and ψ is a conformal representation between Ω and $\mathbb{D}$. It sends the vector field $2i\pi q w(1 - w^q)\frac{\partial}{\partial w}$ to the vector field $2i\pi q\zeta\frac{\partial}{\partial \zeta}$. We define

$$f_\varepsilon = \psi \circ Q_\varepsilon \circ \psi^{-1}$$

on $\psi(\Omega \cap Q_\varepsilon^{-1}(\Omega))$. As $\varepsilon \longrightarrow 0$, f_ε converges uniformly on every compact subset of $\mathbb{D}$ to the rotation of angle p/q. Moreover by (12) we see that when $\varepsilon \longrightarrow 0$,

$$f_\varepsilon^{\circ q}(z) = z + 2i\pi q\varepsilon z + \varepsilon g_\varepsilon(z),$$

with $g_\varepsilon \longrightarrow 0$ uniformly on every compact subset of $\overline{\mathbb{D}}$. Note that when $\alpha' = p/q + \varepsilon$ is a Brjuno number, f_ε has a Siegel disk of conformal radius

$$\rho(\varepsilon) \leq r(\alpha')/|z_\varepsilon|.$$

Let F_ε be the lift of f_ε via $Z \mapsto e^{2i\pi Z}$ which satisfies $|F_\varepsilon(Z) - Z - \alpha'| \longrightarrow 0$ when $\text{Im}(Z) \longrightarrow +\infty$. When $\varepsilon \longrightarrow 0$,

$$F_\varepsilon^{\circ q} \circ T^{-p}(Z) = Z + q\varepsilon + \varepsilon G_\varepsilon(Z)$$

with $G_\varepsilon \longrightarrow 0$ uniformly on every compact subset of $\mathbb{H}$.

Let us fix $\delta \in]0, 1/10[$ and $\eta > 0$. For $n \geq 0$, we will define a sequence of heights t'_n and a sequence of maps $F_{\varepsilon,n+1} \in S(\alpha'_{n+1})$.

As ε tends to 0, F_ε converges uniformly to the translation by p/q on the upper half-plane $\{Z \in \mathbb{C} \mid \mathrm{Im}(Z) \geq \eta/(n_0+1)\}$. Moreover, for $n \leq n_0 - 1$, as $\varepsilon \longrightarrow 0$, $\alpha'_n \longrightarrow \alpha_n \neq 0$. Thus, if ε is sufficiently close to 0, we can take

$$t'_0 = t'_1 = \ldots = t'_{n_0-1} = t \stackrel{\text{def}}{=} \eta/(n_0+1).$$

We will call $\mathcal{W}'_{\varepsilon,n}$ and $K_{\varepsilon,n} : \mathcal{W}'_{\varepsilon,n} \to \mathbb{C}$ the objects corresponding to $\mathcal{W}'_n$ and K_n defined in Section 5.2. When $\varepsilon \longrightarrow 0$, the interior of $\mathcal{W}'_{\varepsilon,n}$ tends to the interior of a set $\mathcal{W}'_{0,n}$ which is the union of two half strips "$-1 \leq \mathrm{Re}(Z) \leq 0$ and $\mathrm{Im}(Z) \geq 4\delta + t$" and "$0 \leq \mathrm{Re}(Z) \leq \alpha_n$ and $\mathrm{Im}(Z) \geq t$". For $n \leq n_0 - 1$, as ε tends to 0, $K_{\varepsilon,n}$ tends to $Z \mapsto (s(Z) - it - i6\delta)/\alpha_n$ uniformly on every compact subset of $\mathcal{W}'_{0,n}$, where $s(Z) = -\overline{Z}$.

Now, when $\varepsilon \longrightarrow 0$, F_{ε,n_0} converges uniformly to the translation $Z \mapsto Z + \alpha_{n_0} = Z + 0$, i.e., to the identity.

Lemma 14. *If ε is small enough, we can take $t'_{n_0} = \eta/(n_0+1)$.*

Proof. Let us now consider the map

$$\Psi_\varepsilon = K_{\varepsilon,n_0-1} \circ \ldots \circ K_{\varepsilon,0}.$$

Its set of definition eventually contains every compact subset of the interior of

$$\mathcal{W}'' = \big\{Z \in \mathbb{C}\ ;\ -\beta_{n_0-1} \leq (-1)^{n_0}\mathrm{Re}(Z) \leq \beta_{n_0-2} \text{ and } \mathrm{Im}(Z) \geq t' - 2\delta\beta_{n_0-2}\big\},$$

with $t' = (t + 6\delta)(1 + \beta_1 + \ldots + \beta_{n_0-2})$. On every of these compact subsets, Ψ_ε eventually conjugates $F_\varepsilon^{\circ q} \circ T^{-p}$ to F_{ε,n_0}.

As ε tends to 0, Ψ_ε converges to $Z \mapsto (s^{n_0}(Z) - it')/\beta_{n_0-1}$, uniformly on every compact subset of the interior of $\mathcal{W}''$. Thus, since $s^{n_0} \circ \Psi_\varepsilon$ is holomorphic, the derivative of $s^{n_0} \circ \Psi_\varepsilon$ converges to $1/\beta_{n_0-1}$, uniformly on every compact subset of the interior of $\mathcal{W}''$. Therefore

$$F_{\varepsilon,n_0}(Z) = Z + \frac{q|\varepsilon|}{\beta_{n_0-1}} + \varepsilon H_\varepsilon(Z)$$

with $H_\varepsilon \longrightarrow 0$ uniformly on every compact subset of $\mathbb{H}$. Since $\alpha'_{n_0} = q|\varepsilon|/\beta'_{n_0-1} = q|\varepsilon|/\beta_{n_0-1} + \mathcal{O}(\varepsilon^2)$, $|F_{\varepsilon,n_0}(Z) - Z - \alpha'_{n_0}| = \alpha'_{n_0} I_\varepsilon(Z)$ with $I_\varepsilon(Z) \longrightarrow 0$ uniformly on every compact subset of $\Psi_0(\mathcal{W}'')$. This set contains "$-1 < \mathrm{Re}(Z) < 1$ and $\mathrm{Im}(Z) > 0$". Since F_{ε,n_0} commutes with T, this implies that $|F_{\varepsilon,n_0}(Z) - Z - \alpha'_{n_0}| = \alpha'_{n_0} I_\varepsilon(Z)$ with $I_\varepsilon(Z) \longrightarrow 0$ uniformly on every compact subset of $\mathbb{H}$. As a consequence $|F'_{\varepsilon,n_0}(Z) - 1| \longrightarrow 0$ uniformly on every compact subset of $\mathbb{H}$. □

Finally, for $n \geq n_0 + 1$, we can take

$$t'_n = \frac{1}{2\pi} \log \frac{1}{\alpha'_n} + C_\delta$$

where C_δ is the constant in Proposition 8. So, if ε is sufficiently small, we have

$$\log \frac{|z_\varepsilon|}{r(\alpha')} \le 2\pi \left(\sum_{n=0}^{\infty} \beta'_{n-1} t'_n + 44\delta \right) \le \Phi(\alpha') - \Phi_{n_0}(\alpha') + 2\pi(\eta + 4\beta'_{n_0} C_\delta + 44\delta).$$

Reordering the terms, we obtain

$$\Phi(\alpha') + \log r(\alpha') \ge \log |z_\varepsilon| + \Phi_{n_0}(\alpha') - 2\pi(\eta + 4\beta'_{n_0} C_\delta + 44\delta).$$

As $\varepsilon \longrightarrow 0$, $\log |z_\varepsilon| + \Phi_{n_0}(\alpha')$ tends to $\Upsilon(p/q)$ and β'_{n_0} tends to 0. We therefore have (see Lemma 6)

$$\liminf_{\alpha' \to p/q,\ \alpha' \in \mathcal{B}} \Phi(\alpha') + \log r(\alpha') \ge \Upsilon\left(\frac{p}{q}\right) - 2\pi(\eta + 44\delta)$$

and the proof of inequality (5) at rational numbers is completed since η and δ can be chosen arbitrarily small.

6.4. *Cremer numbers whose Pérez-Marco sum converges.* It is possible to give a proof that works for all Cremer numbers at the same time, but for clarity, we prefer to study two cases (which overlap) separately. Here, we will assume α is a Cremer number such that

$$\sum_{n=0}^{\infty} \beta_{n-1} \log \log \frac{e}{\alpha_n} < \infty.$$

We will call this sum the Pérez-Marco sum, since it was introduced by Pérez-Marco in [PM]. There, he proves that, under this condition, every germ that fixes 0 with derivative $e^{2i\pi\alpha}$ is linearizable or has small cycles.

Let us fix $\eta > 0$, $\delta \in]0, 1/10[$ and $n_0 \ge 1$. For $n_1 \ge n_0$, we set

$$d_{n_1}(\alpha') = d(0, X_{n_1}(\alpha'))$$

(see Definition 10 for X_n). Since a Cremer point of a polynomial is accumulated by periodic points, and because we defined $X_{n_1}(\alpha)$ as the set of all periodic points of period $\le q_{n_1}$ except 0, we have $d_{n_1}(\alpha) \longrightarrow 0$ when $n_1 \longrightarrow +\infty$. Thus, provided n_1 is big enough, we see that for all α' close enough to α, $F_{\alpha'}$ is injective on $B(0, d_{n_1}(\alpha'))$. Let $F_{\alpha'} \in S(\alpha')$ be the lift of $P_{\alpha'}$ via $Z \mapsto d_{n_1}(\alpha') e^{2i\pi Z}$. This amounts to restrict the polynomial $P_{\alpha'}$ to the disk $B(0, d_{n_1}(\alpha'))$ where there are no periodic cycle of period less than or equal to q_{n_1}, except 0. Note that when α' is a Brjuno number, this restriction has a Siegel disk of conformal radius $\le r(\alpha')$.

For $n \ge 0$, we will define a sequence of heights t'_n and a sequence of maps $F_{\alpha',n+1} \in S(\alpha'_{n+1})$.

LEMMA 15. *If n_1 is sufficiently large and α' is sufficiently close to α, we can take*

$$t'_0 = t'_1 = \ldots = t'_{n_0} = \eta/(n_0 + 1).$$

Proof. Let us choose ε sufficiently small so that $\alpha'_0 \neq 0, \ldots, \alpha'_{n_0} \neq 0$ for all $\alpha' \in [\alpha-\varepsilon, \alpha+\varepsilon]$. As $n_1 \longrightarrow \infty$, $(\alpha', Z) \mapsto F_{\alpha'}(Z) - Z - \alpha'$ converges uniformly to 0 on $[\alpha-\varepsilon, \alpha+\varepsilon] \times \{Z \in \mathbb{C} \mid \mathrm{Im}(Z) \geq \eta/(n_0+1)\}$. If n_1 is sufficiently large, we can therefore take $t'_0 = t'_1 = \ldots = t'_{n_0} = \eta/(n_0+1)$. □

By construction, the maps $F_{\alpha'}$ have no periodic cycle of period less than or equal to q_{n_1}. So, by Proposition 7, for $n \leq n_1$, the renormalizations $F_{\alpha',n}$ have no fixed point in $\mathbb{H}$. Thus, by Proposition 9, we can take

$$t'_{n_0+1} = \frac{1}{2\pi}\log\log\frac{e}{\alpha_{n_0+1}} + C_\delta \quad \ldots \quad t'_{n_1} = \frac{1}{2\pi}\log\log\frac{e}{\alpha_{n_1}} + C_\delta$$

for some constant C_δ which only depends on δ. Finally, by Proposition 8, for $n \geq n_1 + 1$, we can take

$$t'_n = \frac{1}{2\pi}\log\frac{1}{\alpha'_n} + C_\delta.$$

Now, Proposition 6 yields

$$\frac{1}{2\pi}\log\frac{d_{n_1}(\alpha')}{r(\alpha')} \leq \sum_{n=0}^{\infty}\beta'_{n-1}t'_n + 44\delta.$$

Using the value of t'_n chosen above, we get

$$\Phi(\alpha') + \log r(\alpha') \geq \Phi_{n_1}(\alpha') + \log d_{n_1}(\alpha') - \sum_{n=n_0+1}^{n_1}\beta'_{n-1}\log\log\frac{e}{\alpha'_n}$$
$$-2\pi(\eta + 4\beta'_{n_0}C_\delta + 44\delta).$$

Let α' tend to α:

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Phi_{n_1}(\alpha) + \log d_{n_1}(\alpha) - \sum_{n=n_0+1}^{n_1}\beta_{n-1}\log\log\frac{e}{\alpha_n}$$
$$-2\pi(\eta + 4\beta_{n_0}C_\delta + 44\delta).$$

Let n_1 tend to $+\infty$. Recall that $d_{n_1}(\alpha) \sim r_{n_1}(\alpha)$, and by Definition 11 $\Upsilon(\alpha) = \lim_{n_1\longrightarrow+\infty} \Phi_{n_1}(\alpha) + r_{n_1}(\alpha)$. Thus,

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Upsilon(\alpha) - \sum_{n=n_0+1}^{+\infty}\beta_{n-1}\log\log\frac{e}{\alpha_n}$$
$$-2\pi(\eta + 4\beta_{n_0}C_\delta + 44\delta).$$

Let n_0 tend to $+\infty$. Since $\beta_{n_0} \longrightarrow 0$ and the Pérez-Marco sum of α was assumed to be convergent, we have

$$\liminf_{\alpha'\to\alpha,\ \alpha'\in\mathcal{B}} \Phi(\alpha') + \log r(\alpha') \geq \Upsilon(\alpha) - 2\pi(\eta + 44\delta).$$

Since this is valid for arbitrarily small η and δ, this concludes the proof for the case when the Pérez-Marco sum of α converges.

7. Proof of inequality (5) when the Pérez-Marco sum diverges

In this section, we assume that α is a Cremer number such that

$$\sup_n \beta_{n-1} \log \frac{1}{\alpha_n} = \infty.$$

To deal with this case, we will have to combine techniques of parabolic explosion and techniques of renormalization.

Note that if $\beta_{n-1} \log 1/\alpha_n \leq C < \infty$ for all $n \geq 0$, then $\beta_{n-1} \log \log(e/\alpha_n) \leq \beta_{n-1} \log(1 + C/\beta_{n-1})$ decreases exponentially fast, and α belongs to the set of Cremer numbers studied in Section 6.4.

7.1. *Parabolic explosion.* The techniques of parabolic explosion are used to have a precise control on the position of some periodic points of $P_{\alpha'}$ for α' close to α. The maps $P_{\alpha'}$, for α' real, are injective on $B(0, 1/2)$. We let $F_{\alpha'} \in S(\alpha')$ be the lift of $P_{\alpha'}$ via $Z \mapsto \frac{1}{2} e^{2i\pi Z}$. Let us recall that we called a periodic point of a map F that commutes with T, a point Z such that $F^q(Z) = p$ for integers $q \in \mathbb{N}^*$ and $p \in \mathbb{Z}$ (p and q need not be coprime). Then q is called the period, and p/q the rotation number.

LEMMA 16. *There exists a constant $B_\alpha > 0$ such that for all Brjuno number α' sufficiently close to α and all integer $n \geq 2$,*

a) *if $\frac{1}{2\pi}\Phi_n(\alpha') - B_\alpha > 0$, then $P_{\alpha'}$ has a periodic point with period $\leq q_n$ and modulus $\frac{1}{2} e^{-2\pi h_0'}$ with $h_0' \geq \frac{1}{2\pi}\Phi_n(\alpha') - B_\alpha$;*

b) *for all Z in the upper half-plane*

$$\left\{ Z \in \mathbb{C} \mid \operatorname{Im}(Z) \geq \frac{1}{2\pi}\Phi_{n-1}(\alpha') + B_\alpha \right\}$$

the first q_n iterates of Z under iteration of $F_{\alpha'}$ have imaginary part $\geq \frac{1}{2\pi}\Phi_{n-1}(\alpha') + 44\delta$ and if Z is periodic with period $\leq q_n$, then Z comes from $\mathcal{C}_{p_n/q_n}(\alpha')$ (in the sense that $\frac{1}{2} e^{2i\pi Z} \in \mathcal{C}_{p_n/q_n}(\alpha')$).

Proof. For $n \geq 2$ and for $\alpha' \in \mathbb{R} \setminus \mathbb{Q}$, let us define

$$X_n^*(\alpha') = X_n(\alpha') \setminus \mathcal{C}_{p_n/q_n}(\alpha'), \qquad r_n^*(\alpha') = \operatorname{rad}(X_n^*(\alpha')),$$
$$d_n(\alpha') = d(0, X_n(\alpha')), \qquad \text{and} \qquad d_n^*(\alpha') = d(0, X_n^*(\alpha')).$$

By Proposition 2 (since $q_2 \geq 2$), we have for α' close enough to α,

$$\Phi_n(\alpha') + \log r_n(\alpha') \leq \Phi_2(\alpha') + \log r_2(\alpha') + C \sum_{k=3}^{n} \frac{\log q_k}{q_k}.$$

As $\alpha' \longrightarrow \alpha$, the right hand term is bounded independently of n. So, there exists a constant C_α such that for all $n \geq 2$ and all $\alpha' \in \mathcal{B}$ sufficiently close to α,

$$\Phi_n(\alpha') + \log d_n(\alpha') \leq \Phi_n(\alpha') + \log r_n(\alpha') \leq 2\pi C_\alpha.$$

Thus, if α' is sufficiently close to α, $P_{\alpha'}$ has a periodic point with modulus $\frac{1}{2}e^{-2\pi h_0'}$ with $h_0' \geq \dfrac{1}{2\pi}\Phi_n(\alpha') - C_\alpha - \dfrac{\log 2}{2\pi}$ when the right hand is positive. This proves part a).

Part b) follows from [PM, annex 2.f] and the following observation. By Lemma 1, in $B(\alpha', 1/2q_n^3)$, the only cycle of period less than or equal to q_n that does not move holomorphically is the cycle $\mathcal{C}_{p_n/q_n}(\alpha')$. So, as in Lemma 5, for all $n \geq 2$, we have

$$\Phi(\alpha') + \log r(\alpha') \leq \Phi_{n-1}(\alpha') + \log r_n^*(\alpha') + (C-1)\sum_{k\geq n} \frac{\log q_k'}{q_k'},$$

where C is the constant provided by Lemma 2. By inequality (1), $\Phi(\alpha') + \log r(\alpha')$ is universally bounded from below. So, there exists a constant C' such that for all $n \geq 2$ and all α' sufficiently close to α,

$$\Phi_{n-1}(\alpha') + \log r_n^*(\alpha') \geq -C'.$$

Finally, we claim that there exists a constant C_α' such that for all $n \geq 2$ and all α' sufficiently close to α, we have

$$\log d_n^*(\alpha') \geq \log r_n^*(\alpha') - C_\alpha'.$$

Part b) follows easily. To prove the claim, let $\rho' = e^{2i\pi\alpha'}$ and $\rho = e^{2i\pi\alpha}$. Let n_0 be such that $d_{n_0}^*(\alpha) < |\rho - 1|/4$ (this is possible since α is a Cremer number). For α' close enough to α, $d_{n_0}^*(\alpha') < |\rho' - 1|/2$. For each fixed value of $n < n_0$, $\log d_n^*(\alpha') - \log r_n^*(\alpha') \longrightarrow \log d_n^*(\alpha) - \log r_n^*(\alpha)$ when $\alpha' \longrightarrow \alpha$. For $n \geq n_0$, let $z \in X_n^*(\alpha')$ be a point that realizes the distance $d_n^*(\alpha')$ and set $w = P_{\alpha'}(z) = \rho' z + z^2$. Then, $|z| = d_n^*(\alpha') \leq d_{n_0}^*(\alpha') < |\rho' - 1|/2$ and

$$r_n^*(\alpha') \leq \mathrm{rad}(\mathbb{C} \setminus \{z, w\}) = d_n^*(\alpha') \cdot \mathrm{rad}\left(\mathbb{C} \setminus \{1, w/z\}\right).$$

As α' tends to α, $w/z = \rho' + z$ remains in a compact subset of $\mathbb{C} \setminus \{1\}$ and so, $\mathrm{rad}(\mathbb{C} \setminus \{1, w/z\})$ is bounded. □

7.2. *Renormalization.* Let us now fix $\delta \in]0, 1/10[$. For $n \geq 0$, we will define a sequence of heights t_n' and a sequence of maps $F_{\alpha',n} \in S(\alpha_n')$ as in Section 5.2.

Let us set

$$C' = 2\pi(B_\alpha + 4C_\delta + 44\delta),$$

where B_α is the constant in Lemma 16.

Now, let us choose n_0 so that $\beta_{n_0-1}\log 1/\alpha_{n_0} > 4C'$ (this is possible because $\sup \beta_{n-1} \log 1/\alpha_n = \infty$). If α' is sufficiently close to α, we have

$$\beta_{n_0-1}' \log 1/\alpha_{n_0}' > 4C'.$$

By Proposition 8, we can take

$$t_0' = \frac{1}{2\pi}\log\frac{1}{\alpha_0'} + C_\delta \quad \ldots \quad t_{n_0-1}' = \frac{1}{2\pi}\log\frac{1}{\alpha_{n_0-1}'} + C_\delta.$$

By Lemma 16 part a), $P_{\alpha'}$ has a periodic point $\frac{1}{2}e^{2i\pi Z'_0}$ with period $\leq q_{n_0}$ satisfying $\mathrm{Im}(Z'_0) = h'_0 \geq \dfrac{1}{2\pi}\Phi_{n_0}(\alpha') - B_\alpha$. Note that

$$\frac{1}{2\pi}\Phi_{n_0}(\alpha') - B_\alpha \geq \frac{1}{2\pi}\Phi_{n_0-1}(\alpha') + \frac{4C'}{2\pi} - B_\alpha \geq \frac{1}{2\pi}\Phi_{n_0-1}(\alpha') + B_\alpha.$$

By Lemma 16 part b), Z'_0 is periodic for $F_{\alpha'}$ and comes from $\mathcal{C}_{p_{n_0}/q_{n_0}}(\alpha')$, and thus has rotation number p_{n_0}/q_{n_0}. By Proposition 7, F_{α',n_0} has a fixed point Z'_{n_0} with $\mathrm{Im}(Z'_{n_0}) = h'_{n_0}$ satisfying

$$h'_0 - 44\delta < \beta'_{n_0-1}h'_{n_0} + \sum_{n=0}^{n_0-1}\beta'_{n-1}t'_n < h'_0$$

(see inequality (10) page 30). So,

$$h'_{n_0} > \frac{1}{2\pi}\log\frac{1}{\alpha'_{n_0}} - \frac{B_\alpha + 4C_\delta + 44\delta}{\beta'_{n_0-1}} > \frac{3}{4}\cdot\frac{1}{2\pi}\log\frac{1}{\alpha'_{n_0}}.$$

If $Z \neq Z'_{n_0}$ is another fixed point of F_{α',n_0}, then Proposition 7 and Lemma 16 imply that

$$\beta'_{n_0-1}\mathrm{Im}(Z) + \sum_{n=0}^{n_0-1}\beta'_{n-1}t'_n < \frac{1}{2\pi}\sum_{n=0}^{n_0-1}\beta'_{n-1}\log\frac{1}{\alpha'_n} + B_\alpha.$$

Thus,

$$\mathrm{Im}(Z) < \frac{B_\alpha}{\beta'_{n_0-1}} < \frac{1}{4}\cdot\frac{1}{2\pi}\log\frac{1}{\alpha'_{n_0}}.$$

So, there is a gap of height greater than $\dfrac{1}{2}\cdot\dfrac{1}{2\pi}\log\dfrac{1}{\alpha'_{n_0}}$ that separates the fixed point Z'_{n_0} of F_{α',n_0} from the other fixed points of F_{α',n_0}. According to the second remark after Proposition 9, we can therefore take

$$t'_{n_0} = h'_{n_0} + 1 + C_\delta.$$

Finally, for $n \geq n_0 + 1$, we can take

$$t'_n = \frac{1}{2\pi}\log\frac{1}{\alpha'_n} + C_\delta.$$

As in the previous section, Proposition 6 we have

$$\begin{aligned}
\log\frac{1}{2r(\alpha')} &\leq 2\pi\left(\sum_{n=0}^{\infty}\beta'_{n-1}t'_n + 44\delta\right)\\
&\leq 2\pi\left(\sum_{n=0}^{n_0-1}\beta'_{n-1}t'_n + \beta'_{n_0-1}h'_{n_0}\right) + \sum_{n=n_0+1}^{\infty}\beta'_{n-1}\log\frac{1}{\alpha'_n}\\
&\quad +2\pi(\beta'_{n_0-1}(4C_\delta+1)+44\delta)\\
&\leq 2\pi h'_0 + \Phi(\alpha') - \Phi_{n_0}(\alpha') + 2\pi(\beta'_{n_0-1}(4C_\delta+1)+44\delta).
\end{aligned}$$

Note that $2\pi h_0' \le -\log(2d_{n_0}(\alpha'))$ where $d_{n_0}(\alpha') = d(0, X_{n_0}(\alpha'))$. So, reordering the terms and simplifying by $\log 2$, we get

$$\Phi(\alpha') + \log r(\alpha') \ge \Phi_{n_0}(\alpha') + \log d_{n_0}(\alpha') - 2\pi(\beta'_{n_0-1}(4C_\delta + 1) + 44\delta).$$

We can now conclude as in Section 6.4.

Appendix: Extracts from [BC2]

The following proposition is Proposition 10 from [BC2].

Proposition 10. *Assume $U, V \subset \mathbb{C}$ are two hyperbolic domains containing 0 and $\chi : U \to V$ is a holomorphic map fixing 0. Let S be a finite subset of U avoiding 0, such that $\chi(S)$ avoids 0. Then,*

$$\frac{\mathrm{rad}(V \setminus \chi(S))}{\mathrm{rad}(V)} \le \frac{\mathrm{rad}(U \setminus S)}{\mathrm{rad}(U)}.$$

Given an integer $q \ge 1$, set

$$\mathbb{U}_q = \left\{ e^{2i\pi k/q} \mid k = 0, \dots, q-1 \right\}.$$

The following proposition is Proposition 12 from [BC2].

Proposition 11. *There exists a constant $C > 0$ such that for $q \ge 2$ and $r < 1$, we have*

$$\log \mathrm{rad}(\mathbb{D} \setminus r\mathbb{U}_q) \le \log r + \frac{C}{q}.$$

One can take $C = \log 4 + 2\log(1 + \sqrt{2})$.

Let V_λ be hyperbolic subdomains of $\mathbb{C}$ which contain 0 and move holomorphically with respect to $\lambda \in \mathbb{D}$. The following proposition is Proposition 13 from [BC2].

Proposition 12. *There exists a family of simply connected open sets $\widetilde{V}_\lambda$ and of universal coverings $\pi_\lambda : \widetilde{V}_\lambda \to V_\lambda$ such that $\widetilde{V_0} = \mathbb{D}$, the set*

$$\widetilde{\mathcal{V}} = \left\{ (\lambda, z) \in \mathbb{D} \times \mathbb{C} \,\middle|\, z \in \widetilde{V}_\lambda \right\}$$

is open, and $\Pi : (\lambda, z) \in \widetilde{\mathcal{V}} \mapsto \pi_\lambda(z)$ is analytic. For all $\lambda \in \mathbb{D}$,

$$\widetilde{V}_\lambda \subset B(0,\rho) \text{ with } \log \rho = \frac{2 \log 4}{1 + |\lambda|^{-1}}.$$

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