COMPLEX ROTATION NUMBERS

XAVIER BUFF AND NATALIYA B. GONCHARUK

Abstract. We investigate the notion of complex rotation number which was introduced by V. I. Arnold in 1978. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving circle diffeomorphism and let $\omega \in \mathbb{C}/\mathbb{Z}$ be a parameter with positive imaginary part. Construct a complex torus by glueing the two boundary components of the annulus $\{z \in \mathbb{C}/\mathbb{Z} \ : \ 0 < \mathrm{Im}(z) < \mathrm{Im}(\omega)\}$ via the map $f + \omega$. This complex torus is isomorphic to $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for some appropriate $\tau \in \mathbb{C}/\mathbb{Z}$. We study the behavior of τ as $\mathrm{Im}(\omega) \to +\infty$ or as $\mathrm{Im}(\omega) \to 0$. In particular, we show that there is a continuous extension as $\mathrm{Im}(\omega) \to 0$.

Notation:

- $\mathbb{H} = \mathbb{H}^+$ is the set of complex numbers with positive imaginary part.
- $\mathbb{H}^-$ is the set of complex numbers with negative imaginary part.
- If p/q is a rational number, then p and q are assumed to be coprime.
- If x and y are distinct points in $\mathbb{R}/\mathbb{Z}$, then (x, y) denotes the set of points $z \in \mathbb{R}/\mathbb{Z} - \{x, y\}$ such that the three points x, z, y are in increasing order and $[x, y] := (x, y) \cup \{x, y\}$.
- If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a circle diffeomorphism, $D_f := \displaystyle\int_{\mathbb{R}/\mathbb{Z}} \left| \frac{f''(x)}{f'(x)} \right| \, \mathrm{d}x$.

Introduction

Given an orientation preserving analytic circle diffeomorphism $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and a parameter $\omega \in \mathbb{H}/\mathbb{Z}$, set

$$f_\omega := f + \omega : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} + \omega.$$

The circles $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z} + \omega$ bound an annulus $A_\omega \subset \mathbb{C}/\mathbb{Z}$. Glueing the two sides of A_ω via f_ω, we obtain a complex torus $E(f_\omega)$, which may be uniformized as $\mathcal{E}_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for some appropriate $\tau \in \mathbb{H}/\mathbb{Z}$, the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $E(f_\omega)$ corresponding to the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $\mathcal{E}_\tau$. The complex rotation number of f_ω is $\tau_f(\omega) := \tau$. It is the complex analog of the ordinary rotation number of $f + t$ for $t \in \mathbb{R}/\mathbb{Z}$.

V. I. Arnold's problem [1], generalized by R. Fedorov and E. Risler independently, is to study the relation of the ordinary rotation number of the circle diffeomorphism $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and the limit behaviour of the complex rotation number $\tau_f(\omega)$ as ω tends to 0.

According to work of Risler [6, Chapter 2, Proposition 2], the function

$$\tau_f : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$$

The research of the first author was supported by the IUF. The research of the second author was supported by the following grants : RFBR project 10-01-00739-a, RFBR project 12-01-33020 mol_a_ved, joint RFBR/CNRS project 10-01-93115-CNRS_a, Moebius Contest Foundation for Young Scientists, Simons Grant.

is holomorphic. We shall study the behavior of $\tau_f(\omega)$ as the imaginary part of ω tends to $+\infty$, and as the imaginary part of ω tends to 0. In particular, we shall show that there is a continuous extension of τ_f to the boundary $\mathbb{R}/\mathbb{Z}$.

1. Notation and statement of results

The map

$$\mathbb{C}/\mathbb{Z} \ni z \mapsto \exp(2\pi \mathrm{i} z) \in \mathbb{C} - \{0\}$$

is an isomorphism of Riemann surfaces. Thus, $\mathbb{C}/\mathbb{Z}$ may be compactified as a Riemann surface $\overline{\mathbb{C}/\mathbb{Z}}$ isomorphic to the Riemann sphere, by adding two points $+\mathrm{i}\infty$ and $-\mathrm{i}\infty$ (the notation suggests that $\pm\mathrm{i}\infty$ is the limit of points $z \in \mathbb{C}/\mathbb{Z}$ whose imaginary part tends to $\pm\infty$). We shall denote by

$$\overline{\mathbb{H}^\pm/\mathbb{Z}} = \mathbb{H}^\pm/\mathbb{Z} \cup \mathbb{R}/\mathbb{Z} \cup \{\pm\mathrm{i}\infty\}$$

the closure of $\mathbb{H}^\pm/\mathbb{Z}$ in $\overline{\mathbb{C}/\mathbb{Z}}$.

The following construction is usually referred to as *conformal welding*. It is customarily studied in the case of non-smooth circle homeomorphisms and is trivial in the case of analytic circle diffeormorphisms.

The analytic circle diffeomorphism f may be viewed as an analytic diffeomorphism between the boundary of $\overline{\mathbb{H}^+/\mathbb{Z}}$ and the boundary of $\overline{\mathbb{H}^-/\mathbb{Z}}$. If we glue $\overline{\mathbb{H}^+/\mathbb{Z}}$ to $\overline{\mathbb{H}^-/\mathbb{Z}}$ via f, we obtain a Riemann surface which is isomorphic to $\overline{\mathbb{C}/\mathbb{Z}}$. We may choose the isomorphism ϕ such that $\phi(\pm\mathrm{i}\infty) = \pm\mathrm{i}\infty$. Such an isomorphism is not unique, but it is unique up to addition of a constant in $\mathbb{C}/\mathbb{Z}$. It restricts to univalent maps $\phi^\pm : \mathbb{H}^\pm/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ which extend univalently to neighborhoods of $\overline{\mathbb{H}^\pm/\mathbb{Z}}$ and satisfy $\phi^- \circ f = \phi^+$ near the boundary of $\overline{\mathbb{H}^+/\mathbb{Z}}$.

Holomorphy of $\phi^\pm$ near $\pm\mathrm{i}\infty$ yields that

$$\phi^\pm(z) = z + C^\pm + \mathrm{o}(1) \text{ as } z \to \pm\mathrm{i}\infty$$

for appropriate constants $C^\pm \in \mathbb{C}/\mathbb{Z}$. Since ϕ is unique up to addition of a constant, the difference

$$C_f := C^+ - C^-$$

only depends on f and will be referred as the *welding constant* of f.

Our first result, proved in Section 3, concerns the asymptotic behavior of $\tau_f(\omega)$ as $\omega \in \mathbb{C}/\mathbb{Z}$ tends to $+\mathrm{i}\infty$.

Theorem 1.1. *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving analytic circle diffeomorphism and let C_f be its welding constant. As ω tends to $+\mathrm{i}\infty$ in $\mathbb{C}/\mathbb{Z}$,*

$$\tau_f(\omega) = \omega + C_f + \mathrm{o}(1).$$

The ordinary rotation number of a circle homeomorphism $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is defined as follows. Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Such a lift is unique up to addition of an integer. The sequence of functions $\frac{1}{n}\big(F^{\circ n} - \mathrm{id}\big)$ converges uniformly to a constant function Θ. If we replace F by $F + k$ with $k \in \mathbb{Z}$, the limit Θ is replaced by $\Theta + k$, so that the value $\mathrm{rot}(f) \in \mathbb{R}/\mathbb{Z}$ of Θ modulo 1 only depends on f. This is the rotation number of f. Note that the rotation number is rational if and only if the circle homeomorphism has a periodic cycle.

Our second result, proved in Section 4.6, concerns the behavior of $\tau_f(\omega)$ as ω tends to $\mathbb{R}/\mathbb{Z}$. Recall that a periodic cycle of a circle diffeomorphism is called *parabolic* if its multiplier is 1, and it is called *hyperbolic* otherwise. A circle diffeomorphism with periodic cycles is called *hyperbolic* if it has only hyperbolic periodic cycles.

Theorem 1.2. *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving analytic circle diffeomorphism. Then, the function $\tau_f : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ has a continuous extension $\bar{\tau}_f : \overline{\mathbb{H}/\mathbb{Z}} \to \overline{\mathbb{H}/\mathbb{Z}}$. Assume $\omega \in \mathbb{R}/\mathbb{Z}$.*

- *If $\mathrm{rot}(f_\omega)$ is irrational, then $\bar{\tau}_f(\omega) = \mathrm{rot}(f_\omega)$.*
- *If $\mathrm{rot}(f_\omega) = p/q$ is rational, then $\bar{\tau}_f(\omega)$ belongs to the closed disk of radius $D_f/(\pi q^2)$ tangent to $\mathbb{R}/\mathbb{Z}$ at p/q; moreover*
  - *if f_ω has a parabolic cycle, then $\bar{\tau}_f(\omega) = \mathrm{rot}(f_\omega)$.*
  - *if f_ω is hyperbolic, then $\bar{\tau}_f(\omega) \in \mathbb{H}/\mathbb{Z}$, in particular $\bar{\tau}_f(\omega) \neq \mathrm{rot}(f_\omega)$.*

We shall also prove the following result.

Theorem 1.3. *There exist orientation preserving analytic circle diffeomorphisms $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ for which $\tau_f : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ fails to be univalent.*

2. Denjoy's Lemma

Before embarking into the proof of our results, we shall recall a classical result of Denjoy on the dynamics of circle diffeomorphisms.

The *distortion* of a diffeomorphism $f : I \to J$ is

$$\mathrm{dis}_I(f) = \max_{x,y\in I} \log \frac{f'(x)}{f'(y)}.$$

If $f : I \to J$ and $g : J \to K$ are diffeomorphisms, then

$$\mathrm{dis}_J(f^{-1}) = \mathrm{dis}_I(f) \quad \text{and} \quad \mathrm{dis}_I(g \circ f) \le \mathrm{dis}_I(f) + \mathrm{dis}_J(g).$$

Lemma 2.1 (Denjoy)**.** *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving diffeomorphism and $I \subset \mathbb{R}/\mathbb{Z}$ be an interval such that I, $f(I)$, $f^{\circ 2}(I)$, $\ldots$, $f^{\circ n}(I)$ are disjoint. Then,*

$$\mathrm{dis}_I(f^{\circ n}) \le D_f.$$

Proof. Let x and y be points in I. Set $x_k{:=}f^{\circ k}(x)$ and $y_k := f^{\circ k}(y)$. Then,

$$\begin{aligned}\left|\log(f^{\circ n})'(x) - \log(f^{\circ n})'(y)\right| &= \left|\sum_{k=0}^{n-1} \log f'(x_k) - \log f'(y_k)\right| \\ &\le \sum_{k=0}^{n-1} \left|\int_{x_k}^{y_k} \frac{f''(x)}{f'(x)}\,\mathrm{d}x\right| \le \int_{\mathbb{R}/\mathbb{Z}} \left|\frac{f''(x)}{f'(x)}\right|\,\mathrm{d}x = D_f. \ \square\end{aligned}$$

As a corollary, we have the following control on the multipliers of the periodic cycles of f.

Lemma 2.2. *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation preserving diffeomorphism and ρ be the multiplier of a cycle of f. Then, $|\log \rho| \le D_f$.*

Proof. The average of the derivative $(f^{\circ q})'$ along the circle $\mathbb{R}/\mathbb{Z}$ is equal to 1. As a consequence, there exists a point $x_0 \in \mathbb{R}/\mathbb{Z}$ such that $(f^{\circ q})'(x_0) = 1$. Any periodic cycle $\{x, f(x), \ldots, f^{\circ q}(x) = x\}$ divides the circle into disjoint intervals $I_1, \ldots, I_q$

which are permuted by f. Without loss of generality, we may assume that I_1 contains x and x_0. Then, according to the previous Lemma,

$$|\log\rho| = \left|\log(f^{\circ q})'(x)\right| = \left|\log\frac{(f^{\circ q})'(x)}{(f^{\circ q})'(x_0)}\right| \le \mathrm{dis}_{I_1}(f^{\circ q}) \le D_f. \qquad \square$$

3. Behavior of τ_f near $+\mathrm{i}\infty$

The proof of Theorem 1.1 goes as follows.

Step 1. The isomorphism between the complex torus $E(f_\omega)$ and $\mathcal{E}_{\tau_f(\omega)}$ induces a univalent map $\phi_\omega : A_\omega \to \mathbb{C}/\mathbb{Z}$ which extends univalently to a neighborhood of the closed annulus $\overline{A}_\omega$, with $\phi_\omega(f_\omega) = \phi_\omega + \tau_f(\omega)$ in a neighborhood of $\mathbb{R}/\mathbb{Z}$.

Step 2. As $\omega \to +\mathrm{i}\infty$, the sequence of univalent maps

$$\phi^+_\omega : z \mapsto \phi_\omega(z) - \phi_\omega(0)$$

converges locally uniformly in $\mathbb{H}^+/\mathbb{Z}$ to a limit $\phi^+ : \mathbb{H}^+/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$, and the sequence of univalent maps

$$\phi^-_\omega : z \mapsto \phi_\omega(z+\omega) - \phi_\omega\big(f(0)+\omega\big)$$

converges locally uniformly in $\mathbb{H}^-/\mathbb{Z}$ to a limit $\phi^- : \mathbb{H}^-/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$. In addition, the maps $\phi^\pm : \mathbb{H}^+/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ form a pair of univalent maps provided by the welding construction.

Step 3. Comparing constant Fourier coefficients of ϕ_ω, ϕ^+ and ϕ^-, we deduce that as $\omega \to +\mathrm{i}\infty$, we have

$$C^+ + \phi_\omega(0) = -\omega + C^- + \phi_\omega\big(f(0)+\omega\big) + \mathrm{o}(1),$$

whence

$$\tau_f(\omega) = \phi_\omega\big(f(0)+\omega\big) - \phi_\omega(0) = \omega + C^+ - C^- + o(1) = \omega + C_f + \mathrm{o}(1).$$

3.1. **The map ϕ_ω.** Let $\delta > 0$ be sufficiently tiny so that $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ extends univalently to the annulus $B_\delta := \big\{z \in \mathbb{C}/\mathbb{Z} \ :\ \delta > |\operatorname{Im}(z)|\big\}$. Set

$$A^+_\omega := A_\omega \cup B_\delta \cup \big(\omega + f(B_\delta)\big).$$

The complex torus $E(f_\omega)$ is the quotient of A^+_ω where $z \in B_\delta$ is identified to $f_\omega(z) \in f(B_\delta) + \omega$.

An isomorphism between $E(f_\omega)$ and $\mathcal{E}_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ sending the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $E(f_\omega)$ to the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $\mathcal{E}_{\tau_f(\omega)}$ will lift to a univalent map $\phi_\omega : A^+_\omega \to \mathbb{C}/\mathbb{Z}$ sending $\mathbb{R}/\mathbb{Z}$ to a curve homotopic to $\mathbb{R}/\mathbb{Z}$, preserving orientation. The following relation then holds on B_δ:

$$\phi_\omega(f_\omega) = \phi_\omega + \tau_f(\omega).$$

3.2. **Convergence of $\phi^\pm_\omega$.** As $\omega \to +\mathrm{i}\infty$, the open sets A^+_ω eat every compact subset of $\mathbb{H}^+/\mathbb{Z} \cup B_\delta$. The sequence of univalent maps $\phi^+_\omega : A^+_\omega \to \mathbb{C}/\mathbb{Z}$ defined by

$$\phi^+_\omega(z) := \phi_\omega(z) - \phi_\omega(0)$$

is normal and any limit value $\phi^+ : \mathbb{H}^+/\mathbb{Z} \cup B_\delta$ satisfies $\phi^+(0) = 0$. It cannot be constant since each ϕ^+_ω sends $\mathbb{R}/\mathbb{Z}$ to a homotopically nontrivial curve in $\mathbb{C}/\mathbb{Z}$ passing through 0. So, any limit value $\phi^+ : \mathbb{H}^+/\mathbb{Z} \cup B_\delta \to \mathbb{C}/\mathbb{Z}$ is univalent.

Similarly, as $\omega \to +\mathrm{i}\infty$, the open sets

$$A^-_\omega := -\omega + A^+_\omega$$

eat every compact subset of $\mathbb{H}^-/\mathbb{Z}\cup f(B_\delta)$. In addition, the sequence of univalent maps $\phi_\omega^- : A_\omega^- \to \mathbb{C}/\mathbb{Z}$ defined by

$$\phi_\omega^-(z) := \phi_\omega(z+\omega) - \phi_\omega\big(f(0)+\omega\big)$$

is normal and any limit value $\phi^- : \mathbb{H}/\mathbb{Z}\cup f(B_\delta) \to \mathbb{C}/\mathbb{Z}$ is univalent and satisfies $\phi^-\big(f(0)\big) = 0$.

Passing to the limit on the following relation, valid on B_δ:

$$\begin{aligned}\phi_\omega^- \circ f(z) &= \phi_\omega\big(f(z)+\omega\big) - \phi_\omega\big(f(0)+\omega\big)\\ &= \phi_\omega(z) + \tau_f(\omega) - \phi_\omega\big(f(0)+\omega\big) = \phi_\omega(z) - \phi_\omega(0) = \phi_\omega^+(z),\end{aligned}$$

we get the following relation, valid on B_δ:

$$\phi^- \circ f = \phi^+.$$

It follows that the pair (ϕ^-,ϕ^+) induces an isomorphism from $\big(A_\omega^+ \sqcup A_\omega^-\big)/f$ (we identify $z\in B_\delta \subseteq A_\omega^+$ to $f(z)\in f(B_\delta)\subseteq A_\omega^-$) to $\mathbb{C}/\mathbb{Z}$. Therefore, ϕ^- and ϕ^+ coincide with the unique isomorphisms arising from the welding construction, normalized by the conditions $\phi^+(0) = \phi^-\big(f(0)\big) = 0$. This uniqueness shows that there is only one possible pair of limit values. Thus, the sequences $\phi_\omega^- : A_\omega^- \to \mathbb{C}/\mathbb{Z}$ and $\phi_\omega^+ : A_\omega^+ \to \mathbb{C}/\mathbb{Z}$ are convergent.

3.3. **Comparing Fourier coefficients.** Note that $z\mapsto \phi_\omega^\pm(z) - z$ and $z\mapsto\phi^\pm(z)$ are well-defined on $\mathbb{R}/\mathbb{Z}$ with values in $\mathbb{C}$. The previous convergence implies:

$$C_\omega^+ := \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega^+(z) - z\big)\,\mathrm{d}z \underset{\omega\to+\mathrm{i}\infty}{\longrightarrow} C^+ := \int_{\mathbb{R}/\mathbb{Z}}\big(\phi^+(z)-z\big)\,\mathrm{d}z$$

and

$$C_\omega^- := \int_{\mathbb{R}/\mathbb{Z}} \big(\phi_\omega^-(z) - z\big)\,\mathrm{d}z \underset{\omega\to+\mathrm{i}\infty}{\longrightarrow} C^- := \int_{\mathbb{R}/\mathbb{Z}}\big(\phi^-(z)-z\big)\,\mathrm{d}z.$$

Since ϕ_ω is holomorphic on A_ω^+, we have

$$\int_{\mathbb{R}/\mathbb{Z}}\big(\phi_\omega(z)-z\big)\,\mathrm{d}z = \int_{\omega+\mathbb{R}/\mathbb{Z}}\big(\phi_\omega(z)-z\big)\,\mathrm{d}z = \int_{\mathbb{R}/\mathbb{Z}}\big(\phi_\omega(t+\omega)-t\big)\,\mathrm{d}t - \omega.$$

Thus,

$$\begin{aligned}C_\omega^+ &:= \int_{\mathbb{R}/\mathbb{Z}}\big(\phi_\omega^+(z)-z\big)\,\mathrm{d}z\\ &= \int_{\mathbb{R}/\mathbb{Z}}\big(\phi_\omega(z)-z\big)\,\mathrm{d}z - \phi_\omega(0)\\ &= \int_{\mathbb{R}/\mathbb{Z}}\big(\phi_\omega(t+\omega)-t\big)\,\mathrm{d}t - \omega - \phi_\omega(0)\\ &= \int_{\mathbb{R}/\mathbb{Z}}\big(\phi_\omega^-(t)-t\big)\,\mathrm{d}t - \omega + \phi_\omega\big(f(0)+\omega\big) - \phi_\omega(0) = C_\omega^- - \omega + \tau_f(\omega).\end{aligned}$$

As $\omega\to+\mathrm{i}\infty$, we therefore have

$$C^+ + \mathrm{o}(1) = C^- + \mathrm{o}(1) - \omega + \tau_f(\omega)$$

which yields

$$\tau_f(\omega) = \omega + C^+ - C^- + \mathrm{o}(1) = \omega + C_f + \mathrm{o}(1).$$

4. Behavior of τ_f near $\mathbb{R}/\mathbb{Z}$

The proof of Theorem 1.2 goes as follows.

Step 1. Recall that a number $\theta \in \mathbb{R}/\mathbb{Z}$ is *Diophantine* if there are constants $c > 0$ and $\beta > 0$ such that for all rational numbers $p/q \in \mathbb{Q}/\mathbb{Z}$, we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^{2+\beta}}.$$

Theorem 4.1 (V. Moldavskis [5]). *If $\omega \in \mathbb{R}/\mathbb{Z}$ and if $\mathrm{rot}(f_\omega)$ is Diophantine, then*

$$\lim_{\substack{y \to 0 \\ y > 0}} \tau_f(\omega + \mathrm{i}y) = \mathrm{rot}(f_\omega).$$

Step 2. If $\omega \in \mathbb{R}/\mathbb{Z}$ and $\mathrm{rot}(f_\omega)$ is rational, then the conclusion of Theorem 4.1 is not true. This fact was first proved by Yu. Ilyashenko and V. Moldavkis [4]. We do not formulate their result since we will use its later generalized version.

Theorem 4.2 (N. Goncharuk [3]). *If $\omega \in \mathbb{R}/\mathbb{Z}$, if $\mathrm{rot}(f_\omega)$ is rational and if f_ω is hyperbolic, then τ_f extends analytically to a neighborhood of ω.*

In the following, we shall denote by $\bar{\tau}_f(\omega)$ this extension of τ_f at ω.

Step 3. Recall that $\theta \in \mathbb{R}/\mathbb{Z}$ is *Liouville* if it is irrational but not Diophantine. We use the following result of Tsujii.

Theorem 4.3 (M. Tsujii [7]). *The set of $\omega \in \mathbb{R}/\mathbb{Z}$ such that $\mathrm{rot}(f_\omega)$ is Liouville has zero Lebesgue measure.*

It implies that almost every $\omega \in \mathbb{R}/\mathbb{Z}$ satisfies assumptions of either Theorem 4.1, or Theorem 4.2 (note that the set of ω such that f_ω has a parabolic cycle is countable).

Step 4. If f_ω has rational rotation number p/q, we denote by $\mathrm{Per}(f_\omega)$ the set of periodic points of $f_\omega : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. For $x \in \mathrm{Per}(f_\omega)$, we denote by ρ_x the multiplier of f as a fixed point of $f^{\circ q}$. Our contribution starts with the following result. It is an analog of the Yoccoz Inequality which bounds the multiplier of a fixed point of a polynomial in terms of its combinatorial rotation number [2].

Lemma 4.4. *Assume that f_ω is a hyperbolic map with rational rotation number p/q. Then, $\bar{\tau}_f(\omega)$ belongs to the disk tangent to $\mathbb{R}/\mathbb{Z}$ at p/q with radius*

$$R_\omega := \frac{1}{\pi q \cdot \displaystyle\sum_{x \in \mathrm{Per}(f_\omega)} \frac{1}{|\log \rho_x|}}.$$

In addition, $R_\omega \le D_f/(\pi q^2)$.

The cardinal of $\mathrm{Per}(f_\omega)$ is at least q and according to Lemma 2.2, for each $x \in \mathrm{Per}(f_\omega)$ we have $|\log \rho_x| \le D_f$. This yields the upper bound $R_\omega \le D_f/(\pi q^2)$.

Step 5. Let $\bar{\tau}_f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ be defined by

- $\bar{\tau}_f(\omega) := \mathrm{rot}(f_\omega)$ if the rotation number of f_ω is irrational or if f_ω has a parabolic cycle and
- $\bar{\tau}_f(\omega) := \lim\limits_{\substack{y \to 0 \\ y > 0}} \tau_f(\omega + \mathrm{i}y)$ if f_ω is hyperbolic.

Lemma 4.5. *The function $\bar{\tau}_f$ is continuous on $\mathbb{R}/\mathbb{Z}$.*

It is particularly difficult to prove the continuity of $\bar{\tau}_f$ at points $\omega \in \mathbb{R}/\mathbb{Z}$ for which f_ω has hyperbolic and parabolic cycles which bifurcate into complex conjugate cycles. The other cases follow easily from Theorem 4.2 and Lemma 4.4.

Step 6. The holomorphic map $\tau_f : \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ has radial limits on $\mathbb{R}/\mathbb{Z}$ almost everywhere, and those limits coincide with the continuous map $\bar{\tau}_f$. It follows easily that τ_f extends continuously by $\bar{\tau}_f$ to $\mathbb{R}/\mathbb{Z}$.

4.1. **The Diophantine case.** We include a proof of Theorem 4.1. The proof relies on the following lemma on quasiconformal maps which is classical.

Lemma 4.6. *Suppose that there exists a K-quasiconformal map between two complex tori E_1 and E_2. Then*

$$\mathrm{dist}_{\mathbb{H}}(\tau(E_1), \tau(E_2)) \leq \log K$$

where $\mathrm{dist}_{\mathbb{H}}$ is the hyperbolic distance in $\mathbb{H}$, and where $\tau(E_1) \in \mathbb{H}$ and $\tau(E_2) \in \mathbb{H}$ are moduli with respect to corresponding generators in $H_1(E_1)$ and $H_1(E_2)$.

Without loss of generality, we may assume that $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has Diophantine rotation number $\theta \in \mathbb{R}/\mathbb{Z}$. A theorem of Yoccoz (see [8]) asserts that there is an analytic circle diffeomorphism $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ conjugating the rotation of angle θ to f: for all $x \in \mathbb{R}/\mathbb{Z}$, we have

$$\phi(x+\theta) = f \circ \phi(x).$$

Let $\hat{\phi} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ be the homeomorphism defined by

$$\hat{\phi}(z) = \phi\big(\mathrm{Re}(z)\big) + \mathrm{i}\,\mathrm{Im}(z).$$

Then, $\hat{\phi} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ is a K-quasiconformal homeomorphism with

$$K := \max\big(\|\phi'\|_\infty, \|1/\phi'\|_\infty\big).$$

Now, for any $y > 0$,

$$\hat{\phi}(x+\theta+\mathrm{i}y) = f\big(\hat{\phi}(x)\big) + iy,$$

and so, $\hat{\phi}$ induces a K-quasiconformal homeomorphism between the complex tori $\mathbb{C}/\big(\mathbb{Z} + (\theta+\mathrm{i}y)\mathbb{Z}\big)$ and $E(f_{\mathrm{i}y})$. It follows that for $y > 0$, the hyperbolic distance in $\mathbb{H}/\mathbb{Z}$ between $\theta + \mathrm{i}y$ and $\tau_f(\mathrm{i}y)$ is uniformly bounded and thus,

$$\lim_{\substack{y \to 0 \\ y > 0}} \tau_f(\mathrm{i}y) = \theta.$$

4.2. **The hyperbolic case.** We recall the arguments of the proof of Theorem 4.2 given in [3] . It is based on an auxiliary construction of a complex torus $E(f)$ when $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has rational rotation number and is hyperbolic. This construction will be used again in the proofs of Lemmas 4.4 and 4.5.

Let us assume $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has rational rotation number p/q and has only hyperbolic periodic cycles. The number $m \geq 1$ of attracting cycles is equal to the number of repelling cycles. Denote by α_j, $j \in \mathbb{Z}/(2mq)\mathbb{Z}$, the periodic points of f, ordered cyclically; even indices correspond to attracting periodic points and odd indices to repelling periodic points. Note that $f(\alpha_j) = \alpha_{j+2mp}$.

Let ρ_j be the multiplier of α_j as a fixed point of $f^{\circ q}$ and $\phi_j : (\mathbb{C}, 0) \to (\mathbb{C}/\mathbb{Z}, \alpha_j)$ be the linearizing map which conjugates multiplication by ρ_j to $f^{\circ q}$:

$$f^{\circ q} \circ \phi_j(z) = \phi_j(\rho_j z)$$

and is normalized by $\phi_j'(0) = 1$. Then,

$$f \circ \phi_j(z) = \phi_{j+2mp}(\lambda_j \cdot z) \quad \text{with} \quad \lambda_j := f'(\alpha_j).$$

In addition, if $\varepsilon > 0$ is small enough, the linearizing map ϕ_j extends univalently to the strip $\{z \in \mathbb{C} \ : \ |\operatorname{Im}(z)| < \varepsilon\}$ and

$$\phi_j(\mathbb{R}) = (\alpha_{j-1}, \alpha_{j+1}).$$

For each $j \in \mathbb{Z}/(2mq)\mathbb{Z}$, let x_j be a point in (α_j, α_{j+1}), so that

- $f(x_j) \in (\alpha_{j+2pm}, x_{j+2pm})$ if the orbit of α_j attracts (i.e. j is even) and
- $f(x_j) \in (x_{j+2pm}, \alpha_{j+2pm+1})$ if the orbit of α_j repels (i.e. j is odd).

This is possible since $f^{\circ q}(x_j) \in (\alpha_j, x_j)$ when j is even and $f^{\circ q}(x_j) \in (x_j, a_{j+1})$ when j is odd. Similarly, let ε_j be a point on the negative imaginary axis if j is even and on the positive imaginary axis if j is odd, so that for all $j \in \mathbb{Z}/(2mp\mathbb{Z})$,

- $|\varepsilon_j| < \varepsilon$, $|\lambda_j \varepsilon_j| < \varepsilon$ and
- $\lambda_j \varepsilon_j$ is above ε_{j+2mp}.

Let C_j be the arc of circle with endpoints $\phi_j^{-1}(x_{j-1})$ and $\phi_j^{-1}(x_j)$ passing through ε_j and set

$$\gamma := \bigcup_{j \in \mathbb{Z}/(2mq\mathbb{Z})} \phi_j(C_j).$$

Then, γ is a simple closed curve in $\mathbb{C}/\mathbb{Z}$ and f is univalent in a neighborhood of γ.

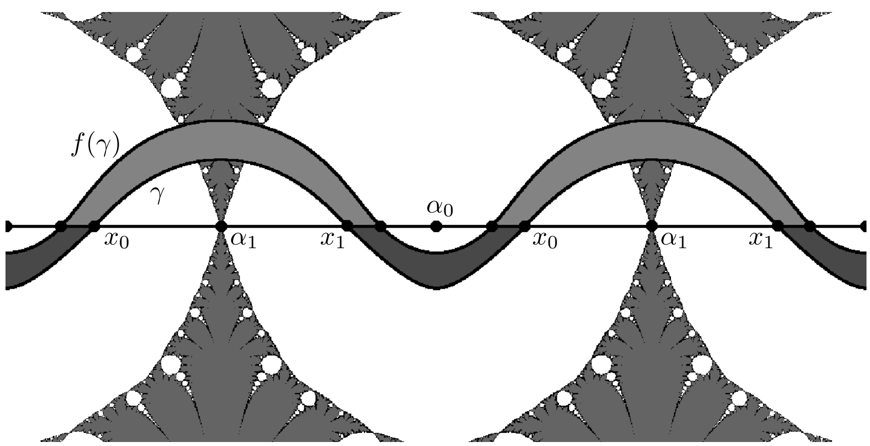

FIGURE 1. A possible choice of curve γ for the circle diffeomorphism $\mathbb{R}/\mathbb{Z} \ni x \mapsto x + \dfrac{1}{4\pi} \sin(2\pi x) \in \mathbb{R}/\mathbb{Z}$. There is an attracting fixed point at $\alpha_0 := 0 \in \mathbb{R}/\mathbb{Z}$ and a repelling fixed point at $\alpha_1 := 1/2 \in \mathbb{R}/\mathbb{Z}$. The basin of attraction of 0 in $\mathbb{C}/\mathbb{Z}$ is white.

The attracting cycles of f are above γ in $\mathbb{C}/\mathbb{Z}$ and the repelling cycles are below γ in $\mathbb{C}/\mathbb{Z}$. In addition,

$$f(\gamma) = \bigcup_{j \in \mathbb{Z}/(2mq\mathbb{Z})} \phi_{j+2mp}(\lambda_j C_j)$$

and so, $f(\gamma)$ lies above γ in $\mathbb{C}/\mathbb{Z}$.

For ω sufficiently close to 0, the curve $f_\omega(\gamma) = f(\gamma) + \omega$ remains above γ in $\mathbb{C}/\mathbb{Z}$. The curves γ and $f_\omega(\gamma)$ bound an essential annulus in $\mathbb{C}/\mathbb{Z}$. Glueing the two sides via f_ω, we obtain a complex torus $\mathfrak{E}(f_\omega)$, which may be uniformized as $\mathcal{E}_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for some appropriate $\tau \in \mathbb{H}/\mathbb{Z}$, the homotopy class of γ in $\mathfrak{E}(f_\omega)$ corresponding to the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $\mathcal{E}_\tau$. We set $\bar{\tau}_f(\omega) := \tau \in \mathbb{H}/\mathbb{Z}$.

According to Risler [6, Chapter 2, Proposition 2], the map $\omega \mapsto \bar{\tau}_f(\omega)$ is holomorphic. When $\omega \in \mathbb{H}/\mathbb{Z}$, the complex torus $\mathfrak{E}(f_\omega)$ is isomorphic to $E(f_\omega)$ and the homotopy class of γ in $\mathfrak{E}(f_\omega)$ corresponds to the homotopy class of $\mathbb{R}/\mathbb{Z}$ in $E(f_\omega)$ (see [3] for details). As a consequence, $\bar{\tau}_f(\omega) = \tau_f(\omega)$ when $\omega \in \mathbb{H}/\mathbb{Z}$ is sufficiently close to 0. This completes the proof of Theorem 4.2 for $\omega = 0$.

4.3. **The Liouville case.** For completeness, we now present a proof of Tsujii's Theorem 4.3 which we believe is a simplification of the original one, although the ideas are essentially the same. The main argument in Tsujii's proof is the following.

Proposition 4.7. *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a $\mathcal{C}^2$-smooth orientation preserving circle diffeomorphism with irrational rotation number $\theta \in \mathbb{R}/\mathbb{Z}$. If p/q is an approximant to θ given by the continued fraction algorithm, then there is an $\omega \in \mathbb{R}/\mathbb{Z}$ satisfying*

$$|\omega| < \mathrm{e}^{D_f} \cdot |\theta - p/q| \quad \textit{and} \quad \mathrm{rot}(f_\omega) = p/q.$$

Proof. According to a Theorem of Denjoy, there is a homeomorphism $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $\phi(x + \theta) = f \circ \phi(x)$ for all $x \in \mathbb{R}/\mathbb{Z}$.

Without loss of generality, let us assume that $\theta < p/q$ and set $\delta := p - q\theta$. Let $T \subset \mathbb{R}/\mathbb{Z}$ be the union of intervals

$$T := \bigcup_{1 \le j \le q} T_j \quad \text{with} \quad T_j := (j\theta, j\theta + \delta).$$

Since p/q is an approximant of θ, this is a disjoint union of q intervals of length δ. According to Lemma 4.8 below, we may choose $t \in \mathbb{R}/\mathbb{Z}$ such that the Lebesgue measure of $\phi(T + t)$ is at most $q\delta$.

Now, set $x := \phi(t)$ and for $j \in \mathbb{Z}$, set

$$x_j := f^{\circ j}(x) = \phi(t + j\theta) \quad \text{and} \quad I_j := (x_j, x_{j-q}) = \phi(T_j).$$

The intervals I_1, $I_2 = f(I_1)$, $\ldots$, $I_q = f^{\circ q}(I_1)$ are disjoint and the sum of their lengths satisfies

$$\sum_{j=1}^{q} |I_j| \le q\delta = q^2 \cdot |\theta - p/q|.$$

As $\omega \in \mathbb{R}/\mathbb{Z}$ increases from 0, the rotation number $\mathrm{rot}(f_\omega) \in \mathbb{R}/\mathbb{Z}$ increases from θ, and there is a first ω_0 such that $\mathrm{rot}(f_{\omega_0}) = p/q$. For $j \in [0, q]$, set

$$y_j := (f_{\omega_0})^{\circ j}(x) \quad \text{and} \quad z_j := f^{\circ(q-j)}(y_j).$$

Finally, for $j \in [1, q]$, set

$$J_j := \big(f(y_{j-1}), y_j\big) = \big(f(y_{j-1}), f(y_{j-1}) + \omega_0\big) \quad \text{and} \quad K_j := (z_{j-1}, z_j).$$

Then, $(z_0, z_1, \ldots, z_q)$ is a subdivision of (z_0, z_q) (see Figure 2).

As ω increases from 0 to ω_0, the point $(f_\omega)^{\circ q}(x)$ increases from x_q to y_q but remains in I_q since $\mathrm{rot}(f_\omega)$ remains less than p/q. Thus, $(z_0, z_q) = (x_q, y_q) \subseteq I_q$ and so,

$$|I_q| \ge |z_q - z_0| = \sum_{j=1}^{q} |K_j|.$$

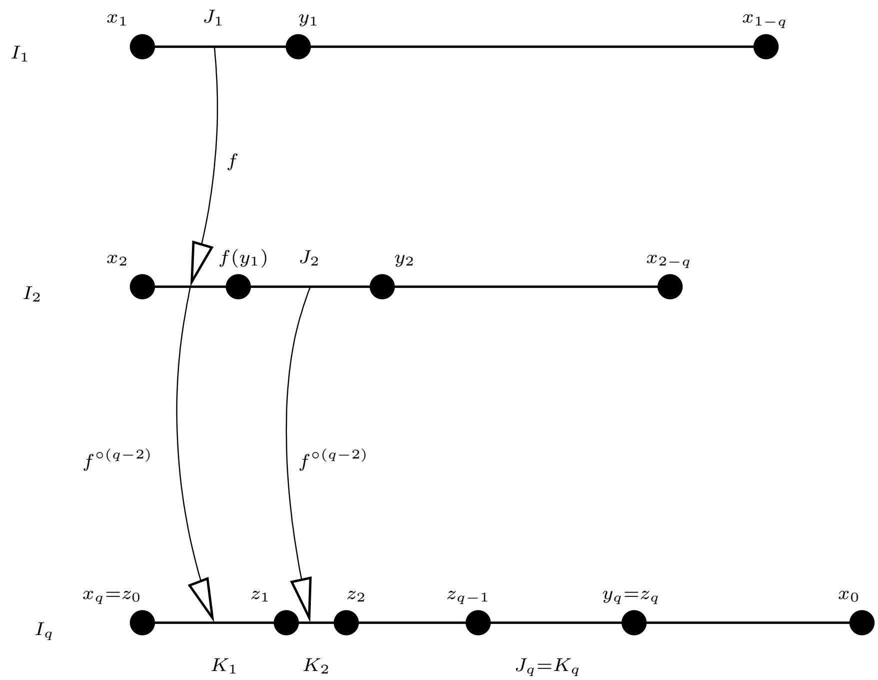

FIGURE 2. The intervals I_j, J_j and K_j.

In addition, $J_j \subset I_j$ and $K_j = f^{\circ(q-j)}(J_j)$. It follows from Denjoy's Lemma 2.1 that

$$\frac{|K_j|}{|I_q|} \geq \mathrm{e}^{-D_f}\frac{|J_j|}{|I_j|} = \mathrm{e}^{-D_f}\frac{\omega_0}{|I_j|}.$$

Now, according to the Cauchy-Schwarz Inequality, we have

$$q^2 = \left(\sum_{j=1}^{q}\sqrt{|I_j|}\cdot\frac{1}{\sqrt{|I_j|}}\right)^2 \leq \left(\sum_{j=1}^{q}|I_j|\right)\cdot\left(\sum_{j=1}^{q}\frac{1}{|I_j|}\right) \leq q^2\cdot|\theta - p/q|\cdot\sum_{j=1}^{q}\frac{1}{|I_j|}.$$

Thus,

$$|I_q| \geq \sum_{j=1}^{q}|K_j| \geq \mathrm{e}^{-D_f}\omega_0|I_q|\cdot\sum_{j=1}^{q}\frac{1}{|I_j|} \geq \frac{\mathrm{e}^{-D_f}\omega_0|I_q|}{|\theta - p/q|}$$

and so,

$$\omega_0 \leq \mathrm{e}^{D_f}\cdot|\theta - p/q|. \qquad \square$$

Lemma 4.8. *Let $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a homeomorphism. Then, for any measurable set $T \subseteq \mathbb{R}/\mathbb{Z}$, there is a $t \in \mathbb{R}/\mathbb{Z}$ such that*

$$\mathrm{Leb}\big(\phi(T+t)\big) \leq \mathrm{Leb}(T).$$

Proof. Let μ be the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. According to Tonelli's theorem,

$$\begin{aligned}\int_{t\in\mathbb{R}/\mathbb{Z}} \mu\big(\phi(T+t)\big)\ \mathrm{d}t &= \int_{t\in\mathbb{R}/\mathbb{Z}} \left(\int_{u\in T+t} \mathrm{d}(\phi^*\mu)\right)\ \mathrm{d}\mu \\ &= \int_{u\in\mathbb{R}/\mathbb{Z}} \left(\int_{t\in -T+u} \mathrm{d}\mu\right)\ \mathrm{d}(\phi^*\mu) \\ &= \int_{u\in\mathbb{R}/\mathbb{Z}} \mu(T)\ \mathrm{d}(\phi^*\mu) \\ &= \mu(T)\cdot\mu\big(\phi(\mathbb{R}/\mathbb{Z})\big) = \mu(T).\end{aligned}$$

So, the average of $\mu\big(\phi(T+t)\big)$ with respect to t is equal to $\mu(T)$ and the result follows. $\square$

Theorem 4.3 follows easily from Proposition 4.7: for $\beta > 0$, let S_β be the set of $\omega \in \mathbb{R}/\mathbb{Z}$ such that $\mathrm{rot}(f_\omega)$ is irrational and such that there are infinitely many $p, q \in \mathbb{Z}$ satisfying $\big|\mathrm{rot}(f_\omega) - p/q\big| < 1/q^{2+\beta}$. The set of $\omega \in \mathbb{R}/\mathbb{Z}$ such that $\mathrm{rot}(f_\omega)$ is Liouville is the intersection of the sets S_β. So, it is sufficient to show that the $\mathrm{Leb}(S_\beta) = 0$ for all $\beta > 0$. Note that

$$S_\beta = \limsup_{q\to+\infty} S_{\beta,q}$$

where $S_{\beta,q}$ is the set of $\omega \in \mathbb{R}/\mathbb{Z}$ such that $\mathrm{rot}(f_\omega)$ is irrational and such that $\big|\mathrm{rot}(f_\omega) - p/q\big| < 1/q^{2+\beta}$ for some approximant p/q of $\mathrm{rot}(f_\omega)$.

Proposition 4.7 implies that $S_{\beta,q}$ is located in the $C/q^{2+\beta}$-neighborhood of the union of q intervals where the rotation number is rational with denominator q, where $C := e^{D_f}$. So,

$$\mathrm{Leb}(S_{\beta,q}) \le 2q\cdot\frac{C}{q^{2+\beta}} = \frac{2C}{q^{1+\beta}}.$$

In particular, for all $\beta > 0$,

$$\mathrm{Leb}(S_\beta) = \mathrm{Leb}\left(\limsup_{q\to+\infty} S_{\beta,q}\right) \le \limsup_{q\to+\infty}\sum_{r\ge q}\frac{2C}{r^{1+\beta}} = 0.$$

4.4. **Back to the hyperbolic case.** We now come to our contribution, starting with the proof of Lemma 4.4. Assume $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has rational rotation number p/q and has only hyperbolic periodic cycles. As in Section 4.2, consider a simple closed curve γ oscillating between the attracting cycles of f (which are above γ in $\mathbb{C}/\mathbb{Z}$) and the repelling cycles of f (which are below γ in $\mathbb{C}/\mathbb{Z}$), so that $f(\gamma)$ lies above γ in $\mathbb{C}/\mathbb{Z}$.

The curves γ and $f(\gamma)$ bound an essential annulus in $\mathbb{C}/\mathbb{Z}$. Glueing the curves via f, we obtain a complex torus $\mathfrak{E}(f)$ isomorphic to $\mathcal{E}_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $\tau := \bar{\tau}_0(f) \in \mathbb{H}/\mathbb{Z}$, the class of γ in $\mathfrak{E}(f)$ corresponding to the class of $\mathbb{R}/\mathbb{Z}$ in $\mathcal{E}_\tau$.

The projection of $\mathbb{R}/\mathbb{Z}$ in $\mathfrak{E}(f)$ consists of $2m$ topological circles cutting $\mathfrak{E}(f)$ into $2m$ annuli associated to the cycles of f. More precisely, each attracting (respectively repelling) cycle c has a basin of attraction B_c for f (respectively for f^{-1}) and the projection of $\mathbb{H}^- \cap B_c$ (respectively $\mathbb{H}^+ \cap B_c$) in $\mathfrak{E}(f)$ is an annulus A_c of modulus

$$\mathrm{mod}\ A_c = \frac{\pi}{|\log\rho_c|},$$

where ρ_c is the multiplier of c as a cycle of f.

Those annuli wind around the class of γ in $\mathfrak{E}(f)$ with combinatorial rotation number $-p/q$. It follows from a classical length-area argument (see [2, Proposition 3.3] for example) that there is a representative $\tilde\tau\in\mathbb{H}$ of $\tau\in\mathbb{H}/\mathbb{Z}$ such that

$$\sum_{c\text{ cycle of }f}\mod A_c\leq\frac{\operatorname{Im}(\tilde\tau)}{|-p+q\tilde\tau|^2}.$$

As a consequence,

$$\frac{|\tilde\tau-p/q|^2}{\operatorname{Im}\tilde\tau}\leq R_\omega:=\frac{1}{\pi q^2\cdot\displaystyle\sum_{c\text{ cycle of }f}\mod A_c},$$

which yields Lemma 4.4 since

$$\sum_{c\text{ cycle of }f}\mod A_c=\sum_{c\text{ cycle of }f}\frac{\pi}{|\log\rho_c|}=\frac{1}{q}\sum_{x\in\operatorname{Per}(f)}\frac{\pi}{|\log\rho_x|}.$$

Before going further, we shall establish a result that will be used in the proof of Lemma 4.5. Recall that the curve γ intersects the interval (α_j,α_{j+1}) at the point x_j, belongs to the lower half-plane below the segment (x_{j-1},x_j) if j is even and to the upper half-plane above the segment (x_{j-1},x_j) if j is odd.

Recall that m is the number of attracting cycles of f. The projection of $\mathbb{R}/\mathbb{Z}$ in $\mathfrak{E}(f^{\circ q})$ cuts the torus in $2mq$ annuli A_j, $j\in\mathbb{Z}/(2mq)\mathbb{Z}$, which wind around the class of γ with combinatorial rotation number 0 and have moduli

$$\mod A_j=m_j:=\frac{\pi}{|\log\rho_j|}.$$

Let $S_j\subset\mathbb{C}$ and $B_j\subset\mathbb{C}/\mathbb{Z}$ be defined by

$$S_j:=\{z\in\mathbb{C}\ :\ 0<\operatorname{Im}(z)<m_j\}\quad\text{and}\quad B_j:=S_j/\mathbb{Z}.$$

Set

$$\tilde r_j:=\frac{\log\phi_j^{-1}(x_j)}{\log\rho_j}\quad\text{and}\quad\tilde s_j:=\frac{\log|\phi_j^{-1}(x_{j-1})|}{\log\rho_j}+\frac{\mathrm{i}\pi}{|\log\rho_j|}.$$

The class r_j of $\tilde r_j$ in $\mathbb{C}/\mathbb{Z}$ belongs to the lower boundary component $C_j^-:=\mathbb{R}/\mathbb{Z}$ of B_j and the class s_j of $\tilde s_j$ in $\mathbb{C}/\mathbb{Z}$ belongs to the upper boundary component $C_j^+:=(\mathbb{R}+\mathrm{i}m_j)/\mathbb{Z}$ of B_j. The map $z\mapsto\phi_j\circ\exp(z\cdot\log\rho_j)$ induces an isomorphism $\chi_j:B_j\to A_j$ which extends analytically to the boundary, sends r_j to the class of x_j in $\mathfrak{E}(f^{\circ q})$ and s_j to the class of x_{j-1} in $\mathfrak{E}(f^{\circ q})$ (see Figure 3).

Lemma 4.9. *We have that*

$$\operatorname{dist}_{\mathbb{H}/\mathbb{Z}}\left(q\tau,-\frac{1}{\sigma}\right)\leq 5D_f\quad\textit{with}\quad\sigma:=\sum_{j\in\mathbb{Z}/2mqZ}\tilde s_j-\tilde r_j.$$

Proof. It will be more convenient to work with $f^{\circ q}$ whose rotation number is $0/1$. The diffeomorphism f induces an automorphism of $\mathfrak{E}(f^{\circ q})$ of order q. The quotient of $\mathfrak{E}(f^{\circ q})$ by this automorphism is isomorphic to $\mathfrak{E}(f)$. The class of γ in $\mathfrak{E}(f)$ has q disjoint preimages in $\mathfrak{E}(f^{\circ q})$ which map with degree 1 to γ. It follows that $\mathfrak{E}(f^{\circ q})$ is isomorphic to $\mathcal{E}_{q\tau}:=\mathbb{C}/(\mathbb{Z}+q\tau\mathbb{Z})$, the class of γ in $\mathfrak{E}(f^{\circ q})$ corresponding to the class of $\mathbb{R}/\mathbb{Z}$ in $\mathcal{E}_{q\tau}$.

Set $\mathcal{E}_\sigma:=\mathbb{C}/(\mathbb{Z}+\sigma\mathbb{Z})$. We will now construct a K-quasiconformal map

$$\psi:\mathfrak{E}(f^{\circ q})\to\mathcal{E}_\sigma$$

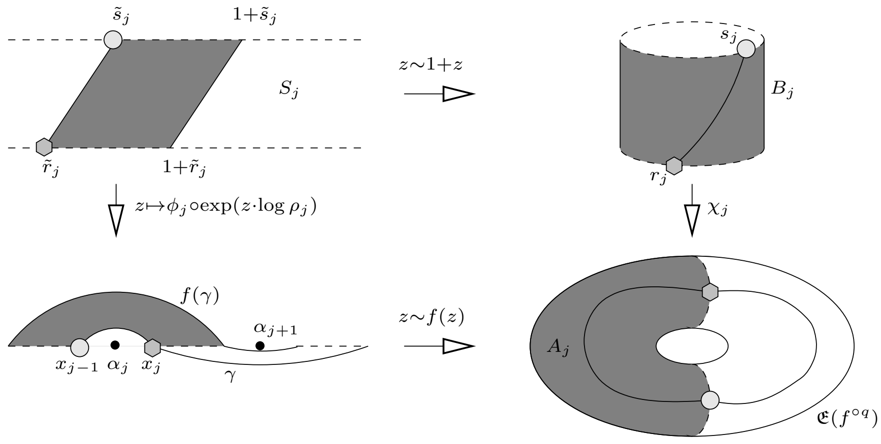

FIGURE 3. The projection of $\mathbb{R}/\mathbb{Z}$ in $\mathfrak{E}(f^{\circ q})$ cuts the torus in $2mq$ annuli A_j, $j \in \mathbb{Z}/(2mq)\mathbb{Z}$.

which sends the class of $\mathbb{R}/\mathbb{Z}$ in $\mathfrak{E}(f^{\circ q})$ to the class of $\sigma\mathbb{R}/\sigma\mathbb{Z}$ in $\mathcal{E}_\sigma$. We will also show that $\log K \le 5D_f$. The result then follows from Lemma 4.6.

On the one hand, glueing the lower boundary component C_j^- of B_j with the upper boundary component C_{j+1}^+ of B_{j+1} via the analytic diffeomorphism

$$\xi_j := \chi_{j+1}^{-1} \circ \chi_j : C_j^- \to C_{j+1}^+,$$

we obtain a complex torus E which is isomorphic to $\mathfrak{E}(f^{\circ q})$. Let δ_j be the projection of the segment $[\tilde{r}_j, \tilde{s}_j]$ to E. The homotopy class of the simple closed curve

$$\delta := \bigcup_{j\in\mathbb{Z}/(2mq)\mathbb{Z}} \delta_j$$

in E corresponds to the homotopy class of γ in $\mathfrak{E}(f^{\circ q})$.

On the other hand, glueing the lower boundary component C_j^- of B_j with the upper boundary component C_{j+1}^+ of B_{j+1} via the translation by $z \mapsto z - r_j + s_{j+1}$, we obtain a complex torus E' which is isomorphic to $\mathcal{E}_\sigma$. Let δ'_j be the projection of the segment $[\tilde{r}_j, \tilde{s}_j]$ to E'. The homotopy class of the simple closed curve

$$\delta' := \bigcup_{j\in\mathbb{Z}/(2mq)\mathbb{Z}} \delta'_j$$

in E' corresponds to the homotopy class of $\sigma\mathbb{R}/\sigma\mathbb{Z}$ in $\mathcal{E}_\sigma$.

The homeomorphism

$$\psi_j := \xi_j - s_{j+1} + r_j : C_j^- \to C_j^-$$

fixes $r_j \in C_j^-$. Let $\tilde{\psi}_j : \mathbb{R} \to \mathbb{R}$ be the lift of $\psi_j : C_j^- \to C_j^-$ which fixes $\tilde{r}_j$ and let $\Psi_j : S_j \to S_j$ be the extension to S_j defined by

$$\Psi_j(x + \mathrm{i}y) := \frac{y}{m_j}(x + \mathrm{i}m_j) + \left(1 - \frac{y}{m_j}\right)\tilde{\psi}_j(x).$$

The homeomorphism $\Psi_j : \overline{S}_j \to \overline{S}_j$ restricts to the identity on $\mathbb{R}+\mathrm{i}m_j$ and descends to a homeomorphism $\psi_j : \overline{B}_j \to \overline{B}_j$. By construction, the following diagram commutes:

$$\begin{tikzcd} C_j^- \arrow[r, "\psi_j"] \arrow[d, "\xi_j"'] & C_j^- \arrow[d, "z\mapsto z-r_j+s_{j+1}"] \\ C_{j+1}^+ \arrow[r, "\psi_{j-1}"'] & C_{j+1}^+. \end{tikzcd}$$

So, the collection of homeomorphisms $\psi_j : \overline{B}_j \to \overline{B}_j$ induces a global homeomorphism $\psi : E \to E'$. Since Ψ_j fixes $\tilde{r}_j$ and $\tilde{s}_j$, the homeomorphism ψ sends the homotopy class of δ in E to the homotopy class of δ' in E'. The proof is completed by Lemma 4.10 below. □

Lemma 4.10. *The homeomorphism $\psi : E \to E'$ is e^{5D_f}-quasiconformal.*

Proof. The image of the curves $C_j^\pm$ in E are analytic (because the glueing map ξ_j is analytic), therefore quasiconformally removable. So, it is enough to prove that each $\psi_j : B_j \to B_j$ is e^{5D_f}-quasiconformal. Equivalently, we must prove that

$$\left\|\frac{\partial\Psi_j/\partial\bar{z}}{\partial\Psi_j/\partial z}\right\|_\infty \le k < 1 \quad \text{with} \quad \mathrm{dist}_{\mathbb{D}}(0,k) < 5D_f,$$

where $\mathrm{dist}_{\mathbb{D}}$ is the hyperbolic distance within the unit disk.

For readibility, we drop the index j in the following computation:

$$\begin{aligned}\frac{\partial\Psi/\partial\bar{z}}{\partial\Psi/\partial z}(x+\mathrm{i}y) &= \frac{\partial\Psi/\partial x + \mathrm{i}\partial\Psi/\partial y}{\partial\Psi/\partial x - \mathrm{i}\partial\Psi/\partial y}(x+\mathrm{i}y)\\ &= \frac{\left(1-\frac{y}{m}\right)\cdot\left(\tilde{\psi}'(x)-1\right) - \frac{\mathrm{i}}{m}\left(\tilde{\psi}(x)-x\right)}{2+\left(1-\frac{y}{m}\right)\cdot\left(\tilde{\psi}'(x)-1\right) + \frac{\mathrm{i}}{m}\left(\tilde{\psi}(x)-x\right)}.\end{aligned}$$

This last quantity is of the form $(a-1)/(\bar{a}+1)$ with

$$\mathrm{Re}(a) = 1 + \left(1-\frac{y}{m}\right)\cdot\left(\tilde{\psi}'(x)-1\right) \quad \text{and} \quad \mathrm{Im}(a) = \frac{\tilde{\psi}(x)-x}{m}.$$

Note that $\left|\dfrac{a-1}{\bar{a}+1}\right| = \left|\dfrac{a-1}{a+1}\right|$ and the Möbius transformation $a \mapsto \dfrac{a-1}{a+1}$ sends the right half-plane into the unit disk. So, it is enough to show that a belongs to the right half-plane $\big\{z \in \mathbb{C}\ ;\ \mathrm{Re}(z) > 0\big\}$ and that the hyperbolic distance within this half-plane between 1 and a is at most $5D_f$.

This hyperbolic distance is bounded from above by $\big|\mathrm{Im}(a)\big| + \big|\log \mathrm{Re}(a)\big|$. Since $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ is an increasing diffeomorphism which fixes $p + \mathbb{Z} \in \mathbb{R}$, we have that $\tilde{\psi}'(x) > 0$ and $\big|\tilde{\psi}(x) - x\big| < 1$. In addition, $0 < 1 - y/m < 1$, and so,

$$0 < \min_{\mathbb{R}} \tilde{\psi}' \le \mathrm{Re}(a) \le \max_{\mathbb{R}} \tilde{\psi}' \quad \text{and} \quad \big|\mathrm{Im}(a)\big| \le \frac{1}{m} = \frac{|\log\rho|}{\pi} \le |\log\rho| \le D_f.$$

The last inequality is given by Lemma 2.2. The average of $\tilde{\psi}'$ on $[0,1]$ is equal to $\tilde{\psi}(1) - \tilde{\psi}(0) = 1$. So, $\tilde{\psi}'$ takes the value 1 and

$$-\mathrm{dis}_{\mathbb{R}}(\xi) = -\mathrm{dis}_{\mathbb{R}}(\tilde{\psi}) < \log\min_{\mathbb{R}}(\tilde{\psi}') \le 0 \le \log\max_{\mathbb{R}}(\tilde{\psi}') < \mathrm{dis}_{\mathbb{R}}(\tilde{\psi}) = \mathrm{dis}_{\mathbb{R}}(\xi).$$

The proof is completed by Lemma 4.11 below. □

Lemma 4.11. *For any $j \in \mathbb{Z}/(2mq)\mathbb{Z}$, the distortion of ξ_j is bounded by $4D_f$.*

Proof. The map $\xi_j : C_j^- \to \mathbb{C}_{j+1}^+$ is induced by the following composition

$$\mathbb{R} \xrightarrow{e_j} (0,+\infty) \xrightarrow{\phi_j} (\alpha_j, \alpha_{j+1}) \xrightarrow{\phi_{j+1}^{-1}} (-\infty, 0) \xrightarrow{e_{j+1}^{-1}} \mathbb{R} + \mathrm{i} m_{j+1}.$$

with

$$e_j(z) := \exp(z \cdot \log \rho_j) \quad \text{and} \quad e_{j+1}(z) = \exp(z \cdot \log \rho_{j+1}).$$

The distortion of e_j on any interval of length 1 is $|\log \rho_j|$ which is at most D_f according to Lemma 2.2. Similarly, the distortion of e_{j+1} on any interval of length 1 is $|\log \rho_{j+1}| \le D_f$.

Let x be any point in (α_j, α_{j+1}) and let $I \subset \mathbb{R}/\mathbb{Z}$ be the interval whose extremities are x and $f(x)$. To complete the proof, it is enough to show that

$$\mathrm{dis}_I(\phi_j^{-1}) \le D_f \quad \text{and} \quad \mathrm{dis}_I(\phi_{j+1}^{-1}) \le D_f.$$

We will only prove this result for ϕ_j in the case where α_j is attracting. The other cases are dealt similarly and left to the reader.

On I, the linearizing map ϕ_j is the limit of the maps $\varphi_n := (f^{\circ qn} - \alpha_j)/\rho_j^n$. Since I is disjoint from all its iterates, Denjoy's Lemma 2.1 yields

$$\mathrm{dis}_I \varphi_n = \mathrm{dis}_I f^{\circ qn} \le D_f.$$

Passing to the limit as n tends to ∞ shows that $\mathrm{dis}_I \phi_j \le D_f$ as required. □

4.5. **Continuity of $\bar{\tau}_f$.** We now prove Lemma 4.5. It is enough to prove that $\bar{\tau}_f$ is continuous at $\omega = 0$. We shall see that when $\mathrm{rot}(f)$ is irrational, the continuity follows from Lemma 4.4, but when $\mathrm{rot}(f)$ is rational, the situation is more subtle.

4.5.1. *Irrational rotation number.* If $\mathrm{rot}(f)$ is irrational, then $\bar{\tau}_f(0) = \mathrm{rot}(f)$ due to the definition of $\bar{\tau}_f$.

Let $I \subset \mathbb{R}/\mathbb{Z}$ be a small neighborhood of 0 such that for $\omega \in I$, the periods of the periodic cycles of f_ω are at least N. For $\omega \in I$, either $\bar{\tau}_f(\omega) = \mathrm{rot}(f_\omega)$, or according to Lemma 4.4,

$$\big|\bar{\tau}_f(\omega) - \mathrm{rot}(f_\omega)\big| \le \frac{D_f}{N^2}.$$

Thus, $\bar{\tau}_f(I)$ is located within D_f/N^2-neighborhood of $\big\{\mathrm{rot}(f_\omega), \omega \in I\big\}$. The result follows since $\omega \mapsto \mathrm{rot}(f_\omega)$ is continuous.

4.5.2. *Rational rotation number.* If f is hyperbolic, then the continuity of $\bar{\tau}_f$ at 0 follows directly from Theorem 4.2.

Let us assume f has at least one parabolic cycle. We will only prove that

$$\lim_{\omega > 0, \omega \to 0} \bar{\tau}_f(\omega) = \frac{p}{q} = \bar{\tau}_f(0).$$

Applying this result to the diffeomorphism $x \mapsto -f(-x)$ yields

$$\lim_{\omega < 0, \omega \to 0} \bar{\tau}_f(\omega) = \frac{p}{q} = \bar{\tau}_f(0).$$

There are three different cases.

(1) All q-periodic orbits of f disappear as ω increases, so that, $\mathrm{rot}(f_\omega) > p/q$ for $\omega > 0$. In this case, the proof is literally the same as in the case of irrational rotation number.
(2) At least one parabolic cycle of f bifurcates into real hyperbolic cycles. In this case, the multipliers of these real hyperbolic cycles tend to 1 as ω tends to 0. The result follows from Lemma 4.4.

(3) All parabolic cycles of f bifurcate into complex conjugate cycles as $\omega > 0$ increases but the rotation number stays unchanged because f has hyperbolic cycles.

The rest of the Section is devoted to the treatment of the third case.

Lemma 4.12. *Under the assumptions of case (3) above, the curve $\bar{\tau}_f(\omega)$ is tangent to the segment $[0, 0+\varepsilon) \subset \mathbb{R}/\mathbb{Z}$; moreover, it is located between two horocycles tangent to $\mathbb{R}/\mathbb{Z}$ at 0.*

Proof. Our proof relies on Lemma 4.9. According to Lemma 4.4, we know that for $\omega > 0$ close to ω, $\bar{\tau}_f(\omega)$ remains in a subdisk of $\mathbb{H}/\mathbb{Z}$ tangent to the real axis at p/q. So, it is enough to prove that $q\bar{\tau}_f(\omega)$ tends to 0 tangentially to the segment $[0, \varepsilon) \in \mathbb{R}/\mathbb{Z}$ and is located in between two horocycles tangent to $\mathbb{R}/\mathbb{Z}$ at the point 0.

The notation we introduce now is similar to that of Section 4.2. The main difference is, that f is *not* hyperbolic.

Let m be the number of attracting hyperbolic cycles of f and order cyclically the hyperbolic periodic points α_j, $j \in \mathbb{Z}/(2mq)\mathbb{Z}$. For each $j \in \mathbb{Z}/(2mq)\mathbb{Z}$, let x_j be a point in (α_j, α_{j+1}), so that

- $f(x_j) \in (\alpha_{j+2pm}, x_{j+2pm})$ if the orbit of α_j attracts (i.e. j is even) and
- $f(x_j) \in (x_{j+2pm}, \alpha_{j+2pm+1})$ if the orbit of α_j repels (i.e. j is odd).

Note that since the parabolic cycles disappear as $\omega > 0$ increases, the graph of $f^{\circ q} - \mathrm{id}$ lies above the diagonal near those points. As a consequence, each parabolic periodic point lies in an interval of the form (α_j, α_{j+1}) with α_j repelling and α_{j+1} attracting.

For $\omega > 0$ close enough to 0, f_ω has a hyperbolic point $\alpha_j(\omega)$ close to α_j. We denote by $\rho_{\omega,j}$ the corresponding multiplier and by $\phi_{\omega,j}$ the corresponding linearizing map. Finally, using the points x_j chosen above which do not depend on ω, set

$$\tilde{r}_{\omega,j} := \frac{\log \phi_{\omega,j}^{-1}(x_j)}{\log \rho_{\omega,j}}, \quad \tilde{s}_{\omega,j} := \frac{\log |\phi_{\omega,j}^{-1}(x_{j-1})|}{\log \rho_{\omega,j}} + \frac{\mathrm{i}\pi}{|\log \rho_{\omega,j}|}$$

and

$$\sigma_\omega := \sum_{j \in \mathbb{Z}/(2mq)\mathbb{Z}} \tilde{s}_{\omega,j} - \tilde{r}_{\omega,j}.$$

According to Lemma 4.9, the hyperbolic distance in $\mathbb{H}/\mathbb{Z}$ between $q\bar{\tau}_f(\omega)$ and $-1/\sigma_\omega$ is uniformly bounded as $\omega > 0$ tends to 0. So, it is enough to show that the imaginary part of σ_ω is bounded and that the real part of σ_ω tends to $-\infty$.

Since

$$\mathrm{Im}(\tilde{r}_{\omega,j}) = 0 \quad \text{and} \quad \mathrm{Im}(\tilde{s}_{\omega,j}) \underset{\omega>0, \omega\to 0}{\longrightarrow} \mathrm{Im}(\tilde{s}_j),$$

we see that the imaginary part remains bounded as $\omega > 0$ tends to 0.

If f has no parabolic periodic point on the interval (α_j, α_{j+1}), then $\phi_{\omega,j}^{-1} \to \phi_j^{-1}$ on the interval (α_j, α_{j+1}). It follows that $\mathrm{Re}(\tilde{r}_{\omega,j})$ and $\mathrm{Re}(\tilde{s}_{\omega,j+1})$ remain bounded. If f has a parabolic periodic point on the interval (α_j, α_{j+1}), then α_j is repelling and α_{j+1} is attracting. Either the two quantities $\log \phi_{\omega,j}^{-1}(x_j)$ and $\log |\phi_{\omega,j+1}^{-1}(x_j)|$ tend to $+\infty$, or one remains bounded and the other tends to $+\infty$. Since $\log \rho_{\omega,j} \to \log \rho_j > 0$ and $\log \rho_{\omega,j+1} \to \log \rho_{j+1} < 0$, in both cases,

$$\mathrm{Re}(\tilde{s}_{\omega,j+1} - \tilde{r}_{\omega,j}) \underset{\omega>0, \omega\to 0}{\longrightarrow} -\infty. \qquad \square$$

We finish with the following corollary that implies Theorem 1.3.

Corollary 4.13. *Assume $x - f(x)$ has two local maxima at points x_1 and x_2 with $x_1 - f(x_1) \neq x_2 - f(x_2)$. Then, τ_f is not injective in the upper half-plane.*

Proof. Let y_1 and y_2 be the respective values of $x - f(x)$ at x_1 and x_2. Suppose that $y_1 < y_2$. Then the map f_ω for $y_1 < \omega < y_2$ has zero rotation number, and it has parabolic fixed points for $\omega = y_1$ and $\omega = y_2$. When ω increases from y_1 to $y_1 + \varepsilon$, the parabolic fixed point disappears, thus due to Lemma 4.12, the curve $\omega \mapsto \bar{\tau}_f(\omega)$ is tangent to $[y_1, y_1 + \varepsilon)$. When $\omega < y_2$ tends to y_2, the two hyperbolic fixed points merge into a parabolic fixed point. Thus, according to Lemma 4.4, the curve $\omega \mapsto \bar{\tau}_f(\omega)$ enters any horocycle as $\omega < y_2$ tends to y_2. But if τ_f were injective, the pair of germs of the curve $\bar{\tau}_f|_{\mathbb{R}/\mathbb{Z}}$ at y_1 and y_2 (both passing through 0) would be oriented clockwise. The contradiction shows that τ_f is not injective in the upper half-plane. □

4.6. **The proof of Theorem 1.2.** Note that $\tau_f \colon \mathbb{H}/\mathbb{Z} \to \mathbb{H}/\mathbb{Z}$ extends holomorphically to $+\mathrm{i}\infty$. Thus, it is a holomorphic function on $\mathbb{H}/\mathbb{Z} \cup \{+\mathrm{i}\infty\}$ which is a Riemann surface isomorphic to the unit disk $\mathbb{D}$. It takes its values in $\overline{\mathbb{H}/\mathbb{Z}}$. Almost everywhere, its radial limits as ω tends to $\mathbb{R}/\mathbb{Z}$ coincide with the value of the continuous function $\bar{\tau}_f : \mathbb{R}/\mathbb{Z} \to \overline{\mathbb{H}/\mathbb{Z}}$. So, Theorem 1.2 is a consequence of the following classical result.

Lemma 4.14. *Let $g : \mathbb{D} \to \mathbb{C}$ be a bounded holomorphic function. Suppose that almost everywhere, its radial limits as z tends to $\partial\mathbb{D}$ are those of a continuous function $h : \partial\mathbb{D} \to \mathbb{C}$:*

$$\text{for almost every } t \in \mathbb{R}/\mathbb{Z}, \qquad \lim_{r \to 1, r<1} g(r\mathrm{e}^{2\pi\mathrm{i}t}) = h(\mathrm{e}^{2\pi\mathrm{i}t}).$$

Then, h extends g continuously to $\overline{\mathbb{D}}$.

Proof. The real and imaginary parts of g are harmonic functions. Due to the Poisson formula (applied to both $\operatorname{Re} g$ and $\operatorname{Im} g$) for $|z| < r$ we have

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\alpha}) P(re^{i\alpha}, z)\, \mathrm{d}\alpha, \tag{4.1}$$

where P is the Poisson kernel,

$$P(re^{i\alpha}, Re^{i\beta}) = \frac{r^2 - R^2}{r^2 + R^2 - 2rR\cos(\alpha - \beta)}.$$

The integrand in (4.1) is bounded as r tends to 1, and it tends to $h(e^{i\alpha})P(e^{i\alpha}, z)$ almost everywhere. Due to the Lebesgue bounded convergence theorem,

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\alpha}) P(e^{i\alpha}, z)\, \mathrm{d}\alpha.$$

Due to the Poisson theorem, the right-hand side provides the solution of the Dirichlet boundary problem for Laplace equation. Thus $\operatorname{Re} g$ and $\operatorname{Im} g$ satisfy

$$\lim_{z \to e^{i\alpha}} \operatorname{Re} g(z) = \operatorname{Re} h(e^{i\alpha}) \qquad \lim_{z \to e^{i\alpha}} \operatorname{Im} g(z) = \operatorname{Im} h(e^{i\alpha}).$$ □

References

[1] V. I. Arnold, *Geometrical Methods In The Theory Of Ordinary Differential Equations*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 250, Springer-Verlag, New York-Berlin, 1983, 334 pp.

[2] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci : three theorems of J. C. Yoccoz*, in Topological Methods in Modern Mathematics, Goldberg and Phillips eds Publish or Perish 1993, p.467–511.

[3] N. B. Goncharuk, *Rotation numbers and moduli of elliptic curves*, Functional Analysis and Its Applications, Volume 46, Issue 1 , pp 11-25.

[4] Y. Ilyashenko & V. Moldavskis, *Morse-Smale circle diffeomorphisms and moduli of complex tori*, Moscow Mathematical Journal, Volume 3, April-June 2003, no 2, p.531–540.

[5] V. S. Moldavskii, *Moduli of Elliptic Curves and Rotation Numbers of Circle Diffeomorphisms*, Functional Analysis and Its Applications, 35:3(2001), p.234–236.

[6] E. Risler, *Linéarisation des perturbations holomorphes des rotations et applications*, Mémoires de la S.M.F. 2^e série, tome 77(1999), p. III–VII +1–102.

[7] M. Tsujii, *Rotation number and one-parameter family of circle diffeomorphisms*, Ergod. th. & Dynam. sys. (1992), 12, 359-363.

[8] J.-C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne*, Ann. Sci. École Norm. Sup. (4) 17 (1984), no. 3, pp. 333-359.

E-mail address: xavier.buff@math.univ-toulouse.fr

Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex, France

E-mail address: natalka@mccme.ru

National Research University Higher School of Economics and Independent University of Moscow