Bifurcation measure and postcritically finite rational maps

Xavier Buff and Adam L. Epstein

Bassanelli and Berteloot [BB] have defined a bifurcation measure μ_{bif} on the moduli space $\mathcal{M}_d$ of rational maps of degree $d \geq 2$. They have proved that it is a positive measure of finite mass and that its support is contained in the closure of the set $\mathcal{Z}_d$ of conjugacy classes of rational maps of degree d having $2d-2$ indifferent cycles.

Denote by $\mathcal{X}_d$ the set of conjugacy classes of strictly postcritically finite rational maps of degree d which are not flexible Lattès maps. We prove that $\text{Supp}(\mu_{\text{bif}}) = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}$. Our proof is based on a transversality result due to the second author.

A similar result was obtained with different techniques by Dujardin and Favre [DF] for the bifurcation measure on moduli spaces of polynomials of degree $d \geq 2$.

Introduction

A result of Lyubich [L] asserts that for each rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$, there is a unique probability measure μ_f of maximal entropy $\log d$. It is ergodic, satisfies $f^*\mu_f = d \cdot \mu_f$ and is carried outside the exceptional set of f. This measure is the *equilibrium measure* of f.

The *Lyapunov exponent* of f with respect to the measure μ_f may be defined by

$$\mathfrak{L}(f) := \int_{\mathbb{P}^1} \log\|Df\| \, \mathrm{d}\mu_f,$$

where $\|\cdot\|$ is any smooth metric on $\mathbb{P}^1$. Since μ_f is ergodic, the quantity $\mathrm{e}^{\mathfrak{L}(f)}$ records the average rate of expansion of f along a typical orbit with respect to μ_f.

The space Rat_d of rational maps of degree d is a smooth complex manifold of dimension $2d+1$. The function $\mathfrak{L} : \text{Rat}_d \to \mathbb{R}$ is continuous [Ma] and plurisubharmonic [DeM2] on Rat_d. The positive $(1,1)$-current

$$T_{\text{bif}} := dd^c \mathfrak{L}$$

is called the *bifurcation current* in Rat_d.

The *bifurcation locus* in Rat_d is the closure of the set of discontinuity of the map $\mathrm{Rat}_d \ni f \mapsto J_f$, where J_f stands for the Julia set of f and the continuity is for the Hausdorff topology for compact subsets of $\mathbb{P}^1$. DeMarco [DeM2] proved that the support of the bifurcation current T_{bif} is equal to the bifurcation locus.

For $f \in \mathrm{Rat}_d$, denote by $\mathcal{O}(f)$ the set of rational maps which are conjugate to f by a Möbius transformation. This set is a 3 dimensional complex analytic submanifold of Rat_d. In fact, $\mathcal{O}(f)$ is biholomorphic to $\mathrm{Aut}(\mathbb{P}^1)/\Gamma$, where $\mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PSL}(2,\mathbb{C})$ is the group of Möbius transformations, and where $\Gamma \subset \mathrm{Aut}(\mathbb{P}^1)$ is the finite subgroup of Möbius transformations which commute with f.

The *moduli space* $\mathcal{M}_d$ is the quotient $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$, where $\mathrm{Aut}(\mathbb{P}^1)$ acts on Rat_d by conjugation. It is an orbifold of complex dimension $2d-2$. It is a normal, quasiprojective variety. We denote by $\mathfrak{p} : \mathrm{Rat}_d \to \mathcal{M}_d$ the canonical projection.

The Lyapunov exponent is invariant under holomorphic conjugacy, hence is constant on the orbits $\mathcal{O}(f)$. The map $\mathfrak{L} : \mathrm{Rat}_d \to \mathbb{R}$ descends to a map $\hat{\mathfrak{L}} : \mathcal{M}_d \to \mathbb{R}$ which is continuous, bounded from below and plurisubharmonic on $\mathcal{M}_d$ (see [BB] proposition 6.2). The measure

$$\mu_{\mathrm{bif}} := (dd^c \hat{\mathfrak{L}})^{\wedge(2d-2)}$$

is called the *bifurcation measure* on $\mathcal{M}_d$. By construction, we have

$$\mathfrak{p}^* \mu_{\mathrm{bif}} = T_{\mathrm{bif}}{}^{\wedge(2d-2)}.$$

We denote by $\mathcal{C}(f)$ the set of critical points of f and by $\mathcal{V}(f) := f\bigl(\mathcal{C}(f)\bigr)$ the set of critical values of f. The *postcritical set* is

$$\mathcal{P}(f) := \bigcup_{c \in \mathcal{C}(f)} \bigcup_{n \geq 1} f^{\circ n}(c).$$

A rational map f is *postcritically finite* if $\mathcal{P}(f)$ is finite. It is *strictly postcritically finite* if $\mathcal{P}(f)$ is finite and if f does not have any superattracting cycle. The map f is a *Lattès map* if it is obtained as the quotient of an affine map $A : z \mapsto az + b$ on a complex torus $\mathbb{C}/\Lambda$: there is a finite-to-one holomorphic map $\Theta : \mathbb{C}/\Lambda \to \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[r, "A"] \arrow[d, "\Theta"'] & \mathbb{C}/\Lambda \arrow[d, "\Theta"] \\ \mathbb{P}^1 \arrow[r, "f"'] & \mathbb{P}^1. \end{tikzcd}$$

A Lattès map is strictly postcritically finite (see [Mi]). It is a *flexible Lattès map* if we can choose Θ with degree 2 and $A(z) = az + b$ with $a > 1$ an integer.

Let us introduce the following notation:

- $\mathcal{X}_d \subset \mathcal{M}_d$ for the set of conjugacy classes of strictly postcritically finite rational maps of degree d which are not flexible Lattès maps;
- $\mathcal{X}_d^* \subset \mathcal{X}_d$ for the subset of conjugacy classes of maps which have only simple critical points and satisfy $\mathcal{C}(f) \cap \mathcal{P}(f) = \emptyset$;
- $\mathcal{Z}_d \subset \mathcal{M}_d$ for the set of conjugacy classes of maps which have $2d-2$ indifferent cycles (we do not count multiplicities).

Bassanelli and Berteloot [BB] proved that

- the bifurcation measure does not vanish identically on $\mathcal{M}_d$ and has finite mass,
- the conjugacy class of any non-flexible Lattès map lies in the support of μ_{bif},
- the support of μ_{bif} is contained in the closure of $\mathcal{Z}_d$.

Our main result is the following.

Main Theorem *The support of the bifurcation measure in $\mathcal{M}_d$ is:*

$$\mathrm{Supp}(\mu_{\mathrm{bif}}) = \overline{\mathcal{X}_d^*} = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}.$$

Remark We shall prove in Section 1 that

$$\overline{\mathcal{X}_d^*} = \overline{\mathcal{X}_d} = \overline{\mathcal{Z}_d}. \tag{0.1}$$

Then, due to the results in [BB], we will only have to prove the inclusion $\mathcal{X}_d^* \subseteq \mathrm{Supp}(\mu_{\mathrm{bif}})$.

Remark We do not know how to prove that the conjugacy classes of flexible Lattès maps lie in the support of μ_{bif}. It would be enough to prove that every flexible Lattès map can be approximated by strictly postcritically finite rational maps which are not Lattès maps.

Remark The underlying idea of the proof is a potential-theoretic interpretation of a result of Tan Lei [T] concerning the similarities between the Mandelbrot set and Julia sets.

Our proof relies on a transversality result that we will now present. Elements of $(\mathbb{P}^1)^{2d-2}$ are denoted $\underline{z} = (z_1, \ldots, z_{2d-2})$.

Let $f \in \mathrm{Rat}_d$ be a postcritically finite rational map with $2d-2$ distinct critical points $c_1, \ldots, c_{2d-2}$, satisfying $\mathcal{C}(f) \cap \mathcal{P}(f) = \emptyset$. There are integers $\ell_j \geq 1$ such that $\alpha_j := f^{\circ \ell_j}(c_j)$ are periodic points of f. Set $\underline{c} := (c_1, \ldots, c_{2d-2})$ and $\underline{\alpha} := (\alpha_1, \ldots, \alpha_{2d-2})$. The critical points are simple and the periodic points are repelling. By the Implicit Function Theorem, there are

- an analytic germ $\underline{\mathfrak{c}} : (\mathrm{Rat}_d, f) \to \big((\mathbb{P}^1)^{2d-2}, \underline{c}\big)$ such that for g near f, $\mathfrak{c}_j(g)$ is a critical point of g and
- an analytic germ $\underline{\mathfrak{a}} : (\mathrm{Rat}_d, f) \to \big((\mathbb{P}^1)^{2d-2}, \underline{\alpha}\big)$ such that for g near f, $\mathfrak{a}_j(g)$ is a periodic point of g.

Let $\underline{\mathfrak{v}} : (\mathrm{Rat}_d, f) \to \big((\mathbb{P}^1)^{2d-2}, \underline{\alpha}\big)$ be defined by

$$\underline{\mathfrak{v}} := (\mathfrak{v}_1, \ldots, \mathfrak{v}_{2d-2}) \quad \text{with} \quad \mathfrak{v}_j(g) := g^{\circ \ell_j}\big(\mathfrak{c}_j(g)\big).$$

Denote by $D_f\underline{\mathfrak{v}}$ and $D_f\underline{\mathfrak{a}}$ the differentials of $\underline{\mathfrak{v}}$ and $\underline{\mathfrak{a}}$ at f. The transversality result we are interested in is the following.

Theorem 1 *If f is not a flexible Lattès map, then the linear map*

$$D_f\underline{\mathfrak{v}} - D_f\underline{\mathfrak{a}} : T_f\mathrm{Rat}_d \to \bigoplus_{j=1}^{2d-2} T_{\alpha_j}\mathbb{P}^1$$

is surjective. The kernel of $D_f\underline{\mathfrak{v}} - D_f\underline{\mathfrak{a}}$ is the tangent space to $\mathcal{O}(f)$ at f.

There are several proofs of this result. Here, we include a proof due to the second author. A different proof was obtained by van Strien [vS]. His proof covers a more general setting where the critical orbits are allowed to be preperiodic to a hyperbolic set. Our proof covers the case of maps commuting with a nontrivial group of Möbius transformations (a slight modification of van Strien's proof probably also covers this case).

Remark A similar transversality theorem holds in the space of polynomials of degree d and the arguments developed in this article lead to an alternative proof of the result of Dujardin and Favre.

Acknowledgements

We would like to thank our colleagues in Toulouse who brought this problem to our attention, in particular François Berteloot, Julien Duval and Vincent Guedj. We would like to thank John Hubbard for his constant support.

1 The sets $\mathcal{X}_d^*$, $\mathcal{X}_d$ and $\mathcal{Z}_d$

In this section, we prove (0.1). Since $\mathcal{X}_d^* \subseteq \mathcal{X}_d$, it is enough to prove that

$$\mathcal{X}_d \subseteq \overline{\mathcal{Z}_d} \quad \text{and} \quad \mathcal{Z}_d \subseteq \overline{\mathcal{X}_d^*}.$$

1.1 Tools

Our proof relies on the following three results.

The first result is an immediate consequence of the Fatou-Shishikura inequality on the number of nonrepelling cycles of a rational map (see [S] and/or [E]). For $f \in \mathrm{Rat}_d$, denote by

- $N_{\mathrm{att}}(f)$ the number of attracting cycles of f,
- $N_{\mathrm{ind}}(f)$ the number of distinct indifferent cycles of f and
- $N_{\mathrm{crit}}(f)$ the number of critical points of f, counting multiplicities, whose orbits are strictly preperiodic to repelling cycles.

Theorem 2 *For any rational map $f \in \mathrm{Rat}_d$, we have:*

$$N_{\mathrm{crit}}(f) + N_{\mathrm{ind}}(f) + N_{\mathrm{att}}(f) \leq 2d - 2.$$

The second result is a characterization of *stability* due to Mañe, Sad and Sullivan [MSS] (compare with [McM2] section 4.1). Let Λ be a complex manifold. A family of rational maps $\Lambda \ni \lambda \mapsto f_\lambda$ is an *analytic family* if the map $\Lambda \times \mathbb{P}^1 \ni (\lambda, z) \mapsto f_\lambda(z) \in \mathbb{P}^1$ is analytic. The family is *stable at* $\lambda_0 \in \Lambda$ if the number of attracting cycles of f_λ is locally constant at λ_0.

Theorem 3 *Let $\Lambda \ni \lambda \mapsto f_\lambda$ be an analytic family of rational maps parametrized by a complex manifold Λ. The following assertions are equivalent.*

- *The family is stable at λ_0.*
- *For all $m \in \mathbb{S}^1$ and $p \geq 1$, the number of cycles of f_λ having multiplier m and period p is locally constant at λ_0.*
- *For all $\ell \geq 1$ and $p \geq 1$, the number of critical points c of f_λ such that $f^{\circ \ell}(c)$ is a repelling periodic point of period p is locally constant at λ_0.*
- *The maximum period of an indifferent cycle of f_λ is locally bounded at λ_0.*
- *The maximum period of a repelling cycle of f_λ contained in the post-critical set $\mathcal{P}(f_\lambda)$ is locally bounded at λ_0.*

The third result is due to McMullen [McM1]. Let Λ be an irreducible quasiprojective complex variety. A family of rational maps $\Lambda \ni \lambda \mapsto f_\lambda$ is an *algebraic family* if the map $\Lambda \times \mathbb{P}^1 \ni (\lambda, z) \mapsto f_\lambda(z) \in \mathbb{P}^1$ is a rational mapping. The family is *trivial* if all its members are conjugate by Möbius transformations. The family is *stable* if it is stable at every $\lambda \in \Lambda$.

Theorem 4 *A stable algebraic family of rational maps is either trivial or it is a family of flexible Lattès maps.*

1.2 Spaces of rational maps with marked critical points and marked periodic points

Here and henceforth, it will be convenient to consider the set $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n}$ of rational maps of degree d with marked critical points and marked periodic points of period dividing n. This set may be defined as follows.

First, the unordered sets of m points in $\mathbb{P}^1$ may be identified with $\mathbb{P}^m$. This yields a rational map $\sigma_m : (\mathbb{P}^1)^m \to \mathbb{P}^m$ which may be defined as follows:

$$\sigma_m\big([x_1 : y_1], \ldots, [x_m : y_m]\big) = [a_0 : \cdots : a_m]$$

with

$$\sum_{j=0}^{m} a_j x^j y^{m-j} = \prod_{i=1}^{m} (x y_i - y x_i).$$

Second, to a rational map $f \in \mathrm{Rat}_d$, we associate the unordered set $\{c_1, \ldots, c_{2d-2}\}$ of its $2d-2$ critical points, listed with repetitions according to their multiplicities. This induces a rational map $\mathfrak{crit} : \mathrm{Rat}_d \to \mathbb{P}^{2d-2}$ which may be defined as follows: if $f\big([x:y]\big) = \big[P(x,y) : Q(x,y)\big]$ with P and Q homogeneous polynomials of degree d, then

$$\mathfrak{crit}(f) = [a_0 : \cdots : a_{2d-2}] \quad \text{with} \quad \sum_{j=0}^{2d-2} a_j x^j y^{2d-2-j} = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}.$$

Third, given an integer $n \geq 1$, to a rational map $f \in \mathrm{Rat}_d$, we associate the unordered set $\{\alpha_1, \ldots, \alpha_{d^n+1}\}$ of the fixed points of $f^{\circ n}$, listed with repetitions according to their multiplicities. This induces a rational map $\mathfrak{per}_n : \mathrm{Rat}_d \to \mathbb{P}^{d^n+1}$ which may be defined as follows:
if $f^{\circ n}\big([x:y]\big) = \big[P_n(x,y) : Q_n(x,y)\big]$ with P_n and Q_n homogeneous polynomials of degree d^n, then

$$\mathfrak{per}_n(f) = [a_0 : \cdots : a_{d^n+1}]$$

with

$$\sum_{j=0}^{d^n+1} a_j x^j y^{d^n+1-j} = y P_n(x,y) - x Q_n(x,y).$$

The space of rational maps of degree d with marked critical points and marked periodic points of period dividing n is the set:

$$\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n} := \left\{ \begin{array}{l} (f,\underline{c},\underline{\alpha}) \in \mathrm{Rat}_d \times (\mathbb{P}^1)^{2d-2} \times (\mathbb{P}^1)^{d^n+1} \text{ such that} \\ \mathfrak{crit}(f) = \sigma_{2d-2}(\underline{c}) \text{ and } \mathfrak{per}_n(f) = \sigma_{d^n+1}(\underline{\alpha}) \end{array} \right\}.$$

Then, $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n}$ is an algebraic subset[1] of $\mathrm{Rat}_d \times (\mathbb{P}^1)^{2d-2} \times (\mathbb{P}^1)^{d^n+1}$. Since there are $2d-2+d^n+1$ equations, the dimension of any irreducible component of $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n}$ is at least that of Rat_d, i.e. $2d+1$. Since the fibers of the projection $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_n} \to \mathrm{Rat}_d$ are finite (a rational map has finitely many critical points and finitely many periodic points of period dividing n), the dimension of any component is exactly $2d+1$.

1.3 The inclusion $\mathcal{X}_d \subseteq \overline{\mathcal{Z}_d}$

Let $f_0 \in \mathrm{Rat}_d$ be a strictly postcritically finite rational map but not a flexible Lattès map. We must show that any neighborhood of f_0 in Rat_d contains a rational map with $2d-2$ indifferent cycles. This is an immediate consequence (by induction) of the following lemma.

Lemma 1 *Let $r \geq 1$ and $s \geq 0$ be integers such that $r+s = 2d-2$. Assume $f_0 \in \mathrm{Rat}_d$ is not a flexible Lattès map, $N_{\mathrm{crit}}(f_0) = r$ and $N_{\mathrm{ind}}(f_0) = s$. Then, arbitrarily close to f_0, we may find a rational map f_1 such that $N_{\mathrm{crit}}(f_1) = r-1$ and $N_{\mathrm{ind}}(f_1) = s+1$.*

Proof: Let $p_1, \ldots, p_s$ be the periods of the indifferent cycles of f_0. Denote by p their least common multiple. A point $\lambda \in \mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_p}$ is of the form

$$\big(g_\lambda, c_1(\lambda), \ldots, c_{2d-2}(\lambda), \alpha_1(\lambda), \ldots \alpha_{d^p+1}(\lambda)\big).$$

We choose $\lambda_0 \in \mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_p}$ so that

- $g_{\lambda_0} = f_0$,
- $c_1(\lambda_0), \ldots c_r(\lambda_0)$ are preperiodic to repelling cycles of f_0 and
- $\alpha_1(\lambda_0), \ldots, \alpha_s(\lambda_0)$ are indifferent periodic points of f_0 belonging to distinct cycles.

For $i \in [1,r]$, let $k_i \geq 1$ be the least integer such that

$$g_{\lambda_0}^{\circ k_i}\big(c_i(\lambda_0)\big) = g_{\lambda_0}^{\circ(k_i+p_i)}\big(c_i(\lambda_0)\big).$$

[1] By algebraic we mean a quasiprojective variety in some projective space. It is classical that any product of projective spaces embeds in some $\mathbb{P}^N$. Thus, it makes sense to speak of algebraic subsets of products of projective spaces

For $j \in [1,s]$, let m_j be the multiplier of $g_{\lambda_0}^{\circ p}$ at $\alpha_j(\lambda_0)$. In particular, $m_j \in \mathbb{S}^1$.

Consider the algebraic subset

$$\left\{ \lambda \in \mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_p} \ ; \quad \begin{array}{l} \forall i \in [1, r-1],\ g_\lambda^{\circ k_i}\big(c_i(\lambda)\big) = g_\lambda^{\circ(k_i+p_i)}\big(c_i(\lambda)\big) \text{ and} \\ \forall j \in [1,s],\ \text{the multiplier of } g_\lambda^{\circ p} \text{ at } \alpha_j(\lambda) \text{ is } m_j \end{array} \right\}$$

and let Λ be an irreducible component containing λ_0. Note that there are $r-1+s = 2d-3$ equations. It follows that the dimension of Λ is at least $(2d+1)-(2d-3) = 4$. In particular, the image of Λ by the projection $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_p} \to \mathrm{Rat}_d$ cannot be contained in $\mathcal{O}(f_0)$.

Embedding Λ in some $\mathbb{P}^N$ (with N sufficiently large) and slicing with an appropriate projective subspace, we deduce that there is an algebraic curve $\Gamma \subseteq \Lambda$ containing λ_0, whose image by the projection $\mathrm{Rat}_d^{\mathrm{crit},\mathrm{per}_p} \to \mathrm{Rat}_d$ is not contained in $\mathcal{O}(f_0)$. Desingularizing the algebraic curve, we obtain a smooth quasiprojective curve Σ (i.e. a Riemann surface of finite type) and an algebraic map $\Sigma \to \Gamma$ which is surjective and generically one-to-one. We let $\sigma_0 \in \Sigma$ be a point which is mapped to $\lambda_0 \in \Gamma$. With an abuse of notation, we write g_σ, $c_i(\sigma)$ and $\alpha_j(\sigma)$ in place of $g_{\lambda(\sigma)}$, $c_i\big(\lambda(\sigma)\big)$ and $\alpha_j\big(\lambda(\sigma)\big)$. Then, we have an algebraic family of rational maps $\Sigma \ni \sigma \mapsto g_\sigma$ parametrized by a smooth quasiprojective curve Σ, coming with marked critical points $c_i(\sigma)$ and marked periodic points $\alpha_j(\sigma)$.

The family is not trivial and g_{σ_0} is not a flexible Lattès map. Assume that the family were stable at σ_0. Then, according to Theorem 3, the number of critical points which are preperiodic to repelling cycles would be locally constant at σ_0. Thus, the critical orbit relation

$$g_\sigma^{\circ k_r}\big(c_r(\sigma)\big) = g_\sigma^{\circ(k_r+p_r)}\big(c_r(\sigma)\big)$$

would hold in a neighborhood of σ_0 in Σ, thus for all $\sigma \in \Sigma$ by analytic continuation. As a consequence, for all $\sigma \in \Sigma$, we would have $N_{\mathrm{crit}}(g_\sigma) + N_{\mathrm{ind}}(g_\sigma) = 2d-2$. According to Theorem 2, this would imply that $N_{\mathrm{att}}(g_\sigma) = 0$ for all $\sigma \in \Sigma$. The family would be stable which would contradict Theorem 4.

Thus, the family $\Sigma \ni \sigma \mapsto g_\sigma$ is not stable at σ_0. According to Theorem 3, the maximum period of an indifferent cycle of g_σ is not locally bounded at σ_0. Thus, we may find a parameter $\sigma_1 \in \Sigma$ arbitrarily close to σ_0 such that g_{σ_1} has at least $s+1$ indifferent cycles. □

1.4 The inclusion $\mathcal{Z}_d \subseteq \overline{\mathcal{X}_d^*}$

The proof follows essentially the same lines as the previous one. We begin with a map $f_0 \in \mathrm{Rat}_d$ having $2d-2$ indifferent cycles. We must show that arbitrarily close to f_0, we may find a postcritically finite map $f_1 \in \mathrm{Rat}_d$

which has only simple critical points, satisfies $\mathcal{C}(f_1) \cap \mathcal{P}(f_1) = \emptyset$ and is not a flexible Lattès map.

The following lemma implies (by induction) that arbitrarily close to f_0, we may find a rational map $f_1 \in \mathrm{Rat}_d$ whose postcritical set contains $2d-2$ distinct repelling cycles. Automatically, the $2d-2$ critical points of f_1 have to be simple, strictly preperiodic, with disjoint orbits. In particular, $\mathcal{C}(f_1) \cap \mathcal{P}(f_1) = \emptyset$. In addition, Lattès maps lie in a Zariski closed subset of Rat_d. Since f_0 has indifferent cycles, it cannot be a Lattès map. So, if f_1 is sufficiently close to f_0, then f_1 is not a Lattès map.

Lemma 2 *Let $r \geq 0$ and $s \geq 1$ be integers such that $r + s = 2d - 2$. Let $f_0 \in \mathrm{Rat}_d$ satisfy $N_{\mathrm{crit}}(f_0) = r$, $N_{\mathrm{ind}}(f_0) = s$, the postcritical set of f_0 containing r distinct repelling cycles. Then, arbitrarily close to f_0, there is a rational map f_1 which satisfies $N_{\mathrm{crit}}(f_1) = r + 1$ and $N_{\mathrm{ind}}(f_1) = s - 1$, the postcritical set of f_1 containing $r + 1$ distinct repelling cycles.*

Proof: The proof is similar to the proof of Lemma 1; we do not give all the details. First, note that since $s \geq 1$, f_0 has an indifferent cycle, and thus, is not a Lattès map. Second, we may find a nontrivial algebraic family of rational maps $\Sigma \ni \sigma \mapsto g_\sigma$ parametrized by a smooth quasiprojective curve Σ with marked critical points $c_1(\sigma), \ldots, c_{2d-2}(\sigma)$ and marked periodic points $\alpha_1(\sigma), \ldots, \alpha_{d^p+1}(\sigma)$ such that

- $g_{\sigma_0} = f_0$,
- for $j \in [1, s]$, the periodic points $\alpha_j(\sigma_0)$ belong to distinct indifferent cycles of f_0,
- for all $\sigma \in \Sigma$ and all $i \in [1, r]$, the critical point $c_i(\sigma)$ is preperiodic to a periodic cycle of g_σ and
- for all $\sigma \in \Sigma$ and all $j \in [1, s-1]$, the periodic points $\alpha_j(\sigma)$ are indifferent periodic points of g_σ.

If the family were stable at σ_0, the indifferent periodic point $\alpha_s(\sigma_0)$ would be persistently indifferent in a neighborhood of σ_0 in Σ, thus in all Σ by analytic continuation. The relation $N_{\mathrm{crit}}(g_\sigma) + N_{\mathrm{ind}}(g_\sigma) = 2d - 2$ would hold throughout Σ. Thus, for all $\sigma \in \Sigma$, we would have $N_{\mathrm{att}}(g_\sigma) = 0$ and the family would be stable. This is not possible since g_{σ_0} is not a Lattès map and since the family $\Sigma \ni \sigma \mapsto g_\sigma$ is not trivial.

It follows that the period of a repelling cycle contained in the postcritical set of g_σ is not locally bounded at σ_0. In addition, for $i \in [1, r]$, the critical points $c_i(\sigma_0)$ of g_{σ_0} are preperiodic to distinct repelling cycles. Thus, we may find a rational map g_{σ_1}, with σ_1 arbitrarily close to σ_0, such that g_{σ_1} has $r + 1$ critical points preperiodic to distinct repelling cycles. $\square$

2 Transversality

Here and henceforth, we will consider various holomorphic families $t \mapsto \gamma_t$ defined near 0 in $\mathbb{C}$. We will employ the notation

$$\gamma := \gamma_0 \quad \text{and} \quad \dot{\gamma} := \frac{d\gamma_t}{dt}\Big|_{t=0}.$$

2.1 The tangent space to $\mathcal{O}(f)$

Here we characterize the vectors $\xi \in T_f \mathrm{Rat}_d$ which are tangent to $\mathcal{O}(f)$ for some rational map $f \in \mathrm{Rat}_d$.

Note that if $t \mapsto f_t$ is a holomorphic family of rational maps, then for every $z \in \mathbb{P}^1$, the vector $\dot{f}(z)$ belongs to the tangent space $T_{f(z)}\mathbb{P}^1$. Thus, if $\xi \in T_f \mathrm{Rat}_d$, then for every $z \in \mathbb{P}^1$, we have $\xi(z) \in T_{f(z)}\mathbb{P}^1$. If $\xi \in T_f \mathrm{Rat}_d$ there is a unique vector field η_ξ, meromorphic on $\mathbb{P}^1$ with poles in $\mathcal{C}(f)$, such that

$$Df \circ \eta_\xi = -\xi.$$

Indeed, if z is not a critical point of f, then $D_z f : T_z\mathbb{P}^1 \to T_{f(z)}\mathbb{P}^1$ is an isomorphism, whence we may define $\eta_\xi(z)$ by

$$\eta_\xi(z) := -\big(D_z f\big)^{-1}\big(\xi(z)\big).$$

Moreover, in this situation, it follows from the Implicit Function Theorem that there is a unique holomorphic germ $t \mapsto z_t$ with $z_0 = z$ such that $f_t(z_t) = f(z)$, and furthermore $\eta_{\dot{f}}(z) = \dot{z} \in T_z\mathbb{P}^1$.

Remark If f has simple critical points, then the vector field η_ξ has simple poles or removable singularities along $\mathcal{C}(f)$. There is a removable singularity at $c \in \mathcal{C}(f)$ if and only if $\xi(c) = 0$. This can be seen by working in coordinates, using the fact that f' has simple zeroes at points of $\mathcal{C}(f)$.

Recalling that $\mathrm{Aut}(\mathbb{P}^1)$ is a Lie group, we denote by $\mathrm{aut}(\mathbb{P}^1)$ the corresponding Lie algebra: that is, the tangent space to $\mathrm{Aut}(\mathbb{P}^1)$ at the identity map. Thus, $\mathrm{aut}(\mathbb{P}^1)$ is canonically isomorphic to the space of globally holomorphic vector fields.

If $X \subseteq \mathbb{P}^1 - \mathcal{C}(f)$ and if θ is a vector field defined on $f(X)$, then the vector field $f^*\theta$ is defined on X by

$$f^*\theta(z) := (D_z f)^{-1}\big(\theta \circ f(z)\big).$$

If $\theta \in \mathrm{aut}(\mathbb{P}^1)$, the vector field $f^*\theta$ is the unique meromorphic vector field on $\mathbb{P}^1$ such that $Df \circ f^*\theta = \theta \circ f$.

Proposition 1 *A vector $\xi \in T_f\mathrm{Rat}_d$ is tangent to $\mathcal{O}(f)$ if and only if*

$$\eta_\xi = \theta - f^*\theta$$

for some $\theta \in \mathrm{aut}(\mathbb{P}^1)$.

Proof: The derivative at the identity of

$$\mathrm{Aut}(\mathbb{P}^1) \ni \phi \mapsto \phi \circ f \circ \phi^{-1} \in \mathrm{Rat}_d$$

is the linear map

$$\mathrm{aut}(\mathbb{P}^1) \ni \theta \mapsto \theta \circ f - Df \circ \theta \in T_f\mathrm{Rat}_d.$$

Thus, $\xi \in T_f\mathrm{Rat}_d$ is tangent to $\mathcal{O}(f)$ if and only if $\xi = \theta \circ f - Df \circ \theta$ for some $\theta \in \mathrm{aut}(\mathbb{P}^1)$. Since $\theta \circ f - Df \circ \theta = Df \circ (f^*\theta - \theta)$, it follows that $\xi \in T_f\mathrm{Rat}_d$ is tangent to $\mathcal{O}(f)$ if and only if $\eta_\xi = \theta - f^*\theta$ for some $\theta \in \mathrm{aut}(\mathbb{P}^1)$. □

2.2 Guided vector fields

Let θ be a vector field, defined and holomorphic on a neighborhood of the critical value set of some rational map $f \in \mathrm{Rat}_d$. Given $\xi \in T_f\mathrm{Rat}_d$, we want to understand under which conditions the vector field $f^*\theta + \eta_\xi$ is holomorphic on a neighborhood of the critical point set of f.

Lemma 3 *Let c be a simple critical point of $f \in \mathrm{Rat}_d$ and let θ be a vector field, holomorphic near $v = f(c)$. For any $\xi \in T_f\mathrm{Rat}_d$, the vector field $f^*\theta + \eta_\xi$ is holomorphic near c if and only if $\theta(v) = \xi(c)$.*

Proof: Since c is a simple critical points of f, it follows from the Implicit Function Theorem that there is a unique holomorphic germ $t \mapsto c_t$ with $c_0 = c$ such that c_t is a critical point of f_t. Let $v_t{:=}f_t(c_t)$ be the corresponding critical values. Note that $\dot v = \dot f(c) + D_cf(\dot c) = \xi(c)$, since $\dot f = \xi$ and $D_cf = 0$.

Let $t \mapsto \phi_t$ be a holomorphic family of Möbius transformations sending v to v_t, with $\phi_0 = \mathrm{Id}$. Note that there is a holomorphic family of local biholomorphisms ψ_t sending c to c_t, with $\psi_0 = \mathrm{Id}$ and

$$\phi_t \circ f = f_t \circ \psi_t.$$

Differentiating this identity with respect to t and evaluating at $t = 0$ yields $\dot\phi \circ f = \xi + Df \circ \dot\psi$, whence $Df \circ (\dot\psi - f^*\dot\phi) = -\xi$. Consequently, $\dot\psi - f^*\dot\phi = \eta_\xi$ whence $f^*\dot\phi + \eta_\xi$ is holomorphic in a neighborhood of c.

It follows that $f^*\theta + \eta_\xi$ is holomorphic near c if and only if $f^*(\theta - \dot\phi)$ is holomorphic near c. Since $\theta - \dot\phi$ is holomorphic near v, this is the case if and only if $\theta - \dot\phi$ vanishes at v, i.e. if and only if $\theta(v) = \dot\phi(v) = \dot v = \xi(c)$. □

For a finite set $X \subset \mathbb{P}^1$, we denote by $\mathcal{T}(X)$ the linear space of vector fields on X; note that $\mathcal{T}(X)$ is canonically isomorphic to $\bigoplus_{x\in X} T_x\mathbb{P}^1$.

Let $f \in \mathrm{Rat}_d$ have $2d-2$ simple critical points, let $A \subset \mathbb{P}^1 - \mathcal{C}(f)$ be finite, and set $B = f(A) \cup \mathcal{V}(f)$. We shall say that a vector field $\tau \in \mathcal{T}(B)$ is *guided* by $\xi \in T_f\mathrm{Rat}_d$ if

$$\tau = f^*\tau + \eta_\xi \text{ on } A \quad \text{and} \quad \tau \circ f = \xi \text{ on } \mathcal{C}(f).$$

Note that a priori, there might be distinct critical points c_1 and c_2 with $f(c_1) = f(c_2)$ but $\xi(c_1) \neq \xi(c_2)$. In this case, no vector field can be guided by ξ.

2.3 Quadratic differentials

Recall that a quadratic differential is a section of the complex line bundle obtained as $\otimes$-square of the holomorphic cotangent bundle. For a finite set $X \subset \mathbb{P}^1$, we denote by $\mathcal{Q}(\mathbb{P}^1, X)$ the set of all meromorphic quadratic differentials whose poles are all simple and lie in X. This is a vector space of dimension $\max\big(|X|-3, 0\big)$.

Given $q \in \mathcal{Q}(\mathbb{P}^1, X)$ and a vector field τ, defined and holomorphic near $x \in X$, we regard the product $q \otimes \tau$ as a meromorphic 1-form defined in a neighborhood of x, whence there is a residue $\mathrm{Res}_x(q\otimes\tau)$ at x. If τ_1 and τ_2 agree at x, then $\mathrm{Res}_x(q\otimes\tau_1) = \mathrm{Res}_x(q\otimes\tau_2)$, since q has at worst a simple pole at x. Thus it makes sense to talk about $\mathrm{Res}_x(q\otimes\tau)$ even when τ is only defined at x.

Given $q \in \mathcal{Q}(\mathbb{P}^1, X)$ and $\tau \in \mathcal{T}(X)$, we define

$$\langle q, \tau\rangle := 2i\pi \sum_{x\in X} \mathrm{Res}_x(q\otimes\tau).$$

Lemma 4 *Given $\tau \in \mathcal{T}(X)$, let θ be a $\mathcal{C}^\infty$ vector field on $\mathbb{P}^1$ which agrees with τ on X and is holomorphic in a neighborhood of X. Then for every $q \in \mathcal{Q}(\mathbb{P}^1, X)$,*

$$\langle q, \tau\rangle = -\int_{\mathbb{P}^1} q \otimes \bar\partial\theta.$$

Proof: Let U be a finite union of smoothly bounded disks, with pairwise disjoint closures, each enclosing a unique point of X, and such that θ is holomorphic in a neighborhood of $\overline{U}$. Then for any $q \in \mathcal{Q}(\mathbb{P}^1, X)$, we have

$$\langle q, \tau\rangle = \int_{\partial U} q\otimes\theta = -\int_{\mathbb{P}^1 - \overline{U}} q\otimes\bar\partial\theta = -\int_{\mathbb{P}^1} q\otimes\bar\partial\theta,$$

where the first equality is due to the Residue Theorem, the second to Stokes' Theorem, and the last from $\bar\partial\theta = 0$ in a neighborhood of $\overline{U}$. □

Given a rational map $f:\mathbb{P}^1\to\mathbb{P}^1$ and a meromorphic quadratic differential q on $\mathbb{P}^1$, we define the pushforward f_*q as follows. At a point $w\in\mathbb{P}^1$ which is neither a critical value nor the image of a pole, we set

$$f_*q(w)(\tau_1,\tau_2)=\sum_{z\in f^{-1}(w)} q(z)\big((D_zf)^{-1}\tau_1,(D_zf)^{-1}\tau_2\big).$$

The resulting quadratic differential f_*q is in fact globally meromorphic. Moreover, if $q\in\mathcal{Q}(\mathbb{P}^1,A)$ then $f_*q\in\mathcal{Q}(\mathbb{P}^1,B)$ with $B=f(A)\cup\mathcal{V}(f)$. Checking that f_*q belongs to $\mathcal{Q}(\mathbb{P}^1,B)$ requires some justifications which can be found in [DH] for example.

We denote by $\nabla:\mathcal{Q}(\mathbb{P}^1,A)\to\mathcal{Q}(\mathbb{P}^1,B)$ the linear map defined by

$$\nabla q{:=}q-f_*q.$$

Lemma 5 *Let $f\in\mathrm{Rat}_d$ be a rational map with all critical points simple and let $A\subset\mathbb{P}^1-\mathcal{C}(f)$ be finite. Set $B=f(A)\cup\mathcal{V}(f)$ and let $\tau\in\mathcal{T}(B)$ be a vector field guided by $\xi\in T_f\mathrm{Rat}_d$. Then, for all $q\in\mathcal{Q}(\mathbb{P}^1,A)$, we have*

$$\langle\nabla q,\tau\rangle=0.$$

Proof: Let θ be a $\mathcal{C}^\infty$ vector field on $\mathbb{P}^1$ which agrees with τ on B and is holomorphic in a neighborhood of B. Since $\tau\circ f=\xi$ on $\mathcal{C}(f)$, Lemma 3 implies that the vector field $f^*\theta+\eta_\xi$ is holomorphic on a neighborhood of $\mathcal{C}(f)$. It follows that $f^*\theta+\eta_\xi$ is $\mathcal{C}^\infty$ on $\mathbb{P}^1$, holomorphic in a neighborhood of A and agrees with $f^*\tau+\eta_\xi=\tau$ on A. Consequently, for any $q\in\mathcal{Q}\big(\mathbb{P}^1,A\big)$

$$\langle f_*q,\tau\rangle=-\int_{\mathbb{P}^1} f_*q\otimes\bar\partial\theta=-\int_{\mathbb{P}^1} q\otimes f^*\bar\partial\theta=-\int_{\mathbb{P}^1} q\otimes\bar\partial\big(f^*\theta+\eta_\xi\big)=\langle q,\tau\rangle$$

where the first and last equalities follow from Lemma 4, the second from a change of variable and the third from $\bar\partial\eta_\xi=0$ on $\mathbb{P}^1-A$. □

Lemma 6 *Let $f\in\mathrm{Rat}_d$ be postcritically finite. If f is not a flexible Lattès map, then the linear endomorphism $\nabla:\mathcal{Q}\big(\mathbb{P}^1,\mathcal{P}(f)\big)\to\mathcal{Q}\big(\mathbb{P}^1,\mathcal{P}(f)\big)$ is injective.*

Proof: See [DH]. □

Proposition 2 *Let $f\in\mathrm{Rat}_d$ be postcritically finite with all critical points simple and $\mathcal{C}(f)\cap\mathcal{P}(f)=\emptyset$. Assume further that f is not a Lattès map. If $\xi\in T_f\mathrm{Rat}_d$ guides $\tau\in\mathcal{T}\big(\mathcal{P}(f)\big)$ then $\xi\in T_f\mathcal{O}(f)$.*

Proof: Since, the vector space $\mathcal{Q}\big(\mathbb{P}^1, \mathcal{P}(f)\big)$ is finite dimensional, the injectivity of $\nabla : \mathcal{Q}\big(\mathbb{P}^1, \mathcal{P}(f)\big) \to \mathcal{Q}\big(\mathbb{P}^1, \mathcal{P}(f)\big)$ implies its surjectivity. Thus, it follows from lemma 5 that $\langle q, \tau\rangle = 0$ for every $q \in \mathcal{Q}\big(\mathbb{P}^1, \mathcal{P}(f)\big)$.

Lemma 7 *A vector field $\tau \in \mathcal{T}(X)$ extends holomorphically to $\mathbb{P}^1$ if and only if $\langle q, \tau\rangle = 0$ for every $q \in \mathcal{Q}(\mathbb{P}^1, X)$.*

Proof: We may clearly assume without loss of generality that X contains at least three distinct points x_1, x_2, x_3. Let θ be the unique holomorphic vector field on $\mathbb{P}^1$ which coincides with τ at x_1, x_2 and x_3. We must show that $\theta(x) = \tau(x)$ for any $x \in X - \{x_1, x_2, x_3\}$. Up to scale, there is a unique meromorphic quadratic differential q with simple poles at x_1, x_2, x_3 and x. The globally meromorphic 1-form $q \otimes \theta$ has only simple poles, and these must lie in $\{x_1, x_2, x_3, x\}$. The sum of residues of a meromorphic 1-form on $\mathbb{P}^1$ is 0. It follows that τ and θ coincide at x if and only if $\mathrm{Res}_x(q \otimes \theta) = \mathrm{Res}_x(q \otimes \theta)$, whence

$$\begin{aligned}\sum_{y \in \{x_1, x_2, x_3, x\}} \mathrm{Res}_y(q \otimes \tau) = &\sum_{y \in \{x_1, x_2, x_3, x\}} \mathrm{Res}_y(q \otimes \theta) \\ &= \sum_{y \in \mathbb{P}^1} \mathrm{Res}_y(q \otimes \theta) = 0. \end{aligned} \qquad \square$$

Consequently, τ admits a globally holomorphic extension $\theta \in \mathrm{aut}(\mathbb{P}^1)$. Since τ is guided by ξ, it follows from Lemma 3 that the vector field $f^*\theta + \eta_\xi$ is holomorphic on $\mathbb{P}^1$. Moreover, $f^*\theta + \eta_\xi$ agrees with θ on $\mathcal{P}(f)$. Since a rational map whose postcritical set contains only two points is conjugate to $z \mapsto z^{\pm d}$, the set $\mathcal{P}(f)$ contains at least three points, whence the globally holomorphic vector fields are equal. That is to say $\eta_\xi = \theta - f^*\theta$ with $\theta \in \mathrm{aut}(\mathbb{P}^1)$. Since $\xi \in T_f\mathcal{O}(f)$ in view of Proposition 1, this completes the proof of Proposition 2. $\square$

2.4 Proof of Theorem 1

In this section, we prove Theorem 1. By assumption, $f \in \mathrm{Rat}_d$ is postcritically finite with $2d-2$ distinct critical points, $\mathcal{C}(f) \cap \mathcal{P}(f) = \emptyset$ and f is not a Lattès map. Let the analytic germs

$$\underline{\mathfrak{v}} : (\mathrm{Rat}_d, f) \to (\mathbb{P}^1)^{2d-2} \quad \text{and} \quad \underline{\mathfrak{a}} : (\mathrm{Rat}_d, f) \to (\mathbb{P}^1)^{2d-2}$$

be defined as in the Introduction. We shall show that the linear map $D_f\underline{\mathfrak{v}} - D_f\underline{\mathfrak{a}}$ is surjective and that its kernel is $T_f\mathcal{O}(f)$.

Note that $T_f\mathrm{Rat}_d$ has complex dimension $2d+1$. The map $D_f\underline{\mathfrak{v}} - D_f\underline{\mathfrak{a}}$ has maximal rank $2d-2$ if and only if the kernel has dimension 3. Now on

$\mathcal{O}(f)$, we have $\underline{\mathfrak{v}} \equiv \underline{\mathfrak{a}}$, whence $T_f\mathcal{O}(f) \subseteq \mathrm{Ker}\big(D_f\underline{\mathfrak{v}} - D_f\underline{\mathfrak{a}}\big)$. Since $\mathcal{O}(f)$ has complex dimension 3, it suffices to show

$$\mathrm{Ker}(D_f\underline{\mathfrak{v}} - D_f\underline{\mathfrak{a}}) \subseteq T_f\mathcal{O}(f).$$

Henceforth, we assume that ξ belongs to $\mathrm{Ker}(D_f\underline{\mathfrak{v}} - D_f\underline{\mathfrak{a}})$. In view of Proposition 2, it suffices to show that ξ guides a vector field $\tau \in \mathcal{T}\big(\mathcal{P}(f)\big)$.

We begin by specifying τ on $\mathcal{P}(f)$. Let $t \mapsto f_t$ be a family of rational maps of degree d such that $f_0 = f$ and $\dot f = \xi$. If α is a repelling periodic point of f then, by the Implicit Function Theorem, there is a unique germ $t \mapsto \alpha_t$ with $\alpha_0 = \alpha$ such that α_t is a periodic point of f_t. For periodic $\alpha \in \mathcal{P}(f)$, we set

$$\tau(\alpha) := \dot\alpha \in T_\alpha\mathbb{P}^1.$$

Note that if $\beta = f(\alpha)$, then $\beta_t = f_t(\alpha_t)$ is a periodic point of f_t. Evaluating derivatives at $t = 0$ yields

$$\tau(\beta) = \dot\beta = \dot f(\alpha) + D_\alpha f(\dot\alpha) = \xi(\alpha) + D_\alpha f\big(\tau(\alpha)\big).$$

Since $D_\alpha f$ is invertible, we deduce that

$$(D_\alpha f)^{-1}\big(\tau(\beta)\big) = (D_\alpha f)^{-1}\big(\xi(\alpha)\big) + \tau(\alpha)$$

whence

$$f^*\tau(\alpha) = -\eta_\xi(\alpha) + \tau(\alpha).$$

Thus, on the set of repelling periodic points contained in $\mathcal{P}(f)$, we have

$$\tau = f^*\tau + \eta_\xi. \tag{2.2}$$

Since there are no critical points in $\mathcal{P}(f)$, there is a unique extension of τ to the whole postcritical set such that (2.2) remains valid. Note that since no $z \in \mathcal{P}(f)$ is precritical, there is a unique analytic germ $t \mapsto z_t$ such that z_t is preperiodic under f_t, and we have $\tau(z) = \dot z$.

To complete the proof of Theorem 1, it suffices to show that $\tau \circ f = \xi$ on $\mathcal{C}(f)$. So, let $c \in \mathcal{C}$ and for $k \geq 1$, define $v_t^k := f_t^{\circ k}(c_t)$. The critical point c is the j-th critical point of f as listed in the Introduction. We have $\mathfrak{v}_j(f_t) = v_t^\ell$ for some integer $\ell \geq 1$ and $\alpha_t := \mathfrak{a}_j(f_t)$ is a periodic point of f_t with $\alpha = v^\ell$. By assumption, $\xi \in \mathrm{Ker}(D_f\underline{\mathfrak{v}} - D_f\underline{\mathfrak{a}})$, whence

$$\dot v^\ell = D_f\mathfrak{v}_j(\xi) = D_f\mathfrak{a}_j(\xi) = \dot\alpha = \tau(\alpha).$$

Differentiating $f_t(v_t^k) = v_t^{k+1}$ with respect to t yields

$$\xi(v^k) + D_{v^k}f(\dot v^k) = \dot v^{k+1}.$$

Applying $(D_{v^k}f)^{-1}$ gives $-\eta_\xi(v^k)+\dot v^k=(D_{v_k}f)^{-1}(\dot v^{k+1})$, whence

$$\dot v^k=(D_{v^k}f)^{-1}(\dot v^{k+1})+\eta_\xi(v^k).$$

We now proceed by decreasing induction on $k\geq 1$. Since (2.2) holds on $\mathcal{P}(f)$, if $\dot v^{k+1}=\tau(v^{k+1})$, then

$$\dot v^k=(D_{v^k}f)^{-1}\big(\tau\circ f(v^k)\big)+\eta_\xi(v^k)=(f^*\tau+\eta_\xi)(v^k)=\tau(v^k).$$

The desired result is obtained by taking $k=1$: $\xi(c)=\dot v^1=\tau(v^1)=\tau\circ f(c)$.

3 The bifurcation measure

We will now prove that $\mathcal{X}_d^*\subseteq \mathrm{Supp}(\mu_{\mathrm{bif}})$. This will complete the proof of our main theorem. Here and henceforth, we let $f\in \mathrm{Rat}_d$ be a postcritically finite map with only simple critical points, which satisfies $\mathcal{C}(f)\cap\mathcal{P}(f)=\emptyset$ and is not a flexible Lattès map. It suffices to exhibit a $2d-2$-dimensional complex manifold $\Sigma\subset\mathrm{Rat}_d$ containing f and a basis of neighborhoods Σ_n of f in Σ, such that

$$\int_{\Sigma_n}(T_{\mathrm{bif}})^{\wedge(2d-2)}>0.$$

3.1 Another definition of the bifurcation current

We will use a second definition of the bifurcation current T_{bif} due to DeMarco [DeM1] (see [DeM2] or [BB] for the equivalence of the two definitions). The current T_{bif} may be defined by considering the behavior of the critical orbits as follows.

Set $\mathcal{J}:=\{1,\dots,2d-2\}$. Let $\pi:\mathbb{C}^2-\{0\}\to\mathbb{P}^1$ be the canonical projection. Denote by $\tilde x:=(x_1,x_2)$ the points in $\mathbb{C}^2$. Let $U\subset\mathrm{Rat}_d$ be a sufficiently small neighborhood of f so that there are:

- holomorphic functions $\{\mathfrak{c}_j:U\to\mathbb{P}^1\}_{j\in\mathcal{J}}$ following the critical points of g as g ranges in U,
- holomorphic functions $\{\tilde{\mathfrak{c}}_j:U\to\mathbb{C}^2-\{0\}\}_{j\in\mathcal{J}}$ such that $\pi\circ\tilde{\mathfrak{c}}_j=\mathfrak{c}_j$ and
- an analytic family $U\ni g\mapsto\tilde g$ of nondegenerate homogeneous polynomials of degree d such that $\pi\circ\tilde g=g\circ\pi$.

The map $\mathcal{G}:U\times\mathbb{C}^2\to\mathbb{R}$ defined by

$$\mathcal{G}(g,\tilde x):=\lim_{n\to+\infty}\frac{1}{d^n}\log\big\|\tilde g^{\circ n}(\tilde x)\big\|$$

is plurisubharmonic on $U \times \mathbb{C}^2$. DeMarco [DeM2] proved that

$$T_{\rm bif}\big|_U = \sum_{j=1}^{2d-2} {\rm dd}^c \mathcal{G}_j$$

where $\mathcal{G}_j : U \to \mathbb{R}$ is defined by $\mathcal{G}_j(g) := \mathcal{G}\big(g, \tilde{\mathfrak{c}}_j(g)\big)$.

3.2 Definition of Σ

Let the analytic germs $\underline{\mathfrak{c}}, \underline{\mathfrak{a}}, \underline{\mathfrak{v}} : ({\rm Rat}_d, f) \to (\mathbb{P}^1)^{2d-2}$ be defined as in the Introduction. Set $\underline{c} := \underline{\mathfrak{c}}(f)$ and $\underline{\alpha} := \underline{\mathfrak{a}}(f) = \underline{\mathfrak{v}}(f)$. Given a rational map $g \in {\rm Rat}_d$, let $\underline{g} : (\mathbb{P}^1)^{2d-2} \to (\mathbb{P}^1)^{2d-2}$ be the map defined by

$$\underline{g}(\underline{z}) := \big(g(z_1), \ldots, g(z_{2d-2})\big).$$

Recall that for g near f, we have $\mathfrak{v}_j(g) = g^{\circ \ell_j} \circ \mathfrak{c}_j(g)$ for some integer ℓ_j. Let p_j be the period of α_j. Let p be the least common multiple of the periods p_j. For g near f, let $\mathfrak{m}_j(g)$ be the multiplier of α_j as a fixed point of $g^{\circ p}$. Denote by $\vec{x} = (x_1, \ldots, x_{2d-2})$ the elements of $\mathbb{C}^{2d-2}$ and let $\overrightarrow{M}_g : \mathbb{C}^{2d-2} \to \mathbb{C}^{2d-2}$ be the linear map defined by

$$\overrightarrow{M}_g(\vec{x}) := \big(\mathfrak{m}_1(g) \cdot x_1, \ldots, \mathfrak{m}_{2d-2}(g) \cdot x_{2d-2}\big).$$

For every $j \in \mathcal{J}$, α_j is repelling. It follows that for g near f, there is a local biholomorphism $\underline{\rm Lin}_g : (\mathbb{C}^{2d-2}, \vec{0}) \to \big((\mathbb{P}^1)^{2d-2}, \underline{\mathfrak{a}}(g)\big)$ linearizing $\underline{g}^{\circ p}$, that is

$$\underline{\rm Lin}_g \circ \overrightarrow{M}_g = \underline{g}^{\circ p} \circ \underline{\rm Lin}_g.$$

In addition, we may choose $\underline{\rm Lin}_g$ such that the germ $(g, \vec{x}) \mapsto \underline{\rm Lin}_g(\vec{x})$ is analytic near $(f, \vec{0})$ and the germ $(g, \underline{z}) \mapsto \underline{\rm Lin}_g^{-1}(\underline{z})$ is analytic near $(f, \underline{\alpha})$.

Lemma 8 *There exists an analytic germ $S : (\mathbb{C}^{2d-2}, \vec{0}) \to ({\rm Rat}_d, f)$ such that for $\vec{x}$ near $\vec{0}$*

$$\underline{\mathfrak{v}} \circ S(\vec{x}) = \underline{\rm Lin}_{S(\vec{x})}.$$

Proof: Let $\vec{\mathfrak{h}} : ({\rm Rat}_d, f) \to (\mathbb{C}^{2d-2}, \vec{0})$ be the analytic germ defined by $\vec{\mathfrak{h}}(g) := \underline{\rm Lin}_g^{-1} \circ \underline{\mathfrak{v}}(g)$. Then $\underline{\mathfrak{v}}(g) = \underline{\rm Lin}_g \circ \vec{\mathfrak{h}}(g)$ and $\underline{\mathfrak{a}}(g) = \underline{\rm Lin}_g(\vec{0})$. Differentiating with respect to g and evaluating at $g = f$ yields

$$D_f \underline{\mathfrak{v}} = D_f \underline{\mathfrak{a}} + D_{\vec{0}} \underline{\rm Lin}_f \circ D_f \vec{\mathfrak{h}}.$$

According to Theorem 1, the linear map $D_f \underline{\mathfrak{v}} - D_f \underline{\mathfrak{a}} : T_f {\rm Rat}_d \to \bigoplus_{j \in \mathcal{J}} T_{\alpha_j} \mathbb{P}^1$ is surjective. Since $D_{\vec{0}} \underline{\rm Lin}_f : \mathbb{C}^{2d-2} \to \bigoplus_{j \in \mathcal{J}} T_{\alpha_j} \mathbb{P}^1$ is invertible. Thus $D_f \vec{\mathfrak{h}} : T_f {\rm Rat}_d \to \mathbb{C}^{2d-2}$ has maximal rank. It follows from the Implicit Function Theorem that there is a section $S : (\mathbb{C}^{2d-2}, \vec{0}) \to ({\rm Rat}_d, f)$ with $\vec{\mathfrak{h}} \circ S = {\rm Id}$. This may be rewritten as $\underline{\mathfrak{v}} \circ S(\vec{x}) = \underline{\rm Lin}_{S(\vec{x})}$. $\square$

For each $j \in \mathcal{J}$, choose a neighborhood V_j of α_j in $\mathbb{P}^1$ such that there is a holomorphic section $\sigma_j : V_j \to \mathbb{C}^2 - \{0\}$ of $\pi : \mathbb{C}^2 - \{0\} \to \mathbb{P}^1$. Fix $r > 0$ small enough that the germ S given by Lemma 8 and the germ $\underline{\mathrm{Lin}}_f$ are both defined and analytic on a neighborhood of $\overline{\Delta}^{2d-2}$ where $\Delta{:=}D(0,r)$, and such that

$$\underline{\mathrm{Lin}}_f(\overline{\Delta}^{2d-2}) \subset \underline{V}{:=} \prod_{j\in\mathcal{J}} V_j.$$

We set

$$\Sigma{:=}S(\Delta^{2d-2}).$$

By definition, this subset of Rat_d is a submanifold of complex dimension $2d-2$.

3.3 Definition of Σ_n

For $n \geq 1$, set

$$S_n{:=}S \circ \overrightarrow{M}_f^{-n} : \Delta^{2d-2} \to \Sigma \quad \text{and} \quad \Sigma_n{:=}S_n(\Delta^{2d-2}) \subset \Sigma.$$

Clearly, the sets Σ_n form a basis of neighborhoods of f in Σ. For $n \geq 1$, let $\underline{\mathfrak{v}}^n : \Sigma \to \mathbb{P}^1$ be the map defined by

$$\underline{\mathfrak{v}}^n(g){:=}\underline{g}^{\circ(np)} \circ \underline{\mathfrak{v}}(g).$$

Note that for $j \in \mathcal{J}$, we have $\mathfrak{v}_j^n(g) = g^{\circ(\ell_j+np)} \circ \mathfrak{c}_j(g)$.

Lemma 9 *The sequence $(\underline{\mathfrak{v}}^n \circ S_n)$ converges uniformly to $\underline{\mathrm{Lin}}_f$ on Δ^{2d-2}.*

Proof: Note that the sequence (S_n) converges to f uniformly and exponentially on $\overline{\Delta}^{2d-2}$ as n tends to ∞. It follows that the sequence $(\vec{x} \mapsto \overrightarrow{M}_{S_n(\vec{x})})$ converges uniformly and exponentially to $\overrightarrow{M}_f$. Consequently the sequence $\big(\vec{x} \mapsto \overrightarrow{M}^n_{S_n(\vec{x})} \circ M_f^{-n}(\vec{x})\big)$ converges uniformly to the identity. Thus, for n large enough and for any $\vec{x} \in \Delta^{2d-2}$, setting $g_n{:=}S_n(\vec{x})$, we have

$$\begin{aligned}\underline{\mathrm{Lin}}_{g_n} \circ \overrightarrow{M}^n_{g_n} \circ \overrightarrow{M}_f^{-n}(\vec{x}) &= \underline{g_n}^{\circ(np)} \circ \underline{\mathrm{Lin}}_{g_n} \circ \overrightarrow{M}_f^{-n}(\vec{x}) \\ &= \underline{g_n}^{\circ(np)} \circ \underline{\mathfrak{v}} \circ S \circ M_f^{-n}(\vec{x}) \\ &= \underline{\mathfrak{v}}^n \circ S_n(\vec{x}).\end{aligned}$$

The result follows easily. $\square$

3.4 The proof

Here, we shall use the notation of Section 3.1. We assume that n is large enough that Σ_n is contained in U, whence every map $g \in \Sigma_n$ has a lift $\tilde{g}$ to homogeneous coordinates and there is a potential function $\mathcal{G}$ defined on $\Sigma_n \times \mathbb{C}^2$ such that

$$\mathcal{G}\big(g, \tilde{g}(\tilde{x})\big) = d \cdot \mathcal{G}(g, \tilde{x}) \quad \text{and} \quad \forall \lambda \in \mathbb{C}^*, \quad \mathcal{G}(g, \lambda\tilde{x}) = \mathcal{G}(g, \tilde{x}) + \log|\lambda|.$$

Recall that by Lemma 9, the sequence $(\underline{\mathfrak{v}}^n \circ S_n)$ converges uniformly to $\underline{\mathrm{Lin}}_f$ on Δ^{2d-2} and by assumption, $\underline{\mathrm{Lin}}_f(\overline{\Delta}^{2d-2}) \subset \underline{V}$. From now on, let n be sufficiently large so that $\underline{\mathfrak{v}}^n(\Sigma_n) \subseteq \underline{V}$. In that case, for each $j \in \mathcal{J}$, the map

$$\tilde{\mathfrak{v}}_j^n := \sigma_j \circ \mathfrak{v}_j^n : \Sigma_n \to \mathbb{C}^2 - \{0\}$$

is-well defined. In this case, we may define plurisubharmonic functions $\mathcal{G}_j^n : \Delta^{2d-2} \to \mathbb{R}$ by

$$\mathcal{G}_j^n(\underline{x}) := \mathcal{G}\big(S_n(\underline{x}), \tilde{\mathfrak{v}}_j^n \circ S_n(\underline{x})\big).$$

Lemma 10 *If n is sufficiently large, then*

$$S_n^*(T_{\mathrm{bif}}) = d^{-np} \sum_{j \in \mathcal{J}} d^{-\ell_j} \mathrm{dd^c} \mathcal{G}_j^n.$$

Proof: Note that for $g \in \Sigma_n$ and $j \in \mathcal{J}$, we have

$$\pi \circ \tilde{g}^{\circ(\ell_j + np)}\big(\tilde{\mathfrak{c}}_j(g)\big) = g^{\circ(\ell_j + np)}\big(\mathfrak{c}_j(g)\big) = \mathfrak{v}_j^n(g) = \pi \circ \tilde{\mathfrak{v}}_j^n(g),$$

whence

$$\tilde{g}^{\circ n}\big(\tilde{\mathfrak{c}}_j(g)\big) = \lambda_j^n(g) \cdot \tilde{\mathfrak{v}}_j^n(g)$$

for some holomorphic functions $\lambda_j^n : \Sigma_n \to \mathbb{C}^*$. Thus, if $\vec{x} \in \Delta^{2d-2}$ and $g_n := S_n(\vec{x})$, then

$$\begin{aligned}
\mathcal{G}_j^n(\vec{x}) &= \mathcal{G}\big(g_n, \tilde{g}_n^{\circ(\ell_j + np)}\big(\tilde{\mathfrak{c}}_j(g_n)\big)\big) - \log\big|\lambda_j^n(g_n)\big| \\
&= d^{\ell_j + np} \mathcal{G}\big(g_n, \tilde{\mathfrak{c}}_j(g_n)\big) - \log\big|\lambda_j^n(g_n)\big| \\
&= d^{\ell_j + np} \mathcal{G}_j(g_n) - \log\big|\lambda_j^n(g_n)\big| \\
&= d^{\ell_j + np} \mathcal{G}_j \circ S_n(\vec{x}) - \log\big|\lambda_j^n \circ S_n(\vec{x})\big|.
\end{aligned}$$

Since $\log|\lambda_j^n \circ S_n|$ is pluriharmonic on Δ^{2d-2}, we have

$$\mathrm{dd^c} \mathcal{G}_j^n = d^{\ell_j + np} \cdot \mathrm{dd^c}(\mathcal{G}_j \circ S_n) = d^{\ell_j + np} \cdot S_n^*(\mathrm{dd^c} \mathcal{G}_j).$$

Consequently, in view of DeMarco's formula, we have

$$S_n^*(T_{\mathrm{bif}}) = \sum_{j \in \mathcal{J}} S_n^*(\mathrm{dd^c} \mathcal{G}_j) = d^{-np} \sum_{j \in \mathcal{J}} d^{-\ell_j} \mathrm{dd^c} \mathcal{G}_j^n.$$

$\square$

Set

$$M_n := \int_{\Sigma_n} T_{\mathrm{bif}}{}^{\wedge(2d-2)}.$$

To conclude the proof of the main theorem, we will now show that $M_n > 0$ for n large enough. In fact, we will show that there is a constant $m > 0$ such that

$$M_n \underset{n\to+\infty}{\sim} \frac{m}{d^{(2d-2)np}}.$$

Set $|\ell| := \sum_{j\in\mathcal{J}} \ell_j$. For $j \in \mathcal{J}$, let $\mathrm{lin}_j : \Delta \to \mathbb{P}^1$ be the map defined by

$$\underline{\mathrm{Lin}}_f(\vec{x}) = \big(\mathrm{lin}_1(x_1), \ldots, \mathrm{lin}_{2d-2}(x_{2d-2})\big)$$

and set

$$W_j := \mathrm{lin}_j(\Delta).$$

Recall that μ_f is the equilibrium measure of f.

Lemma 11 *We have*

$$\lim_{n\to+\infty} d^{(2d-2)np} \cdot M_n = (2d-2)! \cdot d^{-|\ell|} \cdot \prod_{j\in\mathcal{J}} \mu_f(W_j).$$

Proof: By Lemma 9, the sequences of functions $\mathfrak{v}_j^n \circ S_n$ converge uniformly on Δ^{2d-2} to $\underline{x} \mapsto \mathrm{lin}_j(x_j)$. So, the sequences of functions $\mathcal{G}_j^n : \Delta^{2d-2} \to \mathbb{R}$ converge uniformly to

$$\mathcal{G}_j^\infty : \underline{x} \mapsto \mathcal{G}\big(f, \sigma_j \circ \mathrm{lin}_j(x_j)\big).$$

Due to the uniform convergence of the potentials, we may write

$$\lim_{n\to+\infty} \int_{\Delta^{2d-2}} \left(\sum_{j\in\mathcal{J}} d^{-\ell_j} \mathrm{dd}^c \mathcal{G}_j^n\right)^{\wedge(2d-2)} = \int_{\Delta^{2d-2}} \left(\sum_{j\in\mathcal{J}} d^{-\ell_j} \mathrm{dd}^c \mathcal{G}_j^\infty\right)^{\wedge(2d-2)}.$$

Note that $\mathcal{G}_j^\infty$ only depends on the j-th coordinate. It follows that

$$\left(\sum_{j=1}^{2d-2} d^{-\ell_j} \mathrm{dd}^c \mathcal{G}_j^\infty\right)^{\wedge(2d-2)} = (2d-2)! d^{-|\ell|} \cdot \bigwedge_{j\in\mathcal{J}} \mathrm{dd}^c \mathcal{G}_j^\infty.$$

In addition, $\mathcal{G}_j^\infty(\underline{x}) = G_j \circ \mathrm{lin}_j(x_j)$ with $G_j : W_j \to \mathbb{R}$ the subharmonic function defined by $G_j(z) = \mathcal{G}\big(f, \sigma_j(z)\big)$. We have $\mathrm{dd}^c G_j = \mu_f\big|_{W_j}$. Therefore, according to Fubini's theorem,

$$\int_{\Delta^{2d-2}} \left(\bigwedge_{j\in\mathcal{J}} \mathrm{dd}^c \mathcal{G}_j^\infty\right) = \prod_{j\in\mathcal{J}} \left(\int_\Delta \mathrm{dd}^c \left(G_j \circ \mathrm{lin}_j\right)\right) = \prod_{j\in\mathcal{J}} \mu_f(W_j). \quad \square$$

We now complete the proof. Since the periodic points α_j are repelling, they are in the support of the equilibrium measure μ_f. Thus, for every $j \in \mathcal{J}$, $\mu_f(W_j) > 0$. As a consequence,

$$M_n \underset{n\to+\infty}{\sim} \frac{m}{d^{(2d-2)np}} \quad \text{with} \quad m{:=}(2d-2)! \cdot d^{-|\ell|} \cdot \prod_{j=1}^{2d-2} \mu_f(W_j) > 0,$$

f is in the support of ${T_{\mathrm{bif}}}^{\wedge(2d-2)}$ and the conjugacy class of f is in the support of μ_{bif}.

Bibliography

[BB] Giovanni Bassanelli and François Berteloot, *Bifurcation currents in holomorphic dynamics on* $\mathbb{P}^k$, J. Reine Angew. Math. **608** (2007), 201–235.

[DeM1] Laura DeMarco, *Dynamics of rational maps: a current on the bifurcation locus*, Math. Res. Lett. **8** (2001), no. 1-2, 57–66.

[DeM2] Laura DeMarco, *Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity*, Math. Ann. **326** (2003), no. 1, 43–73.

[DF] Romain Dujardin and Charles Favre, *Distribution of rational maps wih a preperiodic critical point*, Amer. Journ. Math., to appear.

[DH] Adrien Douady and John H. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), no. 2, 263–297.

[E] Adam L. Epstein, *Infinitesimal Thurston rigidity and the Fatou-Shishikura inequality*, Stony Brook IMS Preprint (1999).

[HS] John H. Hubbard and Dierk Schleicher, *The spider algorithm*, Complex dynamical systems (Cincinnati, OH, 1994), Proc. Sympos. Appl. Math., vol. 49, Amer. Math. Soc., Providence, RI, 1994, pp. 155–180.

[L] M. Ju. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems **3** (1983), no. 3, 351–385.

[Ma] Ricardo Mañé, *The Hausdorff dimension of invariant probabilities of rational maps*, Dynamical systems, Valparaiso 1986, Lecture Notes in Math., vol. 1331, Springer, Berlin, 1988, pp. 86–117.

[McM1] Curt McMullen, *Families of rational maps and iterative root-finding algorithms*, Ann. of Math. (2) **125** (1987), no. 3, 467–493.

[McM2] Curtis T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies, vol. 135, Princeton University Press, Princeton, NJ, 1994.

[Mi] John Milnor, *On Lattès maps*, Dynamics on the Riemann sphere, Eur. Math. Soc., Zürich, 2006, pp. 9–43.

[MSS] R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 2, 193–217.

[S] Mitsuhiro Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 1, 1–29.

[T] Lei Tan, *Similarity between the Mandelbrot set and Julia sets*, Comm. Math. Phys. **134** (1990), no. 3, 587–617.

[vS] Sebastian van Strien, *Misiurewicz maps unfold generically (even if they are critically non-finite)*, Fund. Math. **163** (2000), no. 1, 39–54.