QUADRATIC JULIA SETS WITH POSITIVE AREA.

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Abstract. We prove the existence of quadratic polynomials having a Julia set with positive Lebesgue measure. We find such examples with a Cremer fixed point, with a Siegel disk, or with infinitely many satellite renormalizations.

To Adrien Douady

Contents

Introduction

Assume $P : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree 2. Its Julia set $J(P)$ is a compact subset of $\mathbb{C}$ with empty interior. Fatou suggested that one should apply to $J(P)$ the methods of Borel-Lebesgue for the measure of sets.

It is known that the area (Lebesgue measure) of $J(P)$ is zero in several cases including:

- if P is hyperbolic;[1]
- if P has a parabolic cycle ([DH1]);
- if P is not infinitely renormalizable ([L] or [Sh1]);

[1] Conjecturally, this is true for a dense and open set of quadratic polynomials. If there were an open set of non-hyperbolic quadratic polynomials, those would have a Julia set of positive area (see [MSS]).

- if P has a (linearizable) indifferent cycle with multiplier $e^{2i\pi\alpha}$ such that $\alpha = \mathrm{a}_0 + \cfrac{1}{\mathrm{a}_1 + \cfrac{1}{\mathrm{a}_2 + \ddots}}$ with $\log \mathrm{a}_n = \mathcal{O}(\sqrt{n})$ ([PZ]).[2]

Recently, we completed a program initiated by Douady with major advances by the second author in [C]: there exist quadratic polynomials with a Cremer fixed point and a Julia set of positive area. In this article, we present a slightly different approach (the general ideas are essentially the same).

Theorem 1. *There exist quadratic polynomials which have a Cremer fixed point and a Julia set of positive area.*

We also have the following two results.

Theorem 2. *There exist quadratic polynomials which have a Siegel disk and a Julia set of positive area.*

Theorem 3. *There exist infinitely satellite renormalizable quadratic polynomials with a Julia set of positive area.*

We will give a detailed proof of Theo. 1 and 2. We will only sketch the proof of Theo. 3.

The proofs are based on

- McMullen's results [McM] regarding the measurable density of the filled-in Julia set near the boundary of a Siegel disk with bounded type rotation number;
- Chéritat's techniques of parabolic explosion [C] and Yoccoz's renormalization techniques [Y] to control the shape of Siegel disks;
- Inou and Shishikura's results [IS] to control the post-critical sets of perturbations of polynomials having an indifferent fixed point.

Acknowledgements

We would like to thank Adrien Douady, John H. Hubbard, Hiroyuki Inou, Curtis T. McMullen, Mitsuhiro Shishikura, Misha Yampolsky and Jean-Christophe Yoccoz whose contributions were decisive in proving these results.

1. The Cremer case

Let us introduce some notations.

Definition 1. *For $\alpha \in \mathbb{C}$, we denote by P_α the quadratic polynomial*

$$P_\alpha : z \mapsto e^{2i\pi\alpha} z + z^2.$$

We denote by K_α the filled-in Julia set of P_α and by J_α its Julia set.

[2] This is true for almost every $\alpha \in \mathbb{R}/\mathbb{Z}$.

1.1. **Strategy of the proof.** The main gear is the following

Proposition 1. *There exists a non empty set $\mathcal{S}$ of bounded type irrationals such that: for all $\alpha \in \mathcal{S}$ and all $\varepsilon > 0$, there exists $\alpha' \in \mathcal{S}$ with*

- $|\alpha' - \alpha| < \varepsilon$,
- $P_{\alpha'}$ *has a cycle in* $D(0, \varepsilon) \setminus \{0\}$ *and*
- $\mathrm{area}(K_{\alpha'}) \geq (1 - \varepsilon)\mathrm{area}(K_\alpha)$.

The proof of Prop. 1 will occupy sections 1.2 to 1.7.

Remark. Since $\alpha \in \mathcal{S}$ has bounded type, K_α contains a Siegel disk [Si] and thus, has positive area.

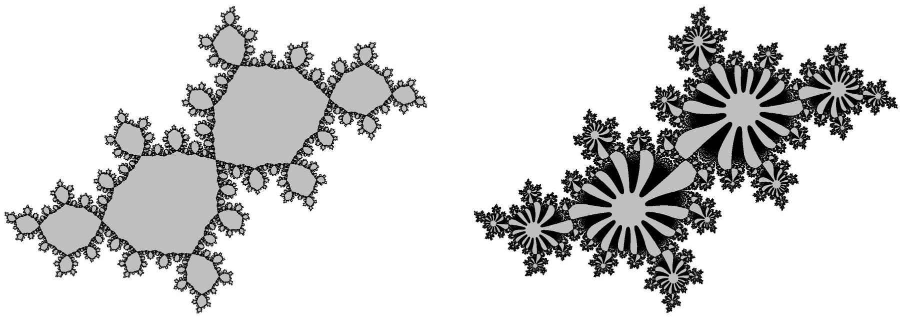


FIGURE 1. Two filled-in Julia sets K_α and $K_{\alpha'}$, with α' a well-chosen perturbation of α as in Prop. 1. This proposition asserts that if α and α' are chosen carefully enough the loss of measure from K_α to $K_{\alpha'}$ is small.

Remark. We do not know what is the largest set $\mathcal{S}$ for which Prop. 1 holds. It might be the set of all bounded type irrationals.

Proposition 2 (Douady)**.** *The function* $\alpha \in \mathbb{C} \mapsto \mathrm{area}(K_\alpha) \in [0, +\infty[$ *is upper semi-continuous.*

Proof. Assume $\alpha_n \to \alpha$. By [D2], for any neighborhood V of K_α, we have $K_{\alpha_n} \subset V$ for n large enough. According to the theory of Lebsegue measure, $\mathrm{area}(K_\alpha)$ is the infimum of the area the open sets containing K_α. Thus,

$$\mathrm{area}(K_\alpha) \geq \limsup_{n \to +\infty} \mathrm{area}(K_{\alpha_n}).$$

$\square$

Proof of Theo. 1 assuming Prop. 1. We choose a sequence of real numbers ε_n in $(0, 1)$ such that $\prod(1 - \varepsilon_n) > 0$. We construct inductively a sequence $\theta_n \in \mathcal{S}$ such that for all $n \geq 1$

- P_{θ_n} has a cycle in $D(0, 1/n) \setminus \{0\}$,
- $\mathrm{area}(K_{\theta_n}) \geq (1 - \varepsilon_n)\mathrm{area}(K_{\theta_{n-1}})$.

Every polynomial P_θ with θ sufficiently close to θ_n has a cycle in $D(0, 1/n) \setminus \{0\}$. By choosing θ_n sufficiently close to θ_{n-1} at each step, we guarantee that

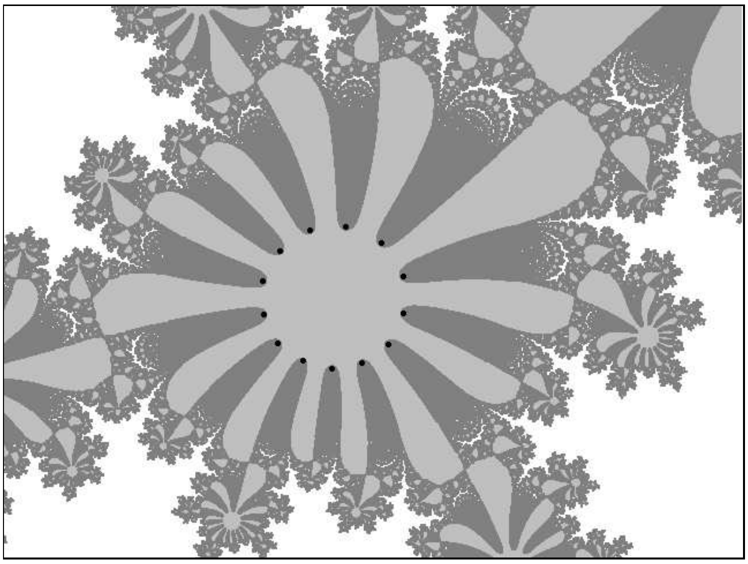


FIGURE 2. A zoom on $K_{\alpha'}$ near its linearizable fixed point. The small cycle is highlighted.

- the sequence (θ_n) is a Cauchy sequence that converges to a limit θ,
- for all $n \geq 1$, P_θ has a cycle in $D(0, 1/n) \setminus \{0\}$.

So, the polynomial P_θ has small cycles and thus is a Cremer polynomial. In that case, $J_\theta = K_\theta$. By Prop. 2:

$$\text{area}(J_\theta) = \text{area}(K_\theta) \geq \limsup_{n\to+\infty} \text{area}(K_{\theta_n}) \geq \text{area}(K_{\theta_0}) \cdot \prod_{n\geq 1}(1-\varepsilon_n) > 0.$$

□

1.2. **A stronger version of Prop. 1.** For a finite or infinite sequence of integers, we will use the following continued fraction notation:

$$[\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots] := \mathrm{a}_0 + \cfrac{1}{\mathrm{a}_1 + \cfrac{1}{\mathrm{a}_2 + \ddots}}.$$

For $\alpha \in \mathbb{R}$, we will denote by $\lfloor \alpha \rfloor$ the integral part of α.

Definition 2. *If $N \geq 1$ is an integer, we set*

$$\mathcal{S}_N := \Big\{\alpha = [\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots] \in \mathbb{R} \setminus \mathbb{Q} \;\Big|\; (\mathrm{a}_k) \textit{ is bounded and } \mathrm{a}_k \geq N \textit{ for all } k \geq 1\Big\}.$$

Note that $\mathcal{S}_{N+1} \subset \mathcal{S}_N \subset \cdots \subset \mathcal{S}_1$ and $\mathcal{S}_1$ is the set of bounded type irrationals. If $\alpha \in \mathcal{S}_1$, the polynomial P_α has a Siegel disk bounded by a quasicircle containing the critical point (see [D1], [He], [Sw]). In particular, the post-critical set of P_α is contained in the boundary of the Siegel disk.

Prop. 1 is an immediate consequence of the following proposition.

Proposition 3. *If N is sufficiently large then the following holds.* [3]
Assume $\alpha \in \mathcal{S}_N$, choose a sequence (A_n) such that

$$\sqrt[q_n]{A_n} \underset{n\to+\infty}{\longrightarrow} +\infty \quad \text{and} \quad \sqrt[q_n]{\log A_n} \underset{n\to+\infty}{\longrightarrow} 1.^4$$

Set

$$\alpha_n := [\mathrm{a}_0, \mathrm{a}_1, \ldots, \mathrm{a}_n, A_n, N, N, N, \ldots].$$

Then, for all $\varepsilon > 0$, if n is sufficiently large,

- *P_{α_n} has a cycle in $D(0,\varepsilon) \setminus \{0\}$ and*
- $\mathrm{area}(K_{\alpha_n}) \geq (1-\varepsilon)\mathrm{area}(K_\alpha)$.

The rest of section 1 is devoted to the proof of Prop. 3. In the sequel, unless otherwise specified,

- α is an irrational number of bounded type,
- p_k/q_k are the approximants to α given by the continued fraction algorithm and
- (α_n) is a sequence converging to α, defined as in Prop. 3.

Note that for $k \leq n$, the approximants p_k/q_k are the same for α and for α_n. The polynomial P_α (resp. P_{α_n}) has a Siegel disk Δ (resp. Δ_n). We let r (resp. r_n) be the conformal radius of Δ (resp. Δ_n) at 0 and we let $\phi : D(0,r) \to \Delta$ (resp. $\phi_n : D(0,r_n) \to \Delta_n$) be the conformal isomorphism which maps 0 to 0 with derivative 1.

1.3. **The control of the cycle.** We first recall results of [C] (see also [BC1] Props. 1 and 2), which we reformulate as follows.

The first proposition asserts that as θ varies in the disk $D(p/q, 1/q^3)$, the polynomial P_θ has a cycle of period q which depends holomorphically on $\sqrt[q]{\theta - p/q}$ and coalesces at $z = 0$ when $\theta = p/q$.

Proposition 4. *For each rational number p/q (with p and q coprime), there exists a holomorphic function*

$$\chi : D(0, 1/q^{3/q}) \to \mathbb{C}$$

with the following properties.

(1) $\chi(0) = 0$.
(2) $\chi'(0) \neq 0$.
(3) If $\delta \in D(0,1/q^{3/q}) \setminus \{0\}$, then $\chi(\delta) \neq 0$.
(4) If $\delta \in D(0,1/q^{3/q}) \setminus \{0\}$ and if we set $\zeta := e^{2i\pi p/q}$ and $\theta := \dfrac{p}{q} + \delta^{q_k}$, then, $\Big\langle \chi(\delta), \chi(\zeta\delta), \ldots, \chi(\zeta^{q-1}\delta) \Big\rangle$ forms a cycle of period q of P_θ. In particular,

$$\forall \delta \in D(0, 1/q^{3/q}), \qquad \chi(\zeta\delta) = P_\theta\big(\chi(\delta)\big).$$

A function $\chi : D(0, 1/q^{3/q}) \to \mathbb{C}$ as in Prop. 4 is called an *explosion function* at p/q. Such a function is not unique. However, if χ_1 and χ_2 are two explosions functions at p/q, they are related by $\chi_1(\delta) = \chi_2(e^{2i\pi kp/q}\delta)$ for some integer $k \in \mathbb{Z}$.

The second proposition studies how the explosion functions behave as p/q ranges in the set of approximants of an irrational number α such that P_α has a Siegel disk.

[3] The choice of N will be specified in equation 3

[4] For example, one can choose $A_n := q_n^{q_n}$. However, we think that the proposition holds for more general sequences (α_n) for instance as soon as $\sqrt[q_n]{A_n} \to +\infty$.

Proposition 5. *Assume $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is an irrational number such that P_α has a Siegel disk Δ. Let p_k/q_k be the approximants to α. Let r be the conformal radius of Δ at 0 and let $\phi : D(0,r) \to \Delta$ be the isomorphism which sends 0 to 0 with derivative 1. For $k \geq 1$, let χ_k be an explosion function at p_k/q_k and set $\lambda_k := \chi_k'(0)$. Then,*

(1) $|\lambda_k| \underset{k\to+\infty}{\longrightarrow} r$ and

(2) the sequence of maps $\psi_k : \delta \mapsto \chi_k(\delta/\lambda_k)$ converges uniformly on every compact subset of $D(0,r)$ to $\phi : D(0,r) \to \Delta$.

Corollary 1. *Let (α_n) be the sequence defined in Prop. 3. Then, for all $\varepsilon > 0$, if n is sufficiently large, P_{α_n} has a cycle in $D(0,\varepsilon)\setminus\{0\}$.*

Proof. Let χ_n be an explosion at p_n/q_n and let C_n be the set of q_n-th roots of

$$\alpha_n - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(q_n A_n' + q_{n-1})} \quad \text{with} \quad A_n' := [A_n, N, N, N, \ldots].$$

Since $\sqrt[q_n]{A_n'} \underset{n\to+\infty}{\longrightarrow} +\infty$, for n large enough, the set C_n is contained in an arbitrarily small neighborhood of 0 and $\chi_n(C_n)$ is a cycle of P_{α_n} contained in an arbitrarily small neighborhood of 0. $\square$

1.4. **Perturbed Siegel disks.**

Definition 3. *If U and X are measurable subsets of $\mathbb{C}$, with $0 < \operatorname{area}(U) < +\infty$, we use the notation*

$$\operatorname{dens}_U(X) := \frac{\operatorname{area}(U\cap X)}{\operatorname{area}(U)}.$$

In the whole section, α is a Bruno number, p_n/q_n are its approximants, and $\chi_n : D_n := D(0, 1/q_n^{3/q_n}) \to \mathbb{C}$ are explosion functions at p_n/q_n.

Proposition 6 (see figure 3)**.** *Assume $\alpha := [\mathrm{a}_0, \mathrm{a}_1, \ldots]$ and $\theta := [0, \mathrm{t}_1, \ldots]$ are Brjuno numbers and let p_n/q_n be the approximants to α. Assume*

$$\alpha_n := [\mathrm{a}_0, \mathrm{a}_1, \ldots, \mathrm{a}_n, A_n, \mathrm{t}_1, \mathrm{t}_2, \ldots]$$

with (A_n) a sequence of positive integers such that

$$\limsup_{n\to+\infty} \sqrt[q_n]{\log A_n} \leq 1.^5 \tag{1}$$

Let Δ be the Siegel disk of P_α and Δ_n' the Siegel disk of the restriction of P_{α_n} to Δ.[6] For all non empty open set $U \subset \Delta$,

$$\liminf_{n\to+\infty} \operatorname{dens}_U(\Delta_n') \geq \frac{1}{2}.$$

Proof. Set

$$\varepsilon_n := \alpha_n - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n^2(A_n+\theta) + q_n q_{n-1}}.$$

Note that

$$\sqrt[q_n]{|\varepsilon_n|} \underset{n\to+\infty}{\sim} \frac{1}{\sqrt[q_n]{A_n}}.$$

[5] We think that the condition $\limsup \sqrt[q_n]{\log A_n} \leq 1$ is not necessary.

[6] Δ_n' is the largest connected open subset of Δ containing 0, on which P_{α_n} is conjugate to a rotation. It is contained in the Siegel disk of P_{α_n}

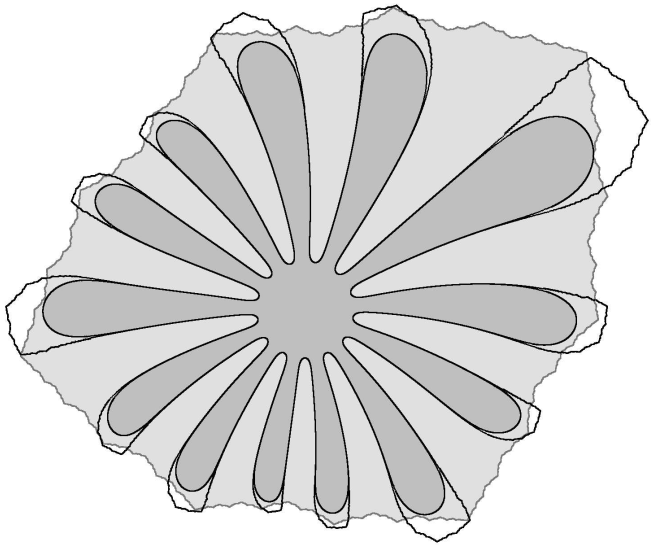


FIGURE 3. Illustration of Prop. 6 for $\alpha = \theta = [0, 1, 1, \ldots]$, $n = 7$ and $A_n = 10^{10}$. We see the Siegel disk Δ of P_α (light grey), the Siegel disk Δ'_n of the restriction of P_{α_n} to Δ (dark grey) and the boundary of the Siegel disk of P_{α_n}.

For $\rho < 1$, define

$$X_n(\rho) := \left\{ z \in \mathbb{C} \ ; \ \frac{z^{q_n}}{z^{q_n} - \varepsilon_n} \in D(0, s_n) \right\} \quad \text{with} \quad s_n := \frac{\rho^{q_n}}{\rho^{q_n} + |\varepsilon_n|}.$$

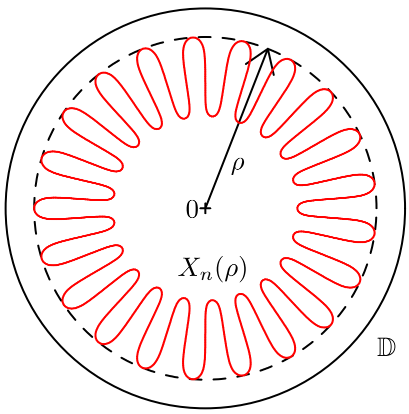


FIGURE 4. The boundary of a set $X_n(\rho)$.

This domain is star-like with respect to 0 and avoids the q_n-th roots of ε_n.[7] It is contained but not relatively compact in $D(0,\rho)$. For all non empty open set U contained in $D(0,\rho)$,

$$\liminf_{n\to+\infty} \mathrm{dens}_U\big(X_n(\rho)\big) \geq \frac{1}{2}.$$

Since the limit values of the sequence $(\chi_n : D_n \to \mathbb{C})$ are isomorphisms $\chi : \mathbb{D} \to \Delta$, Prop. 6 is a corollary of Prop. 7 below. □

Proposition 7. *Under the same assumptions as in Prop. 6, for all $\rho < 1$, if n is large enough, the Siegel disk Δ'_n contains $\chi_n\big(X_n(\rho)\big)$.*

Proof. We will proceed by contradiction. Assume there exist $\rho < 1$ and an increasing sequence of integers n_k such that $\chi_{n_k}\big(X_{n_k}(\rho)\big)$ is not contained in Δ'_{n_k}. Extracting a subsequence, we may assume

$$A_{n_k}^{1/q_{n_k}} \to A \in [1,+\infty].$$

To simplify notations, we will drop the index k.

- Assume $A = 1$. Then, any compact $K \subset \Delta$ is contained in Δ'_n for n large enough (for a proof, see for example in [ABC], Prop. 2, the remark following Prop. 2 and Theo. 3). Note that $X_n(\rho) \subset D(0,\rho)$ and the limit values of the sequence $(\chi_n : D_n \to \mathbb{C})$ are isomorphisms $\chi : \mathbb{D} \to \Delta$. It follows that for n large enough,
$$\chi_n\big(X_n(\rho)\big) \subset \chi_n\big(D(0,\rho)\big) \subset \chi\big(D(0,\sqrt{\rho})\big) \subset \Delta'_n.$$
This contradicts our assumption.
- Assume $A > 1$. Without loss of generality, increasing ρ if necessary, we may assume that $\rho > 1/A$. We will show that for $\rho < \rho' < 1$, if n is large enough, the orbit under iteration of P_{α_n} of any point $z \in \chi_n\big(X_n(\rho)\big)$ remains in $\chi_n\big(D(0,\rho')\big) \subset \Delta$. This will show that $\chi_n\big(X_n(\rho)\big) \subset \Delta'_n$, completing the proof of Prop. 7.

  Since the limit values of the sequence $\chi_n : D_n \to \mathbb{C}$ are isomorphisms $\chi : \mathbb{D} \to \Delta$, there is a sequence r'_n tending to 1 such that χ_n is univalent on $D'_n := D(0,r'_n)$ and the domain of the map
$$f_n := \big(\chi_n|_{D'_n}\big)^{-1} \circ P_{\alpha_n} \circ \chi_n|_{D'_n}$$
eventually contains any compact subset of $\mathbb{D}$. So, Prop. 7 is a corollary of Prop. 7' below.

□

Proposition 7'. *Assume*

$$0 \leq \frac{1}{A} < \rho < \rho' < 1.$$

If n is large enough, the orbit under iteration of f_n of any point $z \in X_n(\rho)$ remains in $D(0,\rho')$.

The rest of section 1.4, is devoted to the proof of Prop. 7'.

[7] It is the preimage by the map $z \mapsto z^{q_n}$ of a disk which is not centered at 0, contains 0 but not ε_n.

1.4.1. *A vector field.* It is not enough to compare the dynamics of f_n with the dynamics of a rotation. Instead, we will compare it with the (real) dynamics of the polynomial vector field

$$\xi_n = \xi_n(z)\frac{\partial}{\partial z} := 2i\pi q_n z(\varepsilon_n - z^{q_n})\frac{\partial}{\partial z}.$$

As we shall see later, the time-1 map of ξ_n very well approximates $f_n^{\circ q_n}$ (the coefficient $2\pi q_n$ has been chosen so that their derivatives coincide at 0). For simplicity we will assume that n is even in which case $\varepsilon_n > 0$.

Note that the polynomial vector field ξ_n is tangent to the boundary of $X_n(\rho)$, which is therefore invariant by the (real) dynamics of ξ_n.

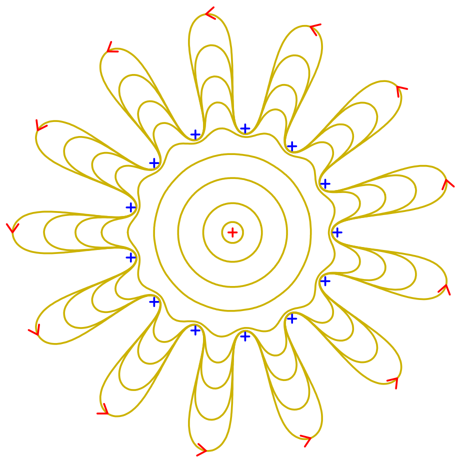


FIGURE 5. Some real trajectories for the vector field ξ_n; zeroes of the vector field are shown.

In order to compare the dynamics of f_n to that of ξ_n, we will work in a coordinate that straightens the vector field ξ_n. Let us first consider the open set

$$\Omega_n := \left\{ z \in \mathbb{C} \ ; \ \frac{z^{q_n}}{z^{q_n} - \varepsilon_n} \in \mathbb{D} \right\}$$

which is invariant by the real flow of the vector field ξ_n.

The map

$$z \mapsto \frac{z^{q_n}}{z^{q_n} - \varepsilon_n} : \Omega_n \to \mathbb{D}$$

is a ramified covering of degree q_n, ramified at 0. Thus, there is an isomorphism $\psi_n : \Omega_n \to \mathbb{D}$ such that

$$\big(\psi_n(z)\big)^{q_n} = \frac{z^{q_n}}{z^{q_n} - \varepsilon_n}.$$

We note $\phi_n : \mathbb{D} \to \Omega_n$ its inverse and $\pi_n : \mathbb{H} \to \Omega_n \setminus \{0\}$ ($\mathbb{H}$ is the upper half-plane) the universal covering given by

$$\pi_n(Z) := \phi_n\big(e^{2i\pi q_n \varepsilon_n Z}\big).$$

Then,

$$\pi_n^* \xi_n = \frac{\partial}{\partial z}.$$

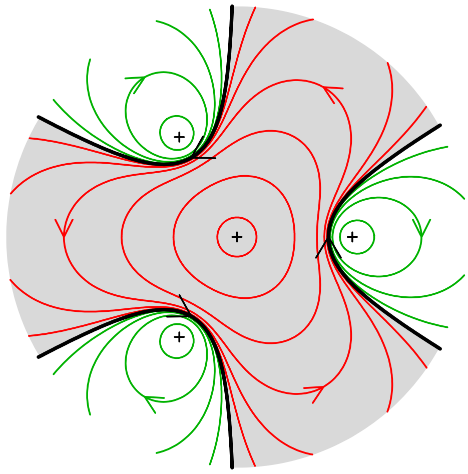


FIGURE 6. An example of open set Ω_n for $q_n = 3$. It is bounded by the black curves. Some trajectories of the vector field ξ_n (red in Ω_n and green outside).

For $r < 1$, we have $X_n(r) \subset \Omega_n$ and the preimage of $X_n(r)$ is the half-plane

$$\mathbb{H}_n(r) := \bigl\{Z \in \mathbb{C}\ ;\ \mathrm{Im}(Z) > \tau_n(r)\bigr\} \quad \text{with} \quad \tau_n(r) := \frac{1}{2\pi q_n^2 \varepsilon_n} \log\left(1 + \frac{\varepsilon_n}{r^{q_n}}\right).$$

The map $\pi_n : \mathbb{H}_n(r) \to X_n(r) \setminus \{0\}$ is a universal covering.

Remark. Note that $\tau_n(r)$ increases exponentially fast with respect to q_n. More precisely,

$$\sqrt[q_n]{\tau_n(r)} \underset{n\to+\infty}{\longrightarrow} \frac{1}{r}.$$

1.4.2. *Working in the coordinate straightening the vector field.*

Definition 4. *We say that a sequence* (B_n) *is* sub-exponential with respect to q_n *if*

$$\limsup_{n\to+\infty} \sqrt[q_n]{|B_n|} \le 1.$$

Proposition 8. *Assume $r < 1$. If n is large enough, there exist holomorphic maps $F_n : \mathbb{H}_n(r) \to \mathbb{H}$ and $G_n : \mathbb{H}_n(r) \to \mathbb{H}$ such that*

- *π_n semi-conjugates F_n to $f_n^{\circ q_n}$ and G_n to $f_n^{\circ q_{n-1}}$:*

$$\pi_n \circ F_n = f_n^{q_n} \circ \pi_n \quad \textit{and} \quad \pi_n \circ G_n = f_n^{q_{n-1}} \circ \pi_n,$$

- *F_n and G_n are periodic of period $1/(q_n\varepsilon_n)$ and*
- *as $\mathrm{Im}(Z) \to +\infty$, we have*

$$F_n(Z) = Z + 1 + o(1) \quad \textit{and} \quad G_n(Z) = Z - (A_n + \theta) + o(1).$$

In addition, the sequences

$$\sup_{Z\in\mathbb{H}_n(r)} \bigl|F_n(Z) - Z - 1\bigr| \quad \textit{and} \quad \sup_{Z\in\mathbb{H}_n(r)} \bigl|G_n(Z) - Z + A_n + \theta\bigr|$$

are sub-exponential with respect to q_n.

Proof. We will use the following theorem of Jellouli (see [J1] or [J2] Theo. 1) to show that the domains of $f_n^{\circ q_n}$ and $f_n^{\circ q_{n-1}}$ eventually contain any compact subset of $\mathbb{D}$.

Theorem (Jellouli)**.** *Assume P_α has a Siegel disk Δ and let $\chi : \mathbb{D} \to \Delta$ be a linearizing isomorphism. For $r < 1$, set $\Delta(r) := \chi\big(D(0,r)\big)$. Assume $\alpha_n \in \mathbb{R}$ and $b_n \in \mathbb{N}$ are such that $b_n \cdot |\alpha_n - \alpha| = o(1)$.*[8] *For all $r_1' < r_2' < 1$, if n is sufficiently large,*

$$\Delta(r_1') \subset \big\{z \in \Delta(r_2')\ ;\ \forall j \le b_n,\ P_{\alpha_n}^{\circ j}(z) \in \Delta(r_2')\big\}.$$

Corollary 2. *For all $r_1 < r_2 < 1$, if n is sufficiently large, then for all $z \in D(0,r_1)$ and for all $j \le q_n$, we have $f_n^{\circ j}(z) \in D(0,r_2)$.*

Proof. Choose r_1' and r_2' such that $r_1 < r_1' < r_2' < r_2$. Let $\chi : \mathbb{D} \to \Delta$ be a linearizing isomorphism of P_α. Set

$$\Delta(r_1') := \chi\big(D(0,r_1')\big) \quad \text{and} \quad \Delta(r_2') := \chi\big(D(0,r_2')\big).$$

Since limit values of the sequence $\chi_n : D_n' \to \mathbb{C}$ are linearizing isomorphisms $\chi : \mathbb{D} \to \Delta$, for n sufficiently large,

$$\chi_n\big(D(0,r_1)\big) \subset \Delta(r_1') \subset \Delta(r_2') \subset \chi_n\big(D(0,r_2)\big).$$

It is therefore enough to show that for n large enough,

$$\Delta(r_1') \subset \big\{z \in \Delta(r_2')\ ;\ \forall j \le q_n,\ P_{\alpha_n}^{\circ j}(z) \in \Delta(r_2')\big\}.$$

This is Jellouli's theorem with $b_n = q_n$ since

$$q_n|\alpha_n - \alpha| \underset{n\to+\infty}{\sim} q_n \left|\frac{p_n}{q_n} - \alpha\right| \underset{n\to+\infty}{=} o(1).$$

□

In particular, for $r < 1$, if n is large enough, then $f_n^{\circ q_n}$ and $f_n^{\circ q_{n-1}}$ are defined on $X_n(r)$. In order to lift them via π_n as required, it is enough to show that if n is large enough, then

$$\forall z \in X_n(r) \setminus \{0\}, \quad f_n^{q_n}(z) \in \Omega_n \setminus \{0\} \quad \text{and} \quad f_n^{q_{n-1}}(z) \in \Omega_n \setminus \{0\}.$$

The periodicity of F_n and G_n then follows from

$$\pi_n\left(Z + \frac{1}{q_n\varepsilon_n}\right) = \pi_n(Z)$$

and the behavior as $\mathrm{Im}(Z) \to +\infty$ follows by computing the derivatives of $f_n^{\circ q_n}$ and $f_n^{\circ q_{n-1}}$ at 0.

Lemma 1 below asserts that $f_n^{\circ q_n}$ is very close to the identity and bounds the difference.

Lemma 1. *There exist a holomorphic function g_n, defined on the same set as $f_n^{\circ q_n}$, such that*

$$f_n^{\circ q_n}(z) = z + \xi_n(z) \cdot g_n(z).$$

For all $r < 1$, the sequence $\sup\limits_{D(0,r)} |g_n|$ is sub-exponential with respect to q_n.

[8] In fact, Jellouli's theorem is stated for the sequence $\alpha_n = p_n/q_n$ and $b_n = o(q_n q_{n+1})$ but the adaptation to $b_n \cdot |\alpha_n - \alpha| = o(1)$ is straightforward.

Proof. The map $f_n^{\circ q_n}$ fixes 0 and the q_n-th roots of ε_n. This shows that $f_n^{\circ q_n}$ can be written as prescribed. To prove the estimate on the modulus of g_n, note that $f_n^{\circ q_n}$ takes its values in $\mathbb{D}$ and thus, $\left|\xi_n(z)\cdot g_n(z)\right| \le 2$. Choose a sequence $r_n \in]0,1[$ tending to 1 so that g_n is defined on $D(0,r_n)$. By the maximum modulus principle, if n is large enough so that $r_n > \max(r, 1/A)$, we have

$$\sup_{|z|\le r}\left|g_n(z)\right| \le \sup_{|z|\le r_n}\left|g_n(z)\right| \le B_n := \sup_{|z|=r_n}\frac{2}{\left|\xi_n(z)\right|}.$$

As $n\to+\infty$,

$$\inf_{|z|=r_n}\left|\xi_n(z)\right| \sim 2\pi q_n r_n^{1+q_n} \quad\text{and thus}\quad \sqrt[q_n]{B_n}\sim r_n \to 1.$$

□

Recall that we assume n even, in which case

$$\varepsilon_n > 0 \quad\text{and}\quad q_{n-1}\cdot\frac{p_n}{q_n} = -\frac{1}{q_n} \bmod (1).$$

Lemma 2 below asserts that $f_n^{\circ q_{n-1}}$ is very close to the rotation of angle $-1/q_n$ and bounds the difference.

Lemma 2. *There exists a holomorphic function h_n, defined on the same set as $f_n^{\circ q_{n-1}}$, such that*

$$e^{2i\pi/q_n} f_n^{\circ q_{n-1}}(z) = z + \xi_n(z)\cdot h_n(z).$$

For all $r<1$, the sequence $\sup\limits_{D(0,r)} |h_n|$ is sub-exponential with respect to q_n.

Proof. The map f_n coincides with the rotation of angle p_n/q_n on the set of q_n-th roots of ε_n and $q_{n-1}\cdot(p_n/q_n) = -1/q_n \bmod(1)$. Thus, $e^{2i\pi/q_n} f_n^{\circ q_{n-1}}(z)$ fixes 0 and the q_n-th roots of ε_n. This shows that $e^{2i\pi/q_n} f_n^{\circ q_{n-1}}$ can be written as prescribed. The same method as in lemma 1 yields the bound on h_n. □

Proof of Prop. 8, continued. Now, given $r<1$, set

$$R_n := \min\left(\frac{1}{q_n\varepsilon_n}, \tau_n(r)\right).$$

Note that

$$\sqrt[q_n]{R_n} \underset{n\to+\infty}{\longrightarrow} \min\left(A, \frac{1}{r}\right).$$

Hence, R_n increases exponentially fast with respect to q_n.

For all n and all $Z\in\mathbb{H}_n(r)$, the map π_n is univalent on $D(Z,R_n)$ and takes its values in $\Omega_n\setminus\{0\}$. By Koebe 1/4-theorem, its image contains a disk centered at $z:=\pi_n(Z)$ with radius

$$\pi_n'(Z)\cdot\frac{R_n}{4} = \xi_n(z)\cdot\frac{R_n}{4}.$$

In particular, if the sequence (B_n) is sub-exponential with respect to q_n and if n is large enough so that $B_n \le R_n/4$, we have

$$\forall z\in X_n(r),\quad D\big(z,\xi_n(z)\cdot B_n\big)\subset \Omega_n\setminus\{0\}.$$

Therefore, it follows from lemmas 1 and 2 that for all $r<1$, if n is large enough, then

$$\forall z\in X_n(r)\setminus\{0\},\quad f_n^{q_n}(z)\in\Omega_n\setminus\{0\} \quad\text{and}\quad f_n^{q_{n-1}}(z)\in\Omega_n\setminus\{0\}.$$

Lemmas 1 and 2 and Koebe distortion theorem applied to $\pi_n : D\big(Z, R_n\big) \to \mathbb{C}$ imply that the sequences

$$\sup_{Z\in\mathbb{H}_n(r)} \big|F_n(Z) - Z - 1\big| \quad \text{and} \quad \sup_{Z\in\mathbb{H}_n(r)} \big|G_n(Z) - Z + A_n + \theta\big|$$

are sub-exponential with respect to q_n.

This completes the proof of Prop. 8. □

We will need the following improved estimate for F_n.

Proposition 9. *Assume $r < 1$. There exists a sequence (B_n), sub-exponential with respect to q_n, such that for all $Z \in \mathbb{H}_n(r)$,*

$$\big|F_n(Z) - Z - 1\big| \leq B_n \cdot \big(|\varepsilon_n| + \big|\varepsilon_n - \pi_n(Z)^{q_n}\big|\big)\,.$$

Proof. Lemma 3 below gives a similar estimate for $f_n^{\circ q_n}$ on $X_n(r)$. This estimate transfers to the required one by Koebe distortion theorem as in the previous proof. □

Lemma 3. *There exist a complex number η_n and a holomorphic function k_n, defined on the same set as $f_n^{\circ q_n}$, such that*

$$f_n^{\circ q_n}(z) = z + \xi_n(z) \cdot \big(1 + \eta_n + (\varepsilon_n - z^{q_n})k_n(z)\big).$$

For all $r < 1$, there exists a sequence (B_n), sub-exponential with respect to q_n, such that

$$|\eta_n| \leq B_n \cdot |\varepsilon_n| \quad \textit{and} \quad \forall z \in D(0, r) \quad \big|k_n(z)\big| \leq B_n.$$

Proof. By lemma 1, we know that

$$f_n^{\circ q_n}(z) = z + \xi_n(z) \cdot h_n(z)$$

with, $B_n := \sup_{D(0,r)} |h_n|$ a sub-exponential sequence with respect to q_n. The map $f_n^{\circ q_n}$ has the same multiplier at each q_n-th roots of ε_n. It follows that

$$h_n(z) = 1 + \eta_n + (\varepsilon_n - z^{q_n})k_n(z)$$

as prescribed. Since $\sqrt[q_n]{\varepsilon_n} \to 1/A < r < 1$, the bound on h_n, taken at any of the q_n-th roots of ε_n, shows that for n large enough,

$$|1 + \eta_n| \leq B_n$$

and thus

$$\forall z \in D(0, r) \quad \big|(\varepsilon_n - z^{q_n})k_n(z)\big| \leq 2B_n.$$

As in lemma 1, we have for some sequence $r_n \to 1$ and for n large enough:

$$\sup_{|z|\leq r} |k_n(z)| \leq B'_n := \frac{2B_n}{r_n^{q_n} - \varepsilon_n}$$

and (B'_n) is sub-exponential with respect to q_n. Looking at $z = 0$ gives:

$$\frac{e^{2i\pi q_n\varepsilon_n} - 1}{2i\pi q_n\varepsilon_n} = 1 + \eta_n + \varepsilon_n k_n(0).$$

As $n \to +\infty$, the left hand of this equality expands to $1 + i\pi q_n\varepsilon_n + o(q_n\varepsilon_n)$. Therefore

$$|\eta_n| \leq \varepsilon_n\big(|k_n(0)| + \pi q_n + o(q_n)\big).$$

Since $|k_n(0)| \leq B'_n$, we get the desired bound on η_n. □

Corollary 3. *Assume $r<1$. Then,*

$$\sup_{Z\in\mathbb{H}_n(r)}\bigl|F_n(Z)-Z-1\bigr|\underset{n\to+\infty}{\longrightarrow}0\quad\textit{and}\quad\sup_{Z\in\mathbb{H}_n(r)}\bigl|F_n'(Z)-1\bigr|\underset{n\to+\infty}{\longrightarrow}0.$$

Proof. The first is an immediate consequence of Prop. 9. For the second, use the first on $\mathbb{H}_n(r')$ with $r<r'<1$. □

1.4.3. *Iterating the commuting pair (F_n,G_n).*

Proposition 10. *Assume $1/A<r_1<r_2<1$. If n is sufficiently large, the following holds. Given any point $Z\in\mathbb{H}_n(r_1)$, there exists a sequence of integers $(j_\ell)_{\ell\geq 0}$ such that for any integer $\ell\geq 0$ and any integer $j\in[0,j_\ell]$, the point*

$$F_n^{\circ j}\circ G_n\circ F_n^{\circ j_{\ell-1}}\circ G_n\circ\cdots\circ F_n^{\circ j_1}\circ G_n\circ F_n^{\circ j_0}(Z)$$

is well defined and belongs to $\mathbb{H}_n(r_2)$.

Proof. We will need to control iterates of F_n for a large number of iterates. We will use the following lemma.

Lemma 4. *Assume $F:\mathbb{H}\to\mathbb{C}$ satisfies*

$$\bigl|F(Z)-Z-1\bigr|<u\bigl(\mathrm{Re}(Z)\bigr)$$

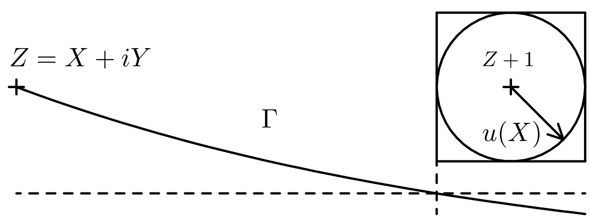
with $u:\mathbb{R}\to]0,1/10[$ a function such that $\log u$ is $1/2$-Lipschitz. Let Γ be the graph of an antiderivative of $-2u$. Then, every $Z\in\mathbb{H}$ which is above Γ has an image above Γ.

Proof. Let U be the antiderivative whose graph is Γ. Let $Z=X+iY\in\mathbb{H}$. The point $Z'=X'+iY'=F(Z)$ satisfies $X'\in[X+\frac{9}{10},X+\frac{11}{10}]$. Note that

$$\forall x\in\left[X,X+\frac{11}{10}\right],\quad \log u(x)\geq\log u(X)-\frac{11}{20}.$$

Therefore, from X to X', U decreases of at least

$$(X'-X)2e^{-11/20}u(X)\geq\frac{18}{10}e^{-11/20}u(X)>u(X)>Y-Y'.$$



□

Lemma 5. *Assume $1/A<r<r'<1$. If n is sufficiently large, then for all $Z\in\mathbb{H}_n(r)$ there exists an integer $j(Z)$ such that*

- *for all $j\leq j(Z)$, we have $F_n^{\circ j}\circ G_n(Z)\in\mathbb{H}_n(r')$ and*
- $\mathrm{Re}\bigl(F_n^{\circ j(Z)}\circ G_n(Z)\bigr)>\mathrm{Re}(Z)$.

Proof. Let us first recall that there exists a sequence (B_n), sub-exponential with respect to q_n, such that for n large enough, for all $Z \in \mathbb{H}_n(r)$,

$$\big|G_n(Z) - Z + A_n + \theta\big| \le B_n.$$

In particular, if n is sufficiently large,

$$\mathrm{Re}\big(G_n(Z)\big) \ge \mathrm{Re}(Z) - A_n - \theta - B_n \quad \text{and} \quad \mathrm{Im}\big(G_n(Z)\big) \ge \tau_n(r) - B_n.$$

We will apply lemma 4 to control the orbit of $G_n(Z)$ under iteration of F_n. More precisely, we will prove the existence of a function u_n such that:

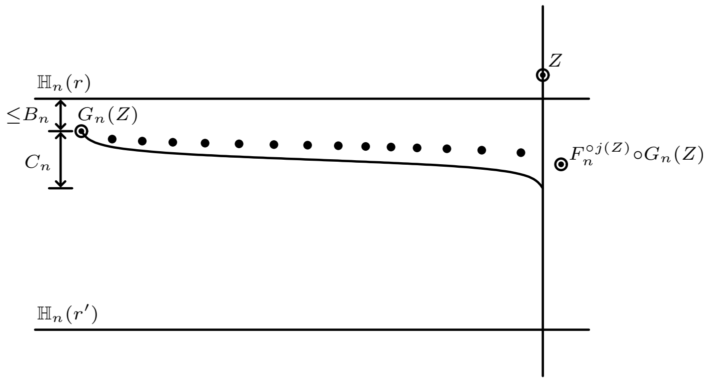
a) $\big|F_n(Z) - Z - 1\big| \le u_n\big(\mathrm{Re}(Z)\big)$,
b) for n large enough $u_n \in \,]0, 1/10[$,
c) for n large enough, $\log u_n$ is $1/2$-Lipschitz and
d) the sequence $C_n := \displaystyle\int_{\mathrm{Re}\left(G_n(Z)\right)}^{\mathrm{Re}(Z)} 2u_n(X)\,\mathrm{d}X$ is sub-exponential with respect to q_n.

If n is taken sufficiently large so as to have

$$\tau_n(r) \ge \tau_n(r') + B_n + C_n + \frac{1}{10},$$

it then follows from lemma 4 that there is an integer $j(Z)$ such that

- for all $j \le j(Z)$, we have $F_n^{\circ j} \circ G_n(Z) \in \mathbb{H}_n(r')$ and
- $\mathrm{Re}\big(F_n^{\circ j(Z)} \circ G_n(Z)\big) > \mathrm{Re}(Z)$.



a) By Prop. 9, there is a sequence (B'_n), sub-exponential with respect to q_n, such that for all $Z \in \mathbb{H}_n(r')$,

$$\big|F_n(Z) - Z - 1\big| \le B'_n\big(\varepsilon_n + \big|\varepsilon_n - \pi_n(Z)^{q_n}\big|\big).$$

Set $T_n := 1/(2\pi q_n^2 \varepsilon_n) \to +\infty$. We have

$$\big(\pi_n(Z)\big)^{q_n} = \frac{\varepsilon_n}{1 - e^{-iZ/T_n}}.$$

Using

$$B'_n\big(\varepsilon_n + \big|\varepsilon_n - \pi_n(Z)^{q_n}\big|\big) \le B'_n\big(2\varepsilon_n + \big|\pi_n(Z)^{q_n}\big|\big)$$

we see that for all $Z \in \mathbb{H}_n(r')$,

$$\begin{aligned}\bigl|F_n(Z) - Z - 1\bigr| &\le B'_n\varepsilon_n\left(2 + \frac{1}{\bigl|1 - e^{-iZ/T_n}\bigr|}\right)\\ &\le B'_n\varepsilon_n\left(2 + \frac{1}{\bigl|s_n e^{i\mathrm{Re}(Z)/T_n} - 1\bigr|}\right)\end{aligned}$$

with

$$s_n = 1 + \frac{\varepsilon_n}{(r')^{q_n}}.$$

Since $1/A < r'$, we have $\varepsilon_n/(r')^{q_n} \to 0$ and thus $s_n \to 1$. Thus, for n large enough

$$\frac{1}{3} \le \frac{1}{|s_n e^{i\mathrm{Re}(Z)/T_n} - 1|},$$

and for all $Z \in \mathbb{H}_n(r')$,

$$\bigl|F_n(Z) - Z - 1\bigr| \le u_n\bigl(\mathrm{Re}(Z)\bigr) \quad \text{with} \quad u_n(X) := \frac{7B'_n\varepsilon_n}{|s_n e^{iX/T_n} - 1|}.$$

b) Let us show that for n large enough $u_n \in \,]0, 1/10[$. Note that

$$\forall X \in \mathbb{R}, \quad \bigl|u_n(X)\bigr| \le \frac{7B'_n\varepsilon_n}{s_n - 1} = 7B'_n(r')^{q_n} \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus u_n tends uniformly to 0 as $n \to +\infty$.

c) Let us now check that for n large enough, $\log u_n$ is 1/2-Lipschitz. One computes

$$\frac{u'_n(X)}{u_n(X)} = -\frac{s_n}{T_n} \cdot \frac{\sin(X/T_n)}{s_n^2 + 1 - 2s_n\cos(X/T_n)}.$$

This function reaches its extrema when $(s_n^2 + 1)\cos(X/T_n) - 2s_n = 0$. It follows that:

$$\left|\frac{u'_n(X)}{u_n(X)}\right| \le \frac{s_n}{T_n(s_n^2 - 1)} \underset{n\to+\infty}{\sim} \pi q_n^2 (r')^{q_n}.$$

Thus, $\dfrac{\partial \log u_n(X)}{\partial X}$ converges uniformly to 0 as $n \to +\infty$, and for n large enough, $\log u_n$ is 1/2-Lipschitz.

d) Let us finally show that the sequence

$$C_n := \int_{\mathrm{Re}\left(G_n(Z)\right)}^{\mathrm{Re}(Z)} 2u_n(X)\,\mathrm{d}X$$

is sub-exponential with respect to q_n. If n is large enough,

$$\mathrm{Re}\bigl(G_n(Z)\bigr) \ge \mathrm{Re}(Z) - A_n - \theta_n - B_n \ge -4\pi T_n.$$

Thus,

$$C_n \le B''_n := \int_{\mathrm{Re}(Z)-4\pi T_n}^{\mathrm{Re}(Z)} 2u_n(X)\,\mathrm{d}X = 4\int_{-\pi T_n}^{\pi T_n} u_n(X)\,\mathrm{d}X.$$

The change of variable $\theta = X/T_n$, which yields

$$B''_n = \frac{14B'_n}{\pi q_n^2}\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{\sqrt{s_n^2 + 1 - 2s_n\cos\theta}}.$$

It follows that

$$B''_n \underset{n\to+\infty}{\sim} \frac{28B'_n}{\pi q_n^2}\log\frac{1}{s_n-1} \underset{n\to+\infty}{\sim} \frac{28B'_n}{\pi q_n}\log(r'A_n).$$

By assumption (condition (1) in the statement of Prop. 6), the sequence $\log A_n$ is sub-exponential with respect to q_n. As a consequence, (B''_n), and thus (C_n), is sub-exponential with respect to q_n. □

Proof of Prop. 10, continued. Remember that we are given r_1 and r_2 with $1/A < r_1 < r_2 < 1$ and we want to prove that for n sufficiently large, any point of $\mathbb{H}_n(r_1)$ has an infinite orbit remaining in $\mathbb{H}_n(r_2)$ along a well chosen composition of F_n and G_n. It is enough to show that this is true for any sequence of points

$$Z_n = X_n + iY_n \in \mathbb{H}_n(r_1).$$

We will use Douady-Ghys-Yoccoz's renormalization techniques and follow the presentation in [ABC] section 3.2.

Step 1. Construction of a Riemann surface: $\mathcal{V}_n$. Choose n sufficiently large so that F_n is defined in the upper half-plane $\{Z \in \mathbb{C}\ ;\ \mathrm{Im}(Z) \geq \tau_n(r_2) - 1/10\}$ with

$$\left|F_n(Z) - Z - 1\right| \leq \frac{1}{10} \quad \text{and} \quad \left|F'_n(Z) - 1\right| \leq \frac{1}{10}.^{9}$$

Set

$$P_n := X_n + i\left(\tau_n(r_2) - \frac{1}{10}\right).$$

Let

$$L_n := \{X_n + it\ ;\ t > \mathrm{Im}(P_n)\}$$

be the vertical half-line starting at P_n and passing through Z_n. The union

$$L_n \cup [P_n, F_n(P_n)] \cup F_n(L_n) \cup \{\infty\}$$

forms a Jordan curve in the Riemann sphere bounding a region U_n such that for $Y > \mathrm{Im}(P_n)$, the segment $[iY, F_n(iY)]$ is contained in $\overline{U}_n$ (see [ABC]). We set $\mathcal{U}_n := U_n \cup L_n$. If we glue the sides L_n and $F_n(L_n)$ of $\overline{\mathcal{U}}_n$ via F_n, we obtain a topological surface $\overline{\mathcal{V}}_n$. We denote by $\iota_n : \overline{\mathcal{U}}_n \to \overline{\mathcal{V}}_n$ the canonical projection. The space $\overline{\mathcal{V}}_n$ is a topological surface with boundary, whose boundary $\iota_n([P_n, F_n(P_n)])$ is denoted $\partial\mathcal{V}_n$. We set $\mathcal{V}_n = \overline{\mathcal{V}}_n \setminus \partial\mathcal{V}_n$. Since the gluing map F_n is analytic, the surface $\mathcal{V}_n$ has a canonical analytic structure induced by the one of $\mathcal{U}_n$. It is possible to show that $\mathcal{V}_n$ is isomorphic to $\mathbb{H}/\mathbb{Z} \simeq \mathbb{D}^*$ (see [Y] for details). Let $\phi_n : \mathcal{V}_n \to \mathbb{D}^*$ be an isomorphism. Hence, we have the following composition:

$$\phi_n \circ \iota_n : \mathcal{U}_n \to \mathbb{D}^*.$$

We set

$$\zeta_n := \phi_n \circ \iota_n(Z_n) \in \mathbb{D}.$$

Step 2. The renormalized map g_n. Choose $r_3 \in\,]r_1, r_2[$. Set

$$P'_n := X_n + i\left(\tau_n(r_3) + \frac{1}{10}\right).$$

[9] This is possible by Cor. 3 applied with $r > r_2$. Indeed, for n large enough, $\tau_n(r_2) > \tau_n(r) + 1/10$.

Let $\mathcal{U}'_n$ be the set of points of $\mathcal{U}_n$ which are above the segment $\big[P'_n, F_n(P'_n)\big]$ and let $\mathcal{V}'_n$ be the image of $\mathcal{U}'_n$ in $\mathcal{V}_n$. Choose n sufficiently large so that lemma 5 can be applied with $r = r_3$ and $r' = r_2$. Then, for all $Z \in \mathcal{U}'_n \subset \mathbb{H}_n(r_3)$, there exists an integer $j(Z)$ such that

$$W := F_n^{\circ j(Z)} \circ G_n(Z) \in \mathcal{U}_n \quad \text{and} \quad \forall j \in \big[0, j(Z)\big] \quad F_n^{\circ j} \circ G_n(Z) \in \mathbb{H}_n(r_2).$$

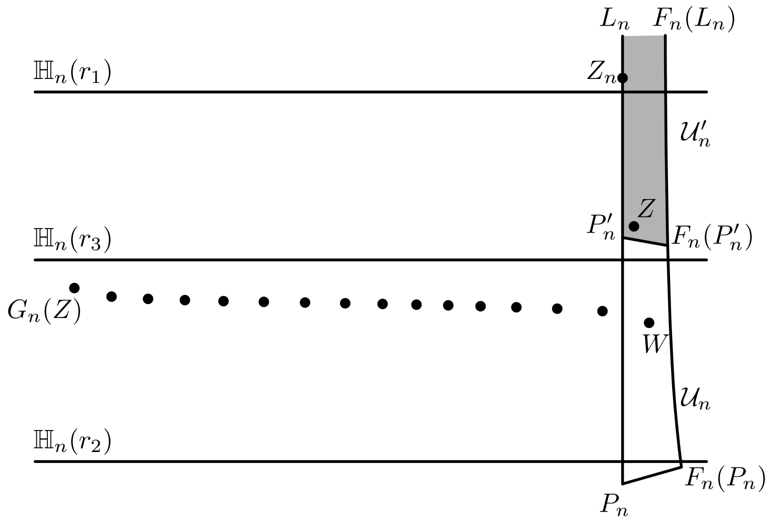
The map $Z \mapsto W$ induces a univalent map $g_n : \phi_n(\mathcal{V}'_n) \to \mathbb{D}^*$. [10] By the removable singularity theorem, this map extends holomorphically to the origin by $g_n(0) = 0$. Since

$$F_n(Z) = Z + 1 + o(1) \quad \text{and} \quad G_n(Z) = Z - A_n - \theta + o(1)$$

as $\mathrm{Im}(Z) \to +\infty$, it is possible to show that

$$g'_n(0) = e^{-2i\pi(A_n+\theta)} = e^{-2i\pi\theta}$$

(again, see [Y] for details).



Step 3. The orbit of ζ_n.

We will show that the orbit of ζ_n under iteration of g_n is infinite. For this, let ρ_n be the radius of the largest disk centered at 0 and contained in $\phi_n(\mathcal{V}'_n)$. We will show that

a) $\exists C > 0$ such that g_n has a Siegel disk which contains $D(0, C\rho_n)$
b) $|\zeta_n| = o(\rho_n)$.

a) The restriction of g_n to $D(0, \rho_n)$ is univalent. It fixes 0 with derivative $e^{-2i\pi\theta}$. Remember that θ is a Brjuno number. It follows (see [Brj] or [Y] for example) that there is a constant $C_\theta > 0$ depending only on θ such that g_n has a Siegel disk containing $D(0, C_\theta\rho_n)$.

b) Denote by B_n the half-strip

$$B_n = \{Z \in \mathbb{C}\ ;\ 0 < \mathrm{Re}(Z) < 1 \text{ and } \mathrm{Im}(Z) > \mathrm{Im}(P_n)\}$$

[10] The fact that $g_n : \phi_n(\mathcal{V}'_n) \to \mathbb{D}^*$ is continuous and univalent is not completely obvious; see [Y] for details.

and consider the map $H_n : \overline{B}_n \to \overline{\mathcal{U}}_n$ defined by

$$H_n(Z) = (1-X)\cdot(X_n+iY) + X\cdot F_n(X_n+iY)$$

where $Z = X+iY$, $(X,Y) \in [0,1]\times\big[\mathrm{Im}(P_n),+\infty\big[$. The map H_n sends each segment $[iY, iY+1]$ to the segment $\big[X_n+iY, F_n(X_n+iY)\big]$. An elementary computation shows that H_n is a 5/4-quasiconformal homeomorphism between $\overline{B}_n$ and $\overline{\mathcal{U}}_n$.[11] Since $H_n(iY+1) = F_n\big(H_n(iY)\big)$, the quasiconformal homeomorphism $H_n : \overline{B}_n \to \overline{\mathcal{U}}_n$ induces a homeomorphism between the half cylinder $\mathbb{H}/\mathbb{Z}$ and the Riemann surface $\mathcal{V}_n$. This homeomorphism is clearly quasiconformal on the image of B_n in $\mathbb{H}/\mathbb{Z}$, i.e., outside a straight line. It is therefore quasiconformal in the whole half cylinder ($\mathbb{R}$-analytic curves are removable for quasiconformal homeomorphisms).

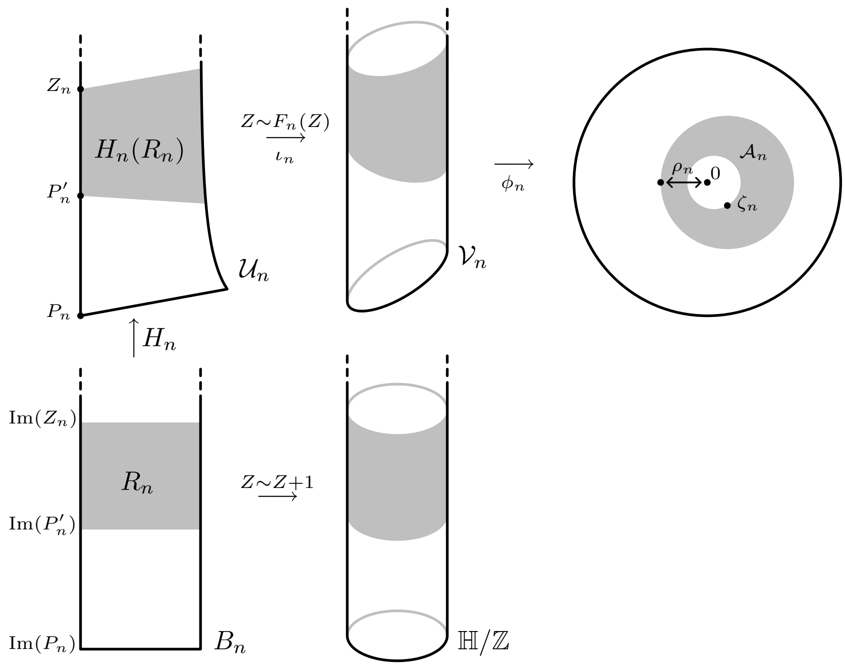
Let R_n be the rectangle

$$R_n := \big\{Z\in\mathbb{C}\ ;\ 0\le \mathrm{Re}(Z) < 1 \text{ and } \mathrm{Im}(P_n') < \mathrm{Im}(Z) < \mathrm{Im}(Z_n)\big\}.$$

Note that $H_n(R_n) \subset \mathcal{U}_n'$ and observe that

$$\mathcal{A}_n := \phi_n \circ \iota_n \circ H_n(R_n)$$

is an annulus contained in $\phi_n(\mathcal{V}_n')$ that surrounds 0 and ζ_n.



The image of R_n in $\mathbb{H}/\mathbb{Z}$ is an annulus of modulus

$$M_n := \mathrm{Im}(Z_n) - \mathrm{Im}(P_n') \ge \tau_n(r_1) - \tau_n(r_3) - \frac{1}{10} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

[11] For a proof that H_n is 5/4-quasiconformal homeomorphism, see for example [ABC] section 3.2 or [Sh2] section 2.5.

Note that H_n induces a 5/4-quasiconformal homeomorphism between this annulus and $\mathcal{A}_n$. It follows that

$$\text{modulus}(\mathcal{A}_n) \geq \frac{4}{5} M_n \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Since A_n separates 0 and ζ_n from ∞ and a point of modulus ρ_n in $\partial\phi_n(\mathcal{V}'_n)$, the claim follows: as $n \to +\infty$, $|\zeta_n| = o(\rho_n)$.

Step 4. Controlling the orbit of Z_n.

We know that the orbit of ζ_n under iteration of g_n is infinite. Thus, we have a sequence

$$\zeta_n \in \mathcal{V}'_n \xrightarrow{g_n} \zeta_n^1 \in \mathcal{V}'_n \xrightarrow{g_n} \zeta_n^2 \in \mathcal{V}'_n \xrightarrow{g_n} \cdots.$$

Now, for each $\ell \geq 0$, we have

$$\zeta_n^\ell = \phi_n \circ \iota_n(Z_n^\ell) \quad \text{for some} \quad Z_n^\ell \in \mathcal{U}'_n.$$

Moreover, by definition of g_n, there exists an integer j_ℓ such that

$$Z_n^{\ell+1} = F_n^{\circ j_\ell} \circ G_n(Z_n^\ell) \quad \text{and} \quad \forall j \in [0, j_\ell] \quad F_n^{\circ j} \circ G_n(Z_n^\ell) \in \mathbb{H}_n(r_2).$$

In other words, $\zeta_n^\ell \in \mathcal{V}'_n \xrightarrow{g_n} \zeta_n^{\ell+1} \in \mathcal{V}'_n$ corresponds to

$$Z_n^\ell \in \mathcal{U}'_n \xrightarrow{G_n} \cdot \in \mathbb{H}_n(r_2) \xrightarrow{F_n} \cdot \in \mathbb{H}_n(r_2) \xrightarrow{F_n} \cdots \xrightarrow{F_n} Z_n^{\ell+1} \in \mathcal{U}'_n.$$

Thus, for n sufficiently large, any point $Z_n \in \mathbb{H}_n(r_1)$ has an infinite orbit remaining in $\mathbb{H}_n(r_2)$ along a well chosen composition of F_n and G_n. This completes the proof of Prop. 10. □

Proof of Prop. 7', continued. Remember that $0 < 1/A < \rho < \rho' < 1$. Choose $r_1 = \rho < r_2 < \rho'$. By Prop. 10, for n sufficiently large, any point $Z \in \mathbb{H}_n(\rho)$ has an infinite orbit remaining in $\mathbb{H}_n(r_2)$ under a well chosen composition of F_n and G_n. This means that any point $z \in X_n(\rho)$ has an infinite orbit remaining in $X_n(r_2)$ under a well chosen composition of $f_n^{\circ q_n}$ and $f_n^{\circ q_{n-1}}$. By Cor. 2, if n is sufficiently large, we know that any point in $X_n(r_2) \subset D(0, r_2)$ has its first q_n iterates in $D(0, \rho')$. This shows that any point $z \in X_n(\rho)$ has an infinite orbit remaining in $D(0, \rho')$ under iteration of f_n, as required.

In other words,

$$\cdot \in \mathbb{H}_n(r_2) \xrightarrow{G_n} \cdot \in \mathbb{H}_n(r_2) \quad \text{corresponds to} \quad \cdot \in X_n(r_2) \xrightarrow{f_n^{\circ q_{n-1}}} \cdot \in X_n(r_2)$$

and

$$\cdot \in \mathbb{H}_n(r_2) \xrightarrow{F_n} \cdot \in \mathbb{H}_n(r_2) \quad \text{corresponds to} \quad \cdot \in X_n(r_2) \xrightarrow{f_n^{\circ q_n}} \cdot \in X_n(r_2).$$

Moreover, for n sufficiently large,

$$\cdot \in X_n(r_2) \xrightarrow{f_n^{\circ q_{n-1}}} \cdot \in X_n(r_2) \quad \text{and} \quad \cdot \in X_n(r_2) \xrightarrow{f_n^{\circ q_n}} \cdot \in X_n(r_2)$$

decompose as

$$\cdot \in X_n(r_2) \subset D(0, r_2) \xrightarrow{f_n} \cdot \in D(0, \rho') \xrightarrow{f_n} \cdots \xrightarrow{f_n} \cdot \in D(0, \rho') \xrightarrow{f_n} \cdot X_n(r_2).$$

This completes the proof of Prop. 7'. □

1.5. **The control of the post-critical set.**

Definition 5. *We denote by ∂ the Hausdorff semi-distance:*

$$\partial(X,Y) = \sup_{x\in X} d(x,Y).$$

Definition 6. *We denote by $\mathcal{PC}(P_\alpha)$ the post-critical set of P_α:*

$$\mathcal{PC}(P_\alpha) := \bigcup_{k\geq 1} P_\alpha^{\circ k}(\omega_\alpha) \quad \textit{with} \quad \omega_\alpha := -\frac{e^{2i\pi\alpha}}{2}.$$

This section is devoted to the proof of the following proposition. Remember that $\mathcal{S}_N$ is the set of irrational numbers of bounded type whose continued fractions have entries greater than or equal to N.

Proposition 11. *There exists N such that as $\alpha' \in \mathcal{S}_N \to \alpha \in \mathcal{S}_N$, we have*

$$\partial\big(\mathcal{PC}(P_{\alpha'}), \overline{\Delta}_\alpha\big) \to 0,$$

with Δ_α the Siegel disk of P_α.

The corollary we will use later is the following.

Corollary 4. *Let (α_n) be the sequence defined in Prop. 3. If n is large enough, the post-critical set of P_{α_n} is contained in the δ-neighborhood of the Siegel disk of P_α.*

The proof of Prop. 11 will rely on some (almost) classical results on Fatou coordinates and perturbed Fatou coordinates. We refer the reader to appendix A and to [Sh2] for more details. The proof will also rely on results of Inou and Shishikura [IS] that we will now recall.

1.5.1. *The class of Inou and Shishikura.* Consider the cubic polynomial

$$P(z) = z(1+z)^2.$$

This polynomial has a multiple fixed point at 0, a critical point at $-1/3$ which is mapped to the critical value at $-4/27$, and a second critical point at -1 which is mapped to 0. We set

$$R := e^{4\pi} \quad \text{and} \quad v := -4/27.$$

Let U be the open set defined by

$$U := P^{-1}\big(D(0,|v|R)\big) \setminus \big(]-\infty,-1] \cup B\big),$$

where B is the connected component of $P^{-1}\big(D(0,|v|/R)\big)$ which contains -1.

Consider the following class of maps (Inou and Shishikura use the notation $\mathcal{F}_2^P$ in [IS]):

$$\mathcal{IS}_0 := \left\{ f = P\circ\varphi^{-1} : U_f \to \mathbb{C} \text{ with } \begin{array}{l} \varphi : U \to U_f \text{ isomorphism such that} \\ \varphi(0) = 0 \text{ and } \varphi'(0) = 1 \end{array} \right\}.$$

Remark. The set $\mathcal{IS}_0$ is identified with the space of univalent maps in U fixing 0 with derivative 1, which is compact. A sequence of univalent maps $(\varphi_n : U \to \mathbb{C})$ satisfying $\varphi_n(0) = 0$ and $\varphi_n'(0) = 1$ converges uniformly to $\varphi : U \to \mathbb{C}$ on every compact subset of U, if and only if the sequence $(f_n = P\circ\varphi_n^{-1})$ converges to $f = P\circ\varphi^{-1}$ on every compact subset of $U_f = \varphi(U)$.

A map $f \in \mathcal{IS}_0$ fixes 0 with multiplier 1. The map $f : U_f \to D\big(0,|v|R\big)$ is surjective. It is not a proper map. Inou and Shishikura call it a *partial covering*. The map f has a critical point $\omega_f := \varphi_f(-1/3)$ which depends on f and a critical value $v := -4/27$ which does not depend on f.

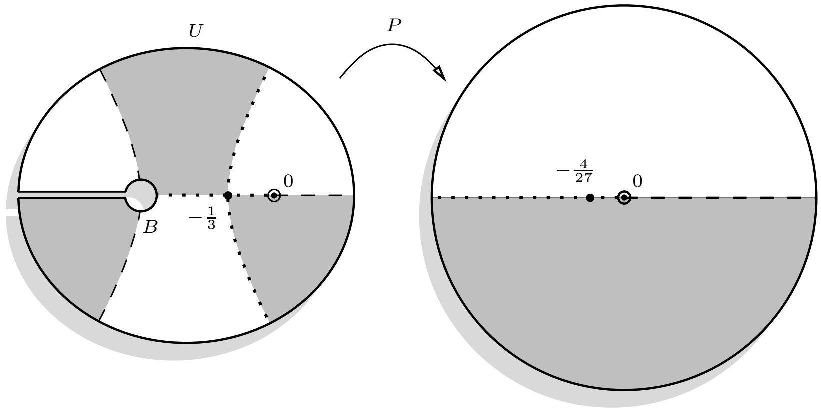


FIGURE 7. A schematic representation of the set U. We colored gray the set of points in U whose image by P is contained in the lower half-plane.

1.5.2. *Fatou coordinates.* Near $z = 0$, elements $f \in \mathcal{I}S_0$ have an expansion of the form

$$f(z) = z + c_f z^2 + \mathcal{O}(z^3).$$

The following result of Inou and Shishikura is an immediate consequence of the Koebe 1/4-theorem.

Result of Inou-Shishikura (Main theorem 1 part a)**.** *The set* $\{c_f\ ;\ f \in \mathcal{I}S_0\}$ *is a compact subset of* $\mathbb{C}^*$*.*

In particular, for all $f \in \mathcal{I}S_0$, $c_f \neq 0$ and f has a multiple fixed point of multiplicity 2 at 0. If we make the change of variables

$$w = \tau_f(z) := -\frac{1}{c_f z},$$

we find $F(w) = w + 1 + o(1)$ near infinity. To lighten notation, we will write f and F for pairs of functions related as above; $\omega_f := \phi_f(-1/3)$ and $\omega_F := \tau_f^{-1}(\omega_f)$ will denote their critical points.

Lemma 6. *There exists* R_0 *such that for all* $f \in \mathcal{I}S_0$

- *F is defined and univalent in a neighborhood of $\mathbb{C} \setminus D(0, R_0)$ and*
- *for all $w \in \mathbb{C} \setminus D(0, R_0)$,*

$$\big|F(w) - w - 1\big| < \frac{1}{4} \quad \text{and} \quad \big|F'(w) - 1\big| < \frac{1}{4}.$$

Proof. This follows from the compactness of $\mathcal{I}S_0$. □

If $R_1 > \sqrt{2}R_0$, the regions

$$\Omega^{\text{att}} := \big\{w \in \mathbb{C}\ ;\ \text{Re}(w) > R_1 - |\text{Im}(w)|\big\}$$

and

$$\Omega^{\text{rep}} := \big\{w \in \mathbb{C}\ ;\ \text{Re}(w) < -R_1 + |\text{Im}(w)|\big\}$$

are contained in $\mathbb{C} \setminus D(0, R_0)$.

Then, for all $f \in \mathcal{I}S_0$,

$$F(\Omega^{\text{att}}) \subset \Omega^{\text{att}} \quad \text{and} \quad F\big(\Omega^{\text{rep}}\big) \supset \Omega^{\text{rep}}.$$

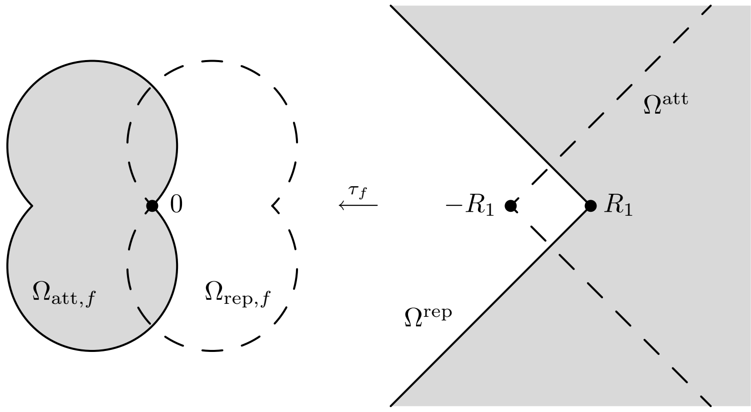


FIGURE 8. Right: the sets $\Omega^{\rm att}$ and $\Omega^{\rm rep}$. Left: the set $\Omega_{{\rm att},f}$ and $\Omega_{{\rm rep},f}$ for a map f with $c_f = 1$. The sets $\Omega^{\rm att}$ and $\Omega_{{\rm att},f}$ are shaded. The boundaries of the sets $\Omega^{\rm rep}$ and $\Omega_{{\rm rep},f}$ are dashed.

In addition, there are univalent maps $\Phi_F^{\rm att} : \Omega^{\rm att} \to \mathbb{C}$ (attracting Fatou coordinate for F) and $\Phi_F^{\rm rep} : \Omega^{\rm rep} \to \mathbb{C}$ (repelling Fatou coordinate for F) such that

$$\Phi_F^{\rm att} \circ F(w) = \Phi_F^{\rm att}(w) + 1 \quad \text{and} \quad \Phi_F^{\rm rep} \circ F(w) = \Phi_F^{\rm rep}(w) + 1$$

when both sides of the equations are defined. The maps $\Phi_F^{\rm att}$ and $\Phi_F^{\rm rep}$ are unique up to an additive constant.

Result of Inou-Shishikura (Main theorem 1 part a)**.** *For all $f \in \mathcal{IS}_0$, the critical point ω_f is attracted to* 0.

The following lemma easily follows, using the compactness of the class $\mathcal{IS}_0$.

Lemma 7. *There exists k such that for all $f \in \mathcal{IS}_0$ we have $F^{\circ k}(\omega_F) \in \Omega^{att}$.*

Proof. By contradiction, suppose that there is a sequence $(f_n) \in \mathcal{IS}_0$ such that for $k \le n$ we have $F_n^{\circ k}(\omega_{F_n}) \notin \Omega^{\rm att}$. By compactness of $\mathcal{IS}_0$ we may assume that the sequence F_n converges to F_∞. But since $f_\infty \in \mathcal{IS}_0$, the orbit of the critical point ω_{f_∞} converges to 0, so for some k we have $F_\infty^{\circ k}(\omega_{F_\infty}) \in \Omega^{\rm att}$. But

$$F_\infty^{\circ k}(\omega_{F_\infty}) = \lim_{n\to\infty} F_n^{\circ k}(\omega_{F_n})$$

and this is a contradiction. $\square$

Since the maps $\Phi_F^{\rm att}$ and $\Phi_F^{\rm rep}$ are only defined up to an additive constant, we can normalize $\Phi_F^{\rm att}$ so that

$$\Phi_F^{\rm att}\big(F^{\circ k}(\omega_F)\big) = k.$$

Then, we can normalize $\Phi_F^{\rm rep}$ so that

$$\Phi_F^{\rm att}(w) - \Phi_F^{\rm rep}(w) \to 0 \quad \text{when} \quad {\rm Im}(w) \to +\infty \quad \text{with} \quad w \in \Omega^{\rm att} \cap \Omega^{\rm rep}.$$

Coming back to the z-coordinate, we define

$$\Omega_{{\rm att},f} := \tau_f(\Omega^{\rm att}) \quad \text{and} \quad \Omega_{{\rm rep},f} := \tau_f(\Omega^{\rm rep})$$

and we set

$$\Phi_{{\rm att},f} := \Phi_F^{\rm att} \circ \tau_f^{-1} \quad \text{and} \quad \Phi_{{\rm rep},f} := \Phi_F^{\rm rep} \circ \tau_f^{-1}.$$

The univalent maps $\Phi_{\mathrm{att},f} : \Omega_{\mathrm{att},f} \to \mathbb{C}$ and $\Phi_{\mathrm{rep},f} : \Omega_{\mathrm{rep},f} \to \mathbb{C}$ are called attracting and repelling Fatou coordinates for f. Note that our normalization of the attracting coordinates is given by

$$\Phi_{\mathrm{att},f}\big(f^{\circ k}(\omega_f)\big) = k.$$

The following result of Inou and Shishikura asserts that the attracting Fatou coordinate can be extended univalently up to the critical point of f. It easily follows from [IS] Prop. 5.6.

Result of Inou-Shishikura (see figure 9)**.** *For all $f \in \mathcal{IS}_0$, there exists an attracting petal $\mathcal{P}_{att,f}$ and an extension of the Fatou coordinate, that we will still denote $\Phi_{att,f} : \mathcal{P}_{att,f} \to \mathbb{C}$, such that*

- $v \in \mathcal{P}_{att,f}$,
- $\Phi_{att,f}(v) = 1$,
- $\Phi_{att,f}$ *is univalent on* $\mathcal{P}_{att,f}$ *and*
- $\Phi_{att,f}(\mathcal{P}_{att,f}) = \big\{w \ ; \ \mathrm{Re}(w) > 0\big\}$.

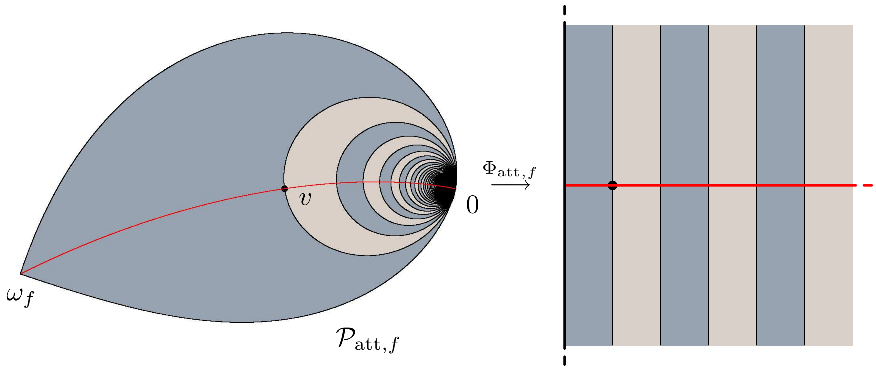


FIGURE 9. Left: the attracting petal $\mathcal{P}_{\mathrm{att},f}$ of some map $f \in \mathcal{IS}_0$; the critical point is ω_f, the critical value v and 0 is a fixed point. Right: its image by $\Phi_{\mathrm{att},f}$; we divided the right half plane $]0, +\infty[\times\mathbb{R}$ into vertical strips of width 1 of alternating color, highlighted the real axis in red, and put a black dot at the point $z = 1$. On the left, we pulled this coloring back by $\Phi_{\mathrm{att},f}$.

Definition 7 (see figure 10)**.** *For $f \in \mathcal{IS}_0$, we set:*

$$V_f := \big\{z \in \mathcal{P}_{att,f} \ ; \ \mathrm{Im}\big(\Phi_{att,f}(z)\big) > 0 \text{ and } 0 < \mathrm{Re}\big(\Phi_{att,f}(z)\big) < 2\big\}$$

and

$$W_f := \big\{z \in \mathcal{P}_{att,f} \ ; \ -2 < \mathrm{Im}\big(\Phi_{att,f}(z)\big) < 2 \text{ and } 0 < \mathrm{Re}\big(\Phi_{att,f}(z)\big) < 2\big\}.$$

We now come to the key result of Inou and Shishikura. The result stated below easily follows from [IS] Prop. 5.7 and 5.8 and section 5.M. Our domain $V_f^{-k} \cup W_f^{-k}$ below corresponds in [IS] to the interior of

$$\overline{D}_{-k} \cup \overline{D}^\sharp_{-k} \cup \overline{D}''_{-k} \cup \overline{D}_{-k+1} \cup \overline{D}^\sharp_{-k+1} \cup \overline{D}'_{-k+1}.$$

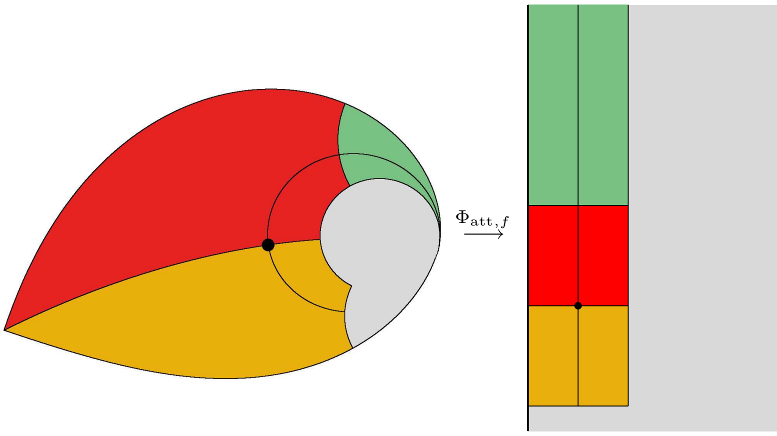


FIGURE 10. On the right, we divided $]0,2[\times]-2,+\infty[$ into 3 regions of different colors. We subdivided each by a vertical line through $z=1$. These 6 pieces were then pulled back on the left by $\Phi_{\mathrm{att},f}$, for the same parabolic $f\in\mathcal{I}S_0$ as in figure 9.

The set W_f^{-k} itself corresponds to the interior of

$$\overline{D}_{-k}\cup\overline{D}''_{-k}\cup\overline{D}_{-k+1}\cup\overline{D}'_{-k+1}.$$

Result of Inou-Shishikura (see figure 11)**.** *For all $f\in\mathcal{I}S_0$ and all $k\geq 0$,*

- *the unique connected component V_f^{-k} of $f^{-k}(V_f)$ which contains 0 in its closure is relatively compact in U_f (the domain of f) and $f^{\circ k}:V_f^{-k}\to V_f$ is an isomorphism and*
- *the unique connected component W_f^{-k} of $f^{-k}(W_f)$ which intersects V_f^{-k} is relatively compact in U_f and $f^{\circ k}:W_f^{-k}\to W_f$ is a covering of degree 2 ramified above v.*

In addition, if k is large enough, then $V_f^{-k}\cup W_f^{-k}\subset\Omega_{rep,f}$.

The following lemma asserts that if k is large enough, then for all map $f\in\mathcal{I}S_0$, the set $V_f^{-k}\cup W_f^{-k}$ is contained in a repelling petal of f, i.e. the preimage of a left half-plane by $\Phi_{\mathrm{rep},f}$.

Lemma 8 (see figure 12)**.** *There is an $R_2>0$ such that for all $f\in\mathcal{I}S_0$, the set $\Phi_f(\Omega_{rep,f})$ contains the half-plane $\{w\in\mathbb{C}\ ;\ \mathrm{Re}\ w<-R_2\}$. There is an integer $k_0>0$ such that for all $k\geq k_0$, we have*

$$V_f^{-k}\cup W_f^{-k}\subset\big\{z\in\Omega_{rep,f}\ ;\ \mathrm{Re}\big(\Phi_{rep,f}(z)\big)<-R_2\big\}.$$

Remark. Of course, R_2 can be replaced by any $R_3\geq R_2$, replacing if necessary k_0 by $k_1:=k_0+\lfloor R_3-R_2\rfloor+1$.

Proof. For all $f\in\mathcal{I}S_0$, $\Phi_f(\Omega_{\mathrm{rep},f})$ contains a left half-plane. The existence of R_2 follows from the compactness of $\mathcal{I}S_0$.

By Inou and Shishikura's result, we know that for all $f\in\mathcal{I}S_0$ there is an integer $k>0$ such that W_f^{-k} is relatively compact in $\Omega_{\mathrm{rep},f}$. It follows from the

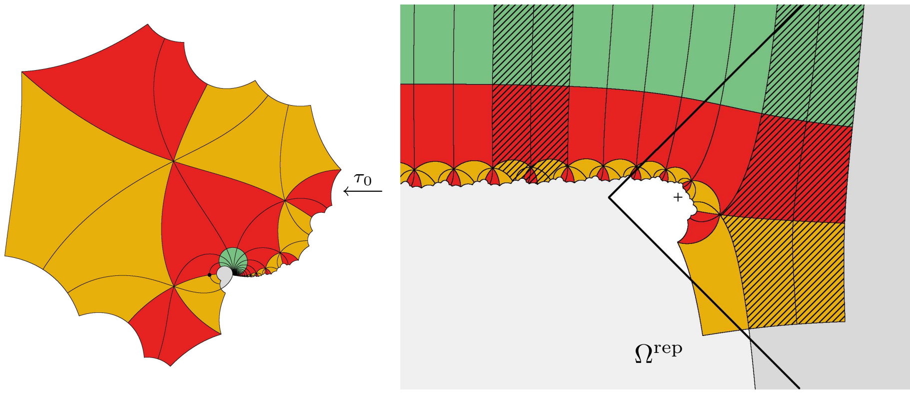


FIGURE 11. Left: among the successive preimages of V_f and W_f by f, those that compose the sets V_f^{-k}, W_f^{-k} are shown. The colors are preserved by f. Right: preimage of the left part by τ_0. We hatched $W_F \cup V_F$ and $W_F^{-7} \cup V_F^{-7}$.

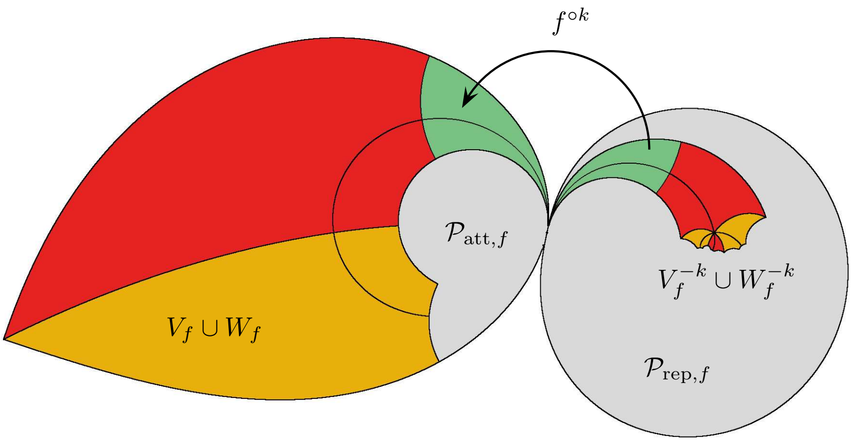


FIGURE 12. If k is large enough, $V_f^{-k} \cup W_f^{-k}$ is contained in the repelling petal $\mathcal{P}_{\mathrm{rep},f}$.

compactness of $\mathcal{IS}_0$ that there is an integer $k_1 > 0$ and a constant M, such that for all $f \in \mathcal{IS}_0$, $W_f^{-k_1} \subset \Omega_{\mathrm{rep},f}$ and

$$\sup_{w \in W_f^{-k_1}} \mathrm{Re}\big(\Phi_{\mathrm{rep},f}(w)\big) < M.$$

Set $k_0 := k_1 + M + \lfloor R_2 \rfloor + 3$. Then,

$$\sup_{w \in W_f^{-k_0}} \mathrm{Re}\big(\Phi_{\mathrm{rep},f}(w)\big) < -R_2 - 2.$$

We will show that we then automatically have

$$V_f^{-k_0} \subset \Omega_{\mathrm{rep},f} \quad \text{and} \quad \sup_{w\in V_f^{-k_0}} \mathrm{Re}\big(\Phi_{\mathrm{rep},f}(w)\big) < -R_2. \tag{2}$$

It will follow immediately that

$$\forall k \geq k_0 \text{ and } \forall w \in V_f^{-k} \cup W_f^{-k}, \qquad \mathrm{Re}\big(\Phi_{\mathrm{rep},f}(w)\big) < -R_2,$$

which will conclude the proof of the lemma.

In order to get (2), we fix $f \in \mathcal{IS}_0$ and consider $k \geq k_0$ large enough so that $V_f^{-k} \subset \Omega_{\mathrm{rep},f}$ (this is possible thanks to Inou and Shishikura). Note that

$$\sup_{w\in W_f^{-k}} \mathrm{Re}\big(\Phi_{\mathrm{rep},f}(w)\big) < -R_2 - 2 - k + k_0.$$

Denote by $g : \overline{V}_f \to \overline{V_f^{-k}}$ the inverse branch of $f^{\circ k} : \overline{V_f^{-k}} \to \overline{V}_f$. Set

$$B := \big\{w \in \mathbb{C}\ ;\ 0 < \mathrm{Re}(w) < 2 \text{ and } 0 < \mathrm{Im}(w)\big\}.$$

Note that $B = \Phi_{\mathrm{att},f}(V_f)$. Consider the map $\Psi : \overline{B} \to \mathbb{C}$ defined by

$$\Psi := \Phi_{rep,f} \circ g \circ \Phi_{\mathrm{att},f}^{-1}.$$

Since Ψ commutes with translation by 1, so that $\Psi(w) - w$ is 1-periodic, the maximum modulus principle yields

$$\sup_{w\in B} \mathrm{Re}\big(\Psi(w) - w\big) = \sup_{w\in[0,2]} \mathrm{Re}\big(\Psi(w) - w\big).$$

Note that

$$g \circ \Phi_{\mathrm{att},f}^{-1}\big([0, 2]\big) \subset W_f^{-k}$$

and thus

$$\sup_{w\in[0,2]} \mathrm{Re}\big(\Psi(w) - w\big) < -R_2 - 2 - k + k_0.$$

Hence,

$$\sup_{w\in V_f^{-k}} \mathrm{Re}\big(\Phi_{\mathrm{rep},f}(w)\big) = \sup_{w\in B} \mathrm{Re}\big(\Psi(w)\big) < -R_2 - k + k_0.$$

It now follows that

$$\sup_{w\in V_f^{-k_0}} \mathrm{Re}\big(\Phi_{\mathrm{rep},f}(w)\big) < -R_2.$$

This completes the proof of (2) and of lemma 8. □

1.5.3. *Perturbed Fatou coordinates.* For $\alpha \in \mathbb{R}$, we denote by $\mathcal{IS}_\alpha$ the set of maps of the form $z \mapsto f(e^{2i\pi\alpha} z)$ with $f \in \mathcal{IS}_0$. If A is a subset of $\mathbb{R}$, we denote by $\mathcal{IS}_A$ the set

$$\mathcal{IS}_A := \bigcup_{\alpha\in A} \mathcal{IS}_\alpha.$$

Note that

$$\mathcal{IS}_\alpha = \left\{ f = P \circ \varphi^{-1} : U_f \to \mathbb{C} \text{ with } \begin{array}{l} \varphi : U \to U_f \text{ isomorphism such that} \\ \varphi(0) = 0 \text{ and } \varphi'(0) = e^{-2i\pi\alpha} \end{array} \right\}$$

and

$$\mathcal{IS}_{\mathbb{R}} = \left\{ f = P \circ \varphi^{-1} : U_f \to \mathbb{C} \text{ with } \begin{array}{l} \varphi : U \to U_f \text{ isomorphism such that} \\ \varphi(0) = 0 \text{ and } \big|\varphi'(0)\big| = 1 \end{array} \right\}.$$

The map f depends on ϕ in a one-to-one way. Thus we get a one-to-one correspondence between $\mathcal{IS}_{\mathbb{R}}$ and the set of univalent maps on U fixing 0 with derivative 1. We put the compact-open topology on this set of univalent maps. This induces a topology on $\mathcal{IS}_{\mathbb{R}}$.

Remark. A sequence $(f_n = P\circ\varphi_n^{-1} \in \mathcal{IS}_{\mathbb{R}})$ converges to $f = P\circ\varphi^{-1} \in \mathcal{IS}_{\mathbb{R}}$ if and only if the sequence (f_n) converges to f on every compact subset of $U_f = \varphi(U)$.

If $f \in \mathcal{IS}_{[0,1[}$, we denote by $\alpha_f \in [0,1[$ the rotation number of f at 0, i.e. the real number $\alpha_f \in [0,1[$ such that

$$f'(0) = e^{2i\pi\alpha_f}.$$

Lemma 9. *There exist $\varepsilon_0 \in]0,1[$ and $r > 0$ such that for all $f \in \mathcal{IS}_{[0,\varepsilon_0[}$, the map f has two fixed points in $D(0,r)$ (counting multiplicities), one at $z = 0$ the other one denoted by σ_f. The map $\sigma : \mathcal{IS}_{[0,\varepsilon_0[} \to D(0,r)$ defined by $f \mapsto \sigma_f$ is continuous.*

Proof. According to Inou and Shishikura, maps $f \in \mathcal{IS}_0$ have a double fixed point at 0. By compactness of $\mathcal{IS}_0$, there is an $r' > 0$ such that maps $f \in \mathcal{IS}_0$ have only 2 fixed points in $D(0,r')$. Choose $r \in]0,r'[$. By Rouché's theorem and by compactness of $\mathcal{IS}_0$, there is an $\varepsilon_0 > 0$ such that maps $f \in \mathcal{IS}_{[0,\varepsilon_0[}$ have exactly two fixed points in $D(0,r)$. The result follows easily. □

The following results are consequences of results in [Sh2], the compactness of the class $\mathcal{IS}_0$ and the results of the previous paragraph.

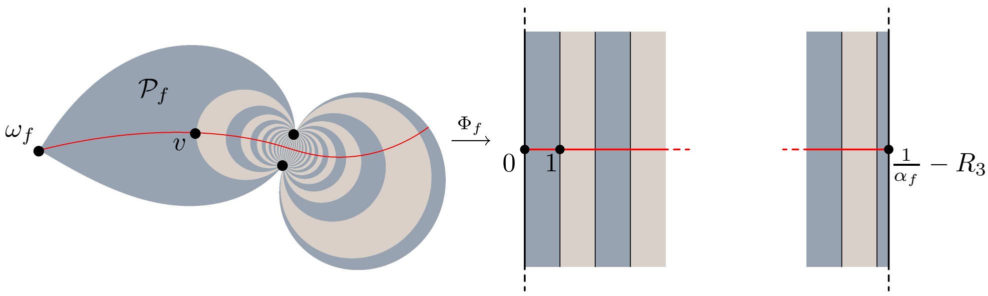


Figure 13. The perturbed petal $\mathcal{P}_f$ whose image by the perturbed Fatou coordinate Φ_f is the strip $\{0 < \mathrm{Re}(w) < 1/\alpha_f - R_3\}$.

Proposition 12 (see figure 13)**.** *There are constants $K > 0$, $\varepsilon_1 > 0$ and $R_3 \geq R_2$ with $1/\varepsilon_1 - R_3 > 1$, such that for all $f \in \mathcal{IS}_{]0,\varepsilon_1[}$ the following holds.*

(1) There is a Jordan domain $\mathcal{P}_f \subset U_f$ (a perturbed petal) containing v, bounded by two arcs joining 0 to σ_f and there is a branch of argument defined on $\mathcal{P}_f$ such that

$$\sup_{z\in\mathcal{P}_f} \arg(z) - \inf_{z\in\mathcal{P}_f} \arg(z) < K.$$

(2) There is a univalent map $\Phi_f : \mathcal{P}_f \to \mathbb{C}$ (a perturbed Fatou coordinate) such that

- *$\Phi_f(v) = 1$,*

- $\Phi_f(\mathcal{P}_f) = \{w \in \mathbb{C}\ ;\ 0 < \mathrm{Re}(w) < 1/\alpha_f - R_3\}$,
- $\mathrm{Im}(\Phi_f(z)) \to +\infty$ *as* $w \to 0$ *and* $\mathrm{Im}(\Phi_f(z)) \to -\infty$ *as* $w \to \sigma_f$ *and*
- *when* $z \in \mathcal{P}_f$ *and* $\mathrm{Re}(\Phi_f(z)) < 1/\alpha_f - R_3 - 1$, $f(z) \in \mathcal{P}_f$ *and* $\Phi_f \circ f(z) = \Phi_f(z) + 1$.

For $f \in \mathcal{IS}_0$, *we set*

$$\mathcal{P}_{rep,f} := \{z \in \Omega_{rep,f}\ ;\ \mathrm{Re}(\Phi_{rep,f}(z)) < -R_3\}.$$

(3) *If* (f_n) *is a sequence of maps in* $\mathcal{IS}_{]0,\varepsilon_1[}$ *converging to a map* $f_0 \in \mathcal{IS}_0$, *then*
   - *any compact* $K \subset \mathcal{P}_{att,f_0}$ *is contained in* $\mathcal{P}_{f_n}$ *for* n *large enough and the sequence* (Φ_{f_n}) *converges to* Φ_{att,f_0} *uniformly on* K, *and*
   - *any compact* $K \subset \mathcal{P}_{rep,f_0}$ *is contained in* $\mathcal{P}_{f_n}$ *for* n *large enough and the sequence* $(\Phi_{f_n} - \frac{1}{\alpha_{f_n}})$ *converges to* Φ_{rep,f_0} *uniformly on* K.

Proof. Thanks to the compactness of the class $\mathcal{IS}_0$, it is enough to show that if (f_n) is a sequence of maps in $\mathcal{IS}_{]0,1[}$ converging to a map $f_0 \in \mathcal{IS}_0$, there is a number $R_3 \geq R_2$ such that properties (1), (2) and (3) hold.

So, assume f_n is such a sequence, and for simplicity, write α_n, σ_n, ... instead of α_{f_n}, σ_{f_n}, ...

Let $\tau_n : \mathbb{C} \to \mathbb{P}^1 \setminus \{0, \sigma_n\}$ be the universal covering given by

$$\tau_n(w) := \frac{\sigma_n}{1 - e^{-2i\pi\alpha_n w}}$$

so that

$$\tau_n(w) \underset{\mathrm{Im}(w)\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \tau_n(w) \underset{\mathrm{Im}(w)\to-\infty}{\longrightarrow} \sigma_n.$$

Denote by $T_n : \mathbb{C} \to \mathbb{C}$ the translation

$$T_n : w \mapsto w - \frac{1}{\alpha_n}.$$

Recall that $f_0(z) = z + c_0 z^2 + \mathcal{O}(z^3)$ with $c_0 \neq 0$, and

$$\tau_0(z) := -\frac{1}{c_0 z}.$$

The following observations follow from [Sh2]. We let R_0 and R_1 be the constants introduced in paragraph 1.5.2.

(1) The sequence (τ_n) converges to τ_0 uniformly on every compact subset of $\mathbb{C}^*$.
(2) If n is sufficiently large, there is a map $F_n : \mathcal{D}_n \to \mathbb{C}$, defined and univalent in

$$\mathcal{D}_n := \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \overline{D}(k/\alpha_n, R_0)$$

which satisfies
   - $f_n \circ \tau_n = \tau_n \circ F_n$,
   - $F_n(w) - w$ is $1/\alpha_n$-periodic (or equivalently, $F_n \circ T_n = T_n \circ F_n$),
   - $F_n(w) - w \to 1$ as $\mathrm{Im}(w) \to +\infty$.

Remark. This lift F_n of f_n may be defined by

$$F_n(w) := w + \frac{1}{2i\pi\alpha_n} \log\left(\frac{f_n(z) - \sigma_n}{f_n(z)} \cdot \frac{z}{z - \sigma_n}\right) \quad \text{with} \quad z = \tau_n(w).$$

(3) As n tends to $+\infty$, the sequence (F_n) converges to F_0 uniformly on every compact subset of $\mathbb{C} \setminus \overline{D}(0, R_0)$.

(4) The set

$$\Omega^n := \left\{ w \in \mathbb{C}\ ;\ \mathrm{Re}(w) > R_1 - \big|\mathrm{Im}(w)\big| \text{ and } \mathrm{Re}(w) < \frac{1}{\alpha_n} - R_1 + \big|\mathrm{Im}(w)\big| \right\}$$

is contained in $\mathcal{D}_n$ (see figure 14).

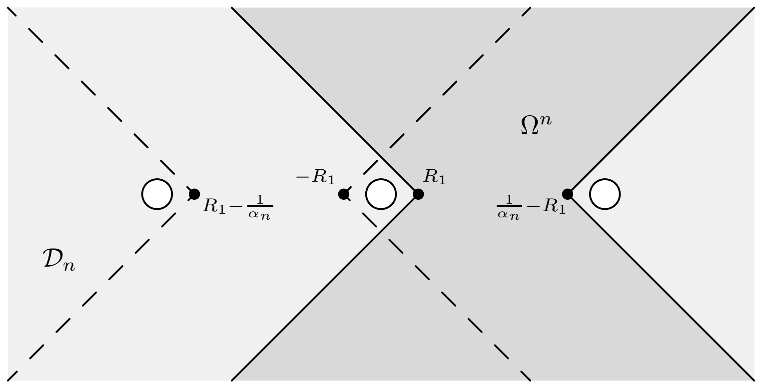


FIGURE 14. The domain $\mathcal{D}_n$ (grey) is the complement of a union of disks and the *hourglass* Ω^n (drak grey) is contained in $\mathcal{D}_n$.

(5) Remember that for all $w \in \mathbb{C} \setminus D(0, R_0)$,

$$\big|F_0(w) - w - 1\big| < \frac{1}{4} \quad \text{and} \quad \big|F_0'(w) - 1\big| < \frac{1}{4}.$$

It follows from the convergence of (F_n) to F_0 that if n is sufficiently large, then for all $w \in \Omega^n$,

$$\big|F_n(w) - w - 1\big| < \frac{1}{4} \quad \text{and} \quad \big|F_n'(w) - 1\big| < \frac{1}{4}.$$

(6) Increasing n if necessary, we may assume that $1/\alpha_n > 2R_1 + 2$. Then, there is a univalent map $\Phi^n : \Omega^n \to \mathbb{C}$, called a perturbed Fatou coordinate for F_n, such that

$$\Phi^n \circ F_n(w) = F_n(w) + 1$$

when $w \in \Omega^n$ and $F_n(w) \in \Omega^n$. This map is unique up to post-composition with a translation.

(7) Remember that there is a k such that $f_0^{\circ k}(\omega_0) \in \Omega_{\mathrm{att}}$, with ω_0 the critical point of f_0. For n large enough, $f_n^{\circ k}(\omega_n)$ is in $\tau_n(\Omega^n)$. There is a point $w_n \in \Omega^n$ such that

$$\tau_n(w_n) = f_n^{\circ k}(\omega_n) \quad \text{with} \quad w_n \underset{n\to+\infty}{\longrightarrow} \tau_0^{-1}\big(f_0^{\circ k}(\omega_0)\big).$$

We can normalize Φ^n by $\Phi^n(w_n) = k$. Then,

$$\Phi^n \underset{n\to+\infty}{\longrightarrow} \Phi_0^{\mathrm{att}}$$

uniformly on every compact subset of $\Omega^{\rm att}$. Due to the normalization $\Phi_0^{\rm att}(w) - \Phi_0^{\rm rep}(w) \to 0$ as $\mathrm{Im}(w) \to +\infty$ with $w \in \Omega^{\rm att} \cap \Omega^{\rm rep}$, we have

$$T_n \circ \Phi^n \circ T_n^{-1} \underset{n\to+\infty}{\longrightarrow} \Phi_0^{\rm rep}$$

uniformly on every compact subset of $\Omega^{\rm rep}$.

Coming back to the z-coordinate is not immediate. Indeed, the map τ_n is not injective on Ω^n and we cannot define a Fatou coordinate for f_n on $\tau_n(\Omega^n)$. We will instead restrict to a subset $\mathcal{P}^n \subset \Omega^n$ whose image by Φ^n is a vertical strip and on which τ_n is injective. The precise statement is the following. The proof is given in appendix A. It is a consequence of results in [Sh2], but is not stated there.

Lemma 10 (see figure 15)**.** *If $K > 0$ and $R \geq R_2$ are sufficiently large, then for n large enough:*

- $\Phi^n(\Omega^n)$ *contains the vertical strip*

$$U^n := \big\{w \in \mathbb{C}\ ;\ R < \mathrm{Re}(w) < 1/\alpha_n - R\big\}$$

and

- τ_n *is injective on* $\mathcal{P}^n := (\Phi^n)^{-1}(U^n)$.
- *there is a branch of argument defined on* $\tau_n(\mathcal{P}^n)$ *such that*

$$\sup_{z\in\tau_n(\mathcal{P}^n)} \arg(z) - \inf_{z\in\tau_n(\mathcal{P}^n)} \arg(z) < K.$$

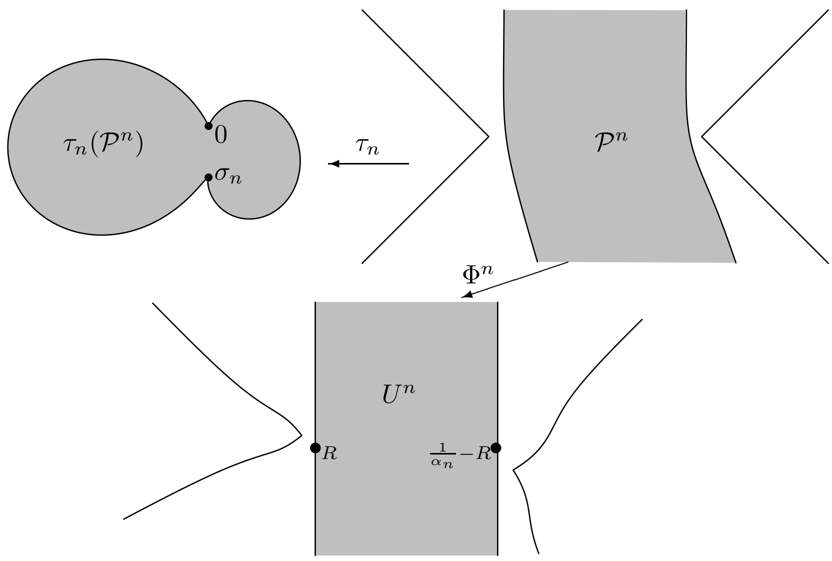


FIGURE 15. The map τ_n is injective on $\mathcal{P}^n := (\Phi^n)^{-1}(U^n)$.

Let $M > R$ be an integer. Note that

$$\big\{w \in \mathbb{C}\ ;\ \mathrm{Re}(w) > M\big\} \subset \Phi_{\rm att,0}(\Omega_{\rm att,0})$$

and

$$\big\{w \in \mathbb{C}\ ;\ \mathrm{Re}(w) < -M\big\} \subset \Phi_{\rm rep,0}(\Omega_{\rm rep,0}).$$

Set

$$\mathcal{P}'_0 := \big\{z \in \Omega_{\mathrm{att},0}\ ;\ \mathrm{Re}\big(\Phi_{\mathrm{att},0}(z)\big) > M\big\} \cup \big\{z \in \Omega_{\mathrm{rep},0}\ ;\ \mathrm{Re}\big(\Phi_{\mathrm{rep},0}(z)\big) < -M\big\}$$

and

$$\mathcal{P}'_n := \tau_n\Big(\big\{w \in \mathcal{P}^n\ ;\ M < \mathrm{Re}\big(\Phi^n(w)\big) < 1/\alpha_n - M\big\}\Big).$$

For any $r > 0$, if n is sufficiently large so that $\sigma_n \in D(0,r)$, then points with large (positive or negative) imaginary part are mapped by τ_n in $D(0,r)$. It therefore follows from point (7) above that $\overline{\mathcal{P}'_n} \to \overline{\mathcal{P}'_0}$ as $n \to +\infty$.

Set

$$\mathcal{P}_0 := \mathcal{P}_{\mathrm{att},0} \cup \big\{z \in \Omega_{\mathrm{rep},0}\ ;\ \mathrm{Re}\big(\Phi_{\mathrm{rep},0}(z)\big) < -2M\big\}.$$

Note that $\mathcal{P}_0$ is compactly contained in the domain of $f_0^{\circ M}$ and that $f_0^{\circ M} : \mathcal{P}_0 \to \mathcal{P}'_0$ is an isomorphism. In addition, for n sufficiently large, $f_n^{\circ M}$ does not have any critical value in $\mathcal{P}'_n$.

It follows from Rouché's theorem that for n large enough, the connected component $\mathcal{P}_n$ of $f_n^{-M}(\mathcal{P}'_n)$ which contains 0 in its boundary is relatively compact in the domain of f_n, and $f_n^{\circ M} : \mathcal{P}_n \to \mathcal{P}'_n$ is an isomorphism. The perturbed Fatou coordinate $\Phi_n : \mathcal{P}_n \to \mathbb{C}$ is defined by

$$\Phi_n(z) := \Phi^n(w) - M \quad \text{where} \quad w \in \mathcal{P}^n \text{ is chosen so that } \tau_n(w) = f_n^{\circ M}(z) \in \mathcal{P}'_n.$$

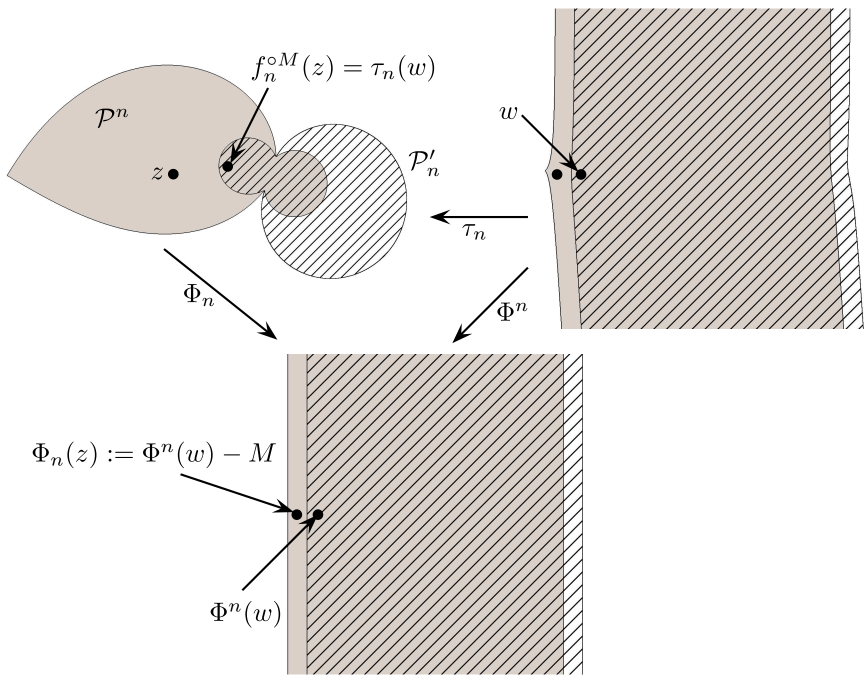


FIGURE 16. Definition of the perturbed Fatou coordinate Φ_n. The perturbed petal $\mathcal{P}_n$ is grey and the set $\mathcal{P}'_n$ is hatched.

In a simply connected neighborhood of $\overline{\mathcal{P}'_0}$, the function $f_0^{\circ M}(z)/z$ does not vanish (and extends by 1 at $z = 0$). It follows that for n large enough, there are branches of argument of $f_n^{\circ M}(z)/z$ which are uniformly bounded on $\mathcal{P}_n$. It is now easy to check that the proposition holds for f_n with n large enough. □

1.5.4. *Renormalization.* Recall that for maps $f \in \mathcal{IS}_0$ we defined sets $V_f \subset \mathcal{P}_{\mathrm{att},f}$ and $W_f \subset \mathcal{P}_{\mathrm{att},f}$. We claimed (see lemma 8) that for $k \geq 0$ there are components V_f^{-k} and W_f^{-k} properly mapped by $f^{\circ k}$ respectively to V_f with degree 1 and W_f with degree 2. In addition, there is an integer $k_0 > 0$ such that

$$\forall f \in \mathcal{IS}_0, \qquad V_f^{-k_0} \cup W_f^{-k_0} \subset \mathcal{P}_{\mathrm{rep},f}.$$

We will now generalize this to maps $f \in \mathcal{IS}_{]0,\varepsilon[}$ with ε sufficiently small. If $f \in \mathcal{IS}_{]0,\varepsilon_1[}$, we set

$$V_f := \big\{z \in \mathcal{P}_f \ ; \ \mathrm{Im}\big(\Phi_f(z)\big) > 0 \text{ and } 0 < \mathrm{Re}\big(\Phi_f(z)\big) < 2\big\}$$

and

$$W_f := \big\{z \in \mathcal{P}_f \ ; \ -2 < \mathrm{Im}\big(\Phi_f(z)\big) < 2 \text{ and } 0 < \mathrm{Re}\big(\Phi_f(z)\big) < 2\big\}.$$

Proposition 13 (see figure 17)**.** *There is a number $\varepsilon_2 > 0$ and an integer $k_1 \geq 1$ such that for all $f \in \mathcal{IS}_{]0,\varepsilon_2[}$ and for all integer $k \in [1, k_1]$,*

(1) the unique connected component V_f^{-k} of $f^{-k}(V_f)$ which contains 0 in its closure is relatively compact in U_f (the domain of f) and $f^{\circ k} : V_f^{-k} \to V_f$ is an isomorphism,
(2) the unique connected component W_f^{-k} of $f^{-k}(W_f)$ which intersects V_f^{-k} is relatively compact in U_f and $f^{\circ k} : W_f^{-k} \to W_f$ is a covering of degree 2 ramified above v.
(3) $V_f^{-k_1} \cup W_f^{-k_1} \subset \big\{z \in \mathcal{P}_f \ ; \ 2 < \mathrm{Re}\big(\Phi_f(z)\big) < \frac{1}{\alpha_f} - R_3 - 5\big\}$.

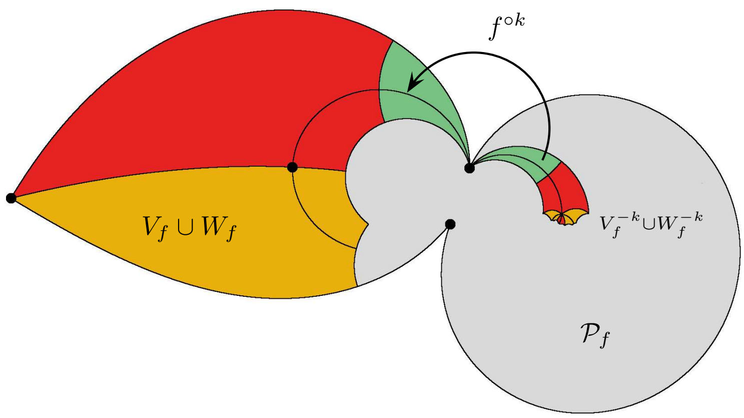


FIGURE 17. If k is large enough, $V_f^{-k} \cup W_f^{-k}$ is contained in the perturbed petal $\mathcal{P}_f$.

Proof. Set $k_1 := k_0 + 7$. By compactness of $\mathcal{IS}_0$, there is an $\varepsilon_2 > 0$ such that for all $f \in \mathcal{IS}_{]0,\varepsilon_2[}$, properties (1) and (2) hold for all integers $k \in [1, k_1]$, and further, $W_f^{-k_1}$ is contained in $\bigl\{z \in \mathcal{P}_f \;;\; 4 < \mathrm{Re}\bigl(\Phi_f(z)\bigr) < \frac{1}{\alpha_f} - R_3 - 7\bigr\}$.

To see that $V_f^{-k_1}$ is a subset of $\bigl\{z \in \mathcal{P}_f \;;\; 2 < \mathrm{Re}\bigl(\Phi_f(z)\bigr) < \frac{1}{\alpha_f} - R_3 - 5\bigr\}$, we proceed as in the proof of lemma 8. □

We now come to the definition of the renormalization of maps $f \in \mathcal{IS}_{]0,\varepsilon_2[}$.

Result of Inou-Shishikura (Main theorem 3 and section 5.M)**.** *If $f \in \mathcal{IS}_{]0,\varepsilon_2[}$, the map*

$$\Phi_f \circ f^{\circ k_1} \circ \Phi_f^{-1} : \Phi_f\bigl(V_f^{-k_1} \cup W_f^{-k_1}\bigr) \to \Phi_f\bigl(V_f \cup W_f\bigr)$$

projects via $w \mapsto -\frac{4}{27}e^{2i\pi w}$ to a map $\mathcal{R}(f) \in \mathcal{IS}_{-1/\alpha_f}$.

Definition 8. *The map $\mathcal{R}(f)$ is called the renormalization of f.*

The polynomial P_α does not belong to the class $\mathcal{IS}_\alpha$. However, the construction we described also works for polynomials P_α with $\alpha > 0$ sufficiently close to 0. In other words, if $\alpha > 0$ is sufficiently close to 0, there are perturbed petals and perturbed Fatou coordinates, and there is a renormalization $\mathcal{R}(P_\alpha)$ which belongs to $\mathcal{IS}_{-1/\alpha}$. In the sequel, $\varepsilon_2 > 0$ is chosen sufficiently small so that for $\alpha \in \,]0, \varepsilon_2[$, a map f which either is a polynomial P_α, or belongs to $\mathcal{IS}_\alpha$, has a renormalization $\mathcal{R}(f) \in \mathcal{IS}_{-1/\alpha}$.

1.5.5. *Renormalization tower.* Assume $1/N < \varepsilon_2$. Denote by $\mathrm{Irrat}_{\geq N}$ the set:

$$\mathrm{Irrat}_{\geq N} := \bigl\{\alpha = [\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots] \in \mathbb{R} \setminus \mathbb{Q} \;;\; \mathrm{a}_k \geq N \text{ for all } k \geq 1\bigr\}.$$

Assume $\alpha = [\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots] \in \mathrm{Irrat}_{\geq N}$. For $j \geq 0$, set

$$\alpha_j := [0, \mathrm{a}_{j+1}, \mathrm{a}_{j+2}, \ldots].$$

Note that for all $j \geq 1$,

$$\alpha_{j+1} = \frac{1}{\alpha_j} - \left\lfloor \frac{1}{\alpha_j} \right\rfloor.$$

The requirement $\alpha \in \mathrm{Irrat}_{\geq N}$ translates into

$$\forall j, \qquad \alpha_j \in]0, 1/N[.$$

Denote by p_j/q_j the approximants to α_0 given by the continued fraction algorithm.

Now, if either $f_0 = P_\alpha$ or $f_0 \in \mathcal{IS}_\alpha$, we can define inductively an infinite sequence of renormalizations, also called a *renormalization tower*, by

$$f_{j+1} := s \circ \mathcal{R}(f_j) \circ s^{-1},$$

the conjugacy by $s : z \mapsto \bar{z}$ being introduced so that

$$f_j'(0) = e^{2i\pi\alpha_j}.$$

It will be convenient to define

$$\begin{array}{rccc} \mathrm{Exp}: & \mathbb{C} & \to & \mathbb{C}^* \\ & w & \mapsto & -\frac{4}{27}s\bigl(e^{2i\pi w}\bigr). \end{array}$$

For $j \geq 0$, we define

$$\phi_j := \mathrm{Exp} \circ \Phi_{f_j} : \mathcal{P}_{f_j} \to \mathbb{C}.$$

The map ϕ_j goes from the j-th level of the renormalization tower to the next level.

We now want to relate the dynamics of maps at different levels of the renormalization tower. For this purpose, we will use the following lemma.

Lemma 11. *There is a constant $K > 0$ such that for all $f \in \mathcal{IS}_{]0,\varepsilon_2[}$, there is an inverse branch of* Exp *which is defined on $\mathcal{P}_f$ and takes its values in the strip* $\{w \in \mathbb{C}\ ;\ 0 < \mathrm{Re}(w) < K\}$.

Proof. This is an immediate consequence of Prop. 12 part (1). □

From now on, we assume that N is sufficiently large so that

$$\frac{1}{N} < \varepsilon_2 \quad \text{and} \quad \frac{1}{N} - R_3 > K. \tag{3}$$

Then, according to lemma 11, for all $j \geq 1$, there is an inverse branch ψ_j of ϕ_{j-1} defined on the perturbed petal $\mathcal{P}_{f_j}$ with values in $\mathcal{P}_{f_{j-1}}$ (there are several possible choices, we choose any one).

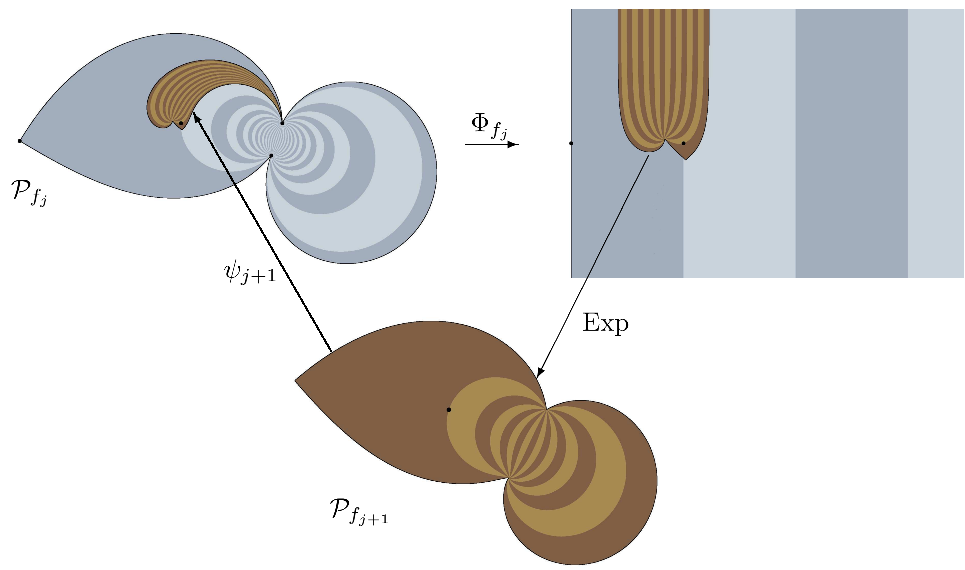


FIGURE 18. The branch ψ_{j+1} maps $\mathcal{P}_{f_{j+1}}$ univalently into $\mathcal{P}_{f_j}$.

The map

$$\Psi_j := \psi_1 \circ \psi_2 \circ \ldots \circ \psi_j$$

is then defined and univalent on $\mathcal{P}_{f_j}$ with values in the dynamical plane of the polynomial f_0.

Remember that

$$\Phi_{f_j}(\mathcal{P}_{f_j}) = \{w \in \mathbb{C}\ ;\ 0 < \mathrm{Re}(w) < 1/\alpha_j - R_3\}.$$

Define $\mathcal{P}_j \subset \mathcal{P}_{f_j}$ and $\mathcal{P}'_j \subset \mathcal{P}_{f_j}$ by

$$\mathcal{P}_j := \{z \in \mathcal{P}_{f_j}\ ;\ 0 < \mathrm{Re}\big(\Phi_{f_j}(w)\big) < 1/\alpha_j - R_3 - 1\}$$

and

$$\mathcal{P}'_j := \{z \in \mathcal{P}_{f_j}\ ;\ 1 < \mathrm{Re}\big(\Phi_{f_j}(w)\big) < 1/\alpha_j - R_3\}.$$

Note that f_j maps $\mathcal{P}_j$ to $\mathcal{P}'_j$ isomorphically. Set

$$\mathcal{Q}_j := \Psi_j(\mathcal{P}_j) \quad \text{and} \quad \mathcal{Q}'_j := \Psi_j(\mathcal{P}'_j).$$

Proposition 14. *The map Ψ_j conjugates $f_j : \mathcal{P}_j \to \mathcal{P}'_j$ to $f_0^{\circ q_j} : \mathcal{Q}_j \to \mathcal{Q}'_j$.*

In other words, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{Q}_j \subset \Psi_j(\mathcal{P}_{f_j}) \arrow[r, "f_0^{\circ q_j}"] & \mathcal{Q}'_j \subset \Psi_j(\mathcal{P}_{f_j}) \\
\mathcal{P}_j \subset \mathcal{P}_{f_j} \arrow[u, "\Psi_j"'] \arrow[r, "f_j"'] & \mathcal{P}'_j \subset \mathcal{P}_{f_j}. \arrow[u, "\Psi_j"']
\end{tikzcd}$$

Proof. We must show that if $z_j \in \mathcal{P}_j$ and $z'_j := f_j(z_j) \in \mathcal{P}'_j$, then the points $z_0 := \Psi_j(z_j)$ and $z'_0 := \Phi_j(z'_j)$ are related by

$$z'_0 = f_0^{\circ q_j}(z_0).$$

Let us first show that there is an integer k such that $z'_0 = f_0^{\circ k}(z_0)$. Our proof is based on the following lemma.

Lemma 12. *Assume $\ell \geq 0$, $w \in U_{f_{\ell+1}}$ and $w' := f_{\ell+1}(w)$. Let $z \in \mathcal{P}_{f_\ell}$ and $z' \in \mathcal{P}_{f_\ell}$ be such that*

$$\operatorname{Exp} \circ \Phi_{f_\ell}(z) = w \quad \textit{and} \quad \operatorname{Exp} \circ \Phi_{f_\ell}(z') = w'.$$

Then, there is an integer $k \geq 1$ such that $z' = f_\ell^{\circ k}(z)$.

Proof. Let $z'_1 \in \mathcal{P}_{f_\ell}$ be the unique point such that

$$\operatorname{Re}\bigl(\Phi_{f_\ell}(z'_1)\bigr) \in]0,1] \quad \text{and} \quad \operatorname{Exp} \circ \Phi_{f_\ell}(z'_1) = w'.$$

By definition of the renormalization $f_{\ell+1}$, there is a point $z_1 \in V_{f_\ell}^{-k_1} \cup W_{f_\ell}^{-k_1}$ such that

$$\operatorname{Exp} \circ \Phi_{f_\ell}(z_1) = w \quad \text{and} \quad f_\ell^{\circ k_1}(z_1) = z'_1.$$

We then have

$$\Phi_{f_\ell}(z_1) = \Phi_{f_\ell}(z) + m_1 \quad \text{and} \quad \Phi_{f_\ell}(z') = \Phi_{f_\ell}(z'_1) + m'_1$$

with $m_1 \in \mathbb{Z}$ and $m'_1 \in \mathbb{N}$. If $m_1 \geq 0$, we have

$$z_1 = f_\ell^{\circ m_1}(z) \quad \text{and} z' = f_\ell^{\circ m'_1}(z'_1).$$

Since $k_1 \geq 0$, we then have

$$z' = f^{\circ k}(z) \quad \text{with} \quad k := k_1 + m_1 + m'_1 \geq 1.$$

If $m_1 < 0$, then $z = f_\ell^{\circ - m_1}(z'_1)$. However, for $m \leq -m_1$, we have $f_\ell^{\circ m}(z'_1) \in \mathcal{P}_{f_\ell}$, and so, $k_1 \geq -m_1 + 1$. Thus, we can write

$$z'_1 = f_\ell^{\circ m_2}(z) \quad \text{with} \quad m_2 := k_1 + m_1 \geq 1.$$

In that case,

$$z' = f^{\circ k}(z) \quad \text{with} \quad k := m_2 + m'_1 \geq 1.$$

□

It follows by decreasing induction on ℓ from j to 0 that for all $z_j \in \mathcal{P}_j$, there is an integer $k \geq 1$ such that

$$z'_0 = f_0^{\circ k}(z_0).$$

We will now show that we have a common integer k, valid for all points $z_j \in \mathcal{P}_j$.

Lemma 13. *There is an integer $k_0 \geq 1$ such that for all point $z_j \in \mathcal{P}_j$, we have*

$$z_0' = f_0^{\circ k_0}(z_0).$$

Proof. We will use the connectivity of $\mathcal{P}_j$. For $k \geq 1$, set

$$\mathcal{O}_k := \{z \in \mathcal{P}_j \ ; \ f_0^{\circ k}\big(\Psi_j(z)\big) \text{ is defined}\}$$

This is an open set. Set

$$X_k := \big\{z \in \mathcal{O}_k \ ; \ f_0^{\circ k}\big(\Psi_j(z)\big) = \Psi_j\big(f_j(z)\big)\big\}.$$

Note that for every component O of $\mathcal{O}_k$, either $X_k \cap \mathcal{O} = O$, or X_k is discrete in O, in particular countable. Indeed, X_k is the set of zeroes of the holomorphic function $f_0^{\circ k} \circ \Psi_j - \Psi_j \circ f_j : \mathcal{O}_k \to \mathbb{C}$.

Since

$$\mathcal{P}_j = \bigcup_{k \geq 1} X_k$$

there is a smallest integer $k_0 \geq 1$ such that X_{k_0} is not countable. Then, there is a component O of $\mathcal{O}_{k_0}$ such that on O, we have $f_0^{\circ k_0} \circ \Psi_j = \Psi_j \circ f_j$.

Since O is a component of $\mathcal{O}_{k_0}$, we have

$$\partial O \cap \mathcal{P}_j \subset \mathbb{C} \setminus \mathcal{O}_{k_0}.$$

It follows that

$$\partial O \cap \mathcal{P}_j \subset X_1 \cup \ldots X_{k_0 - 1}$$

since the remaining X_k's are contained in $\mathcal{O}_{k_0}$. So, $\partial O \cap \mathcal{P}_j$ is countable. This is only possible if $\partial O \cap \mathcal{P}_j = \emptyset$ since in any neighborhood of a point $z \in \mathbb{C} \setminus \mathcal{O}_{k_0}$, there are uncountably many points in $\mathbb{C} \setminus \mathcal{O}_{k_0}$. As a consequence, $O = \mathcal{P}_j$, which concludes the proof of the lemma. □

We must now show that $k_0 = q_j$. Let $L_j \subset \mathcal{P}_j$ be the curve defined by

$$L_j := \big\{z \in \mathcal{P}_j \ ; \ \mathrm{Re}\big(\Phi_{f_j}(z)\big) = 1\big\}.$$

Set $L_j' := f_j(L_j)$, i.e. the curve

$$L_j' := \big\{z \in \mathcal{P}_j \ ; \ \mathrm{Re}\big(\Phi_{f_j}(z)\big) = 2\big\}.$$

Those curves both have an end point at $z = 0$. They both have tangents at $z = 0$. Since the linear part of f_j at $z = 0$ is the rotation of angle α_j, the angle between L_j and L_j' at $z = 0$ is α_j. It follows that the curves $\Psi_j(L_j)$ and $\Psi_j(L_j')$ have tangents at $z = 0$ and the angle between those curves is $\alpha_0\alpha_1 \cdots \alpha_j$. So, the linear part of $f_0^{\circ k_0}$ at $z = 0$ is the rotation of angle $\alpha_0\alpha_1 \cdots \alpha_j$. It follows that $k_0 = q_j$. □

Set

$$D_j := V_{f_j}^{-k_1} \cup W_{f_j}^{-k_1}, \quad D_j' := V_{f_j} \cup W_{f_j},$$
$$C_j := \Psi_j(D_j) \quad \text{and} \quad C_j' := \Psi_j(D_j').$$

Note that $f_j^{\circ k_1}$ maps D_j to D_j'.

Proposition 15. *The map Ψ_j conjugates the map $f_j^{\circ k_1} : D_j \to D_j'$ to the map $f_0^{\circ(k_1 q_j + q_{j-1})} : C_j \to C_j'$.*

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} C_j \subset \Psi_j(\mathcal{P}_{f_j}) & \xrightarrow{f_0^{\circ(k_1 q_j + q_{j-1})}} & C'_j \subset \Psi_j(\mathcal{P}_{f_j}) \\ \uparrow \Psi_j & & \uparrow \Psi_j \\ D_j \subset \mathcal{P}_{f_j} & \xrightarrow[f_j^{\circ k_1}]{} & D'_j \subset \mathcal{P}_{f_j}. \end{array}$$

Proof. The proof is similar to the one of Prop. 14. □

1.5.6. *Neighborhoods of the postcritical set.* We can now see that the post-critical set of maps $f \in \mathcal{IS}_\alpha$ with $\alpha \in \mathrm{Irrat}_{\geq N}$ is infinite.

Proposition 16 (Inou-Shishikura Cor. 4.2)**.** *For all $\alpha \in \mathrm{Irrat}_{\geq N}$ and all $f \in \mathcal{IS}_\alpha$, the postcritical set of f is infinite.*

Proof. For $j \geq 1$, the map $f_j^{\circ k_1} : W_{f_j}^{-k_1} \to W_{f_j}$ is a ramified covering of degree 2, ramified above v. Denote by w_j the critical point of this ramified covering. Set $w_0 := \Psi_j(w_j)$. According to Prop. 15, we can iterate f_0 at least $k_1 q_j + q_{j-1}$ times at w_0, w_0 is a critical point of $f_0^{\circ(k_1 q_j + q_{j-1})}$ and its critical value is $\Psi_j(v)$. In particular, $\Psi_j(v)$ is a point of the postcritical set of f_0.

Note that $v \in \mathcal{P}_j$. According to Prop. 14, we can iterate f_0 at least q_j times at $\Psi_j(v)$. This shows that we can iterate f_0 at least q_j times at v. Since $j \geq 1$ is arbitrary, the postcritical set of f_0 is infinite. □

For every $\alpha \in \mathrm{Irrat}_{\geq N}$, we are going to define a sequence (U_j) of open sets containing the post-critical set of P_α. We still use the notations of the previous paragraph. In particular, for $j \geq 1$, the j-th renormalization of $f_0 := P_\alpha$ has a perturbed petal $\mathcal{P}_{f_j}$, a perturbed Fatou coordinate

$$\Phi_{f_j} : \mathcal{P}_{f_j} \to \big\{ w \in \mathbb{C} \; ; \; 0 < \mathrm{Re}(w) < 1/\alpha_j - R_3 \big\}.$$

The set

$$D_j := V_{f_j}^{-k_1} \cup W_{f_j}^{-k_1} \subset \mathcal{P}_{f_j}$$

is mapped by $f_j^{\circ k_1}$ to

$$D'_j := \big\{ z \in \mathcal{P}_{f_j} \; ; \; 0 < \mathrm{Re}\big(\Phi_{f_j}(z)\big) < 2 \text{ and } \mathrm{Im}\big(\Phi_{f_j}(z)\big) > -2 \big\}.$$

There is a map Ψ_j, univalent on $\mathcal{P}_{f_j}$, with values in the dynamical plane of P_α, conjugating $f_j^{\circ k_1} : D_j \to D'_j$ to $P_\alpha^{\circ(k_1 q_j + q_{j-1})} : C_j \to C'_j$ with

$$C_j := \Psi_j(D_j) \quad \text{and} \quad C'_j := \Psi_j(D'_j).$$

Definition 9. *For $\alpha \in \mathrm{Irrat}_{\geq N}$ and $j \geq 1$ we set*

$$U_j(\alpha) := \bigcup_{k=0}^{q_{j+1} + \ell q_j} P_\alpha^{\circ k}(C_j)$$

where $\ell := k_1 - \lfloor R_3 \rfloor - 4 \in \mathbb{N}$.

Figure 19 shows the open set $U_1(\alpha)$ for an α of bounded type.

Proposition 17. *For all $\alpha \in \mathrm{Irrat}_{\geq N}$ and all $j \geq 1$, the post-critical set $\mathcal{PC}(P_\alpha)$ is contained in $U_j(\alpha)$.*

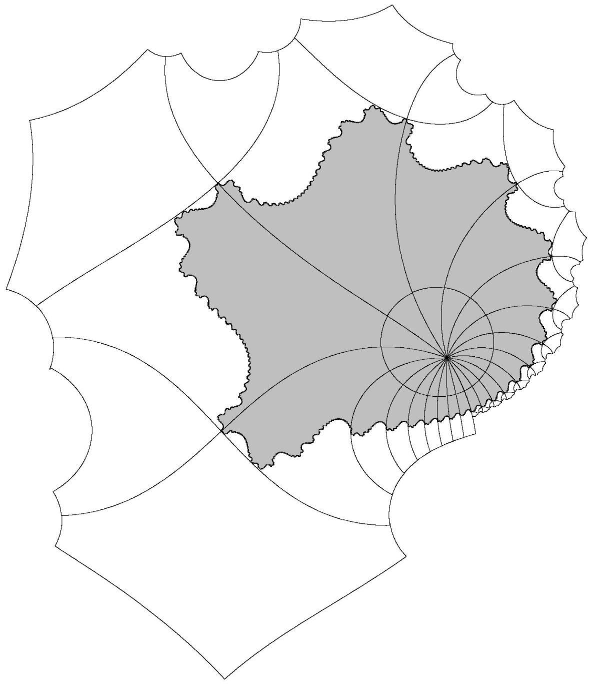


FIGURE 19. If $f \in \mathcal{IS}_\alpha$ with $\alpha \in \mathrm{Irrat}_{\geq N}$, the set $U_1(f)$ contains the postcritical set $\mathcal{PC}(f)$. If α is of bounded type, this post-critical set is dense in the boundary of the Siegel disk of f.

Proof. We will show that for all $j \geq 1$, there is a point $z_0 \in \mathbb{C}_j$ which is a precritical point of P_α, and a sequence of positive integers with $t_1 < t_2 < t_2 < \dots$ such that

- for all $m \geq 1$, $t_{m+1} - t_m < q_{j+1} + \big(k_1 + \lfloor R_3 \rfloor - 4\big) q_j$ and
- $P_\alpha^{\circ t_m}(z_0) \in C_j$.

The proof follows immediately.

Denote by ω_{j+1} the critical point of f_{j+1}. According to Prop. 16 the orbit of ω_{j+1} under iteration of f_{j+1} is infinite. In particular, for all $m \geq 0$, $f_{j+1}^{\circ m}(\omega_{j+1})$ is in the domain $U_{f_{j+1}}$ of f_{j+1}. Remember that the map $\phi_j := \mathrm{Exp} \circ \Phi_{f_j} : D_j \to U_{f_{j+1}}$ is surjective. So, for all $m \geq 0$, we can find a point $w_m \in D_j$ such that

$$\phi_j(w_m) = f_{j+1}^{\circ m}(\omega_{j+1}).$$

Set

$$z_m := \Psi_j(w_m) \in C_j.$$

Then, z_0 is a precritical point of P_α and according to lemma 12, there is an increasing sequence (t_m) such that $z_m = P_\alpha^{\circ t_m}(z_0)$. It is therefore enough to show that for all $m \geq 1$, $t_{m+1} - t_m < q_{j+1} + \big(k_1 + \lfloor R_3 \rfloor - 4\big) q_j$.

Note that for $m \geq 0$, $w_m \in D_j$, $w'_m := f_j^{\circ k_1}(w_m) \in D'_j$. By definition of the renormalization f_{j+1}, we have

$$\phi_j(w'_m) = f_{j+1}\big(\phi_j(w_m)\big) = f_{j+1}^{\circ(m+1)}(\omega_{j+1}) = \phi_j(w_m).$$

Thus, $\Phi_{f_j}(w_{m+1}) - \Phi_{f_j}(w'_m)$ is a positive integer ℓ_m. Then,

$$w_{m+1} = f_j^{\circ \ell_m)}(w'_m).$$

We have

$$\mathrm{Re}\big(\Phi_{f_j}(w'_m)\big) \geq 0 \quad \text{and} \quad \mathrm{Re}\big(\Phi_{f_j}(w_{m+1})\big) < \frac{1}{\alpha_j} - R_3 - 5.$$

Remember $\mathrm{a}_{j+1} = \lfloor 1/\alpha_j \rfloor$. Thus,

$$\ell_m \leq \mathrm{a}_{j+1} - \lfloor R_3 \rfloor - 4.$$

Set $z'_m := \Psi_j(w'_m)$. According to Prop. 14 and 15, we have

$$z'_m = P_\alpha^{\circ(k_1 q_j + q_{j-1})}(z_m) \quad \text{and} \quad z_{m+1} = P_\alpha^{\circ \ell_m q_j}(z'_m).$$

Thus,

$$t_{m+1} - t_m = k_1 q_j + q_{j-1} + \ell_m q_j \leq (\mathrm{a}_{j+1} + k_1 + \lfloor R_3 \rfloor - 4) q_j + q_{j-1}.$$

The result now follows immediately from $q_{j+1} = \mathrm{a}_{j+1} q_j + q_{j-1}$. □

We will now assume that $\alpha \in \mathcal{S}_N$, i.e. $\alpha \in \mathrm{Irrat}_{\geq N}$ is a bounded type irrational number (the coefficients of the continued fraction are bounded). We will use the additional hypothesis that α has bounded type in order to obtain the following result.

Proposition 18. *For all $\alpha \in \mathcal{S}_N$, for all $\varepsilon > 0$, if j is large enough, the set $U_j(\alpha)$ is contained in the ε-neighborhood of the Siegel disk Δ_α.*

Proof. Consider the renormalization tower associated to $f_0 := P_\alpha$ and let us keep the notations we have introduced so far. Set

$$D''_j := f_j^{\circ(a_{j+1}+\ell)}(D_j).$$

Define

$$N_j := \mathrm{a}_{j+1} - \lfloor R_3 \rfloor - 1 < \frac{1}{\alpha_j} - R_3.$$

Note that

$$D''_j = \big\{ z \in \mathbb{C} \ ; \ N_j - 3 < \mathrm{Re}\big(\Phi_{f_j}(z)\big) < N_j - 1 \text{ and } \mathrm{Im}(w) > -2 \big\}.$$

In particular, $D''_j \subset \mathcal{P}_{f_j}$. Set

$$C''_j := \Psi_j(D''_j).$$

According to Prop. 14 and 15,

$$C''_j = P_\alpha^{\circ(q_{j+1}+\ell q_j)}(C_j).$$

Lemma 14. *There exists M such that for all $j \geq 1$, the disk $D\big(0, |v|e^{-2\pi M}\big)$ is contained in the Siegel disk of f_j.*

Proof. Let $B(\alpha_j)$ be the Brjuno sum defined by Yoccoz as

$$B(\alpha_j) := \sum_{k=0}^{+\infty} \alpha_j \cdots \alpha_{j+k-1} \log \frac{1}{\alpha_{j+k}}.$$

Since α is of bounded type, there is a constant B such that for all $j \geq 1$, $B(\alpha_j) \leq B$.

The map f_j has a univalent inverse branch $g_j : D\big(0, |v|\big) \to \mathbb{C}$ fixing 0 with derivative $e^{-2i\pi\alpha_j}$. According to a theorem of Yoccoz [Y], there is a constant C, which does not depend on j, such that the Siegel disk of g_j contains the disk centered at 0 with radius

$$|v|e^{-2\pi(B(\alpha_j)+C)} \geq |v|e^{-2\pi(B+C)}.$$

The lemma is proved with $M := B + C$. $\square$

Let us now show that for any $\varepsilon > 0$, for j large enough, C''_j is contained in the ε-neighborhood of Δ_α. Denote by $D''^{\sharp}_j$ the set of points in D''_j which are mapped by $\phi_j = \mathrm{Exp} \circ \Phi_{f_j}$ in $D\big(0, |v|e^{-2\pi M}\big)$ and set $D''^{\flat}_j := D''_j \setminus D''^{\sharp}_j$. In addition, set

$$C''^{\sharp}_j := \Psi_j\big(D''^{\sharp}_j\big) \quad \text{and} \quad C''^{\flat}_j := \Psi_j\big(D''^{\flat}_j\big).$$

Points in $D\big(0, |v|e^{-2\pi M}\big)$ have an infinite orbit under iteration of f_{j+1}. It follows that points in $D''^{\sharp}_j$ have an infinite orbit under iteration of f_j. Thus, the orbit of points in $C''^{\sharp}_j$ remains in $U_j(\alpha)$, thus is bounded. As a consequence, $C''^{\sharp}_j$ (which is open) is contained in the Fatou set of P_α, and since it contains 0 in its boundary, $C''^{\sharp}_j$ is contained in the Siegel disk of P_α.

So, in order to show that C''_j is contained in the ε-neighborhood of Δ_α, it is enough to show that $C''^{\flat}_j$ is contained in the ε-neighborhood of Δ_α. Note that $D''^{\flat}_j$ is the image of the rectangle

$$\big\{w \in \mathbb{C}\ ;\ N_j - 3 < \mathrm{Re}(w) < N_j - 1 \text{ and } -2 < \mathrm{Im}(w) \leq M\big\}$$

by the map $\Phi_{f_j}^{-1}$ which is univalent on the strip

$$\big\{w \in \mathbb{C}\ ;\ 0 < \mathrm{Re}(w) < 1/\alpha_j - R_3\big\}.$$

Since

$$1 < N_j - 3 < N_j < 1/\alpha_j - R_3,$$

the modulus of the annulus $\mathcal{P}_{f_j} \setminus \overline{D''^{\flat}_j}$ is bounded from below independently of j.

It follows from Koebe's distortion lemma that there is a constant K such that

$$\mathrm{diam}(C''^{\flat}_j) \leq K \cdot d(z_j, z'_j)$$

where

$$z_j := \Psi_j \circ \Phi_{f_j}^{-1}(N_j - 3) \quad \text{and} \quad z'_j := \Psi_j \circ \Phi_{f_j}^{-1}(N_j - 2).$$

According to Prop. 14,

$$z_j = P_\alpha^{\circ(N_j-3)q_j}(\omega_\alpha) \quad \text{and} \quad z'_j = P_\alpha^{\circ q_j}(z_j).$$

The boundary of P_α is a Jordan curve, and $P_\alpha : \partial\Delta_\alpha \to \partial\Delta_\alpha$ is conjugate to the rotation of angle α on $\mathbb{R}/\mathbb{Z}$. It follows that

$$\mathrm{diam}(C''^{\flat}_j) \leq K \cdot \max_{z \in \partial\Delta_\alpha} \big|P_\alpha^{\circ q_j}(z) - z\big|.$$

Without loss of generality, we may assume that $M \geq 2$. If $z \in U_j(\alpha)$, then there is a $k \leq q_{j+1} + \ell q_j$ such that $P_\alpha^{\circ k}(z) \in C''_j$. Then,

- either $P_\alpha^{\circ k}(z) \in C''^{\sharp}_j$ in which case $z \in \Delta_\alpha$,
- or $P_\alpha^{\circ k}(z) \in C''^{\flat}_j$ in which case z belongs to the connected component O_j^{-k} of $P_\alpha^{-k}(C''^{\flat}_j)$ intersecting Δ_α.

In the second case, O_j^{-k} contains two points z_j^{-k} and $z_j'^{-k}$ which are in the boundary of Δ_α and which are respectively mapped to z_j and z'_j by P_α^k. We have $z_j'^{-k} = P_\alpha^{\circ q_j}(z_j^{-k})$.

Note that since α is of bounded type, there is a constant A such that

$$\forall j \geq 1 \qquad q_{j+1} + \ell q_j \leq A \cdot q_j.$$

According to lemma 15 below, there is a constant K' such that for all $j \geq 1$ and all $k \leq q_{j+1} + \ell q_j$

$$\operatorname{diam}(O_j^{-k}) \leq K' \cdot \left|z_j'^{-k} - z_j^{-k}\right| \leq K' \cdot \max_{z \in \partial\Delta_\alpha} \left|P_\alpha^{\circ q_j}(z) - z\right|.$$

So, we see that

$$\sup_{z \in U_j(\alpha)} d(z, \Delta_\alpha) \leq \max(K, K') \cdot \max_{z \in \partial\Delta_\alpha} \left|P_\alpha^{\circ q_j}(z) - z\right| \underset{j \to +\infty}{\longrightarrow} 0.$$

This completes the proof of Prop. 18. $\square$

Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is of bounded type. If $z \in \partial\Delta_\alpha$, we set

$$r_j(z) = \left|P_\alpha^{\circ q_j}(z) - z\right|.$$

Lemma 15. *For all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ of bounded type, all $A \geq 1$ and all $K \geq 1$, there exists a K' such that the following holds. If $j \geq 1$, if $k \leq A \cdot q_j$, if $z_0 \in \partial\Delta_\alpha$, if $z_k = P_\alpha^{\circ k}(z_0)$ and if O is the connected component of $P_\alpha^{-k}\big(D(z_k, K \cdot r_j(z_k))\big)$ containing z_0, then*

$$\operatorname{diam}(O) \leq K' \cdot r_j(z_0).$$

Proof. The constants M_1, M_2 and m which will be introduced in the proof depend on α, A and K, but they do not depend on j, k or z.

Set

$$D := D\big(z_k, K \cdot r_j(z_k)\big) \quad \text{and} \quad \widehat{D} := D\big(z_k, 2K \cdot r_j(z_k)\big).$$

Since $\partial\Delta_\alpha$ is a quasicircle and since $P_\alpha : \partial\Delta_\alpha \to \partial\Delta_\alpha$ is conjugate to the rotation of angle α on $\mathbb{R}/\mathbb{Z}$, the number of critical values of $P_\alpha^{\circ k}$ in $\widehat{D}$ is bounded by a constant M_1 which only depends on α, A and K.

Let O (respectively $\widehat{O}$) be the connected component of $P_\alpha^{-k}(D)$ (respectively $P_\alpha^{-k}(\widehat{D})$) containing z_0. The degree of $P_\alpha^{\circ k} : \widehat{O} \to \widehat{D}$ is bounded by 2^{M_1}.

On the one hand, it easily follows from the Grötzsch inequality that the modulus of the annulus $\widehat{O} \setminus \overline{O}$ is bounded from below by $\log 2/(2\pi 2^M)$ (see for example [ShT] lemma 2.1).

On the other hand, it follows from Schwarz's lemma that the hyperbolic distance in $\widehat{O}$ between z_0 and $P_\alpha^{\circ q_j}(z_0)$ is greater than the hyperbolic distance in $\widehat{D}$ between z_k and $P_\alpha^{\circ q_j}(z_j)$, i.e. a constant m which only depends on α, A and K.

Lemma 15 now follows easily from the Koebe distortion lemma. $\square$

Note that for each fixed j, the set $U_j(\alpha)$ depends continuously on α as long as the first $j + 1$ approximants remain unchanged. Hence, given $\alpha \in \mathcal{S}_N$ and $\delta > 0$, if $\alpha' \in \mathrm{Irrat}_{\geq N}$ is sufficiently close to α (in particular, the first j entries of the continued fractions of α and α' coincide), then $\overline{U}_j(\alpha')$ is contained in the δ-neighborhood of $\overline{U}_j(\alpha)$. This completes the proof of Prop. 11.

1.6. **Lebesgue density near the boundary of a Siegel disk.**

Definition 10. *If α is a Brjuno number and if $\delta > 0$, we denote by Δ the Siegel disk of P_α and by $K(\delta)$ the set of points whose orbit under iteration of P_α remains at distance less than δ of Δ.*

Our proof will be based on the following theorem of Curtis T. McMullen [McM].

Theorem 4 (McMullen)**.** *Assume α is a bounded type irrational and $\delta > 0$. Then, every point $z \in \partial\Delta$ is a Lebesgue density point of $K(\delta)$.*

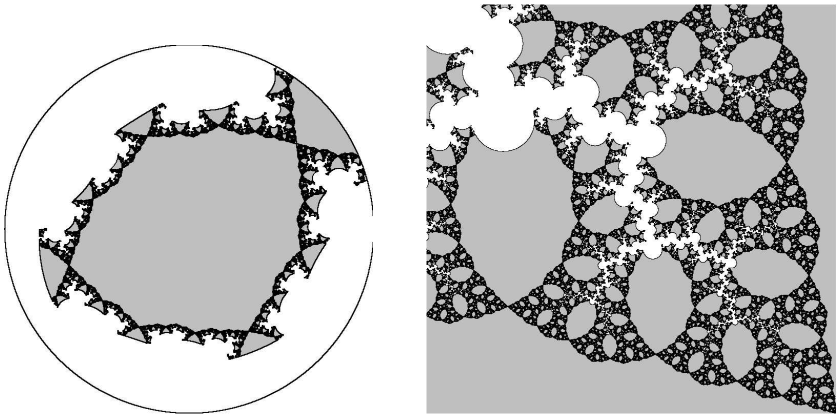


FIGURE 20. If $\alpha = (\sqrt{5}-1)/2$, the critical point of P_α is a Lebesgue density point of the set of points whose orbit remain in $D(0,1)$. Left: the set of points whose orbit remains in $D(0,1)$. Right: a zoom near the critical point.

Corollary 5. *Assume α is a bounded type irrational and $\delta > 0$. Then*

$$d := d(z, \partial\Delta) \to 0 \quad \text{with } z \notin \overline{\Delta} \quad \Longrightarrow \quad \mathrm{dens}_{D(z,d)}\big(\mathbb{C} \setminus K(\delta)\big) \to 0.$$

Proof. We proceed by contradiction. Assume we can find a sequence (z_j) such that
- $d_j := d(z_j, \partial\Delta) \to 0$ and
- $\rho_j := \mathrm{dens}_{D(z_j,d_j)}\big(\mathbb{C} \setminus K(\delta)\big) \not\to 0$.

Extracting a subsequence if necessary, we may assume that the sequence (z_j) converges to a point $z_0 \in \partial\Delta$ and that $\lim \rho_j = \rho > 0$.

Set $\eta := \rho/5$ and for $i \geq 1$, set

$$X_i := \big\{ w \in \partial\Delta \;\big|\; (\forall r \leq 1/i)\ \mathrm{dens}_{D(w,r)}\big(\mathbb{C} \setminus K(\delta)\big) \leq \eta \big\}.$$

The sets X_i are closed. By McMullen's Theo. 4, $\bigcup X_i = \partial\Delta$. By Baire category, one of these sets X_i contains an open subset W of $\partial\Delta$. Then, for all sequence of points $w_j \in W$ and all sequence of real number r_j converging to 0, we have

$$\limsup_{j \to +\infty} \mathrm{dens}_{D(w_j,r_j)}\big(\mathbb{C} \setminus K(\delta)\big) \leq \eta = \frac{\rho}{5}. \tag{4}$$

We claim that there is a map g defined and univalent in a neighborhood U of z_0, such that
- $g(z_0) = w_0 \in W$,

- $g\big(K(\delta)\cap U\big) = K(\delta)\cap g(U)$ and
- $g(\partial\Delta\cap U) = \partial\Delta\cap g(U)$.

Indeed, if z_0 is not precritical, we can find an integer $k\geq 0$ such that $f^{\circ k}(z_0)\in W$ and we let g be the restriction of $f^{\circ k}$ to a sufficiently small neighborhood of z_0. If z_0 is precritical, we can find a point $w_0\in W$ and an integer $k\geq 0$ such that $f^{\circ k}(w_0)=z_0$ and we let g coincide the restriction of the branch of f^{-k} sending z_0 to w_0, to a sufficiently small neighborhood of z_0.

Let $z'_j\in\partial\Delta$ be such that $|z_j - z'_j| = d_j$. Then, $z'_j \underset{j\to+\infty}{\longrightarrow} z_0$. Let j be sufficiently large so that $z'_j\in U$ and set $w_j := g(z'_j)$. On the one hand, $w_j \underset{j\to+\infty}{\longrightarrow} w_0$. Thus, $w_j\in W$ for j large enough. On the other hand,

$$\mathrm{dens}_{D(z'_j,2d_j)}\big(\mathbb{C}\setminus K(\delta)\big)\geq \frac{1}{4}\mathrm{dens}_{D(z_j,d_j)}\big(\mathbb{C}\setminus K(\delta)\big)$$

and so

$$\liminf_{j\to+\infty}\mathrm{dens}_{D(z'_j,2d_j)}\big(\mathbb{C}\setminus K(\delta)\big)\geq\frac{\rho}{4}.$$

Since g is holomorphic at z_0,

$$\liminf_{j\to+\infty}\mathrm{dens}_{D(w_j,r_j)}\big(\mathbb{C}\setminus K(\delta)\big)\geq\frac{\rho}{4}\quad\text{with}\quad r_j := \big|g'(w_0)\big|\cdot 2d_j \underset{j\to+\infty}{\longrightarrow} 0.$$

This contradicts (4). □

1.7. **The proof.** We will now prove Prop. 3. We let N be sufficiently large so that the conclusions of Prop. 11 and Cor. 4 apply. Assume $\alpha\in\mathcal{S}_N$ and choose a sequence (A_n) such that

$$\sqrt[q_n]{A_n}\underset{n\to+\infty}{\longrightarrow}+\infty\quad\text{and}\quad\sqrt[q_n]{\log A_n}\underset{n\to+\infty}{\longrightarrow}1.$$

Set

$$\alpha_n := [\mathrm{a}_0,\mathrm{a}_1,\ldots,\mathrm{a}_n,A_n,N,N,N,\ldots].$$

Note that since α is of bounded type, the Julia set J_α has zero Lebesgue measure (see [P]). Prop. 6 then easily implies that

$$\liminf \mathrm{area}(K_{\alpha_n})\geq\frac{1}{2}\mathrm{area}(K_\alpha).$$

Everything relies on our ability to promote the coefficient 1/2 to a coefficient 1.

Denote by K (resp. K_n) the filled-in Julia set of P_α (resp. P_{α_n}) and by Δ (resp. Δ_n) its Siegel disk. For $\delta>0$, set

$$\begin{aligned} V(\delta) &:= \big\{z\in\mathbb{C}\ \big|\ d(z,\partial\Delta)<\delta\big\},\\ K(\delta) &:= \big\{z\in V(\delta)\ \big|\ (\forall k\geq 0)\ P_\alpha^{\circ k}(z)\in V(\delta)\big\}\quad\text{and}\\ K_n(\delta) &:= \big\{z\in V(\delta)\ \big|\ (\forall k\geq 0)\ P_{\alpha_n}^{\circ k}(z)\in V(\delta)\big\}.\end{aligned}$$

Define $\rho_n :]0,+\infty[\ \to [0,1]$ by

$$\rho_n(\delta) := \mathrm{dens}_\Delta\big(\mathbb{C}\setminus K_n(\delta)\big).$$

Lemma 16. *For all $\delta>0$, there exist $\delta'>0$ (with $\delta'<\delta$) and a sequence $(c_n>0)$ converging to 0, such that*

$$\rho_n(\delta)\leq\frac{3}{4}\rho_n(\delta')+c_n.\text{[12]}$$

[12] The coefficient $\frac{3}{4}$ could have been replaced by any $\lambda>\frac{1}{2}$

This lemma enables us to complete the proof of Prop. 3 as follows. We set

$$\rho(\delta) := \limsup_{n\to+\infty} \rho_n(\delta) \quad (\leq 1).$$

Then, for all $\delta > 0$, there is a $\delta' > 0$ such that $\rho(\delta) \leq \frac{3}{4}\rho(\delta')$. Since ρ is bounded from above by 1, this implies that ρ identically vanishes. In other words

$$(\forall \delta > 0) \quad \mathrm{dens}_\Delta\big(K_n(\delta)\big) \underset{n\to+\infty}{\longrightarrow} 1. \tag{5}$$

Since $K_n(\delta) \subset K_{\alpha_n}$, we deduce that $\mathrm{dens}_\Delta(K_{\alpha_n}) \underset{n\to+\infty}{\longrightarrow} 1$. We know that

- P_{α_n} converges locally uniformly to P_α,
- the orbit under iteration of P_α of any point in $K_\alpha \setminus J_\alpha$ eventually lands in Δ and
- $P_{\alpha_n}^{-1}(K_{\alpha_n}) = K_{\alpha_n}$.

It follows that $\mathrm{dens}_{K_\alpha\setminus J_\alpha}(K_{\alpha_n}) \underset{n\to+\infty}{\longrightarrow} 1$. Since the Julia set J_α has Lebesgue measure zero, this implies that $\liminf \mathrm{area}(K_{\alpha_n}) \geq \mathrm{area}(K_\alpha)$. This completes the proof of Prop. 3 up to Lemma 16.

Proof of Lemma 16. Let us sum up what we obtained in sections 1.4, 1.5 and 1.6.

(A) For all open set $U \subset \Delta$ and all $\delta > 0$, $\liminf_{n\to+\infty} \mathrm{dens}_U\big(K_n(\delta)\big) \geq \dfrac{1}{2}$. This is an immediate consequence of Prop. 6 in section 1.4.
(B) For all $\delta > 0$, if n is sufficiently large, the post-critical set of P_{α_n} is contained in $V(\delta)$. This is just a restatement of Cor. 4 in section 1.5.
(C) For all $\eta > 0$ and all $\delta > 0$, there exists $\delta_0' > 0$ such that if $\delta' < \delta_0'$ and if $z \in \overline{V(8\delta')}\setminus V(2\delta')$, then $\mathrm{dens}_{D(z,\delta')}\big(\mathbb{C}\setminus K(\delta)\big) < \eta$. This is an easy consequence of Cor. 5 in section 1.6.

Step 1. By Koebe distortion theorem, there exists a constant κ such that for all map $\phi : D := D(a,r) \to \mathbb{C}$ which extends univalently to $D(a, 3r/2)$, we have

$$\sup_D |\phi'| \leq \kappa \inf_D |\phi'|.$$

We choose $\eta > 0$ such that

$$8\pi\kappa^2\eta < \frac{1}{4}.$$

Step 2. Fix $\delta > 0$. We claim that there exists $\delta' > 0$ such that:
(i) $9\delta' < \delta$ and $(2 + 3\kappa)\cdot\delta' < \delta$,[13]
(ii) if $d(z,\Delta) < 2\delta'$, then $d\big(P_\alpha(z),\Delta\big) < 8\delta'$ and
(iii) if $z \in \overline{V(8\delta')} \setminus V(2\delta')$, then $\mathrm{dens}_{D(z,\delta')}\big(\mathbb{C}\setminus K(\delta)\big) < \eta$.

Indeed, it is well-known that for $\alpha \in \mathbb{R}$, $\left|P_\alpha'\right| < 4$ on K_α. As a consequence, if $\delta' > 0$ is sufficiently small, then $\left|P_\alpha'\right| < 4$ on $V(2\delta')$. It follows that (ii) holds for $\delta' > 0$ sufficiently small. Claim (iii) follows from the aforementioned point (C).

From now on, we assume that δ' is chosen so that the above claims hold and we set

$$W := \overline{V(8\delta')} \setminus V(2\delta').$$

[13] Those requirements will be used in step 9.

Step 3. Set

$$Y^\ell := \big\{z \in K(\delta) \bigm| P_\alpha^{\circ\ell}(z) \in \Delta\big\}.$$

The set of points in $K(\delta)$ whose orbits do not intersect Δ, is contained in the Julia set of P_α. This set has zero Lebesgue measure. Thus, $K(\delta)$ and $\bigcup Y^\ell$ coincide up to a set of zero Lebesgue measure. The sequence $(Y^\ell)_{\ell\geq 0}$ is increasing. From now on, we assume that ℓ is sufficiently large so that

$$\big(\forall w \in W\big) \quad \mathrm{dens}_{D(w,\delta')}(\mathbb{C} \setminus Y^\ell) < \eta.$$

Step 4. Assume ϕ is univalent on $D(w, 3\delta'/2)$ with $w \in W$, r is the radius of the largest disk centered at $\phi(w)$ and contained in $\phi\big(D(w,\delta')\big)$ and Q is a square contained in $\phi\big(D(w,\delta')\big)$ with side length at least $r/\sqrt{8}$. Set $D := D(w,\delta')$. Then, $r \geq \inf_D |\phi'| \cdot \delta'$ and thus,

$$\mathrm{area}(Q) \geq \inf_D |\phi'|^2 \cdot \frac{(\delta')^2}{8}.$$

In addition, $\sup\limits_D |\phi'| \leq \kappa \inf\limits_D |\phi'|$ and so,

$$\mathrm{dens}_Q\big(\mathbb{C} \setminus \phi(Y^\ell)\big) \leq \frac{\mathrm{area}\big(\phi(D \setminus Y^\ell)\big)}{\mathrm{area}(Q)} \leq \frac{\sup\limits_D |\phi'|^2 \cdot \pi(\delta')^2 \cdot \eta}{\inf\limits_D |\phi'|^2 \cdot (\delta')^2/8} \leq 8\pi\kappa^2\eta < \frac{1}{4}.$$

As a consequence,

$$\mathrm{dens}_Q\big(\phi(Y^\ell)\big) > \frac{3}{4}.$$

Step 5. If $X \subset \mathbb{C}$ is a measurable set, we use the notation $m|_X$ for the Lebesgue measure on X, extended by 0 outside X. If $U \subset \mathbb{C}$ is an open set, (X_n) is a sequence of measurable subsets of $\mathbb{C}$ and $\lambda \in [0,1]$, we say that

$$\liminf_{n\to+\infty} m|_{X_n} \geq \lambda \cdot m|_U$$

if for all non empty open set U' relatively compact in U, we have

$$\liminf_{n\to+\infty} \mathrm{dens}_{U'}(X_n) \geq \lambda.\text{[14]}$$

Assume $f : V \to U$ is a holomorphic map, nowhere locally constant, and $(f_n : V_n \to \mathbb{C})$ is a sequence of holomorphic maps such that

- every compact subset of V is eventually contained in V_n and
- the sequence (f_n) converges uniformly to f on every compact subset of V.

Then,

$$\liminf_{n\to+\infty} m|_{X_n} \geq \lambda \cdot m|_U \quad \Longrightarrow \quad \liminf_{n\to+\infty} m|_{f_n^{-1}(X_n)} \geq \lambda \cdot m|_V.$$

Step 6. Set

$$Y_n^\ell := \big\{z \in V(\delta) \bigm| (\forall j \leq \ell)\ P_{\alpha_n}^{\circ j}(z) \in V(\delta) \text{ and } P_{\alpha_n}^{\circ\ell}(z) \in \Delta\big\}.$$

On the one hand, if $z \in Y_n^\ell$ and $P_{\alpha_n}^\ell(z) \in K_n(\delta)$, then $z \in K_n(\delta)$. On the other hand, every compact subset of Y^ℓ is eventually contained in Y_n^ℓ and the sequence

[14]Equivalently, for all non empty open set $U' \subset \mathbb{C}$ with finite area, $\liminf\limits_{n\to+\infty} \mathrm{dens}_{U'}(X_n) \geq \lambda \cdot \mathrm{dens}_{U'}(U)$.

$(P_{\alpha_n}^{\circ\ell})$ converges uniformly to P_α^ℓ on every compact subset of Y^ℓ. By the aforementioned point (A), we have

$$\liminf_{n\to+\infty} m|_{K_n(\delta)} \geq \frac{1}{2} m|_\Delta.$$

So, according to step 5,

$$\liminf_{n\to+\infty} m|_{K_n(\delta)} \geq \frac{1}{2} m|_{Y^\ell}.$$

Step 7. Assume ϕ_n is univalent on $D(w_n, 3\delta'/2)$ with $w_n \in W$, r_n is the radius of the largest disk centered at $\phi_n(w_n)$ and contained in $\phi_n\big(D(w_n,\delta')\big)$ and Q_n is a square contained in $\phi_n\big(D(w_n,\delta')\big)$ with side length at least $r_n/\sqrt{8}$. Then,

$$\liminf_{n\to+\infty} \mathrm{dens}_{Q_n}\big(\phi_n\big(K_n(\delta)\big)\big) \geq \frac{3}{8}.$$

Indeed, assume λ is a limit value of the sequence

$$\mathrm{dens}_{Q_n}\big(\phi_n\big(K_n(\delta)\big)\big).$$

Post-composing the maps ϕ_n with affine maps and extracting a subsequence if necessary, we may assume that (w_n) converges to $w \in W$, (ϕ_n) converges locally uniformly to $\phi : D(w, 3\delta'/2) \to \mathbb{C}$, r_n converges to the radius r of the largest disk centered at $\phi(w)$ and contained in $\phi\big(D(w,\delta')\big)$ and Q_n converges to a square Q with side length at least $r/\sqrt{8}$. According to steps 5 and 6,

$$\liminf_{n\to+\infty} m|_{\phi_n(K_n(\delta))} \geq \frac{1}{2} m|_{\phi(Y^\ell)}.$$

According to step 4, it follows that

$$\lambda \geq \frac{1}{2}\mathrm{dens}_Q\big(\phi(Y^\ell)\big) \geq \frac{3}{8}.$$

Step 8. From now on, we assume that n is sufficiently large, so that:

(i) $\Delta \setminus K_n(\delta) \subset X_n \subset \Delta \setminus K_n(\delta')$ with

$$X_n := \big\{z \in \Delta \;\big|\; (\exists k)\ P_{\alpha_n}^{\circ k}(z) \in W\big\}$$

(this is possible by step 2);

(ii) $s_n < \delta'$ with

$$s_n := \sup_{z\in\Delta} d\big(z, K_n(\delta')\big)$$

(this is possible since $s_n \underset{n\to+\infty}{\longrightarrow} 0$ in order for the aforementioned point (A) to hold);

(iii) the post-critical set of P_{α_n} is contained in $V(\delta'/2)$ (this is possible by the aforementioned point (B));

(iv) if ϕ is univalent on $D(w, 3\delta'/2)$ with $w \in W$, if r is the radius of the largest disk centered at $\phi(w)$ and contained in $\phi\big(D(w,\delta')\big)$ and if Q is a square contained in $\phi\big(D(w,\delta')\big)$ with side length at least $r/\sqrt{8}$, then

$$\mathrm{dens}_Q\big(\phi\big(K_n(\delta)\big)\big) \geq \frac{1}{4}$$

(this is easily follows from step 7 by contradiction).

Step 9. Assume $z_0 \in X_n$. Then, we have

$$z_0 \in X_n \overset{P_{\alpha_n}}{\mapsto} z_1 \in V(2\delta') \overset{P_{\alpha_n}}{\mapsto} \cdots \overset{P_{\alpha_n}}{\mapsto} z_{k-1} \in V(2\delta') \overset{P_{\alpha_n}}{\mapsto} z_k \in W$$

for some integer $k > 0$. Since the post-critical set of P_{α_n} is contained in $V(\delta'/2)$, for $j \leq k$ there exists a univalent map $\phi_j : D := D(z_k, \delta') \to \mathbb{C}$ such that

- ϕ_j is the inverse branch of $P_{\alpha_n}^{\circ k-j}$ which maps z_k to z_j and
- ϕ_j extends univalently to $D(z_k, 3\delta'/2)$.

In particular,

$$\sup_D |\phi_j'| \leq \kappa \inf_D |\phi_j'|.$$

Let $D(z_j, r_j)$ be the largest disk centered at z_j and contained in $\phi_j(D)$ and $D(z_j, R_j)$ be the smallest disk centered at z_j and containing $\phi_j(D)$. Note that D is contained in $\mathbb{C} \setminus V(\delta')$ and so, for $j \leq k-1$, $D(z_j, r_j) \subset \phi_j(D) \subset \mathbb{C} \setminus K_n(\delta')$. On the one hand, $d(z_j, \Delta) < 2\delta'$ and on the other hand, every point of Δ is at distance at most s_n from a point of $K_n(\delta')$. It follows that

$$R_j \leq \kappa r_j \leq \kappa \cdot (s_n + 2\delta').$$

If $w_0 \in \phi_0(D)$ and $w_j := P_{\alpha_n}^{\circ j}(w_0)$, then for $j \leq k-1$,

$$d(w_j, \Delta) \leq d(w_j, z_j) + d(z_j, \Delta) \leq \kappa \cdot (s_n + 2\delta') + 2\delta' < (2 + 3\kappa) \cdot \delta' < \delta$$

and for $j = k$,

$$d(w_k, \Delta) \leq d(w_k, z_k) + d(z_k, \Delta) \leq 9\delta' < \delta.$$

In other words, w_0, w_1, …, w_k all belong to $V(\delta)$. As a consequence,

$$\phi_0\big(K_n(\delta)\big) \subset K_n(\delta).$$

Step 10. Continuing with the notations of step 9, we denote by Q_{z_0} the largest douadic square containing z_0 and contained in $D(z_0, r_0)$. On the one hand, since $z_0 \in \Delta$ and since $\phi_0(D) \subset \mathbb{C} \setminus K_n(\delta')$, we have $r_0 \leq s_n$, and so

$$Q_{z_0} \subset D(z_0, r_0) \subset V(s_n) \setminus K_n(\delta').$$

On the other hand, Q_{z_0} has an edge of length greater than $r_0/2\sqrt{2}$ and so, according to step 8 point (iv),

$$\mathrm{dens}_{Q_{z_0}}\big(K_n(\delta)\big) > \frac{1}{4}.$$

As a consequence

$$\mathrm{dens}_{Q_{z_0}}\big(\mathbb{C} \setminus K_n(\delta)\big) < \frac{3}{4}.$$

Given two douadic squares Q and Q', either $Q \cap Q' = \emptyset$, or $Q \subset Q'$ or $Q' \subset Q$. It follows that

$$\begin{aligned}
\mathrm{area}\big(\Delta \setminus K_n(\delta)\big) &\leq \frac{3}{4}\mathrm{area}\left(\bigcup_{z \in X_n} Q_z\right) \\
&\leq \frac{3}{4}\mathrm{area}\big(V(s_n) \setminus K_n(\delta')\big) \\
&\leq \frac{3}{4}\mathrm{area}\big(\Delta \setminus K_n(\delta')\big) + \frac{3}{4}\mathrm{area}\big(V(s_n) \setminus \Delta\big) \\
&= \frac{3}{4}\mathrm{area}\big(\Delta \setminus K_n(\delta')\big) + c_n \cdot \mathrm{area}(\Delta)
\end{aligned}$$

with

$$c_n := \frac{3}{4}\frac{\operatorname{area}\bigl(V(s_n)\setminus\Delta\bigr)}{\operatorname{area}(\Delta)}.$$

Step 11. Since $s_n \to 0$ and since the boundary of Δ has zero Lebesgue measure,

$$\operatorname{area}\bigl(V(s_n)\setminus\Delta\bigr) \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus,

$$\operatorname{dens}_\Delta\bigl(\mathbb{C}\setminus K_n(\delta)\bigr) < \frac{3}{4}\operatorname{dens}_\Delta\bigl(\mathbb{C}\setminus K_n(\delta')\bigr) + c_n \quad \text{with} \quad c_n \underset{n\to+\infty}{\longrightarrow} 0.$$

This completes the proof of Lemma 16. □

2. The linearizable case

In order to find a quadratic polynomial with a linearizable fixed point and a Julia set of positive area, we need to modify the argument.

Definition 11. *If α is a Brjuno number, we denote by Δ_α the Siegel disk of P_α and by r_α its conformal radius. For $\rho \le r_\alpha$, we denote by $\Delta_\alpha(\rho)$ the invariant sub-disk with conformal radius ρ and by $L_\alpha(\rho)$ the set of points in K_α whose orbits do not intersect $\Delta_\alpha(\rho)$.*

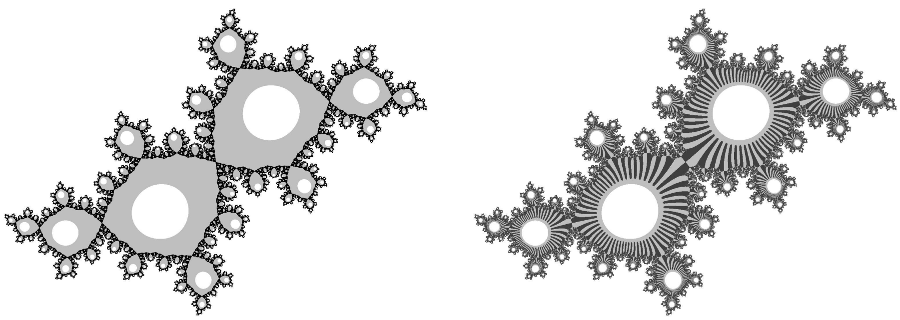


Figure 21. Two sets $L_\alpha\rho)$ and $L_{\alpha'}(\rho)$, with α' a well-chosen perturbation of α as in Prop. 19. This proposition asserts that if α and α' are chosen carefully enough, the loss of measure from $L_\alpha(\rho)$ to $L_{\alpha'}(\rho)$ is small. We colored white the basin of infinity, the invariant subdisks $\Delta_\alpha(\rho)$ and $\Delta_{\alpha'}(\rho)$ and their preimages; we colored light grey the remaining parts of the Siegel disks and their preimages; we colored dark grey the pixels where the preimages are too small to be drawn.

Proposition 19. *There exists a set $\mathcal{S}$ of bounded type irrationals such that for all $\alpha \in \mathcal{S}$, all $\rho < \rho' < r_\alpha$ and all $\varepsilon > 0$, there exists $\alpha' \in \mathcal{S}$ with*

- $|\alpha' - \alpha| < \varepsilon$,
- $\max\bigl(\rho, (1-\varepsilon)\rho'\bigr) < r_{\alpha'} < (1+\varepsilon)\rho'$ *and*
- $\operatorname{area}\bigl(L_{\alpha'}(\rho)\bigr) \ge (1-\varepsilon)\operatorname{area}\bigl(L_\alpha(\rho)\bigr)$.

Proof. We let N be sufficiently large so that the conclusions of Prop. 11 and Cor. 4 apply. We will work with $\mathcal{S} = \mathcal{S}_N$. Assume $\alpha \in \mathcal{S}_N$ and choose a sequence (A_n) such that

$$\lim_{n\to+\infty} \sqrt[q_n]{A_n} = \frac{r_\alpha}{\rho'}.$$

Set

$$\alpha_n := [\mathrm{a}_0, \mathrm{a}_1, \ldots, \mathrm{a}_n, A_n, N, N, N, \ldots].$$

This guaranties that $r_{\alpha_n} \underset{n\to+\infty}{\longrightarrow} \rho'$ (see [ABC]).

Let Δ be the Siegel disk of P_α. Let us use the notations $V(\delta)$, $K(\delta)$ and $K_n(\delta)$ introduced in section 1.7. With an abuse of notations, set $\Delta(\rho) := \Delta_\alpha(\rho)$ and $\Delta_n(\rho) := \Delta_{\alpha_n}(\rho)$. Set

$$\Delta'(\rho) := P_\alpha^{-1}\big(\Delta(\rho)\big) \setminus \Delta(\rho).$$

Then, $\Delta(\rho)$ and $\Delta'(\rho)$ are symmetric with respect to the critical point of P_α. The orbit under iteration of P_α of a point $z \notin \Delta(\rho)$ lands in $\Delta(\rho)$ if and only if it passes through $\Delta'(\rho)$. We have a similar property for

$$\Delta'_n(\rho) := P_{\alpha_n}^{-1}\big(\Delta_n(\rho)\big) \setminus \Delta_n(\rho).$$

We have proved – see equation (5) – that

$$(\forall \delta > 0) \quad \mathrm{dens}_\Delta\big(K_n(\delta)\big) \underset{n\to+\infty}{\longrightarrow} 1.$$

The sequence of compact sets $\big(\overline{\Delta}_n(\rho)\big)$ converges to $\overline{\Delta}(\rho)$ for the Hausdorff topology on compact subsets of $\mathbb{C}$, because $\lim r_{\alpha_n} > \rho$. It immediately follows that for all $\delta > 0$,

$$\mathrm{dens}_{\Delta\setminus\overline{\Delta}(\rho)}\big(K_n(\delta) \setminus \Delta_n(\rho)\big) \underset{n\to+\infty}{\longrightarrow} 1.$$

Choose δ sufficiently small so that $V(\delta)$ does not intersect $\overline{\Delta}'(\rho)$. Then, for n large enough $V(\delta)$ does not intersect $\overline{\Delta}'_n(\rho)$. In that case, the orbit under iteration of P_{α_n} of a point in $K_n(\delta) \setminus \Delta_n(\rho)$ cannot pass through $\Delta'_n(\rho)$ and so,

$$K_n(\delta) \setminus \Delta_n(\rho) \subset L_{\alpha_n}(\rho).$$

Thus,

$$\mathrm{dens}_{\Delta\setminus\overline{\Delta}(\rho)}\big(L_{\alpha_n}(\rho)\big) \underset{n\to+\infty}{\longrightarrow} 1.$$

The points of $L_\alpha(\rho)$ whose orbits do not intersect $\Delta \setminus \overline{\Delta}(\rho)$ are contained in the union of the Julia set J_α and the countably many preimages of $\partial\Delta(\rho)$. Thus, they form a set of zero Lebesgue measure. It follows that

$$\mathrm{area}\big(L_{\alpha_n}(\rho)\big) \underset{n\to+\infty}{\longrightarrow} \mathrm{area}\big(L_\alpha(\rho)\big).$$

□

Proof of Theo. 2. We start with $\alpha_0 \in \mathcal{S}$ and set $\rho_0 := r_{\alpha_0}$. We then choose $\rho \in \,]0, \rho_0[$ and two sequences of real numbers ε_n in $(0,1)$ and ρ_n in $(0, \rho_0)$ such that $\prod(1-\varepsilon_n) > 0$ and $\rho_n \searrow \rho > 0$. We can construct inductively a Cauchy sequence $(\alpha_n \in \mathcal{S})$ such that for all $n \geq 1$,

- $r_{\alpha_n} \in (\rho_n, \rho_{n-1})$ and
- $\mathrm{area}\big(L_{\alpha_n}(\rho)\big) \geq (1-\varepsilon_n)\mathrm{area}\big(L_{\alpha_{n-1}}(\rho)\big)$.

Let α be the limit of the sequence (α_n). The conformal radius of a fixed Siegel disk depends upper semi-continuously on the polynomial (a limit of linearizations linearizes the limit). So, $r_\alpha \geq \lim r_{\alpha_n} = \rho$. Also, by choosing α_n sufficiently close to α_{n-1} at each step, we can guaranty that $r_\alpha \leq \rho$, in which case $r_\alpha = \rho$.

In addition, the sequence of pointed domains $\big(\Delta_{\alpha_n}(\rho), 0\big)$ converges for the Carathéodory topology to $(\Delta_\alpha, 0)$. In particular, every compact subset of Δ_α is contained in $\Delta_{\alpha_n}(\rho)$ for n large enough. Similarly, every compact subset of $\mathbb{C} \setminus K_\alpha$ is contained in $\mathbb{C} \setminus K_{\alpha_n}$ for n large enough. It follows that

$$\limsup L_{\alpha_n}(\rho) := \bigcap_m \overline{\bigcup_{n \geq m} L_{\alpha_n}(\rho)} \subset L_\alpha(\rho).$$

Since $r_\alpha = \rho$, $\Delta_\alpha(\rho) = \Delta_\alpha$ and $L_\alpha(\rho) = J_\alpha$. Thus, $\limsup L_{\alpha_n}(\rho) \subset J_\alpha$ and

$$\text{area}(J_\alpha) \geq \text{area}(\limsup L_{\alpha_n}(\rho)) \geq \text{area}\big(L_{\alpha_0}(\rho)\big) \cdot \prod (1 - \varepsilon_n) > 0.$$

□

3. The infinitely renormalizable case

In order to find an infinitely renormalizable quadratic polynomial with a Julia set of positive area, we need a modification based on Sørensen's construction of an infinitely renormalizable quadratic polynomial with a non-locally connected Julia set.

Proposition 20. *There exists a set $\mathcal{S}$ of bounded type irrationals such that for all $\alpha \in \mathcal{S}$ and all $\varepsilon > 0$, there exists $\alpha' \in \mathbb{C} \setminus \mathbb{R}$ with*

- $|\alpha' - \alpha| < \varepsilon$,
- *$P_{\alpha'}$ has a periodic Siegel disk with period > 1 and rotation number in $\mathcal{S}$ and*
- $\text{area}(K_{\alpha'}) \geq (1 - \varepsilon)\text{area}(K_\alpha)$.

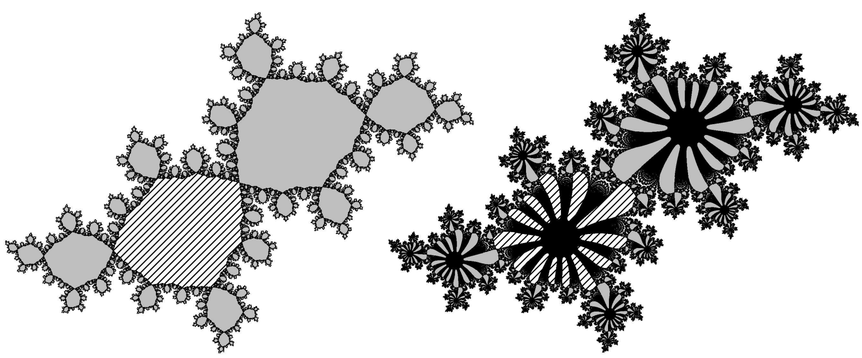


FIGURE 22. Two filled-in Julia sets K_α and $K_{\alpha'}$, with α' a well-chosen perturbation of α as in Prop. 20. This proposition asserts that if α and α' are chosen carefully enough, $P_{\alpha'}$ has a periodic Siegel disk and the loss of measure from K_α to $K_{\alpha'}$ is small. Left: we hatched the fixed Siegel disk. Right: we hatched the cycle of Siegel disks.

Proof. We can choose $\mathcal{S} = \mathcal{S}_N$ with N large enough (in order to be able to apply Inou and Shishikura techniques). The proof essentially goes as in the Cremer case

Given $\alpha \in \mathcal{S}$, we let p_k/q_k be its approximants, and we consider the functions of explosion χ_k given by Prop. 4. If α' belongs to the disk centered at p_k/q_k with radius $1/q_k^3$, the set

$$\mathcal{C}_k(\alpha') := \chi_k\left\{\sqrt[q_k]{\alpha_k - p_k/q_k}\right\}$$

is a cycle of $P_{\alpha'}$. Its multiplier is $e^{2i\pi\theta_k(\alpha')}$ with $\theta_k : D(p_k/q_k, 1/q_k^3) \to \mathbb{C}$ a non-constant holomorphic function which vanishes at p_k/q_k.

We consider a sequence (α_n) converging to α so that

- $\limsup\limits_{n\to+\infty} \sqrt[q_n]{\alpha_n - p_n/q_n} = +\infty$ and
- $\theta_n(\alpha_n) := [A_n, N, N, N, \ldots]$ with
$$\lim_{n\to+\infty} \sqrt[q_n]{A_n} = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \sqrt[q_n]{\log A_n} = 1.$$

We control the shape of the cycle of Siegel disk as in the Cremer case. For all $\rho < 1$ and all n sufficiently large, the cycle of Siegel disks contains the $\chi_n\bigl(Y_n(\rho)\bigr)$ with

$$Y_n(\rho) := \left\{z \in \mathbb{C} \ ; \ \frac{z^{q_n} - \varepsilon_n}{z^{q_n}} \in D(0, s_n)\right\} \quad \text{with} \quad s_n := \frac{\rho^{q_n} - |\varepsilon_n|}{\rho^{q_n}}.$$

For this purpose, we work in the coordinate given by χ_n and compare the dynamics of the conjugated map to the flow of a vector field.

We control the post-critical set of P_{α_n} via Inou-Shishikura's techniques.

We then control the loss of area as in the Cremer case. □

Definition 12. *For $c \in \mathbb{C}$, we denote by Q_c the quadratic polynomial $Q_c : z \mapsto z^2 + c$. With an abuse of notations, we denote by K_c its filled-in Julia set and by J_c its Julia set. We denote by M the Mandelbrot set, i.e. the set of parameters c for which K_c is connected.*

The previous proposition can be restated as follows.

Proposition 21. *Assume P_c has a fixed Siegel disk with rotation number in $\mathcal{S}$. Then, for all $\varepsilon > 0$, there exists c' such that*

- $|c' - c| < \varepsilon$,
- *$P_{c'}$ has a periodic Siegel disk with period > 1 and rotation number in $\mathcal{S}$ and*
- $\text{area}(K_{c'}) > (1 - \varepsilon)\text{area}(K_c)$.

In fact, such a c is on the boundary of the main cardioid of M and the proof we proposed yields a c' which is on the boundary of a satellite component of the main cardioid of M.

Using the theory of quadratic-like maps introduced by Douady and Hubbard [DH2], we can transfer this statement to perturbations of quadratic polynomials having periodic Siegel disks. We will use the notions of renormalization and tuning (see for example [Ha]).

If 0 is periodic of period p under iteration of Q_{c_0}, then c_0 is the center of a hyperbolic component Ω of the Mandelbrot set. This component Ω has a root: the parameter $c_1 \in \partial\Omega$ such that Q_{c_1} has an indifferent cycle with multiplier 1. In addition, there exist

- a compact set $M' \subset M$ such that $\partial M' \subset \partial M$,

- a simply connected neighborhood Λ of $M' \setminus \{c_1\}$,
- a continuous map $\chi : \Lambda \cup \{c_1\} \to \mathbb{C}$ and
- two families of open sets $(U'_\lambda)_{\lambda\in\Lambda}$ and $(U_\lambda)_{\lambda\in\Lambda}$,

such that

- $\big(f_\lambda := Q_\lambda^{\circ p} : U'_\lambda \to U_\lambda\big)_{\lambda\in\Lambda}$ is an analytic family of quadratic-like maps
- for all $\lambda \in \Lambda$, f_λ is hybrid conjugate to $Q_{\chi(\lambda)}$,
- the Julia set of f_λ is connected if and only if $\lambda \in M'$ and
- $\chi : M' \to M$ is a homeomorphism (sending c_0 to 0 and c_1 to 1/4).

We denote by $c_0 \perp \cdot : M \to M'$ the homeomorphism $(\chi|_{M'})^{-1}$. We say that $c_0 \perp c$ is c_0 is tuned by c and that $\big(f_\lambda := Q_\lambda^{\circ p} : U'_\lambda \to U_\lambda\big)_{\lambda\in\Lambda}$ is a Mandelbrot-like family centered at c_0.

Proposition 22. *Assume* 0 *is periodic under iteration of* Q_{c_0} *and* $c' \in M \to c \in M$ *with* $\mathrm{area}(K_{c'}) \to \mathrm{area}(K_c)$. *Then*

$$\mathrm{area}(K_{c_0\perp c'}) \to \mathrm{area}(K_{c_0\perp c}).$$

Proof. Let p be the period of 0 under iteration of Q_{c_0} and let $\big(f_\lambda := Q_\lambda^{\circ p} : U'_\lambda \to U_\lambda\big)_{\lambda\in\Lambda}$ be a Mandelbrot-like family centered at c_0.

Let $\phi_{c'} : U_{c_0\perp c'} \to \mathbb{C}$ be hybrid conjugacies. As $c' \to c$, the modulus of the annulus $U_{c_0\perp c'} \setminus \overline{U}'_{c_0\perp c'}$ is bounded from below. So, the $\phi_{c'}$ can be chosen to have a uniformly bounded quasiconformal dilatation. It follows that if $c' \in M \to c \in M$ with $\mathrm{area}(K_{c'}) \to \mathrm{area}(K_c)$, we have

$$\mathrm{area}\left(\phi_{c'}^{-1}(K_{c'})\right) \underset{c'\to c}{\longrightarrow} \mathrm{area}\left(\phi_c^{-1}(K_c)\right).$$

It follows easily that $\mathrm{area}(K_{c_0\perp c'}) \to \mathrm{area}(K_{c_0\perp c})$ since almost every point in $K_{c_0\perp c}$ has an orbit terminating in $\phi_c^{-1}(K_c)$. □

Proof of Theo. 3. If P_c has a periodic Siegel disk then c is on the boundary of a hyperbolic component with center c_0. We denote by Ω_c this hyperbolic component and we set $M_c := c_0 \perp M$.

We will denote by S the image of $\mathcal{S}$ by the map $\alpha \mapsto e^{2i\pi\alpha}/2 - e^{4i\pi\alpha}/4$. Then, $c \in S$ if and only if P_c has a fixed Siegel disk with rotation number in $\mathcal{S}$. Moreover, P_c has a periodic Siegel disk with rotation number in $\mathcal{S}$ if and only if $c = c_0 \perp s$ with c_0 the center of the hyperbolic component containing c in its boundary and $s \in S$.

It follows from Prop. 21 and 22 that if Q_c has a periodic Siegel disk with rotation number in $\mathcal{S}$, then for all $\varepsilon > 0$, we can find $c' \in M_c \setminus \overline{\Omega}_c$ such that

- $|c' - c| < \varepsilon$,
- $P_{c'}$ has a periodic Siegel disk with rotation number in $\mathcal{S}$ and
- $\mathrm{area}(K_{c'}) > (1-\varepsilon)\mathrm{area}(K_c)$.

Let us choose a parameter $c_0 \in S$ and a sequence of real number ε_n in $(0,1)$ such that $\prod(1-\varepsilon_n) > 0$. We can construct inductively a sequence (c_n) such that

- (c_n) is a Cauchy sequence that converges to a parameter c,
- Q_{c_n} has a periodic Siegel disk with rotation number in $\mathcal{S}$,
- for $n \geq 1$, $c_n \in M_{c_{n-1}} \setminus \overline{\Omega}_{c_{n-1}}$ and
- $\mathrm{area}(K_{c_n}) > (1-\varepsilon_n)\mathrm{area}(K_{c_{n-1}})$.

Then, P_c is infinitely renormalizable (it is in the intersection of the nested copies M_{c_n}). Thus, $J_c = K_c = \lim K_{c_n}$. Finally,

$$\mathrm{area}(J_c) = \mathrm{area}(K_c) \geq \mathrm{area}(K_{c_0}) \cdot \prod (1 - \varepsilon_n) > 0.$$

□

Appendix A. Parabolic implosion and perturbed petals

The notations used in this appendix are those of section 1.5.3. We postponed the proof of the following lemma to this appendix.

Lemma 17. *If $R > 0$ and $K > 0$ are sufficiently large, then for n large enough:*

(1) $\Phi^n(\Omega^n)$ contains the vertical strip

$$U^n := \big\{ w \in \mathbb{C} \; ; \; R < \mathrm{Re}(w) < 1/\alpha_n - R \big\},$$

(2) τ_n is injective on $\mathcal{P}^n := (\Phi^n)^{-1}(U^n)$ and

(3) there is a branch of argument defined on $\tau_n(\mathcal{P}^n)$ such that

$$\sup_{z \in \tau_n(\mathcal{P}^n)} \arg(z) - \inf_{z \in \tau_n(\mathcal{P}^n)} \arg(z) < K.$$

Proof. As in [Sh2], the argument consists in comparing the Fatou coordinate Φ^n to the Fatou coordinate Ψ^n of the time one map of the vector field ζ_n defined on $\mathcal{D}_n$ by

$$\zeta_n = \zeta_n(w) \frac{\partial}{\partial w} := \big(F_n(w) - w \big) \frac{\partial}{\partial w}.$$

In other words, set $w_n := \dfrac{1}{2\alpha_n}$ and let $\Psi^n : \Omega_n \to \mathbb{C}$ be defined by

$$\Psi^n(w) = \Phi^n(w_n) + \int_{w_n}^{w} \frac{du}{F_n(u) - u}.$$

Claim 1. Increasing R_1 if necessary, there is a constant $C > 0$ such that for all n sufficiently large

$$\sup_{w \in \Omega^n} \big| \Phi^n(w) - \Psi^n(w) \big| < C.$$

Proof of Claim 1. According to Prop. 2.6.2 in [Sh2], there are constants R and C such that for all sufficiently large n and for all $w \in \Omega^n$ with $d(w, \partial\Omega^n) \geq R$, we have

$$\big| (\Phi^n)'(w) - (\Psi^n)'(w) \big| \leq C \left(\frac{1}{d(w, \partial\Omega^n)^2} + \big| F_n'(w) - 1 \big| \right).$$

We will first show that we can get rid of $\big|F_n'(w) - 1\big|$. Set

$$G_n(w) := F_n'(w) - 1 \quad \text{and} \quad S_n(w) := \left(\frac{\pi\alpha_n}{\sin(\pi\alpha_n w)} \right)^2.$$

Those functions are $1/\alpha_n$ periodic. On the one hand, as $n \to +\infty$,

- the functions G_n are uniformly bounded by $1/4$ on $\partial\Omega^n$ and
- the sequence (S_n) converges uniformly to $w \mapsto 1/w^2$ on $\partial\Omega^n$, and thus, the functions S_n are uniformly bounded away from 0 on $\partial\Omega^n$.

As a consequence, the functions G_n/S_n are uniformly bounded on $\partial\Omega^n$. On the other hand, as $\mathrm{Im}(w) \to \pm\infty$, $G_n(w) \to 0$. Thus, in $\mathbb{C}/\frac{1}{\alpha_n}\mathbb{Z}$, G_n has removable singularities at $\pm i\infty$ and vanishes at those points. Since in $\mathbb{C}/\frac{1}{\alpha_n}\mathbb{Z}$, S_n has simple zeros at $\pm i\infty$, the function G_n/S_n has removable singularities at $\pm i\infty$ in $\mathbb{C}/\frac{1}{\alpha_n}\mathbb{Z}$. It follows that from the maximum modulus principle that there is a constant C_1 such that for all sufficiently large n and all $w \in \Omega^n$, we have

$$\left|F_n'(w) - 1\right| \leq C_1 \left|\frac{\pi\alpha_n}{\sin(\pi\alpha_n w)}\right|^2.$$

Note that there is a constant $C_2 > 0$ such that

$$\forall w \in \mathbb{C}, \quad d(w, \mathbb{Z}) \leq C_2\left|\sin(\pi w)\right|.$$

Indeed, the quotient $\dfrac{d(w,\mathbb{Z})}{\left|\sin(\pi w)\right|}$ extends continuously to $(\mathbb{C}/\mathbb{Z}) \cup \{\pm i\infty\}$ which is compact. It follows that for all $w \in \Omega^n$,

$$\left|\frac{\pi\alpha_n}{\sin(\pi\alpha_n w)}\right|^2 \leq \frac{C_2^2\pi^2|\alpha_n|^2}{d(\alpha_n w, \mathbb{Z})^2} \leq \frac{C_2^2\pi^2}{d(w, \partial\Omega^n)^2}.$$

Thus, there is a constant C' such that for all sufficiently large n and for all $w \in \Omega^n$ with $d(w, \partial\Omega^n) \geq R$, we have

$$\left|(\Phi^n)'(w) - (\Psi^n)'(w)\right| \leq \frac{C'}{d(w, \partial\Omega^n)^2}.$$

Taking $R \geq 1$ and replacing R_1 by $R_1 + \sqrt{2}R$, this can be rewritten as: there is a constant C such that for all sufficiently large n and for all $w \in \Omega^n$

$$\left|(\Phi^n)'(w) - (\Psi^n)'(w)\right| \leq \frac{C'}{\big(1 + d(w, \partial\Omega^n)\big)^2}.$$

Let us now assume n is sufficiently large, so that

$$X_n := \frac{1}{2\alpha_n} - R_1 > 0.$$

Then, $w_n := \dfrac{1}{2\alpha_n}$ belongs to Ω^n. Fix $w := w_n + x + iy \in \Omega^n$. Note that

$$|x| < X_n + |y| \quad \text{and} \quad d(w, \partial\Omega^n) > \sqrt{2}\big(X_n + |y| - |x|\big).$$

It follows that

$$\begin{aligned}
\left|\Phi^n(w) - \Psi^n(w)\right| &\leq \int_{[w_n, w_n+iy]\cup[w_n+iy, w]} \frac{C'|du|}{\big(1 + d(u, \partial\Omega^n)\big)^2} \\
&\leq \int_0^{+\infty} \frac{C'ds}{\big(1 + \sqrt{2}(X_n + s)\big)^2} + \int_0^{X_n+|y|} \frac{C'dt}{\big(1 + \sqrt{2}(X_n + |y| - t)\big)^2} \\
&\leq 2C'.
\end{aligned}$$

This completes the proof of Claim 1. □

Claim 2. The map Ψ^n is univalent on Ω^n, $\Psi^n(\Omega^n)$ contains the vertical strip

$$V^n := \big\{w \in \mathbb{C}\ ;\ \mathrm{Re}\big(\Psi^n(R_1)\big) < \mathrm{Re}(w) < \mathrm{Re}\big(\Psi^n(1/\alpha_n - R_1)\big)\big\}$$

and τ_n is injective on $\mathcal{Q}^n := (\Psi^n)^{-1}(V^n)$.

Proof of Claim 2. Note that Ψ^n is a straightening map for the vector field ζ_n:

$$(\Psi^n)_*\zeta_n = \frac{\partial}{\partial w}.$$

Since $F_n(w) - w \in D(1, 1/4)$ on Ω^n, the trajectories of the vector field ζ_n are curves which enter Ω^n through its left boundary and exit Ω^n through the right boundary. In particular, no trajectory is periodic. Since two distinct trajectories cannot intersect, the map Ψ^n is injective.

Observe that for $w \in \partial\Omega^n$,

$$\arg\big((\Psi^n)'(w)\big) = -\arg\big(F_n(w)-w\big) \in \big]-\arcsin(1/4), \arcsin(1/4)\big[\subset \big]-\pi/12, \pi/12\big[.$$

Integrating $(\Psi^n)'(w)$ along $\partial\Omega^n$, we conclude that

$$\frac{2\pi}{3} < \arg\big(\Psi^n(w) - \Psi^n(R_1)\big) < \frac{4\pi}{3}$$

on the left boundary of Ω^n and that

$$-\frac{\pi}{3} < \arg\big(\Psi^n(w) - \Psi^n(1/\alpha_n - R_1)\big) < \frac{\pi}{3}$$

on the right boundary of Ω^n. This proves that $\Psi^n(\Omega^n)$ contains the vertical strip V^n.

Assume by contradiction that τ_n is not injective on V^n. Then, there is an integer $k \in \mathbb{Z}\setminus\{0\}$ and a point $w \in V^n$ such that $w+k/\alpha_n$ is in V^n. Note that V^n is a union of trajectories for the rotated vector field $i\zeta_n$. As w runs along those trajectories, the imaginary part of w increases from $-i\infty$ to $+i\infty$. In particular, every trajectory intersects $\mathbb{R}$. Since for all $w \in \mathcal{D}_n$, we have $i\zeta_n(w) = i\zeta_n(w + 1/\alpha_n)$, the trajectory for $i\zeta_n$ passing through $w + k/\alpha_n$ is obtained from the trajectory passing through w by translation by k/α_n. This is not possible since the intersection of those trajectories with $\mathbb{R}$ is contained in $\Omega^n \cap \mathbb{R} =]R_1, 1/\alpha_n - R_1[$. This completes the proof of Claim 2. $\square$

Let us now come to the proof of parts (1) and (2) of lemma 17. Assume n is sufficiently large, so that

$$\sup_{w\in\Omega^n} \big|\Phi^n(w) - \Psi^n(w)\big| \le C.$$

Then, $\Phi^n(\mathcal{Q}^n)$ contains the vertical strip

$$\big\{w \in \mathbb{C}\ ;\ \mathrm{Re}\big(\Psi^n(R_1)\big) + C < \mathrm{Re}(w) < \mathrm{Re}\big(\Psi^n(1/\alpha_n - R_1)\big) - C\big\}.$$

Note that

$$\Psi^n(R_1) = \Phi^n(R_1) + \mathcal{O}(1) = \mathcal{O}(1)$$

and

$$\Psi^n(1/\alpha_n - R_1) = \Phi^n(1/\alpha_n - R_1) + \mathcal{O}(1) = 1/\alpha_n + \mathcal{O}(1).$$

Thus, if R is large enough and if n is sufficiently large, then $\Phi^n(\mathcal{Q}^n)$ contains the vertical strip

$$U^n := \big\{w \in \mathbb{C}\ ;\ R < \mathrm{Re}(w) < 1/\alpha_n - R\big\}.$$

Since τ_n is injective on $\mathcal{Q}^n$, this proves parts (1) and (2) of lemma 17.

Let us now come to the proof of part (3) of lemma 17. Note that τ_n sends the segment $]0, 1/\alpha_n[$ to the perpendicular bisector of the segment $[0, \sigma_n]$. The map τ_n sends the lower half-plane $\mathbb{H}^- := \big\{w \in \mathbb{C}\ ;\ \mathrm{Im}(w) < 0\big\}$ in the half-plane $\big\{z \in \mathbb{C}\ ;\ |z| > |z - \sigma_n|\big\}$. This takes care of $\tau_n(\mathcal{P}^n \cap \mathbb{H}^-)$.

The map τ_n is a universal covering from the upper half-plane

$$\mathbb{H}^+ := \big\{w \in \mathbb{C}\ ;\ \mathrm{Im}(w) > 0\big\}$$

to the punctured half-plane $\big\{z \in \mathbb{C}\ ;\ 0 < |z| < |z - \sigma_n|\big\}$, with covering transformation group generated by the translation $T_n : w \mapsto w + 1/\alpha_n$. It sends the lines

$$L_k := \left\{w \in \mathbb{C}\ ;\ \mathrm{Re}(w) = \frac{2k+1}{2\alpha_n}\right\}, \quad k \in \mathbb{Z}$$

to the segment $]0, \sigma_n[$. It is therefore enough to show that there is a constant M such that for n large enough, $\mathcal{P}^n \cap \mathbb{H}^+$ is contained in the vertical strip

$$\left\{w \in \mathbb{C}\ ;\ -\frac{M}{\alpha_n} < \mathrm{Re}(w) < \frac{M}{\alpha_n}\right\}.$$

For all $w \in \mathcal{P}^n$, we have

$$R \le \mathrm{Re}\big(\Phi^n(w)\big) \le \frac{1}{\alpha_n} - R.$$

It is therefore enough to show that

$$\sup_{w \in \Omega^n \cap \mathbb{H}^+} \big|\Phi^n(w) - w\big| = \mathcal{O}\left(\frac{1}{\alpha_n}\right)$$

or equivalently that

$$\sup_{w \in \Omega^n \cap \mathbb{H}^+} \big|\Psi^n(w) - w\big| = \mathcal{O}\left(\frac{1}{\alpha_n}\right).$$

Note that $\dfrac{1}{F_n(w) - w} - 1$ is periodic of period $1/\alpha_n$, bounded by $1/3$ in Ω^n and tends to 0 as $\mathrm{Im}(w)$ tends to $+\infty$. It follows from the maximum modulus principle that

$$\left|\frac{1}{F_n(w) - w} - 1\right| < \frac{1}{3} \cdot \left(\inf_{w \in \partial(\Omega^n \cap \mathbb{H}^+)} |e^{2i\pi\alpha_n w}|\right) \cdot |e^{2i\pi\alpha_n w}| \le C e^{-2\pi\alpha_n \mathrm{Im}(w)}$$

for some constant C which does not depend on n. If $w := R + x + iy \in \Omega^n \cap \mathbb{H}^+$, then $|x| < y + 1/\alpha_n$. So

$$\begin{aligned}
\sup_{w \in \Omega^n \cap \mathbb{H}^+} \big|\Psi^n(w) - w\big| &\le \big|\Psi^n(R) - R\big| \\
&\quad + \sup_{\substack{y>0 \\ |x| < y + 1/\alpha_n}} \left(\int_0^y C e^{-2\pi\alpha_n t} dt + \int_0^{|x|} C e^{-2\pi\alpha_n y} dt\right) \\
&= C\left(\frac{1 - e^{-2\pi\alpha_n y}}{2\pi\alpha_n} + e^{-2\pi\alpha_n y} \cdot (y + 1/\alpha_n)\right) + \mathcal{O}(1) \\
&\le \frac{C}{\alpha_n}\left(\frac{1}{2\pi} + \frac{e^{-1}}{2\pi} + 1\right) + \mathcal{O}(1) \\
&= \mathcal{O}\left(\frac{1}{\alpha_n}\right).
\end{aligned}$$

This completes the proof of part (3) of lemma 17. □

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