The homotopy theory of dg-categories and derived Morita theory

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September 2006

Abstract

The main purpose of this work is to study the homotopy theory of dg-categories up to quasi-equivalences. Our main result is a description of the mapping spaces between two dg-categories C and D in terms of the nerve of a certain category of (C, D)-bimodules. We also prove that the homotopy category $Ho(dg-Cat)$ possesses internal Hom's relative to the (derived) tensor product of dg-categories. We use these two results in order to prove a derived version of Morita theory, describing the morphisms between dg-categories of modules over two dg-categories C and D as the dg-category of (C, D)-bi-modules. Finally, we give three applications of our results. The first one expresses Hochschild cohomology as endomorphisms of the identity functor, as well as higher homotopy groups of the *classifying space of dg-categories* (i.e. the nerve of the category of dg-categories and quasi-equivalences between them). The second application is the existence of a good theory of localization for dg-categories, defined in terms of a natural universal property. Our last application states that the dg-category of (continuous) morphisms between the dg-categories of quasi-coherent (resp. perfect) complexes on two schemes (resp. smooth and proper schemes) is quasi-equivalent to the dg-category of quasi-coherent (resp. perfect) complexes on their product.

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1 Introduction

Let A and B be two associative algebras (over some field k), and $A-Mod$ and $B-Mod$ be their categories of right modules. It is well known that any functor $A-Mod \longrightarrow B-Mod$ which commutes with colimits is of the form

$$\begin{array}{ccc} A-Mod & \longrightarrow & B-Mod \\ M & \mapsto & M\otimes_A P, \end{array}$$

for some $A^{op}\otimes B$-module P. More generally, there exists a natural equivalence of categories between $(A^{op}\otimes B)-Mod$ and the category of all colimit preserving functors $A-Mod \longrightarrow B-Mod$. This is known as Morita theory for rings.

Now, let A and B be two associative dg-algebras (say over some field k), together with their triangulated derived category of right (unbounded) dg-modules $D(A)$ and $D(B)$. A natural way of constructing triangulated functors from $D(A)$ to $D(B)$ is by choosing P a left $A^{op}\otimes B$-dg-module, and considering the derived functor

$$\begin{array}{ccc} D(A) & \longrightarrow & D(B) \\ M & \mapsto & M\otimes^{\mathbb{L}}_A P. \end{array}$$

However, it is well known that there exist triangulated functors $D(A) \longrightarrow D(B)$ that does not arise from a $A^{op}\otimes B$-dg-module (see e.g. [Du-Sh, 2.5, 6.8]). The situation is even worse, as the functor

$$D(A^{op}\otimes B) \longrightarrow Hom_{tr}(D(A), D(B))$$

is not expected to be reasonable in any sense as the right hand side simply does not possess a natural triangulated structure. Therefore, triangulated categories do not appear as the right object to consider if one is looking for an extension of Morita theory to dg-algebras. The main purpose of this work is to provide a solution to this problem by replacing the notion of triangulated categories by the notion of dg-categories.

dg-Categories

A dg-category is a category which is enriched over the monoidal category of complexes over some base ring k. It consists of a set of objects together with complexes $C(x, y)$ for two any objects x and y, and composition morphisms $C(x, y) \otimes C(y, z) \longrightarrow C(x, z)$ (assumed to be associative and unital). As linear categories can be understood as *rings with several objects*, dg-categories can be thought as *dg-algebras with several objects*, the precise statement being that dg-algebras are exactly dg-categories having a unique object.

From a dg-category C one can form a genuine category $[C]$ by keeping the same set of objects and defining the set of morphisms between x and y in $[C]$ to be $H^0(C(x, y))$. In turns out that a lot of triangulated categories appearing in geometric contexts are of the form $[C]$ for some natural dg-category C (this is for example the case for the derived category of a reasonable abelian category, as well as for the derived category of dg-modules over some dg-algebra). The new feature of dg-categories is the notion of *quasi-equivalences*, a mixture between quasi-isomorphisms and categorical equivalences and which turns out to be the right notion of equivalences between dg-categories. Precisely, a morphism $f : C \longrightarrow D$ between two dg-categories is a quasi-equivalence if it satisfies the following two conditions

- For any objects x and y in C the induced morphism $C(x, y) \longrightarrow D(f(x), f(y))$ is a quasi-isomorphism.
- The induced functor $[C] \longrightarrow [D]$ is an equivalence of categories.

In practice we are only interested in dg-categories up to quasi-equivalences, and the main object of study is thus the localized category $Ho(dg - Cat)$ of dg-categories with respect to quasi-equivalences, or better its refined simplicial version $L(dg - Cat)$ of Dwyer and Kan (see [D-K2]). The main purpose of this paper is to study the simplicial category $L(dg - Cat)$, and to show that a derived version of Morita theory can be extracted from it. The key tool for us will be the existence of a model structure on the category of dg-categories (see [Tab]), which will allow us to use standard constructions of homotopical algebra (mapping spaces, homotopy limits and colimits ...) in order to describe $L(dg - Cat)$.

Statement of the results

Let C and D be two dg-categories, considered as objects in $L(dg - Cat)$. A first invariant is the homotopy type of the simplicial set of morphism $L(dg - Cat)(C, D)$, which is well known to be weakly equivalent to the mapping space $Map(C, D)$ computed in the model category of dg-categories (see [D-K1, D-K2]). From C and D one can form the tensor product $C \otimes D^{op}$ (suitably derived if necessary), as well as the category $(C \otimes D^{op}) - Mod$ of $C \otimes D^{op}$-modules (these are enriched functors from $C \otimes D^{op}$ to the category of complexes). There exists an obvious notion of quasi-isomorphism between $C \otimes D^{op}$-modules, and thus a homotopy category $Ho((C \otimes D^{op}) - Mod)$. Finally, inside $Ho((C \otimes D^{op}) - Mod)$ is a certain full sub-category of *right quasi-representable objects*, consisting of modules F such that for any $x \in C$ the induced D^{op}-module $F(x, -)$ is quasi-isomorphic to a D^{op}-module of the form $D(-, y)$ for some $y \in D$ (see §3 for details). One can then consider the category $\mathcal{F}(C, D)$ consisting of all right quasi-representable $C \otimes D^{op}$-modules and quasi-isomorphisms between them. The main result of this work is the following.

Theorem 1.1 *(See Thm. 4.2) There exists a natural weak equivalence of simplicial sets*

$$Map(C,D) \simeq N(\mathcal{F}(C,D))$$

where $N(\mathcal{F}(C,D))$ *is the nerve of the category* $\mathcal{F}(C,D)$.

We would like to mention that this theorem does not simply follow from the existence of the model structure on dg-categories. Indeed, this model structure is not simplicially enriched (even in some weak sense, as the model category of complexes is for example), and there is no obvious manner to compute the mapping spaces $Map(C,D)$.

As an important corollary one gets the following result.

Corollary 1.2

1. *There is a natural bijection between* $[C,D]$, *the set of morphisms between* C *and* D *in* $Ho(dg-Cat)$, *and the isomorphism classes of right quasi-representable objects in* $Ho((C\otimes D^{op})-Mod)$.

2. *For two morphism* $f,g : C \longrightarrow D$ *there is a natural weak equivalence*

$$\Omega_{f,g}Map(C,D) \simeq Map(\phi(f),\phi(g))$$

where $Map(\phi(f),\phi(g))$ *is the mapping space between the* $C\otimes D^{op}$*-modules corresponding to* f *and* g.

The tensor product of dg-categories, suitably derived, induces a symmetric monoidal structure on $Ho(dg-Cat)$. Our second main result states that this monoidal structure is closed.

Theorem 1.3 *(See Thm. 6.1) The symmetric monoidal category* $Ho(dg-Cat)$ *is closed. More precisely, for any three dg-categories* A, B *and* C, *there exists a dg-category* $\mathbb{R}\underline{Hom}(B,C)$ *and functorial isomorphisms in* $Ho(SSet)$

$$Map(A,\mathbb{R}\underline{Hom}(B,C)) \simeq Map(A\otimes^{\mathbb{L}} B, C).$$

Furthermore, $\mathbb{R}\underline{Hom}(B,C)$ *is naturally isomorphic in* $Ho(dg-Cat)$ *to the dg-category of cofibrant right quasi-representable* $B\otimes C^{op}$*-modules.*

Finally, Morita theory can be expressed in the following terms. Let us use the notation $\widehat{C} := \mathbb{R}\underline{Hom}(C^{op}, Int(C(k)))$, where $Int(C(k))$ is the dg-category of cofibrant complexes. Note that by our theorem 1.3 $\widehat{C}$ is also quasi-equivalent to the dg-category of cofibrant C^{op}-modules.

Theorem 1.4 *(See Thm. 7.2 and Cor. 7.6) There exists a natural isomorphism in* $Ho(dg-Cat)$

$$\mathbb{R}\underline{Hom}_c(\widehat{C},\widehat{D}) \simeq \widehat{C^{op}\otimes^{\mathbb{L}}} D,$$

where $\mathbb{R}\underline{Hom}_c(\widehat{C},\widehat{D})$ *is the full sub-dg-category of* $\mathbb{R}\underline{Hom}(\widehat{C},\widehat{D})$ *consisting of morphisms commuting with infinite direct sums.*

As a corollary we obtain the following result.

Corollary 1.5 *There is natural bijection between* $[\widehat{C}, \widehat{D}]_c$*, the sub-set of* $[\widehat{C}, \widehat{D}]$ *consisting of direct sums preserving morphisms, and the isomorphism classes in* $Ho((C \otimes^{\mathbb{L}} D^{op}) - Mod)$.

Three applications

We will give three applications of our general results. The first one is a description of the homotopy groups of the classifying space of dg-categories $|dg - Cat|$, defined as the nerve of the category of quasi-equivalences between dg-categories. For this, recall that the Hochschild cohomology of a dg-category C is defined by

$$\mathbb{HH}^i := [C, C[i]]_{C \otimes^{\mathbb{L}} C^{op} - Mod},$$

where C is the $C \otimes^{\mathbb{L}} C^{op}$-module sending $(x, y) \in C \otimes C^{op}$ to $C(y, x)$.

Corollary 1.6 *(See Cor. 8.4, 8.6)*

For any dg-category C *one has*

1. $$\mathbb{HH}^*(C) \simeq H^*(\mathbb{R}\underline{Hom}(C, C)(Id, Id)).$$

2. $$\pi_i(|dg - Cat|, C) \simeq \mathbb{HH}^{2-i}(C) \qquad \forall i > 2.$$

3. $$\pi_2((|dg - Cat|, C) \simeq Aut_{Ho(C \otimes C^{op} - Mod)}(C) \simeq \mathbb{HH}^0(C)^*$$

4. $$\pi_1(|dg - Cat|, \widehat{BA}) \simeq RPic(A),$$

   where A *is a dg-algebra,* BA *the dg-category with a unique object and* A *as its endomorphism, and where* $RPic(A)$ *is the derived Picard group of* A *as defined for example in [Ro-Zi, Ke2, Ye].*

Our second application is the existence of localization for dg-categories. For this, let C be any dg-category and S be a set of morphisms in $[C]$. For any dg-category D we define $Map_S(C, D)$ as the sub-simplicial set of $Map(C, D)$ consisting of morphisms sending S to isomorphisms in $[D]$.

Corollary 1.7 *(See Cor. 8.7) The* $Ho(SSet_{\mathbb{U}})$*-enriched functor*

$$Map_S(C, -) : Ho(dg - Cat_{\mathbb{U}}) \longrightarrow Ho(SSet_{\mathbb{U}})$$

is co-represented by an object $L_S(C) \in Ho(dg - Cat_{\mathbb{U}})$.

Our final application will provide a proof of the following fact, which can be considered as a possible answer to a folklore question to know whether or not all triangulated functors between derived categories of varieties are induced by some object in the derived category of their product (see e.g. [O] where this is proved for triangulated equivalences between derived categories of smooth projective varieties).

Corollary 1.8 *(See Thm. 8.9) Let X and Y be two quasi-compact and separated k-schemes, one of them being flat over $Spec\, k$, and let $L_{qcoh}(X)$ and $L_{qcoh}(Y)$ their dg-categories of (fibrant) quasi-coherent complexes. Then, one has a natural isomorphism in $Ho(dg-Cat)$*

$$L_{qcoh}(X \times_k Y) \simeq \mathbb{R}\underline{Hom}_c(L_{qcoh}(X), L_{qcoh}(Y)).$$

In particular, there is a natural bijection between $[L_{qcoh}(X), L_{qcoh}(Y)]_c$ and set of isomorphism classes of objects in the category $D_{qcoh}(X \times Y)$.

If furthermore X and Y are smooth and proper over $Spec\, k$, then one has a natural isomorphism in $Ho(dg-Cat)$

$$L_{parf}(X \times_k Y) \simeq \mathbb{R}\underline{Hom}(L_{parf}(X), L_{parf}(Y)),$$

where $L_{parf}(X)$ (resp. $L_{parf}(Y)$) is the full sub-dg-category of $L_{qcoh}(X)$ (resp. of $L_{qcoh}(Y)$) consisting of perfect complexes.

Related works

The fact that dg-categories provide natural and interesting enhancement of derived categories has been recognized for some times, and in particular in [B-K]. They have been used more recently in [B-L-L] in which a very special case of our theorem 8.9 is proved for smooth projective varieties. The present work follows the same philosophy that dg-categories are the *true derived categories* (though I do not like very much this expression).

Derived equivalences between (non-dg) algebras have been heavily studied by J. Rickard (see e.g. [Ri1, Ri2]), and the results obtained have been commonly called *Morita theory for derived categories*. The present work can be considered as a continuation of this fundamental work, though our techniques and our purposes are rather different. Indeed, in our mind the word *derived* appearing in our title does not refer to generalizing Morita theory from module categories to derived categories, but to generalizing Morita theory from algebras to dg-algebras.

Morita theory for dg-algebras and ring spectra has been approached recently using model category techniques in [S-S]. The results obtained this way state in particular that two ring spectra have Quillen equivalent model categories of modules if and only if a certain bi-module exists. This approach, however, does not say anything about *higher homotopies*, in the sense that it seems hard (or even impossible) to compare the whole model category of bi-modules with the category of Quillen equivalences, already simply because a model category of Quillen functors does not seem to exist in any reasonable sense. This is another incarnation of the principle that model category theory does not work very well as soon as categories of functors are involved, and that some sort of higher categorical structures are then often needed (see e.g. [T2, §1]).

A relation between the derived Picard group and Hochschild cohomology is given in [Ke2], and is somehow close to our Corollary 8.4. An interpretation of Hochschild cohomology as first order deformations of dg-categories is also given in [HAGII].

There has been many works on dg-categories (as well as its weakened, but after all equivalent, notion of A_∞-categories) in which several universal constructions, such as reasonable dg-categories of dg-functors or quotient and localization of dg-categories, have been studied (see for example [Dr, Ke1, Ly1, Ly2]). Of course, when compared in a correct way, our constructions give back the same objects as the ones considered in these papers, but I would like to point out that the two approaches are different and that our results can not be deduced from these previous works. Indeed, the universal properties of the constructions of [Dr, Ke1, Ly1, Ly2] are expressed in a somehow un-satisfactory manner (at least for my personal taste) as they are stated in terms of certain dg-categories of dg-functors that are not themselves defined by some universal properties (except an obvious one with respect to themselves !)[1]. In some sense, the results proved in these papers are more properties satisfied by certain constructions rather than existence theorems. On the contrary our results truly are existence theorems and our dg-categories of dg-functors, or our localized dg-categories, are constructed as solution to a universal problem inside the category $Ho(dg-Cat)$ (or rather inside the simplicial category $L(dg-Cat)$). As far as I know, these universal properties were not known to be satisfied by the constructions of [Dr, Ke1, Ly1, Ly2].

The results of the present work can also be generalized in an obvious way to other contexts, as for example simplicially enriched categories, or even spectral categories. Indeed, the key tool that makes the proofs working is the existence of a nice model category structure on enriched categories. For simplicial categories this model structure is known to exist by a recent work of J. Bergner, and our theorems 4.2 and 6.1 can be easily shown to be true in this setting (essentially the same proofs work). Theorem 7.2 also stays correct for simplicial categories except that one needs to replace the notion of continuous morphisms by the more elaborated notion of colimit preserving morphisms. More recently, J. Tapia has done some progress for proving the existence of a model category structure on M-enriched categories for very general monoidal model categories M, including for example spectral categories (i.e. categories enriched in symmetric spectra). I am convinced that theorems 4.2 and 6.1, as well as the correct modification of theorem 7.2, stay correct in this general setting. As a consequence one would get a Morita theory for symmetric ring spectra.

Finally, I did not investigate at all the question of the behavior of the equivalence of theorem 4.2 with respect to composition of morphisms. Of course, on the level of bi-modules composition is given by the tensor product, but the combinatorics of these compositions are not an easy question. This is related to the question: *What do dg-categories form ?* It is commonly expected that the answer is *an E_2-category*, whatever this means. The point of view of this work is to avoid this difficulty by stating that another possible answer is *a simplicially enriched category* (precisely the Dwyer-Kan localization $L(dg-Cat)$), which is a perfectly well understood struc-

[1] The situation is very comparable to the situation where one tries to explain why categories of functors give the *right notion*: expressing universal properties using itself categories of functors is not helpful.

ture. Our theorem 6.1, as well as its corollary 6.4 state that the simplicial category $L(dg-Cat)$ is enriched over itself in a rather strong sense. In fact, one can show that $L(dg-Cat)$ is a *symmetric monoidal simplicial category* in the sense of Segal monoids explained in [K-T], and I believe that another equivalent way to talk about E_2-categories is by considering $L(dg-Cat)$-enriched simplicial categories, again in some Segal style of definitions (see for example [T1]). In other words, I think the E_2-category of dg-categories should be completely determined by the symmetric monoidal simplicial category $L(dg-Cat)$.

Acknowledgments: I am very grateful to M. Anel, C. Barwick, L. Katzarkov, T. Pantev, M. Spitzweck, J. Tapia, M. Vaquié and G. Vezzosi for their participation in the small workshop on non-abelian Hodge theory which took place in Toulouse during the spring 2004. I also would like to thank warmly C. Simpson for his participation to this workshop via some tricky but enjoyable video-conference meeting. It has been during one of the informal conversation of this workshop that the general ideas for a proof of theorem 8.9 have been found, and I think this particular theorem should be attributed to the all of us.

I would like to thank B. Keller for several comments on earlier versions of this work, and for pointing out to me related references. I also thank the referee for his careful reading.

Conventions: All along this work universes will be denoted by $\mathbb{U} \in \mathbb{V} \in \mathbb{W} \ldots$. We will always assume that they satisfy the infinite axiom.

We use the notion of model categories in the sense of [Ho1]. The expression *equivalence* always refer to weak equivalence in a model category. For a model category M, we will denote by Map_M (or Map if M is clear) its mapping spaces as defined in [Ho1]. We will always consider $Map_M(x,y)$ as an object in the homotopy category $Ho(SSet)$. In the same way, the set of morphisms in the homotopy category $Ho(M)$ will be denoted by $[-,-]_M$, or by $[-,-]$ if M is clear. The natural $Ho(SSet)$-tensor structure on $Ho(M)$ will be denoted by $K \otimes^{\mathbb{L}} X$, for K a simplicial set and X an object in M. In the same way, the $Ho(SSet)$-cotensor structure will be denoted by $X^{\mathbb{R}K}$. The homotopy fiber products will be denoted by $x \times_z^h y$, and dually the homotopy push-outs will be denoted by $x \coprod_z^{\mathbb{L}} y$.

For all along this work, we fix an associative, unital and commutative ring k. We denote by $C(k)_{\mathbb{U}}$ the category of $\mathbb{U}$-small (un-bounded) complexes of k-modules, for some universe $\mathbb{U}$ with $k \in \mathbb{U}$. The category $C(k)_{\mathbb{U}}$ is a symmetric monoidal model category, where one uses the projective model structures for which fibrations are epimorphisms and equivalences are quasi-isomorphisms (see e.g. [Ho1]). When the universe $\mathbb{U}$ is irrelevant we will simply write $C(k)$ for $C(k)_{\mathbb{U}}$. The monoidal structure on $C(k)$ is the usual tensor product of complexes over k, and will be denoted by $\otimes$. Its derived version will be denoted by $\otimes^{\mathbb{L}}$.

2 The model structure

Recall that a $\mathbb{U}$-small dg-category C consists of the following data.

- A $\mathbb{U}$-small set of objects $Ob(C)$, also sometimes denoted by C itself.
- For any pair of objects $(x,y) \in Ob(C)^2$ a complex $C(x,y) \in C(k)$.

- For any triple $(x, y, z) \in Ob(C)^3$ a composition morphism $C(x, y) \otimes C(y, z) \longrightarrow C(x, z)$, satisfying the usual associativity condition.

- For any object $x \in Ob(C)$, a morphism $k \longrightarrow C(x, x)$, satisfying the usual unit condition with respect to the above composition.

For two dg-categories C and D, a morphism of dg-categories (or simply a dg-functor) $f : C \longrightarrow D$ consists of the following data.

- A map of sets $f : Ob(C) \longrightarrow Ob(D)$.

- For any pair of objects $(x, y) \in Ob(C)^2$, a morphism in $C(k)$

$$f_{x,y} : C(x, y) \longrightarrow D(f(x), f(y))$$

  satisfying the usual unit and associativity conditions.

The $\mathbb{U}$-small dg-categories and dg-functors do form a category $dg-Cat_{\mathbb{U}}$. When the universe $\mathbb{U}$ is irrelevant, we will simply write $dg - Cat$ for $dg - Cat_{\mathbb{U}}$.

We define a functor

$$[-] : dg - Cat_{\mathbb{U}} \longrightarrow Cat_{\mathbb{U}},$$

from $dg - Cat_{\mathbb{U}}$ to the category of $\mathbb{U}$-small categories by the following construction. For $C \in dg - Cat_{\mathbb{U}}$, the set of object of $[C]$ is simply the set of object of C. For two object x and y in $[C]$, the set of morphisms from x to y in $[C]$ is defined by

$$[C](x, y) := H^0(C(x, y)).$$

Composition of morphisms in $[C]$ is given by the natural morphism

$$[C](x, y)\times[C](y, z) = H^0(C(x, y))\times H^0(C(x, y)) \longrightarrow H^0(C(x, y)\otimes^{\mathbb{L}}C(y, z)) \longrightarrow H^0(C(x, z)) = [C](x, z).$$

The unit of an object x in $[C]$ is simply given by the point in $[k, C(x, x)] = H^0(C(x, x))$ image of the unit morphism $k \longrightarrow C(x, x)$ in M. This construction, provides a functor $C \mapsto [C]$ from $dg - Cat_{\mathbb{U}}$ to the category of $\mathbb{U}$-small categories. For a morphism $f : C \longrightarrow D$ in $dg - Cat$, we will denote by $[f] : [C] \longrightarrow [D]$ the corresponding morphism in Cat.

Definition 2.1 *Let $f : C \longrightarrow D$ be a morphism in $dg - Cat$.*

1. *The morphism f is* quasi-fully faithful *if for any two objects x and y in C the morphism $f_{x,y} : C(x, y) \longrightarrow D(f(x), f(y))$ is a quasi-isomorphism.*

2. *The morphism f is* quasi-essentially surjective *if the induced functor $[f] : [C] \longrightarrow [D]$ is essentially surjective.*

3. *The morphism f is a* quasi-equivalence *if it is quasi-fully faithful and quasi-essentially surjective.*

4. *The morphism f is a* fibration *if it satisfies the following two conditions.*

   (a) *For any x and y in C the morphism $f_{x,y} : C(x,y) \longrightarrow D(f(x), f(y))$ is a fibration in $C(k)$ (i.e. is an epimorphism).*

   (b) *For any $x \in C$, and any isomorphism $v : [f](x) \to y'$ in $[D]$, there exists an isomorphism $u : x \to y$ in $[C]$ such that $[f](u) = v$.*

In [Tab] it is proved that the above notions of fibrations and quasi-equivalences in $dg - Cat$ form a model category structure. The model category $dg - Cat_{\mathbb{U}}$ is furthermore $\mathbb{U}$-cofibrantly generated in the sense of [HAGI, Appendix]. Moreover, for $\mathbb{U} \in \mathbb{V}$, the set of generators for the cofibrations and trivial cofibrations can be chosen to be the same for $dg - Cat_{\mathbb{U}}$ and for $dg - Cat_{\mathbb{V}}$. As a consequence we get that the natural inclusion functor

$$Ho(dg - Cat_{\mathbb{U}}) \longrightarrow Ho(dg - Cat_{\mathbb{V}})$$

is fully faithful. This inclusion functor also induces natural equivalences on mapping spaces

$$Map_{dg-Cat_{\mathbb{U}}}(C, D) \simeq Map_{dg-Cat_{\mathbb{V}}}(C, D),$$

for two $\mathbb{U}$-small dg-categories C and D. As a consequence we see that we can change our universe without any serious harm.

Note also that the functor

$$[-] : dg - Cat \longrightarrow Cat$$

induces a functor

$$Ho(dg - Cat) \longrightarrow Ho(Cat),$$

where $Ho(Cat)$ is the category of small categories and isomorphism classes of functors between them. In other words, any morphism $C \to D$ in $Ho(dg - Cat)$ induces a functor $[C] \to [D]$ well defined up to a non-unique isomorphism. This lack of uniqueness will not be so much of a trouble as we will essentially be interested in properties of functors which are invariant by isomorphisms (e.g. being fully faithful, being an equivalence ...).

Definition 2.2 *Let $f : C \longrightarrow D$ be a morphism of dg-categories. The* quasi-essential image *of f is the full sub-dg-category of D consisting of all objects $x \in D$ whose image in $[D]$ lies in the essential image of the functor $[f] : [C] \to [D]$.*

The model category $dg - Cat$ also satisfies the following additional properties.

Proposition 2.3 1. *Any object $C \in dg - Cat$ is fibrant.*

2. *There exists a cofibrant replacement functor Q on $dg - Cat$, such that for any $C \in dg - Cat$ the natural morphism $Q(C) \longrightarrow C$ induces the identity of the sets of objects.*

3. *If C is a cofibrant object in $dg - Cat$ and x and y are two objects in C, then $C(x,y)$ is a cofibrant object in $C(k)$.*

Sketch of proof: (1) is clear by definition. (2) simply follows from the fact that one can choose the generating cofibrations $A \to B$ to induce the identity on the set of objects (see [Tab] for details). Finally, for (3), one uses that any cofibrant object can be written as a transfinite composition of push-outs along the generating cofibrations. As the functor $C \mapsto C(x,y)$ commutes with filtered colimits, and that a filtered colimit of cofibrations stays a cofibration, one sees that it is enough to prove that the property (3) is preserved by push-outs along a generating cofibration. But this can be easily checked by an explicit description of such a push-out (see [Tab] proof of Lem. 2.2. for more details). $\Box$

To finish this paragraph, recall that a morphism $x \to y$ in a model category M is called a *homotopy monomorphism* if for any $z \in M$ the induced morphism

$$Map_M(z,x) \longrightarrow Map_M(z,y)$$

induces an injection on π_0 and isomorphisms on all π_i for $i > 0$ (for all base points). This is also equivalent to say that the natural morphism

$$x \longrightarrow x \times^h_y x$$

is an isomorphism in $Ho(M)$. The following lemma will be used implicitly in the sequel.

Lemma 2.4 *A morphism $f : C \longrightarrow D$ in $dg - Cat$ is a homotopy monomorphism if and only if it is quasi-fully faithful.*

Proof: We can of course suppose that the morphism f is a fibration in $dg - Cat$. Then, f is a homotopy monomorphism if and only if the induced morphism

$$\Delta : C \longrightarrow C \times_D C$$

is a quasi-equivalence.

Let us first assume that f is quasi-fully faithful. For any x and y in C the induced morphism by Δ is the diagonal of $C(x,y)$

$$\Delta(x,y) : C(x,y) \longrightarrow C(x,y) \times_{D(f(x),f(y))} C(x,y).$$

As f is a fibration, the morphism $C(x,y) \longrightarrow D(f(x),f(y))$ is a trivial fibration, and thus the morphism $\Delta(x,y)$ is a quasi-isomorphism. This shows that Δ is quasi-fully faithful. Now, let t be an object in $C \times_D C$, corresponding to two points x and y in C such that $f(x) = f(y)$. We consider the identity morphism $f(x) \to f(y)$ in $[D]$. As $[C] \to [D]$ is fully faithful, the identity can be lifted to an isomorphism in $[C]$ $u : x \to y$. Furthermore, as $C(x,y) \longrightarrow D(f(x),f(y))$ is a fibration, the morphism u can be represented by a zero cycle $u \in Z^0(C(x,y))$ whose image by f is the identity. This implies that the point t is isomorphic in $[C \times_D C]$ to the image of the point $x \in C$ by Δ, and thus that Δ is quasi-essentially surjective. We have shown that Δ is a quasi-equivalence and therefore that f is a homotopy monomorphism.

Conversely, let us assume that f is a homotopy monomorphism. Then, for any x and y in C the natural morphism

$$C(x,y) \longrightarrow C(x,y) \times_{D(f(x),f(y))} C(x,y)$$

is a quasi-isomorphism, and thus the morphism $C(x,y) \longrightarrow D(f(x),f(y))$ is a homotopy monomorphism in $C(k)$. As $C(k)$ is a stable model category (see [Ho1, §7])this clearly implies that $C(x,y) \longrightarrow D(f(x),f(y))$ is in fact a quasi-isomorphism. $\Box$

Corollary 2.5 *Let $C \longrightarrow D$ be a quasi-fully faithful morphism in $dg-Cat$ and B be any dg-category. Then, the induced morphism*

$$Map(B,C) \longrightarrow Map(B,D)$$

induces an injection on π_0 and an isomorphism on π_i for $i>0$. Furthermore, the image of

$$\pi_0(Map(B,C)) = [B,C] \longrightarrow [B,D] = \pi_0(Map(B,D))$$

consists of all morphism such that the induced functor $[B] \to [D]$ factors through the essential image of $[C] \to [D]$.

Proof: Only the last statement requires a proof. For this we can of course assume that B is cofibrant. Furthermore, one can replace C by its quasi-essential image in D. The statement is then clear by the description of $[B,C]$ and $[B,D]$ as homotopy classes of morphisms between B and C or D. $\Box$

3 Modules over dg-categories

Let $C \in dg-Cat_{\mathbb{U}}$ be a fixed $\mathbb{U}$-small dg-category. Recall that a $\mathbb{U}$-small C-dg-module F (or simply a C-module) consists of the following data.

- For any object $x \in C$ a complex $F(x) \in C(k)_{\mathbb{U}}$.
- For any two objects x and y in C, a morphism of complexes

  $$C(x,y) \otimes F(x) \longrightarrow F(y),$$

  satisfying the usual associativity and unit conditions.

Note that a C-module is nothing else than a morphism of dg-categories $F : C \longrightarrow C(k)$, where $C(k)$ is a dg-category in the obvious way, or equivalently as a $C(k)$-enriched functor from C to $C(k)$. For two C-dg-modules F and G, a morphism from F to G is simply the data of morphisms $f_x : F(x) \longrightarrow G(x)$ commuting with the structure morphisms. This is nothing else than a $C(k)$-enriched natural transformation between the corresponding $C(k)$-enriched functors. The $\mathbb{U}$-small C-modules and morphisms between them form a category, denoted by $C-Mod_{\mathbb{U}}$. Once again, when the universe $\mathbb{U}$ is irrelevant we will simply write $C-Mod$ for $C-Mod_{\mathbb{U}}$.

Let $z \in C$ be an object in C. One defines a C-module $\underline{h}^z \in C-Mod$, by the formula $\underline{h}^z(x) := C(z,x)$, and with structure morphisms

$$C(z,x) \otimes C(x,y) \longrightarrow C(z,y)$$

being the composition in C.

Definition 3.1 *Let $C \in dg-Cat$ and $f : F \longrightarrow G$ be a morphism of C-modules. The morphism f is an* equivalence *(resp. a* fibration*) if for any $x \in C$ the morphism*

$$f_x : F(x) \longrightarrow G(x)$$

is an equivalence (resp. a fibration) in $C(k)$.

We recall that as $C(k)$ is cofibrantly generated, the above definition endows $C - Mod$ with a structure of a cofibrantly generated model category (see for example [Hi, §11]). The natural $C(k)$-enrichment of $C - Mod$ endows furthermore $C - Mod$ with a structure of a $C(k)$-model category in the sense of [Ho1, 4.2.18]. The $C(k)$-enriched Hom's of the category $C - Mod$ will be denoted by $\underline{Hom}$, and its derived version by

$$\mathbb{R}\underline{Hom} : Ho(C - Mod)^{op} \times Ho(C - Mod) \longrightarrow Ho(C(k)).$$

The notion of modules over dg-categories has the following natural generalization. Let M be a $C(k)_{\mathbb{U}}$-model category in the sense of [Ho1, 4.2.18], and let us suppose that it is $\mathbb{U}$-cofibrantly generated in the sense of [HAGI, Appendix A]. Then, for a $\mathbb{U}$-small dg-category C one has a category of $C(k)$-enriched functors M^C from C to M. Furthermore, it can be endowed with a structure of a $\mathbb{U}$-cofibrantly generated model category for which equivalences and fibrations are defined levelwise in M (see e.g. [Hi, 11.6]). The category M^C has itself a natural $C(k)$-enrichment induced from the one on M, making it into a $C(k)$-model category. When $M = C(k)_{\mathbb{U}}$ itself, the model category M^C can be identified with $C - Mod_{\mathbb{U}}$.

Let $f : C \longrightarrow D$ be a morphism in $dg - Cat$. Composing with f gives a restriction functor

$$f^* : M^D \longrightarrow M^C.$$

This functor has a left adjoint

$$f_! : M^C \longrightarrow M^D.$$

The adjunction $(f_!, f^*)$ is clearly a Quillen adjunction, compatible with the $C(k)$-enrichment.

Proposition 3.2 *Let $f : C \longrightarrow D$ be a quasi-equivalence between $\mathbb{U}$-small dg-categories. Let M be a $\mathbb{U}$-cofibrantly generated $C(k)$-model category, such that the domain and codomain of a set of generating cofibrations are cofibrant objects in M. We assume that one of the following conditions is satisfied.*

1. *For any cofibrant object $A \in M$, and any quasi-isomorphism $X \longrightarrow Y$ in $C(k)$, the induced morphism*

$$X \otimes A \longrightarrow Y \otimes A$$

is an equivalence in M.

2. *All the complexes of morphisms of C and D are cofibrant objects in $C(k)$.*

Then the Quillen adjunction $(f_!, f^*)$ is a Quillen equivalence.

Proof: The functor f^* clearly preserves equivalences. Furthermore, as f is quasi-essentially surjective, the functor $f^* : Ho(M^D) \longrightarrow Ho(M^C)$ is easily seen to be conservative. Therefore, one is reduced to check that the adjunction morphism $Id \Rightarrow f^*\mathbb{L}f_!$ is an isomorphism.

For $x \in C$, and $A \in M$, one writes $\underline{h}^x \otimes A \in M^C$ for the object defined by

$$\begin{array}{cccc} \underline{h}^x \otimes A & C & \longrightarrow & M \\ & y & \mapsto & C(x,y) \otimes A. \end{array}$$

The model category M^C is itself cofibrantly generated, and a set of generating cofibration can be chosen to consist of morphisms of the form

$$\underline{h}^x \otimes A \longrightarrow \underline{h}^x \otimes B$$

for some generating cofibration $A \longrightarrow B$ in M. By assumption on M, any object $F \in Ho(M^C)$ can thus be written as a homotopy colimit of objects of the form $\underline{h}^x \otimes A$, for certain cofibrant $A \in M$, and certain $x \in C$. As the two functors f^* and $\mathbb{L}f_!$ commute with homotopy colimits it is then enough to show that the natural morphism

$$\underline{h}^x \otimes A \longrightarrow f^*\mathbb{L}f_!(\underline{h}^x \otimes A)$$

is an isomorphism in $Ho(M^C)$. By adjunction, one clearly has $\mathbb{L}f_!(\underline{h}^x \otimes A) \simeq \underline{h}^{f(x)} \otimes A$. Therefore, the adjunction morphism

$$\underline{h}^x \otimes A \longrightarrow f^*\mathbb{L}f_!(\underline{h}^x \otimes A) \simeq f^*(\underline{h}^{f(x)} \otimes A)$$

evaluated at $y \in C$ is the morphism

$$f_{x,y} \otimes Id_A : C(x,y) \otimes A \longrightarrow D(f(x), f(y)) \otimes A.$$

The fact that this is an isomorphism in $Ho(M)$ follows from the fact that f is quasi-fully faithful, one of our hypothesis (1) and (2), and the fact that M is a $C(k)$-model category. $\Box$

Another important property of the model category M^C is the following.

Proposition 3.3 *Let C be a $\mathbb{U}$-small dg-category with cofibrant complexes of morphisms (i.e. $C(x,y)$ is cofibrant in $C(k)$ for all x and y), and M be a $\mathbb{U}$-cofibrantly generated $C(k)$-model category. Then, for any $x \in C$ the evaluation functor*

$$\begin{array}{cccc} x^* : & M^C & \longrightarrow & M \\ & F & \mapsto & F(x) \end{array}$$

preserves fibrations, cofibrations and equivalences.

Proof: For fibrations and equivalences this is clear by definition. The functor x^* commutes with colimits, and thus by a small object argument one is reduced to show that x^* sends generating cofibrations to cofibrations. One knows that the generating set of cofibrations in

M^C can be chosen to consist of morphisms of the form $\underline{h}^z \otimes A \longrightarrow \underline{h}^z \otimes B$ for some cofibration $A \longrightarrow B$ in M. The image by x^* of such a morphism is

$$C(z,x) \otimes A \longrightarrow C(z,x) \otimes B.$$

As by assumption $C(z,x)$ is a cofibrant object in $C(k)$, one sees that this morphism is a cofibration in M. $\Box$

Two important cases of application of proposition 3.3 is when C itself is a cofibrant dg-category (see Prop. 2.3), or when k is a field.

Corollary 3.4 *The conclusion of Prop. 3.2 is satisfied when M is of the form $D - Mod_{\mathbb{U}}$, for a $\mathbb{U}$-small dg-category D with cofibrant complexes of morphisms (in particular for $M = C(k)$).*

Proof: This follows easily from Prop. 3.3 and the fact that $C(k)$ itself satisfies the hypothesis (1) of Prop. 3.2. $\Box$

Let $\mathbb{U} \in \mathbb{V}$ be two universes. Let M be a $C(k)_{\mathbb{U}}$-model category which is supposed to be furthermore $\mathbb{V}$-small. We define a $\mathbb{V}$-small dg-category $Int(M)$ in the following way[2]. The set of objects of $Int(M)$ is the set of fibrant and cofibrant objects in M. For two such objects F and E one sets

$$Int(M)(E,F) := \underline{Hom}(E,F) \in C(k)_{\mathbb{U}},$$

where $\underline{Hom}(E,F)$ is the $C(k)$-valued Hom of the category M. The dg-category $Int(M)$ is of course only $\mathbb{V}$-small as its sets of objects is only $\mathbb{V}$-small. However, for any E and F in $Int(M)$ the complex $Int(M)(E,F)$ is in fact $\mathbb{U}$-small.

The following is a general fact about $C(k)$-enriched model categories.

Proposition 3.5 *There exists a natural equivalence of categories*

$$[Int(M)] \simeq Ho(M).$$

Proof: This follows from the formula

$$H^0(\mathbb{R}\underline{Hom}(X,Y)) \simeq [k, \mathbb{R}\underline{Hom}(X,Y)]_{C(k)} \simeq [X,Y]_M,$$

for two objects X and Y in M. $\Box$

For $x \in C$, the object $\underline{h}^x \in C - Mod_{\mathbb{U}}$ is cofibrant and fibrant, and therefore the construction $x \mapsto \underline{h}^x$, provides a morphism of dg-categories

$$\underline{h}^- : C^{op} \longrightarrow Int(C - Mod_{\mathbb{U}}),$$

[2]The notation *Int* is taken from [Hir-Si]. As far as I understand it stands for *internal*.

where C^{op} is the opposite dg-category of C (C^{op} has the same set of objects than C and $C^{op}(x,y) := C(y,x)$). The morphism $\underline{h}^{-}$ can also be written dually as

$$\underline{h}_{-} : C \longrightarrow Int(C^{op} - Mod_{\mathbb{U}}).$$

The dg-functor $\underline{h}^{-}$ will be considered as a morphism in $dg - Cat_{\mathbb{V}}$, and is clearly quasi-fully faithful by an application of the $C(k)$-enriched Yoneda lemma.

Definition 3.6 *1. Let $C \in dg - Cat_{\mathbb{U}}$, and $F \in C^{op} - Mod_{\mathbb{U}}$ be a C^{op}-module. The object F is called* representable *(resp.* quasi-representable*) if it is isomorphic in $C^{op} - Mod_{\mathbb{U}}$ (resp. in $Ho(C^{op} - Mod_{\mathbb{U}})$) to $\underline{h}_x$ for some object $x \in C$.*

2. Dually, let $C \in dg - Cat_{\mathbb{U}}$, and $F \in C - Mod_{\mathbb{U}}$ be a C-module. The object F is called corepresentable *(resp.* quasi-corepresentable*) if it is isomorphic in $C - Mod_{\mathbb{U}}$ (resp. in $Ho(C - Mod_{\mathbb{U}})$) to $\underline{h}^x$ for some object $x \in C$.*

As the morphism $\underline{h}_{-}$ is quasi-fully faithful, it induces a quasi-equivalence between C and the full dg-category of $Int(C^{op} - Mod_{\mathbb{U}})$ consisting of quasi-representable objects. This quasi-equivalence is a morphism in $dg - Cat_{\mathbb{V}}$.

4 Mapping spaces and bi-modules

Let C and D be two objects in $dg - Cat$. One has a tensor product $C \otimes D \in dg - Cat$ defined in the following way. The set of objects of $C \otimes D$ is $Ob(C) \times Ob(D)$, and for (x,y) and (x',y') two objects in $Ob(C \otimes D)$ one sets

$$(C \otimes D)((x,y),(x',y')) := C(x,y) \otimes D(x',y').$$

Composition in $C \otimes D$ is given by the obvious formula. This defines a symmetric monoidal structure on $dg - Cat$, which is easily seen to be closed. The unit of this structure will be denoted by **1**, and is the dg-category with a unique object and k as its endomorphism ring.

The model category $dg - Cat$ together with the symmetric monoidal structure $- \otimes -$ is *not* a symmetric monoidal model category, as the tensor product of two cofibrant objects in $dg - Cat$ is not cofibrant in general. A direct consequence of this fact is that the internal Hom object between cofibrant-fibrant objects in $dg - Cat$ can not be invariant by quasi-equivalences, and thus does not provide internal Hom's for the homotopy categories $Ho(dg - Cat)$. This fact is the main difficulty in computing the mapping spaces in $dg - Cat$, as the naive approach simply does not work.

However, it is true that the monoidal structure $\otimes$ on $dg - Cat$ is closed, and that $dg - Cat$ has corresponding internal Hom objects C^D satisfying the usual adjunction rule

$$Hom_{dg-Cat}(A \otimes B, C) \simeq Hom(A, C^B).$$

This gives a natural equivalence of categories

$$M^{C \otimes D} \simeq (M^C)^D$$

for any $C(k)$-enriched category M. Furthermore, when M is a $\mathbb{U}$-cofibrantly generated model category, this last equivalence is compatible with the model structures on both sides.

The functor $-\otimes-$ can be derived into a functor

$$-\otimes^{\mathbb{L}}-: dg-Cat \times dg-Cat \longrightarrow dg-Cat$$

defined by the formula

$$C\otimes^{\mathbb{L}} D := Q(C)\otimes D$$

where Q is a cofibrant replacement in $dg-Cat$ which acts by the identity on the sets of objects. Clearly, the functor $-\otimes^{\mathbb{L}}-$ preserves quasi-equivalences and passes through the homotopy categories

$$-\otimes^{\mathbb{L}}-: Ho(dg-Cat)\times Ho(dg-Cat) \longrightarrow Ho(dg-Cat).$$

Note that when C is cofibrant, one has a natural quasi-equivalence $C\otimes^{\mathbb{L}} D \longrightarrow C\otimes D$.

We now consider $(C\otimes D^{op})-Mod$, the category of $(C\otimes D^{op})$-modules. For any object $x\in C$, there exists a natural morphism of dg-categories $D^{op}\longrightarrow (C\otimes D^{op})$ sending $y\in D$ to the object (x,y), and

$$D^{op}(y,z)\longrightarrow (C\otimes D^{op})((x,y),(x,z)) = C(x,x)\otimes D^{op}(y,z)$$

being the tensor product of the unit $k\longrightarrow C(x,x)$ and the identity on $D^{op}(y,z)$. As C and $Q(C)$ has the same set of objects, one sees that for any $x\in C$ one also gets a natural morphism of dg-categories

$$i_x : D^{op}\longrightarrow Q(C)\otimes D^{op} = C\otimes^{\mathbb{L}} D^{op}.$$

Definition 4.1 *Let C and D be two dg-categories. An object $F\in(C\otimes^{\mathbb{L}} D^{op})-Mod$ is called* right quasi-representable*, if for any $x\in C$, the D^{op}-module $i_x^*(F)\in D^{op}-Mod$ is quasi-representable in the sense of Def. 3.6.*

We now let $\mathbb{U}\in\mathbb{V}$ be two universes, and let C and D be two $\mathbb{U}$-small dg-categories. Let Γ^* be a co-simplicial resolution functor in $dg-Cat_{\mathbb{U}}$ in the sense of [Hi, §16.1]. Recall that Γ^* is a functor from $dg-Cat_{\mathbb{U}}$ to $dg-Cat_{\mathbb{U}}^{\Delta}$, equipped with a natural augmentation $\Gamma^0\longrightarrow Id$, and such the following two conditions are satisfied.

- For any n, and any $C\in dg-Cat_{\mathbb{U}}$ the morphism $\Gamma^n(C)\to C$ is a quasi-equivalence.
- For any $C\in dg-Cat_{\mathbb{U}}$, the object $\Gamma^*(C)\in dg-Cat_{\mathbb{U}}^{\Delta}$ is cofibrant for the Reedy model structure.
- The morphism $\Gamma^0(C)\longrightarrow C$ is equal to $Q(C)\longrightarrow C$.

The left mapping space between C and D is by definition the $\mathbb{U}$-small simplicial set

$$Map^l(C,D) := Hom(\Gamma^*(C),D): \begin{array}{ccc} \Delta^{op} & \longrightarrow & Set_{\mathbb{U}} \\ [n] & \mapsto & Hom(\Gamma^n(C),D). \end{array}$$

Note that the mapping space $Map^l(C,D)$ defined above has the correct homotopy type as all objects are fibrant in $dg-Cat_{\mathbb{U}}$.

For any $[n]\in\Delta$, one considers the (non-full) sub-category $\mathcal{M}(\Gamma^n(C),D)$ of $(\Gamma^n(C)\otimes D^{op})-Mod_{\mathbb{U}}$ defined in the following way. The objects of $\mathcal{M}(\Gamma^n(C),D)$ are the $(\Gamma^n(C)\otimes D^{op})$-modules F such that F is right quasi-representable, and for any $x\in\Gamma^n(C)$ the D^{op}-module $F(x,-)$ is cofibrant in $D^{op}-Mod_{\mathbb{U}}$. The morphisms in $\mathcal{M}(\Gamma^n(C),D)$ are simply the equivalences in $(\Gamma^n(C)\otimes D^{op})-Mod_{\mathbb{U}}$. The nerve of the category $\mathcal{M}(\Gamma^n(C),D)$ gives a $\mathbb{V}$-small simplicial set $N(\mathcal{M}(\Gamma^n(C),D))$. For $[n]\to[m]$ a morphism in Δ, one has a natural morphism of dg-categories $\Gamma^n(C)\otimes D^{op}\longrightarrow\Gamma^m(C)\otimes D^{op}$, and thus a well defined morphism of simplicial sets

$$N(\mathcal{M}(\Gamma^m(C),D))\longrightarrow N(\mathcal{M}(\Gamma^n(C),D))$$

obtained by pulling back the modules from $\Gamma^m(C)\otimes D^{op}$ to $\Gamma^n(C)\otimes D^{op}$. This defines a functor

$$\begin{array}{rccc} N(\mathcal{M}(\Gamma^*(C),D)): & \Delta^{op} & \longrightarrow & SSet_{\mathbb{V}} \\ & [n] & \mapsto & N(\mathcal{M}(\Gamma^n(C),D)). \end{array}$$

The set of zero simplices in $N(\mathcal{M}(\Gamma^n(C),D))$ is the set of all objects in the category $\mathcal{M}(\Gamma^n(C),D)$. Therefore, one defines a natural morphism of sets

$$Hom(\Gamma^n(C),D)\longrightarrow N(\mathcal{M}(\Gamma^n(C),D))_0$$

by sending a morphism of dg-categories $f:\Gamma^n(C)\longrightarrow D$, to the $(\Gamma^n(C)\otimes D^{op})$-module $\phi(f)$ defined by $\phi(f)(x,y):=D(y,f(x))$ and the natural transition morphisms. Note that $\phi(f)$ belongs to the sub-category $\mathcal{M}(\Gamma^n(C),D)$ as for any $x\in\Gamma^n(C)$ the D^{op}-module $\phi(f)(x,-)=\underline{h}_{f(x)}$ is representable and thus quasi-representable and cofibrant. By adjunction, this morphism of sets can also be considered as a morphism of simplicial sets

$$\phi: Hom(\Gamma^n(C),D)\longrightarrow N(\mathcal{M}(\Gamma^n(C),D)),$$

where the set $Hom(\Gamma^n(C),D)$ is considered as a constant simplicial set. This construction is clearly functorial in n, and gives a well defined morphism of bi-simplicial sets

$$\phi: Hom(\Gamma^*(C),D)\longrightarrow N(\mathcal{M}(\Gamma^*(C),D)).$$

Passing to the diagonal one gets a morphism in $SSet_{\mathbb{V}}$

$$\phi: Map^l(C,D)\longrightarrow d(N(\mathcal{M}(\Gamma^*(C),D))).$$

Finally, the diagonal $d(N(\mathcal{M}(\Gamma^*(C),D)))$ receives a natural morphism

$$\psi: N(\mathcal{M}(\Gamma^0(C),D))=N(\mathcal{M}(Q(C),D))\longrightarrow d(N(\mathcal{M}(\Gamma^*(C),D))).$$

Clearly, the diagram of simplicial sets

$$Map^l(C,D)\xrightarrow{\phi} d(N(\mathcal{M}(\Gamma^*(C),D)))\xleftarrow{\psi} N(\mathcal{M}(Q(C),D))$$

is functorial in C.

The main theorem of this work is the following.

Theorem 4.2 *The two morphisms in $SSet_{\mathbb{V}}$*

$$Map^l(C,D) \xrightarrow{\phi} d(N(\mathcal{M}(\Gamma^*(C),D))) \xleftarrow{\psi} N(\mathcal{M}(Q(C),D))$$

are weak equivalences.

Proof: For any n, the morphism $\Gamma^n(C) \otimes D^{op} \longrightarrow Q(C) \otimes D^{op}$ is a quasi-equivalence of dg-categories. Therefore, Prop. 3.4 implies that the pull-back functor

$$(Q(C) \otimes D^{op}) - Mod \longrightarrow (\Gamma^n(C) \otimes D^{op}) - Mod$$

is the right adjoint of a Quillen equivalence. As these functors obviously preserve the notion of being right quasi-representable, one finds that the induced morphism

$$N(\mathcal{M}(Q(C),D)) \longrightarrow N(\mathcal{M}(\Gamma^n(C),D))$$

is a weak equivalence. This clearly implies that the morphism ψ is a weak equivalence.

It remains to show that the morphism ϕ is also a weak equivalence. For this, we start by proving that it induces an isomorphism on connected components.

Lemma 4.3 *The induced morphism*

$$\pi_0(\phi) : [C,D] \simeq \pi_0(Map^l(C,D)) \longrightarrow \pi_0(d(N(\mathcal{M}(\Gamma^*(C),D))))$$

is an isomorphism.

Proof: First of all, replacing C by $Q(C)$ one can suppose that $Q(C) = C$ (one can do this because of Prop. 3.4). One then has $\pi_0(Map^l(C,D)) \simeq [C,D]$, and $\pi_0(d(N(\mathcal{M}(\Gamma^*(C),D)))) \simeq \pi_0(N(\mathcal{M}(C,D)))$ is the set of isomorphism classes in $Ho((C \otimes D^{op}) - Mod)^{rqr}$, the full sub-category of $Ho((C \otimes D^{op}) - Mod)$ consisting of all right quasi-representable objects. The morphism ϕ naturally gives a morphism

$$\phi : [C,D] \longrightarrow Iso(Ho((C \otimes D^{op}) - Mod)^{rqr})$$

which can be described as follows. For any $f \in [C,D]$, represented by $f : C \longrightarrow D$ in $Ho(dg - Cat)$, $\phi(f)$ is the $C \otimes D^{op}$-module defined by $\phi(f)(x,y) := D(y, f(x))$.

Sub-lemma 4.4 *With the same notations as above, let M be a $\mathbb{U}$-cofibrantly generated $C(k)_{\mathbb{U}}$-model category, which is furthermore $\mathbb{V}$-small. Let $Iso(Ho(M^C))$ be the set of isomorphism classes of objects in $Ho(M^C)$. Then, the natural morphism*

$$Hom(C, Int(M)) \longrightarrow Iso(Ho(M^C))$$

is surjective.

Proof of sub-lemma 4.4: Of course, the morphism

$$Hom(C, Int(M)) \longrightarrow Iso(Ho(M^C))$$

sends a morphism of dg-categories $C \longrightarrow Int(M)$ to the corresponding object in M^C. Let $F \in Ho(M^C)$ be a any cofibrant and fibrant object. This object is given by a $C(k)$-enriched functor $F : C \longrightarrow M$. Furthermore, as F is fibrant and cofibrant, Prop. 3.3 tells us that $F(x)$ is fibrant and cofibrant in M for any $x \in C$. The object F can therefore be naturally considered as a morphism of $\mathbb{V}$-small dg-categories

$$F : C \longrightarrow Int(M),$$

which gives an element in $Hom(C, Int(M))$ sent to F by the map of the lemma. $\square$

Let us now prove that the morphism ϕ is surjective on connected component. For this, let $F \in Ho((C \otimes D^{op}) - Mod_{\mathbb{U}})$ be a right quasi-representable object. One needs to show that F is isomorphic to some $\phi(f)$ for some morphism of dg-categories $f : C \longrightarrow D$. Sub-lemma 4.4 applied to $M = D^{op} - Mod_{\mathbb{U}}$ implies that F corresponds to a morphism of $\mathbb{V}$-small dg-categories

$$F : C \longrightarrow Int(D^{op} - Mod_{\mathbb{U}})^{qr},$$

where $Int(D^{op} - Mod)^{qr}$ is the full sub-dg-category of $Int(D^{op} - Mod_{\mathbb{U}})$ consisting of all quasi-representable objects.

One has a diagram in $dg - Cat_{\mathbb{V}}$

$$\begin{array}{ccc} C & \xrightarrow{F} & Int(D^{op} - Mod_{\mathbb{U}})^{qr} \\ & & \uparrow \underline{h} \\ & & D. \end{array}$$

As the morphism $\underline{h}$ is a quasi-equivalence, and as C is cofibrant, one finds a morphism of dg-categories $f : C \longrightarrow D$, such that the two morphisms

$$F : C \longrightarrow Int(D^{op} - Mod_{\mathbb{U}})^{qr} \qquad \underline{h}_{f(-)} = \phi(f) : C \longrightarrow Int(D^{op} - Mod_{\mathbb{U}})^{qr}$$

are homotopic in $dg - Cat_{\mathbb{V}}$. Let

$$\begin{array}{ccc} C & & \\ {\scriptstyle i_0}\downarrow & \searrow^{F} & \\ C' & \xrightarrow{H} & Int(D^{op} - Mod_{\mathbb{U}})^{qr} \\ {\scriptstyle i_1}\uparrow & \nearrow_{\phi(f)} & \\ C & & \end{array}$$

be a homotopy in $dg - Cat_{\mathbb{V}}$. Note that C' is a cylinder object for C, and thus can be chosen to be cofibrant and $\mathbb{U}$-small. We let $p : C' \longrightarrow C$ the natural projection, such that $p \circ i_0 = p \circ i_1 = Id$. This diagram gives rise to an equivalence of categories (by Prop. 3.4)

$$i_0^* \simeq i_1^* \simeq (p^*)^{-1} : Ho((C' \otimes D^{op}) - Mod_{\mathbb{U}}) \longrightarrow Ho((C \otimes D^{op}) - Mod_{\mathbb{U}}).$$

Furthermore, one has

$$F \simeq i_0^*(H) \simeq i_1^*(H) \simeq \phi(f).$$

This shows that the two $C \otimes D^{op}$-modules F and $\phi(f)$ are isomorphic in $Ho((C \otimes D^{op}) - Mod_{\mathbb{U}})$, or in other words that $\phi(f) = F$ in $Iso(Ho((C \otimes D^{op}) - Mod_{\mathbb{U}}))$. This finishes the proof of the surjectivity part of the lemma 4.3.

Let us now prove that ϕ is injective. For this, let $f, g : C \longrightarrow D$ be two morphisms of dg-categories, such that the two $(C \otimes D^{op})$-modules $\phi(f)$ and $\phi(g)$ are isomorphic in $Ho((C \otimes D^{op}) - Mod_{\mathbb{U}}))$. Composing f and g with

$$\underline{h} : D \longrightarrow Int(D^{op} - Mod_{\mathbb{U}})$$

one gets two new morphisms of dg-categories

$$f', g' : C \longrightarrow Int(D^{op} - Mod_{\mathbb{U}}).$$

Using that $\underline{h}$ is quasi-fully faithful Cor. 2.5 implies that if f' and g' are homotopic morphisms in $dg - Cat_{\mathbb{V}}$, then f and g are equal as morphisms in $Ho(dg - Cat_{\mathbb{V}})$. As the inclusion $Ho(dg - Cat_{\mathbb{U}}) \longrightarrow Ho(dg - Cat_{\mathbb{V}})$ is fully faithful (see remark after Def. 2.1), we see it is enough to show that f' and g' are homotopic in $dg - Cat_{\mathbb{V}}$.

Sub-lemma 4.5 *Let M be a $C(k)_{\mathbb{U}}$-model category which is $\mathbb{U}$-cofibrantly generated and $\mathbb{V}$-small. Let u and v be two morphisms in $dg - Cat_{\mathbb{V}}$*

$$u, v : C \longrightarrow Int(M)$$

such that the corresponding objects in $Ho(M^C)$ are isomorphic. Then u and v are homotopic as morphisms in $dg - Cat_{\mathbb{V}}$.

Proof of sub-lemma 4.5: First of all, any isomorphism in $Ho(M^C)$ can be represented as a string of trivial cofibrations and trivial fibrations between cofibrant and fibrant objects. Therefore, sub-lemma 4.4 shows that one is reduced to the case where there exists an equivalence $\alpha : u \longrightarrow v$ in M^C which is either a fibration or a cofibration.

Let us start with the case where α is a cofibration in M^C. The morphism α can also be considered as an object in $(M^I)^C$, where I is the category with two objects 0 and 1 and a unique morphism $0 \to 1$. The category M^I, which is the category of morphisms in M, is endowed with its projective model structure, for which fibrations and equivalences are defined on the underlying objects in M. As the morphism α is a cofibration in M^C, we see that for $x \in C$ the corresponding morphism $\alpha_x : u(x) \to v(x)$ is a cofibration in M, and thus is a cofibrant (and fibrant) object in M^I because of proposition 3.3. This implies that α gives rise to a morphism of dg-categories

$$\alpha : C \longrightarrow Int(M^I).$$

Now, let $Int(M) \longrightarrow Int(M^I)$ be the natural inclusion morphism, sending a cofibrant and fibrant object in M to the identity morphism. This a morphism in $dg - Cat_{\mathbb{V}}$ which is easily seen to be quasi-fully faithful. We let $C' \subset Int(M^I)$ be the quasi-essential image of $Int(M)$

in $Int(M^I)$. It is easy to check that C' is the full sub-dg-category of $Int(M^I)$ consisting of all objects in M^I corresponding to equivalences in M. The morphism $\alpha : C \longrightarrow Int(M^I)$ thus factors through the sub-dg-category $C' \subset Int(M^I)$. The two objects 0 and 1 of I give two projections

$$C' \subset Int(M^I) \rightrightarrows Int(M),$$

both of them having the natural inclusion $Int(M) \longrightarrow Int(M^I)$ as a section. We have thus constructed a commutative diagram in $dg - Cat_{\mathbb{V}}$

$$\begin{array}{ccc} & & Int(M) \\ & \overset{u}{\nearrow} & \uparrow \\ C & \xrightarrow{\alpha} & C' \\ & \underset{v}{\searrow} & \downarrow \\ & & Int(M) \end{array}$$

which provides a homotopy between u and v in $dg - Cat_{\mathbb{V}}$.

For the case where α is a fibration in M^C, one uses the same argument, but endowing M^I with its injective model structure, for which equivalences and cofibrations are defined levelwise. We leave the details to the reader. $\Box$

We have finished the proof of sub-lemma 4.5, which applied to $M = D^{op} - Mod_{\mathbb{U}}$ finishes the proof of the injectivity on connected components, and thus of lemma 4.3. $\Box$

In order to finish the proof of the theorem, one uses the functoriality of the morphisms ϕ and ψ with respect to D. First of all, the simplicial set $Map^l(C, D) = Hom(\Gamma^*(C), D)$ is obviously functorial in D. One thus has a functor

$$\begin{array}{rccc} Map^l(C,-): & dg - Cat_{\mathbb{U}} & \longrightarrow & SSet_{\mathbb{V}} \\ & D & \mapsto & Map^l(C,D). \end{array}$$

The functoriality of $N(\mathcal{M}(C, D))$ in D is slightly more complicated. Let $u : D \longrightarrow E$ a morphism in $dg - Cat_{\mathbb{U}}$. One has a functor

$$(Id \otimes u_!) : (C \otimes D^{op}) - Mod_{\mathbb{U}} \longrightarrow (C \otimes E^{op}) - Mod_{\mathbb{U}}.$$

This functor can also be described as

$$(u_!)^C : (D^{op} - Mod_{\mathbb{U}})^C \longrightarrow (E^{op} - Mod_{\mathbb{U}})^C,$$

the natural extension of the functor $u_! : D^{op} - Mod_{\mathbb{U}} \longrightarrow E^{op} - Mod_{\mathbb{U}}$. Clearly, the functor $(u_!)^C$ sends the sub-category $\mathcal{M}(C, D)$ to the sub-category $\mathcal{M}(C, E)$ (here one uses that $(u_!)^C$ preserves equivalences because the object $F \in \mathcal{M}(C, D)$ are such that $F(x, -)$ is cofibrant in $D^{op} - Mod_{\mathbb{U}}$). Unfortunately, this does not define a presheaf of categories $\mathcal{M}(C, -)$ on $dg - Cat_{\mathbb{U}}$, as for two morphisms

$$D \xrightarrow{u} E \xrightarrow{v} F$$

of dg-categories one only has a natural isomorphism $(v \circ u)_! \simeq (v_!) \circ u_!$ which in general is not an identity. However, these natural isomorphisms makes $D \mapsto \mathcal{M}(C, D)$ into a lax functor from $dg - Cat_{\mathbb{U}}$ to $Cat_{\mathbb{V}}$. Using the standard rectification procedure, one can replace up to a natural equivalence the lax functor $\mathcal{M}(C, -)$ by a true presheaf of categories $\mathcal{M}'(C, -)$. Furthermore, the natural morphism

$$Hom(C, D) \longrightarrow \mathcal{M}(C, D)$$

from the set of morphisms $Hom(C, D)$, considered as a discrete category, to the category $\mathcal{M}(C, D)$ clearly gives a morphism of lax functors

$$Hom(C, -) \longrightarrow \mathcal{M}(C, -).$$

By rectification this also induces a natural morphism of presheaves of categories

$$Hom(C, -) \longrightarrow \mathcal{M}'(C, -).$$

Passing to the nerve one gets a morphism of functors from $dg - Cat_{\mathbb{U}}$ to $SSet_{\mathbb{V}}$

$$Hom(C, -) \longrightarrow N(\mathcal{M}'(C, -)).$$

This morphism being functorial in C give a diagram in $(SSet_{\mathbb{V}})^{dg-Cat_{\mathbb{U}}}$

$$Map^l(C, -) = Hom(\Gamma^*(C), -) \xrightarrow{\phi'} d(N(\mathcal{M}'(\Gamma^*(C), -))) \xleftarrow{\psi'} N(\mathcal{M}'(Q(C), -)).$$

These morphisms, evaluated at an object $D \in dg - Cat_{\mathbb{U}}$ gives a diagram of simplicial sets

$$Map^l(C, D) \longrightarrow d(N(\mathcal{M}'(\Gamma^*(C), D)) \longleftarrow N(\mathcal{M}'(Q(C), D)),$$

weakly equivalent to the diagram

$$Map^l(C, D) \longrightarrow d(N(\mathcal{M}(\Gamma^*(C), D)) \longleftarrow N(\mathcal{M}(Q(C), D)).$$

In order to finish the proof of the theorem it is therefore enough to show that the two morphism ϕ' and ψ' are weak equivalences of diagrams of simplicial sets. We already know that ψ' is a weak equivalence, and thus we obtain a morphism well defined in $Ho((SSet_{\mathbb{V}})^{dg-Cat_{\mathbb{U}}})$

$$k : (\psi')^{-1} \circ \phi : Map^l(C, -) \longrightarrow N(\mathcal{M}'(Q(C), -)).$$

Using our corollary 3.4 it is easy to see that the functor $N(\mathcal{M}'(Q(C), -))$ sends quasi-equivalences to weak equivalences. Furthermore, the standard properties of mapping spaces imply that so does the functor $Map^l(C, -)$.

Sub-lemma 4.6 *Let $k : F \longrightarrow G$ be a morphism in $(SSet_{\mathbb{V}})^{dg-Cat_{\mathbb{U}}}$. Assume the following conditions are satisfied.*

1. *Both functors F and G send quasi-equivalences to weak equivalences.*

2. *For any diagram in* $dg - Cat_{\mathbb{U}}$

$$\begin{array}{ccc} & & C \\ & & \downarrow p \\ D & \longrightarrow & E \end{array}$$

with p *a fibration, the commutative diagrams*

$$\begin{array}{ccc} F(C \times_E D) & \longrightarrow & F(C) \\ \downarrow & & \downarrow \\ F(D) & \longrightarrow & F(E) \end{array} \qquad \begin{array}{ccc} G(C \times_E D) & \longrightarrow & G(C) \\ \downarrow & & \downarrow \\ G(D) & \longrightarrow & G(E) \end{array}$$

are homotopy cartesian.

3. $F(*) \simeq G(*) \simeq *$*, where* $*$ *is the final object in* $dg - Cat$*.*

4. *For any* $C \in dg - Cat_{\mathbb{U}}$ *the morphism* $k_C : \pi_0(F(C)) \longrightarrow \pi_0(G(C))$ *is an isomorphism.*

Then, for any $C \in dg - Cat_{\mathbb{U}}$ *the natural morphism*

$$k_C : F(C) \longrightarrow G(C)$$

is a weak equivalence.

Proof of sub-lemma 4.6: Condition (1) implies that the induced functors

$$Ho(F), Ho(G) : Ho(dg - Cat_{\mathbb{U}}) \longrightarrow Ho(SSet_{\mathbb{V}})$$

have natural structures of $Ho(SSet_{\mathbb{U}})$-enriched functors (see for example [HAGI, Thm. 2.3.5]). In particular, for any $K \in Ho(SSet_{\mathbb{U}})$, and any $C \in Ho(dg - Cat_{\mathbb{U}})$ one has natural morphisms in $Ho(SSet_{\mathbb{U}})$

$$F(C^{\mathbb{R}K}) \longrightarrow Map(K, F(C)) \qquad G(C^{\mathbb{R}K}) \longrightarrow Map(K, G(C)).$$

Our hypothesis (2) and (3) tells us that when K is a finite simplicial set, these morphisms are in fact isomorphisms, as the object $C^{\mathbb{R}K}$ can be functorially constructed using successive homotopy products and homotopy fiber products. Therefore, conditions (4) implies that for any finite $K \in Ho(SSet_{\mathbb{U}})$ and any $C \in dg - Cat_{\mathbb{U}}$, the morphism k_C induces an isomorphism

$$k_{C^{\mathbb{R}K}} : \pi_0(F(C^{\mathbb{R}K})) \simeq [K, F(C)] \longrightarrow [K, G(C)] \simeq \pi_0(G(C^{\mathbb{R}K})).$$

This of course implies that $F(C) \longrightarrow G(C)$ is a weak equivalence. □

In order to finish the proof of theorem 4.2 it remains to show that the two functors $Map^l(C, -)$ and $N(\mathcal{M}'(Q(C), -))$ satisfy the conditions of sub-lemma 4.6. The case of $Map^l(C, -)$ is clear by the standard properties of mapping spaces (see [Ho1, §5.4] or [Hi, §17]). It only remains to show property (2) of sub-lemma 4.6 for the functor $N(\mathcal{M}'(Q(C), -))$.

Sub-lemma 4.7 *Let C be a cofibrant $\mathbb{U}$-small dg-category, and let*

$$\begin{array}{ccc} D & \xrightarrow{u} & D_1 \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle p} \\ D_2 & \xrightarrow[q]{} & D_3 \end{array}$$

be a cartesian diagram in $dg-Cat_{\mathbb{U}}$ with p a fibration. Then, the square

$$\begin{array}{ccc} N(\mathcal{M}'(C,D)) & \longrightarrow & N(\mathcal{M}'(C,D_1)) \\ \downarrow & & \downarrow \\ N(\mathcal{M}'(C,D_2)) & \longrightarrow & N(\mathcal{M}'(C,D_3)) \end{array}$$

is homotopy cartesian.

Proof: We start by showing that the morphism

$$N(\mathcal{M}'(C,D)) \longrightarrow N(\mathcal{M}'(C,D_1)) \times^h_{N(\mathcal{M}'(C,D_3))} N(\mathcal{M}'(C,D_2))$$

induces an injection on π_0 and an isomorphism on all π_i for $i > 0$. For this, we consider the induced diagram of dg-categories

$$\begin{array}{ccc} C \otimes D^{op} & \xrightarrow{u} & C \otimes D_1^{op} \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle p} \\ C \otimes D_2^{op} & \xrightarrow[q]{} & C \otimes D_3^{op}, \end{array}$$

where we keep the same names for the induced morphisms after tensoring with C. It is then enough to show that for F and G in $\mathcal{M}(C,D)$ the square of path spaces

$$\begin{array}{ccc} \Omega_{F,G}N(\mathcal{M}'(C,D)) & \longrightarrow & \Omega_{u_!F,u_!G}N(\mathcal{M}'(C,D_1)) \\ \downarrow & & \downarrow \\ \Omega_{v_!F,v_!G}N(\mathcal{M}'(C,D_2)) & \longrightarrow & \Omega_{w_!F,w_!G}N(\mathcal{M}'(C,D_3)), \end{array}$$

is homotopy cartesian (where $w = p \circ u$). Using the natural equivalence between path spaces in nerves of sub-categories of equivalences in model categories and mapping spaces of equivalences (see [D-K1], and also [HAGII, Appendix A]), one finds that the previous diagram is in fact equivalent to the following one

$$\begin{array}{ccc} Map^{eq}(F,G) & \longrightarrow & Map^{eq}(u_!F,u_!G) \\ \downarrow & & \downarrow \\ Map^{eq}(v_!F,v_!G) & \longrightarrow & Map^{eq}(w_!F,w_!G), \end{array}$$

where Map^{eq} denotes the sub-simplicial set of the mapping spaces consisting of all connected components corresponding to equivalences. By adjunction, this last diagram is equivalent to

$$\begin{array}{ccc} Map^{eq}(F,G) & \longrightarrow & Map^{eq}(F,u^*u_!G) \\ \downarrow & & \downarrow \\ Map^{eq}(F,v^*v_!G) & \longrightarrow & Map^{eq}(F,w^*w_!G). \end{array}$$

Therefore, to show that this last square is homotopy cartesian, it is enough to prove that for any $G \in \mathcal{M}(C,D)$ the natural morphism

$$G \longrightarrow u^*u_!G \times^h_{w^*w_!G} v^*v_!G$$

is an equivalence in $C \otimes D^{op} - Mod_{\mathbb{U}}$. As this can be tested by fixing some object $x \in C$ and considering the corresponding morphism

$$G(x,-) \longrightarrow (u^*u_!G \times^h_{w^*w_!G} v^*v_!G)(x,-)$$

in $D^{op} - Mod_{\mathbb{U}}$, we see that one can assume that $C = \mathbf{1}$. One can then write $G = \underline{h}_x$ for some point $x \in D$. For $z \in D$, one has natural isomorphisms

$$u^*u_!G(z) = D_1(u(z),u(x)) \qquad v^*v_!G(z) = D_2(v(z),v(x)) \qquad w^*w_!G(z) = D_3(w(z),w(x)).$$

We therefore find that for any $z \in D$ the morphism

$$G(z) \longrightarrow (u^*u_!G \times^h_{w^*w_!G} v^*v_!G)(z)$$

can be written as

$$D(z,x) \longrightarrow D_1(u(z),u(x)) \times^h_{D_3(w(z),w(x))} D_2(v(z),v(x)),$$

which by assumption on the morphism p is a quasi-isomorphism of complexes. This implies that the morphism

$$G \longrightarrow u^*u_!G \times^h_{w^*w_!G} v^*v_!G$$

is an equivalence, and thus that

$$N(\mathcal{M}'(C,D)) \longrightarrow N(\mathcal{M}'(C,D_1)) \times^h_{N(\mathcal{M}'(C,D_3))} N(\mathcal{M}'(C,D_2))$$

induces an injection on π_0 and an isomorphisms on all π_i for $i > 0$. It only remains to show that the above morphism is also surjective on connected components.

The set $\pi_0(N(\mathcal{M}'(C,D_1)) \times^h_{N(\mathcal{M}'(C,D_3))} N(\mathcal{M}'(C,D_2)))$ can be described in the following way. We consider a category $\mathcal{N}$ whose objects are 5-tuples $(F_1,F_2,F_3;a,b)$, with $F_i \in \mathcal{M}(C,D_i)$ and where a and b are two morphisms in $\mathcal{M}(C,D_3)$

$$a : p_!(F_1) \longrightarrow F_3 \longleftarrow q_!(F_2) : b.$$

Morphisms in $\mathcal{N}$ are defined in the obvious way, as morphisms $F_i \to G_i$ in $\mathcal{M}(C, D_i)$, commuting with the morphisms a and b. It is not hard to check that $\pi_0(N(\mathcal{N}))$ is naturally isomorphic to $\pi_0(N(\mathcal{M}'(C, D_1)) \times^h_{N(\mathcal{M}'(C,D_3))} N(\mathcal{M}'(C, D_2)))$. Furthermore, the natural map

$$\pi_0(N(\mathcal{M}(C, D))) \longrightarrow \pi_0(N(\mathcal{N}))$$

is induced by the functor $\mathcal{M}(C, D) \longrightarrow \mathcal{N}$ that sends an object $F \in \mathcal{M}(C, D)$ to $(u_!F, v_!F, w_!F; a, b)$ where a and b are the two natural isomorphisms

$$p_!u_!(F) \simeq w_!(F) \simeq q_!v_!(F).$$

Now, let $(F_1, F_2, F_3; a, b) \in \mathcal{N}$, and let us define an object $F \in Ho((C \otimes D^{op}) - Mod_{\mathbb{U}})$ by the following formula

$$F := u^*(F_1) \times^h_{w^*(F_3)} v^*(F_1).$$

Clearly, one has natural morphisms in $Ho(C \otimes D_i^{op} - Mod_{\mathbb{U}})$

$$\mathbb{L}u_!(F) \to F_1 \qquad \mathbb{L}v_!(F) \to F_2 \qquad \mathbb{L}w_!(F) \to F_3.$$

We claim that F is right quasi-representable and that these morphisms are in fact isomorphisms. This will clearly finish the proof of the surjectivity on connected components. For this one can clearly assume that $C = \mathbf{1}$. One can then write $F_i = \underline{h}_{x_i}$, for some $x_i \in D_i$. As p is a fibration, the equivalence

$$a : p_!(\underline{h}_{x_1}) = \underline{h}_{p(x_1)} \longrightarrow \underline{h}_{x_3}$$

can be lifted to an equivalence $\underline{h}_{x_1} \longrightarrow \underline{h}_{x_1'}$ in $D_1^{op} - Mod$. Replacing x_1 by x_1' one can suppose that $p(x_1) = x_3$ and $a = id$. In the same way, the equivalence

$$b : q_!(\underline{h}_{x_2}) \longrightarrow \underline{h}_{p(x_1)}$$

can be lifted to an equivalence $\underline{h}_{x_1''} \longrightarrow \underline{h}_{x_1}$ in $D_1^{op} - Mod$. Thus, replacing x_1 by x_1'' one can suppose that $q(x_2) = p(x_1) = x_3$ and that a and b are the identity morphisms. Then, clearly $F \simeq \underline{h}_x$, where $x \in D$ is the point given by (x_1, x_2, x_3). This shows that F is right quasi-representable, and also that the natural morphisms

$$u_!(F) \to F_1 \qquad v_!(F) \to F_2 \qquad w_!(F) \to F_3$$

are equivalences. □

We have now finished the proof of sub-lemma 4.7 and thus of theorem 4.2. □

Recall that $\mathcal{M}(Q(C), D)$ has been defined as the category of equivalences between right quasi-representable $Q(C) \otimes D^{op}$-modules F such that $F(x, -)$ is cofibrant in $D^{op} - Mod$ for any $x \in C$. This last condition is only technical and useful for functorial reasons and does not affect the nerve. Indeed, let $\mathcal{F}(Q(C), D)$ be the category of all equivalences between right quasi-representable $(Q(C) \otimes D^{op})$-modules. The natural inclusion functor

$$\mathcal{M}(Q(C), D) \longrightarrow \mathcal{F}(Q(C), D)$$

induces a weak equivalence on the corresponding nerves as there exists a functor in the other direction just by taking a cofibrant replacement (note that a cofibrant $(Q(C) \otimes D^{op})$-module F is such that $F(x,-)$ is cofibrant for any $x \in Q(C)$, because of Prop. 3.3). In particular, theorem 4.2 implies the existence of a string of weak equivalences

$$Map^l(C,D) \longrightarrow d(N(\mathcal{M}(\Gamma^*(C),D))) \longleftarrow N(\mathcal{M}(Q(C),D)) \longrightarrow N(\mathcal{F}(Q(C),D)).$$

The following corollary is a direct consequence of theorem 4.2 and the above remark.

Corollary 4.8 *Let C and D be two $\mathbb{U}$-small dg-categories. Then, there exists a functorial bijection between the set of maps $[C,D]$ in $Ho(dg-Cat_{\mathbb{U}})$, and the set of isomorphism classes of right quasi-representable objects in $Ho((C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}})$.*

Another important corollary of Theorem 4.2 is the following.

Corollary 4.9 *Let C be a $\mathbb{U}$-small dg-categories. Then, there exists a functorial isomorphism between the set $[\mathbf{1}, C]$ and the set of isomorphism classes of the category $[C]$.*

Proof: The Yoneda embedding $\underline{h} : C \longrightarrow Int(C^{op} - Mod_{\mathbb{U}})$ induces a fully faithful functor

$$[C] \longrightarrow [Int(C^{op} - Mod_{\mathbb{U}})].$$

The essential image of this functor clearly is the sub-category of quasi-representable C^{op}-modules. Therefore, $[\underline{h}]$ induces a natural bijection between the isomorphism classes of $[C]$ and the isomorphism classes of quasi-representable objects in $[Int(C^{op} - Mod_{\mathbb{U}})]$. As one has a natural equivalence $[Int(C^{op} - Mod_{\mathbb{U}})] \simeq Ho(C^{op} - Mod_{\mathbb{U}})$ corollary 4.8 implies the result. $\square$

More generally, one can describe the higher homotopy groups of the mapping spaces by the following formula.

Corollary 4.10 *Let C be a $\mathbb{U}$-small dg-category, and $x \in C$ be an object. Then, one has natural isomorphisms of groups*

$$\pi_1(Map(\mathbf{1}, C), x) \simeq Aut_{[C]}(x) \qquad \pi_i(Map(\mathbf{1}, C), x) \simeq H^{1-i}(C(x,x)) \; \forall\, i > 1.$$

Proof: We use the general formula

$$\pi_1(N(W), x) \simeq Aut_{Ho(M)}(x) \qquad \pi_i(N(W), x) \simeq \pi_{i-1}(Map_M(x,x), Id) \; \forall\, i > 1,$$

for a model category M, its sub-category of equivalences W and a point $x \in M$ (see e.g. [HAGII, Cor. A.0.4]). Applied to $M = C^{op} - Mod_{\mathbb{U}}$ and using theorem 4.2 one finds

$$\pi_1(Map(\mathbf{1}, C), x) \simeq Aut_{Ho(C^{op}-Mod)}(\underline{h}_x) \qquad \pi_i(Map(\mathbf{1}, C), x) \simeq \pi_{i-1}(Map_{C^{op}-Mod}(\underline{h}_x, \underline{h}_x), Id) \; \forall\, i > 1.$$

Using that the morphism $\underline{h}$ is quasi-fully faithful one finds

$$Aut_{Ho(C^{op}-Mod)}(\underline{h}_x) \simeq Aut_{[C]}(x) \qquad \pi_{i-1}(Map_{C^{op}-Mod}(\underline{h}_x, \underline{h}_x), Id) \simeq H^{1-i}(C(x,x)).$$

$\square$

Corollary 4.11 *Let C and D be two $\mathbb{U}$-small dg-categories. Let $Int((C\otimes^{\mathbb{L}} D^{op})-Mod_{\mathbb{U}}^{rqr})$ be the full sub-dg-category of $Int((C\otimes^{\mathbb{L}} D^{op})-Mod_{\mathbb{U}})$ consisting of all right quasi-representable objects. Then, $Int((C\otimes^{\mathbb{L}} D^{op})-Mod_{\mathbb{U}}^{rqr})$ is isomorphic in $Ho(dg-Cat_{\mathbb{V}})$ to a $\mathbb{U}$-small dg-category.*

Proof: Indeed, we know by corollary 4.8 that the set of isomorphism classes of $[Int((C\otimes^{\mathbb{L}} D^{op})-Mod_{\mathbb{U}}^{rqr})]$ is essentially $\mathbb{U}$-small, as it is in bijection with $[C,D]$. Let us choose an essentially $\mathbb{U}$-small full sub-dg-category E in $Int((C\otimes^{\mathbb{L}} D^{op})-Mod_{\mathbb{U}}^{rqr})$ which contains a set of representatives of isomorphism classes of objects. As we already know that the complexes of morphisms in $Int((C\otimes^{\mathbb{L}} D^{op})-Mod_{\mathbb{U}}^{rqr})$ are $\mathbb{U}$-small, the dg-category E is essentially $\mathbb{U}$-small, and thus isomorphic to a $\mathbb{U}$-small dg-category. As E is quasi-equivalent to $Int((C\otimes^{\mathbb{L}} D^{op})-Mod_{\mathbb{U}}^{rqr})$ this implies the result. $\Box$

We finish by the following last corollary.

Corollary 4.12 *Let C and D be two $\mathbb{U}$-small dg-categories, and let $f,g : C \longrightarrow D$ be two morphisms with corresponding $(C\otimes^{\mathbb{L}} D^{op})$-modules $\phi(f)$ and $\phi(g)$. Then, there exists a natural weak equivalence of simplicial sets*

$$\Omega_{f,g}Map_{dg-Cat}(C,D)\simeq Map^{eq}_{(C\otimes^{\mathbb{L}} D^{op})-Mod}(\phi(f),\phi(g)),$$

where $Map^{eq}(\phi(f),\phi(g))$ is the sub-simplicial set of $Map(\phi(f),\phi(g))$ consisting of equivalences.

Proof: This follows immediately from theorem 4.2 and the standard relations between path spaces of nerves of equivalences in a model category and its mapping spaces (see e.g. [HAGII, Appendix A]). $\Box$

5 The simplicial structure

Let $K\in SSet_{\mathbb{U}}$ be a $\mathbb{U}$-small simplicial set and $C\in dg-Cat_{\mathbb{U}}$. One can form the derived tensor product $K\otimes^{\mathbb{L}} C\in Ho(dg-Cat_{\mathbb{U}})$, as well as the derived exponential $C^{\mathbb{R}K}$. One has the usual adjunction isomorphism

$$[K\otimes^{\mathbb{L}} C,D]\simeq [C,D^{\mathbb{R}K}]\simeq [K,Map(C,D)].$$

Let $\Delta(K)$ be the simplex category of K. An object of $\Delta(K)$ is therefore a pair (n,a) with $n\in\Delta$ and $x\in K_n$. A morphism $(n,x)\to(m,y)$ is the data of a morphism $u:[n]\to[m]$ in Δ such that $u^*(y)=x$. The simplicial set K is then naturally weakly equivalent to the homotopy colimit of the constant diagram

$$\Delta(K)\longrightarrow *\in SSet.$$

In other words, one has a natural weak equivalence

$$N(\Delta(K))\simeq K.$$

We now consider $\Delta(K)_k$ the k-linear category freely generated by the category $\Delta(K)$, and consider $\Delta(K)_k$ as an object in $dg-Cat_{\mathbb{U}}$.

Theorem 5.1 *Let C and D be two $\mathbb{U}$-small dg-categories, and $K \in SSet_{\mathbb{U}}$. Then, there exists a functorial injective map*

$$[K \otimes^{\mathbb{L}} C, D] \longrightarrow [\Delta(K)_k \otimes^{\mathbb{L}} C, D].$$

Moreover, the image of this map consists exactly of all morphism $\Delta(K)_k \otimes^{\mathbb{L}} C \longrightarrow D$ in $Ho(dg-Cat_{\mathbb{U}})$ such that for any $c \in C$ the induced functor

$$\Delta(K)_k \longrightarrow [D]$$

sends all morphisms in $\Delta(K)_k$ to isomorphisms in $[D]$.

Proof: Using our theorem 4.2 one finds natural equivalences

$$[K \otimes^{\mathbb{L}} C, D] \simeq [K, Map(C, D)] \simeq [K, N(\mathcal{M}(Q(C), D))].$$

We then use the next technical lemma.

Lemma 5.2 *Let M be a $\mathbb{V}$-small $\mathbb{U}$-combinatorial model category and $K \in SSet_{\mathbb{U}}$. Let $W \subset M$ be the sub-category of equivalences in M. Then, there exists a natural bijection between $[K, N(W)]_{SSet_{\mathbb{V}}}$ and the set of isomorphism classes of objects $F \in Ho(M^{\Delta(K)})$ corresponding to functors $F : \Delta(K) \longrightarrow M$ sending all morphisms of $\Delta(K)$ to equivalences in M.*

Proof: First of all, the lemma is invariant by changing M up to a Quillen equivalence, and thus by [Du] one can suppose that M is a simplicial model category. The proof of the lemma will use some techniques of simplicial localizations à la Dwyer-Kan, as well as some result about S-categories. We start by a short digression on the subject.

We recall the existence of a model category of S-categories, as shown in [Be], and which is similar to the one we use on dg-categories. This model category will be denoted by $S-Cat$ (or $S-Cat_{\mathbb{V}}$ if one needs to specify the universe). For any $\mathbb{V}$-small category C with a sub-category $S \subset C$, one can form a $\mathbb{V}$-small S-category $L(C, S)$ by formally inverting the morphisms in S in a homotopy meaningful way (see e.g. [D-K2]). Using the language of model categories, this means that for any $\mathbb{V}$-small S-category T, there exists functorial isomorphisms between $[L(C, S), T]_{S-Cat}$ and the subset of $[C, T]_{S-Cat}$ consisting of all morphisms sending S to isomorphisms in $[T]$ (the category $[T]$ is defined by taking connected component of simplicial sets of morphisms in T). Finally, one can define a functor $N : Ho(S-Cat_{\mathbb{V}}) \longrightarrow Ho(SSet_{\mathbb{V}})$ by sending an S-category to its nerve. It is well known that the functor N becomes an equivalence when restricted to S-categories T such that $[T]$ is a groupoid (this is just another way to state delooping theory). Finally, for any category C with a sub-category $S \subset C$, one has a natural weak equivalence $N(L(C, S)) \simeq N(C)$.

Now, as explained in [HAGII, Prop. A.0.6], $N(W)$ can be also interpreted as the nerve of the S-category $\mathcal{G}(M)$, of cofibrant and fibrant objects in M together with their simplicial sets of equivalences. One therefore has natural isomorphism

$$[K, N(W)] \simeq [N(\Delta(K)), N(\mathcal{G}(M))] \simeq [L(\Delta(K), \Delta(K)), \mathcal{G}(M)].$$

Furthermore, as all morphisms in $[\mathcal{G}(M)]$ are isomorphisms one finds a bijection between $[K, N(W)]$ and $[\Delta(K), \mathcal{G}(M)]$. Let $Int(M)$ be the S-category of fibrant and cofibrant objects in M together with their simplicial sets of morphisms. Then, as $\mathcal{G}(M)$ is precisely the sub-S-category of $Int(M)$ consisting of equivalences, the set $[\Delta(K), \mathcal{G}(M)]$ is also the subset of $[\Delta(K), Int(M)]$ consisting of all morphisms such that the induced functor $\Delta(K) \longrightarrow [Int(M)] \simeq Ho(M)$ sends all morphisms to isomorphisms. Finally, it turns out that the same results as our lemmas 4.4 and 4.5 are valid in the context of S-categories (their proofs are exactly the same). Therefore, we see that $[\Delta(K), Int(M)]$ is in a natural bijection with isomorphism classes of objects in $Ho(M^{\Delta(K)})$. Putting all of this together gives the lemma. $\Box$

We apply the previous lemma to the case where $M := (C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}$, and we find a natural injection $[K, N(W)] \hookrightarrow Iso(Ho(M^{\Delta(K)}))$, whose image consists of all functors $\Delta(K) \rightarrow M$ sending all morphisms of $\Delta(K)$ to equivalences in M. Composing with the natural inclusion $\mathcal{M}(Q(C), D) \subset M$ provides a natural injection of

$$[K, N(\mathcal{M}(Q(C), D))] \subset [K, N(W)] \subset Iso(Ho(M^{\Delta(K)})).$$

By the construction of the bijection of lemma 5.2 one easily sees that the image of this inclusion consists of all functors $F : \Delta(K) \longrightarrow W$ such that for any $k \in K$ one has $F(k) \in \mathcal{M}(Q(C), D)$. Finally, one clearly has a natural equivalence of categories, compatible with the model structures

$$M^{\Delta(K)} \simeq (C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}^{\Delta(K)_k} \simeq (\Delta(K)_k \otimes C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}},$$

inducing a bijection between $Iso(Ho(M^{\Delta(K)}))$ and the isomorphism classes of objects in $Ho((\Delta(K)_k \otimes C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}})$. Another application of theorem 4.2 easily implies the result. $\Box$

6 Internal Hom's

Let us recall that $Ho(dg - Cat_{\mathbb{U}})$ is endowed with the symmetric monoidal structure $\otimes^{\mathbb{L}}$. Recall that the monoidal structure $\otimes^{\mathbb{L}}$ is said to be closed if for any two objects C and D in $Ho(dg - Cat_{\mathbb{U}})$ the functor $A \mapsto [A \otimes^{\mathbb{L}} C, D]$ is representable by an object $\mathbb{R}\underline{Hom}(C, D) \in Ho(dg - Cat_{\mathbb{U}})$. Recall also from corollary 4.11 that the $\mathbb{V}$-small dg-category $Int((C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}^{rqr})$ is essentially $\mathbb{U}$-small and therefore can be considered as an object in $Ho(dg - Cat_{\mathbb{U}})$.

Theorem 6.1 *The monoidal category $(Ho(dg - Cat_{\mathbb{U}}), \otimes^{\mathbb{L}})$ is closed. Furthermore, for any two $\mathbb{U}$-small dg-categories C and D one has a natural isomorphism in $Ho(dg - Cat_{\mathbb{U}})$*

$$\mathbb{R}\underline{Hom}(C, D) \simeq Int((C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}^{rqr}).$$

Proof: The proof is essentially the same as for theorem 4.2 and is also based on the same lemmas 4.4 and 4.5. Indeed, from these two lemmas one extracts the following result.

Lemma 6.2 *Let M be $C(k)_{\mathbb{U}}$-enriched $\mathbb{U}$-cofibrantly generated model category which is $\mathbb{V}$-small. We assume that the domain and codomain of a set of generating cofibrations are cofibrant in*

M. Let M_0 be a full sub-category of M which is closed by equivalences, and $Int(M_0)$ be the full sub-dg-category of $Int(M)$ consisting of all objects belonging to M_0. Let A be a cofibrant and $\mathbb{U}$-small dg-category, and let $Ho(M_0^A)$ be the full sub-category of $Ho(M^A)$ consisting of objects $F \in Ho(M^A)$ such that $F(a) \in M_0$ for any $a \in A$. Then, one has a natural isomorphism

$$\phi : [A, Int(M_0)] \simeq Iso(Ho(M_0^A)).$$

Proof: The morphism

$$\phi : [A, Int(M_0)] \longrightarrow Iso(Ho(M_0^A))$$

simply sends a morphism $A \longrightarrow Int(M_0)$ to the corresponding object in M_0^A. Using our proposition 3.2 it is easy to see that this maps sends homotopic morphisms to isomorphic objects in $Ho(M_0^A)$, and is therefore well defined. As for the proof of lemma 4.4, the morphism ϕ is clearly surjective. Let $u, v : A \longrightarrow Int(M_0)$ be two morphisms of dg-categories such that the corresponding objects in $Ho(M_0^A)$ are isomorphic. Then, these objects are isomorphic in $Ho(M^A)$, which implies by lemma 4.5 that the two compositions

$$u', v' : A \longrightarrow Int(M_0) \longrightarrow Int(M)$$

are homotopic in $dg - Cat_{\mathbb{V}}$. Let

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow^{u'} & \\ A' & \xrightarrow{H} & Int(M) \\ \uparrow & \nearrow_{v'} & \\ A & & \end{array}$$

be a homotopy between u' and v'. As M_0 is closed by equivalences in M one clearly sees that the morphism H factors through the sub-dg-category $Int(M_0)$, showing that u and v are homotopic.
$\Box$

We come back to the proof of theorem 6.1. Using our theorem 4.2 one has a natural isomorphism

$$[A \otimes^{\mathbb{L}} C, D] \simeq Iso(Ho(((A \otimes^{\mathbb{L}} C) \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}^{rqr})) \simeq Iso(Ho(((C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}^{rqr})^A)).$$

An application of lemma 6.2 (with $M = (C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}$ and M_0 the full sub-category of right quasi-representable objects) shows that one has a natural isomorphism

$$[A, Int((C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}^{rqr})] \simeq Iso(Ho(((C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}^{rqr})^A)).$$

Putting this together one finds a natural isomorphism

$$[A \otimes^{\mathbb{L}} C, D] \simeq [A, Int((C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}^{rqr})]$$

showing the theorem. $\Box$

Corollary 6.3 *For any C and D two $\mathbb{U}$-small dg-categories, and any $K \in SSet_{\mathbb{U}}$, one has a functorial isomorphism in $Ho(dg-Cat_{\mathbb{U}})$*

$$K \otimes^{\mathbb{L}} (C \otimes^{\mathbb{L}} D) \simeq (K \otimes^{\mathbb{L}} C) \otimes^{\mathbb{L}} D.$$

Proof: This follows easily from Thm. 5.1, Thm. 6.1 and the Yoneda lemma applied to $Ho(dg-Cat_{\mathbb{U}})$. $\Box$

Corollary 6.4 *For any C, D and E three $\mathbb{U}$-small dg-categories one has a functorial isomorphism in $Ho(SSet_{\mathbb{U}})$*

$$Map(C \otimes^{\mathbb{L}} D, E) \simeq Map(C, \mathbb{R}\underline{Hom}(D, E)).$$

Proof: By Cor. 6.3, for any $K \in SSet_{\mathbb{U}}$, one has functorial isomorphisms

$$[K, Map(C \otimes^{\mathbb{L}} D, E)] \simeq [K \otimes^{\mathbb{L}} (C \otimes^{\mathbb{L}} D), E] \simeq [(K \otimes^{\mathbb{L}} C) \otimes^{\mathbb{L}} D, E] \simeq$$

$$[K \otimes^{\mathbb{L}} C, \mathbb{R}\underline{Hom}(D, E)] \simeq [K, Map(C, \mathbb{R}\underline{Hom}(D, E))].$$

$\Box$

Corollary 6.5 *Let $C \in dg-Cat_{\mathbb{U}}$ be a dg-category. Then the functor*

$$- \otimes^{\mathbb{L}} C : dg-Cat_{\mathbb{U}} \longrightarrow dg-Cat_{\mathbb{U}}$$

commutes with homotopy colimits.

Proof: This follows formally from Cor. 6.4. $\Box$

Corollary 6.6 *Let $C \longrightarrow D$ be a quasi-fully faithful morphism in $dg-Cat_{\mathbb{U}}$. Then, for any $B \in dg-Cat_{\mathbb{U}}$ the induced morphism*

$$\mathbb{R}\underline{Hom}(B, C) \longrightarrow \mathbb{R}\underline{Hom}(B, D)$$

is quasi-fully faithful.

Proof: Using Lem. 2.4 it is enough to show that $\mathbb{R}\underline{Hom}(B, -)$ preserves homotopy monomorphisms. But this follows formally from Cor. 6.4. $\Box$

7 Morita morphisms and bi-modules

In this paragraph we will use the following notations. For any $C \in dg - Cat_{\mathbb{U}}$ one sets

$$\widehat{C} := Int(C^{op} - Mod_{\mathbb{U}}) \in dg - Cat_{\mathbb{V}}.$$

By theorem 6.1 and lemma 6.2, one has an isomorphism in $Ho(dg - Cat_{\mathbb{V}})$

$$\widehat{C} \simeq \mathbb{R}\underline{Hom}(C^{op}, Int(C(k)_{\mathbb{U}})) \in Ho(dg - Cat_{\mathbb{V}}).$$

Indeed, lemma 6.2 implies that for any $A \in dg - Cat_{\mathbb{U}}$ one has

$$[A, \widehat{C}] \simeq Iso(Ho((A \otimes^{\mathbb{L}} C^{op}) - Mod_{\mathbb{U}})) \simeq [A \otimes^{\mathbb{L}} C^{op}, \widehat{\mathbf{1}}].$$

Note also that

$$Int(C(k)_{\mathbb{U}}) \simeq \widehat{\mathbf{1}}.$$

We will also consider $\widehat{C}_{pe}$ the full sub-dg-category of $\widehat{C}$ consisting of C^{op}-modules which are homotopically finitely presented. In other words, a C^{op}-module F is in $\widehat{C}_{pe}$ if for any filtered diagram of objects G_i in $C^{op} - Mod_{\mathbb{U}}$, the natural morphism

$$Colim_i Map(F, G_i) \longrightarrow Map(F, Colim_i G_i)$$

is a weak equivalence. It is easy to check that the objects in $\widehat{C}_{pe}$ are precisely the objects equivalent to retracts of finite cell C^{op}-modules. To be more precise, an object $F \in Ho(\widehat{C})$ is in $Ho(\widehat{C}_{pe})$ if and only if it is a retract in $Ho(\widehat{C})$ of an object G for which there exists a finite sequence of morphisms of C^{op}-modules

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots \longrightarrow G_n = G,$$

in such a way that for any i there exists a push-out square

$$\begin{array}{ccc} G_i & \longrightarrow & G_{i+1} \\ \uparrow & & \uparrow \\ A \otimes \underline{h}_x & \longrightarrow & B \otimes \underline{h}_x \end{array}$$

for some $x \in C$, and some cofibration $A \to B$ in $C(k)$ with A and B bounded complexes of projective modules of finite type.

Objects in $\widehat{C}_{pe}$ will also be called *compact* or *perfect* (note that they are precisely the compact objects in the triangulated category $[\widehat{C}]$, in the usual sense). More generally, for any dg-category T, we will write T_{pe} for the full sub-dg-category of T consisting of compact objects (i.e. the objects x such that $[T](x, -)$ commutes with (infinite) direct sums).

Let us consider C and D two $\mathbb{U}$-small dg-categories, and $u : \widehat{C} \longrightarrow \widehat{D}$ a morphism in $Ho(dg - Cat_{\mathbb{V}})$. Then, u induces a functor, well defined up to an (non-unique) isomorphism

$$[u] : [\widehat{C}] \longrightarrow [\widehat{D}].$$

We will say that the morphism u is *continuous* if the functor $[u]$ commutes with $\mathbb{U}$-small direct sums. Note that $[\widehat{C}]$ and $[\widehat{D}]$ are the homotopy categories of the model categories of C^{op}-modules and D^{op}-modules, and thus these two categories always have direct sums. More generally, we will denote by $\mathbb{R}\underline{Hom}_c(\widehat{C}, \widehat{D})$ the full sub-dg-category of $\mathbb{R}\underline{Hom}(\widehat{C}, \widehat{D})$ consisting of continuous morphisms.

Definition 7.1 *Let C and D be two $\mathbb{U}$-small dg-categories.*

1. *The dg-category of* Morita morphisms *from C to D is $\mathbb{R}\underline{Hom}_c(\widehat{C}, \widehat{D})$.*
2. *The dg-category of* perfect Morita morphisms *from C to D is $\mathbb{R}\underline{Hom}(\widehat{C}_{pe}, \widehat{D}_{pe})$.*

We warn the reader that there are in general no relations between the dg-category $\mathbb{R}\underline{Hom}(\widehat{C}_{pe}, \widehat{D}_{pe})$ and $\mathbb{R}\underline{Hom}_c(\widehat{C}, \widehat{D})_{pe}$. An example where these two objects agree will be given in Thm. 8.15.

Theorem 7.2 *Let $C \in dg-Cat_{\mathbb{U}}$, and let us consider the Yoneda embedding $\underline{h} : C \longrightarrow \widehat{C}$. Let D be any $\mathbb{U}$-small dg-category.*

1. *The pull-back functor*
$$\underline{h}^* : \mathbb{R}\underline{Hom}_c(\widehat{C}, \widehat{D}) \longrightarrow \mathbb{R}\underline{Hom}(C, \widehat{D})$$
is an isomorphism in $Ho(dg-Cat_{\mathbb{V}})$.
2. *The pull-back functor*
$$\underline{h}^* : \mathbb{R}\underline{Hom}(\widehat{C}_{pe}, \widehat{D}_{pe}) \longrightarrow \mathbb{R}\underline{Hom}(C, \widehat{D}_{pe})$$
is an isomorphism in $Ho(dg-Cat_{\mathbb{V}})$.

Proof: We start by proving (1).

Using the universal properties of internal Hom's one reduces the problem to show that for any $A \in dg-Cat_{\mathbb{U}}$, the morphism[3]

$$l := \underline{h} : C \longrightarrow \widehat{C}$$

induces a bijective morphism

$$l^* : [\widehat{C} \otimes^{\mathbb{L}} A, \widehat{D}]_c \longrightarrow [C \otimes^{\mathbb{L}} A, \widehat{D}],$$

where by definition $[\widehat{C} \otimes^{\mathbb{L}} A, \widehat{D}]_c$ is the subset of $[\widehat{C} \otimes^{\mathbb{L}} A, \widehat{D}]$ consisting of morphisms $f : \widehat{C} \otimes^{\mathbb{L}} A \longrightarrow \widehat{D}$ such that for any object $a \in A$ the induced morphism $f(-, a) : \widehat{C} \longrightarrow \widehat{D}$ is continuous. Now, as $\widehat{D} = \mathbb{R}\underline{Hom}(D^{op}, \widehat{\mathbf{1}})$, one has natural bijections

$$[C \otimes^{\mathbb{L}} A, \widehat{D}] \simeq [C, \widehat{A^{op} \otimes^{\mathbb{L}} D}] \qquad [\widehat{C} \otimes^{\mathbb{L}} A, \widehat{D}]_c \simeq [\widehat{C}, \widehat{A^{op} \otimes^{\mathbb{L}} D}]_c.$$

[3]We prefer to change notation from $\underline{h}$ to l during the proof, just in order to avoid future confusions.

Therefore, we have to prove that for any A the induced morphism

$$l^* : [\widehat{C}, \widehat{A^{op} \otimes^{\mathbb{L}} D}]_c \longrightarrow [C, \widehat{A^{op} \otimes^{\mathbb{L}} D}],$$

is bijective. For this, we consider the quasi-fully faithful morphism in $dg - Cat_{\mathbb{W}}$ for some universe $\mathbb{V} \in \mathbb{W}$

$$\widehat{A^{op} \otimes^{\mathbb{L}} D} \simeq Int((A \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}) \longrightarrow \widehat{A^{op} \otimes^{\mathbb{L}} D}_{\mathbb{V}} := Int((A \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{V}}).$$

One has a commutative square

$$\begin{array}{ccc} [\widehat{C}, \widehat{A^{op} \otimes^{\mathbb{L}} D}]_c & \longrightarrow & [\widehat{C}, \widehat{A^{op} \otimes^{\mathbb{L}} D}_{\mathbb{V}}]_c \\ \downarrow & & \downarrow \\ [C, \widehat{A^{op} \otimes^{\mathbb{L}} D}] & \longrightarrow & [C, \widehat{A^{op} \otimes^{\mathbb{L}} D}_{\mathbb{V}}]. \end{array}$$

We claim that the right vertical morphism is bijective. For this, we use lemma 6.2 which implies that it is enough to show the following lemma.

Lemma 7.3 *Let C be a $\mathbb{U}$-small dg-category and M a $\mathbb{V}$-combinatorial $C(k)_{\mathbb{V}}$-model category which is $\mathbb{W}$-small for some $\mathbb{V} \in \mathbb{W}$. We assume that the domain and codomain of a set of generating cofibrations are cofibrant in M. We also assume that for any cofibrant object $X \in M$, and any quasi-isomorphism $Z \longrightarrow Z'$ in $C(k)$, the induced morphism*

$$Z \otimes X \longrightarrow Z' \otimes X$$

is an equivalence in M. Then, the Quillen adjunction

$$l_! : M^C \longrightarrow M^{\widehat{C}} \qquad M^C \longleftarrow M^{\widehat{C}} : l^*$$

induces a fully faithful functor

$$\mathbb{L}l_! : Ho(M^C) \longrightarrow Ho(M^{\widehat{C}})$$

whose essential image consists of all $\widehat{C}$-modules corresponding to continuous morphisms in $Ho(dg - Cat_{\mathbb{W}})$.

Proof: First of all, the modules $F \in Ho(M^{\widehat{C}})$ corresponding to continuous morphisms are precisely the ones for which for any $\mathbb{U}$-small family of objects $x_i \in \widehat{C}$, the natural morphism

$$\bigoplus^{\mathbb{L}} F(x_i) \longrightarrow F(\oplus_i x_i)$$

is an isomorphism in $Ho(M)$.

We start by showing that $\mathbb{L}l_!$ is fully faithful. As both functors $\mathbb{L}l_!$ and l^* commute with homotopy colimits, it is enough to show that for any $x \in C$ and any $X \in M$, the adjunction morphism

$$X \otimes^{\mathbb{L}} \underline{h}^x \longrightarrow l^* \mathbb{L}l_!(X \otimes^{\mathbb{L}} \underline{h}^x)$$

is an isomorphism in $Ho(M^C)$. But this follows immediately from the fact that the morphism of dg-categories l is fully faithful and our hypothesis on M.

It remains to show that for any $F \in Ho(M^{\widehat{C}})$, corresponding to a continuous morphism, the adjunction morphism

$$\mathbb{L}l_!l^*(F) \longrightarrow F$$

is an isomorphism in $Ho(M^{\widehat{C}})$. As we already know that $\mathbb{L}l_!$ is fully faithful it is enough to show that the functor l^* is conservative when restricted to the sub-category of modules corresponding to continuous functors. Let $u : F \longrightarrow G$ be morphism between such modules, and let us assume that $l^*(F) \longrightarrow l^*(G)$ is an isomorphism in $Ho(M^C)$. We need to show that u itself is an isomorphism in $Ho(M^{\widehat{C}})$.

Sub-lemma 7.4 *Let $F : \widehat{C} \longrightarrow M$ be a morphism of dg-categories corresponding to a continuous morphism.*

1. *Let $X : I \longrightarrow C^{op} - Mod_{\mathbb{U}}$ be a $\mathbb{U}$-small diagram of cofibrant objects in $C^{op} - Mod_{\mathbb{U}}$. Then, the natural morphism*
$$Hocolim_i F(X_i) \longrightarrow F(Hocolim_i X_i)$$
is an isomorphism in $Ho(M)$.

2. *Let $Z \in C(k)_{\mathbb{U}}$ and $X \in M$. Then, the natural morphism*
$$Z \otimes^{\mathbb{L}} F(X) \longrightarrow F(Z \otimes^{\mathbb{L}} X)$$
is an isomorphism in $Ho(M)$.

Proof of sub-lemma 7.4: (1) As any homotopy colimit is a composition of homotopy push-outs and infinite (homotopy) sums, it is enough to check the sub-lemma for one of these colimits. For the direct sum case this is our hypothesis on F. It remains to show that F commutes with homotopy push-outs. For this we assume that F is fibrant and cofibrant, and thus is given by a morphism of dg-categories $\widehat{C} \longrightarrow Int(M)$.

We consider the commutative diagram of dg-categories

$$\begin{array}{ccc} (\widehat{C})^{op} & \xrightarrow{F} & Int(M)^{op} \\ \downarrow & & \downarrow \\ Int(\widehat{C} - Mod_{\mathbb{V}}) & \xrightarrow[F_!]{} & Int(Int(M) - Mod_{\mathbb{V}}), \end{array}$$

where the vertical morphisms are the dual Yoneda embeddings $\underline{h}^{(-)}$. The functor $F_!$ being left Quillen clearly commutes, up to equivalences, with homotopy push-outs. Furthermore, as the model categories $\widehat{C} - Mod_{\mathbb{V}}$ and $Int(M) - Mod_{\mathbb{V}}$ are stable model categories, this implies that $F_!$ also commutes, up to equivalence, with homotopy pull-backs. Furthermore, the morphism $\underline{h}^{(-)}$ sends homotopy push-out squares to homotopy pull-back squares, and moreover a square in $Int(M)$ is a homotopy push-out square if and only if its image by $\underline{h}$ is a homotopy pull-back square in $Int(M) - Mod_{\mathbb{V}}$. We deduce from these remarks that F preserves homotopy push-out squares.

(2) Any complex Z can be constructed from the trivial complex k using homotopy colimits and loop objects. As we already know that F commutes with homotopy colimits, it is enough to see that it also commutes with loop objects. But the loop functor is inverse, up to equivalence, to the suspension functor. The suspension being a homotopy push-out, F commutes with it, and therefore F commutes with the loop functor. □

Now, let us come back to our morphism $u : F \longrightarrow G$ such that $l^*(u)$ is an equivalence. Let X be an object in $\widehat{C}$. We know that X can be written as the homotopy colimit of objects of the form $Z \otimes^{\mathbb{L}} \underline{h}_x$ with $x \in C$ and $Z \in C(k)$. Therefore, one has a commutative diagram in $Ho(M)$

$$\begin{array}{ccc} Hocolim_i F(Z_i \otimes^{\mathbb{L}} \underline{h}_{x_i}) & \xrightarrow{u} & Hocolim_i G(Z_i \otimes^{\mathbb{L}} \underline{h}_{x_i}) \\ \downarrow & & \downarrow \\ F(X) & \xrightarrow{u} & G(X). \end{array}$$

By the sub-lemma (1) the vertical morphisms are isomorphisms in $Ho(M)$, and the top horizontal morphism is also by hypothesis and the sub-lemma (2). Thus, the bottom horizontal morphism is an isomorphism in $Ho(M)$, and this for any $X \in \widehat{C}$. This shows that l^* is conservative when restricted to continuous morphisms, and thus finishes the proof of the lemma 7.3. □

We come back to our commutative diagram

$$\begin{array}{ccc} [\widehat{C}, \widehat{A^{op} \otimes^{\mathbb{L}} D}]_c & \longrightarrow & [\widehat{C}, \widehat{A^{op} \otimes^{\mathbb{L}} D_{\mathbb{V}}}]_c \\ \downarrow & & \downarrow \\ [C, \widehat{A^{op} \otimes^{\mathbb{L}} D}] & \longrightarrow & [C, \widehat{A^{op} \otimes^{\mathbb{L}} D_{\mathbb{V}}}]. \end{array}$$

Lemma 7.3 shows that the right vertical morphism is bijective, and corollary 2.5 implies that the horizontal morphisms are injective. It remains to show that a morphism $u \in [\widehat{C}, \widehat{A^{op} \otimes^{\mathbb{L}} D_{\mathbb{V}}}]_c$, whose restriction $C \longrightarrow \widehat{A^{op} \otimes^{\mathbb{L}} D_{\mathbb{V}}}$ factors thought $\widehat{A^{op} \otimes^{\mathbb{L}} D}$, itself factors through $\widehat{A^{op} \otimes^{\mathbb{L}} D}$. But this is true as by sub-lemma 7.4 the image by u of any C^{op}-module can be written as a $\mathbb{U}$-small homotopy colimit of objects of the form $Z \otimes^{\mathbb{L}} u(l(x))$ for $Z \in C(k)_{\mathbb{U}}$ and $x \in C$. Therefore, if the restriction of u to C has $\mathbb{U}$-small images, then so does u itself. This finishes the proof of theorem 7.2 (1).

(2) We consider the quasi-fully faithful morphism $\widehat{D}_{pe} \longrightarrow \widehat{D}$. We therefore have a homotopy commutative diagram

$$\begin{array}{ccc} \mathbb{R}\underline{Hom}(\widehat{C}_{pe}, \widehat{D}_{pe}) & \longrightarrow & \mathbb{R}\underline{Hom}(\widehat{C}_{pe}, \widehat{D}) \\ \downarrow & & \downarrow \\ \mathbb{R}\underline{Hom}(C, \widehat{D}_{pe}) & \longrightarrow & \mathbb{R}\underline{Hom}(C, \widehat{D}), \end{array}$$

where the horizontal morphisms are quasi-fully faithful by Cor. 6.6. We claim that the right vertical morphism is a quasi-equivalence. For this, using the universal properties of internal Hom's, it is enough to show that the induced morphism

$$[\widehat{C}_{pe}, \widehat{D}] \longrightarrow [C, \widehat{D}]$$

is bijective for any D. Using our lemma 6.2 one sees that it is enough to prove the following lemma.

Lemma 7.5 *Let C be a cofibrant and $\mathbb{U}$-small dg-category and M a $\mathbb{V}$-combinatorial $C(k)_{\mathbb{V}}$-model category satisfying the same assumption as in lemma 7.3.*

1. *Then, the Quillen adjunction*

$$l_! : M^C \longrightarrow M^{\widehat{C}_{pe}} \qquad M^C \longleftarrow M^{\widehat{C}_{pe}} : l^*$$

   is a Quillen equivalence.

2. *For any $F \in M^{\widehat{C}_{pe}}$, and any a $\mathbb{U}$-small diagram of perfect and cofibrant objects in $C^{op} - Mod_{\mathbb{U}}$, $X : I \longrightarrow C^{op} - Mod_{\mathbb{U}}$, the natural morphism*

$$Hocolim_i F(X_i) \longrightarrow F(Hocolim_i X_i)$$

   is an isomorphism in $Ho(M)$.

3. *For any $F \in M^{\widehat{C}_{pe}}$, and any perfect complex $Z \in C(k)_{\mathbb{U}}$ and any $X \in M$, the natural morphism*

$$Z \otimes^{\mathbb{L}} F(X) \longrightarrow F(Z \otimes^{\mathbb{L}} X)$$

   is an isomorphism in $Ho(M)$.

Proof: This is the same as for lemma 7.3 and sub-lemma 7.4. □

Coming back to our square of dg-categories one sees that the horizontal morphisms are quasi-fully faithful and that the right vertical morphism is a quasi-equivalence. This formally implies that the left vertical morphism is quasi-fully faithful. We now consider the square of sets

$$\begin{array}{ccc} [\widehat{C}_{pe}, \widehat{D}_{pe}] & \longrightarrow & [\widehat{C}_{pe}, \widehat{D}] \\ \downarrow & & \downarrow \\ [C, \widehat{D}_{pe}] & \longrightarrow & [C, \widehat{D}], \end{array}$$

obtained from the square of dg-categories by passing to equivalence classes of objects. Again, the right vertical morphism is a bijection and the horizontal morphisms are injective. For $u \in [C, \widehat{D}_{pe}]$, its image in $[C, \widehat{D}]$ comes from an element $v \in [\widehat{C}_{pe}, \widehat{D}]$. For any $x \in C$, $v(l(x)) \in \widehat{D}$ is a perfect D^{op}-module, and thus so is $v(Z \otimes^{\mathbb{L}} l(x)) \simeq Z \otimes^{\mathbb{L}} v(l(x))$ for any perfect complex Z of k-modules. As any perfect C^{op}-module is constructed as a retract of a finite homotopy colimit of objects of the form $Z \otimes^{\mathbb{L}} l(x)$, we deduce that $v(X)$ is a perfect D^{op}-module for any $X \in \widehat{C}_{pe}$. Therefore, Cor. 2.5 implies that v comes in fact from an element in $[\widehat{C}_{pe}, \widehat{D}_{pe}]$. This shows that $[\widehat{C}_{pe}, \widehat{D}_{pe}] \longrightarrow [C, \widehat{D}_{pe}]$ is surjective, and thus that

$$\mathbb{R}\underline{Hom}(\widehat{C}_{pe}, \widehat{D}_{pe}) \longrightarrow \mathbb{R}\underline{Hom}(C, \widehat{D}_{pe})$$

is quasi-essentially surjective. This finishes the proof of the theorem. □

The following corollary is the promised derived version of Morita theory.

Corollary 7.6 *Let C and D be two $\mathbb{U}$-small dg-categories, then there exists a natural isomorphism in $Ho(dg - Cat_{\mathbb{V}})$*

$$\mathbb{R}\underline{Hom}_c(\widehat{C}, \widehat{D}) \simeq \widehat{C^{op} \otimes^{\mathbb{L}} D} \simeq Int((C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}).$$

In particular, there exists a natural weak equivalence

$$Map_c(\widehat{C}, \widehat{D}) \simeq |(C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}|,$$

where $Map_c(\widehat{C}, \widehat{D})$ is the sub-simplicial set of continuous morphisms in $Map(\widehat{C}, \widehat{D})$ and where $|(C \otimes^{\mathbb{L}} D^{op}) - Mod_{\mathbb{U}}|$ is the nerve of the sub-category of equivalences between $C \otimes^{\mathbb{L}} D^{op}$-modules.

Proof: The first part follows from the universal properties of internal Hom's, as by theorem 7.2

$$\mathbb{R}\underline{Hom}_c(\widehat{C}, \widehat{D}) \simeq \mathbb{R}\underline{Hom}(C, \mathbb{R}\underline{Hom}(D^{op}, \widehat{\mathbf{1}})) \simeq \mathbb{R}\underline{Hom}(C \otimes^{\mathbb{L}} D^{op}, \widehat{\mathbf{1}}) \simeq \widehat{C^{op} \otimes^{\mathbb{L}} D}.$$

The second part follows from the relation between mapping spaces and internal Hom's, as well as Prop. 6.4. Indeed, one has

$$Map_c(\widehat{C}, \widehat{D}) \simeq Map(\mathbf{1}, \mathbb{R}\underline{Hom}_c(\widehat{C}, \widehat{D})) \simeq Map(\mathbf{1}, \mathbb{R}\underline{Hom}(C \otimes^{\mathbb{L}} D^{op}, \widehat{\mathbf{1}})) \simeq Map(\mathbf{1}, \widehat{C^{op} \otimes^{\mathbb{L}} D}).$$

By theorem 4.2 this last simplicial set is weakly equivalent to the nerve of the category of equivalences between quasi-representable $\mathbb{V}$-small $\widehat{C^{op} \otimes^{\mathbb{L}} D}$-modules. The enriched Yoneda lemma for the model category $C \otimes^{\mathbb{L}} D^{op} - Mod$ easily implies that this nerve is weakly equivalent to the nerve of equivalences between $\mathbb{U}$-small $C \otimes^{\mathbb{L}} D^{op}$-modules. □

8 Applications

In this last section we present three kinds of applications of our main results. A first application explains the relation between Hochschild cohomology and internal Hom's of dg-categories. In the same spirit, we present a relation between the negative part of Hochschild cohomology and the higher homotopy groups of the *classifying space of dg-categories*, as well as an interpretation of the fundamental group of this space as the so-called *derived Picard group*. As a second application, we present a proof of the existence of a good localization functor for dg-categories.

This implies for example the existence of a quotient of a dg-category by a full sub-dg-category, satisfying the required universal property. Finally, our last application states that the (derived) dg-category of morphisms between the dg-categories of quasi-coherent complexes over some (reasonable) schemes is naturally equivalent to the dg-category of quasi-coherent complexes over their product. Under smoothness and properness conditions the same statement stays correct when one replaces *quasi-coherent* by *perfect*. This last result can be considered as a solution to a question of D. Orlov, concerning the existence of representative objects for triangulated functors between derived categories of smooth projective varieties.

8.1 Hochschild cohomology, classifying space of dg-categories, and derived Picard groups

As a first application we give a formula relating higher homotopy groups of mapping spaces between dg-categories and Hochschild cohomology. For this, let us recall that for any $\mathbb{U}$-small dg-category C, one defines its Hochschild cohomology groups as

$$\mathbb{HH}^i(C) := H^i(\mathbb{R}\underline{Hom}_{C\otimes^{\mathbb{L}}C^{op}}(C,C)),$$

where C is the $C\otimes^{\mathbb{L}}C^{op}$-module defined by the trivial formula $C(x,y) := C(x,y)$, and where $\mathbb{R}\underline{Hom}_{C\otimes^{\mathbb{L}}C^{op}}$ are the $Ho(C(k))$-enriched Hom's of the category $Ho(C\otimes^{\mathbb{L}}C^{op} - Mod_{\mathbb{U}})$. More generally, the Hochschild complex of C is defined by

$$\mathbb{HH}(C) := \mathbb{R}\underline{Hom}_{C\otimes^{\mathbb{L}}C^{op}}(C,C)),$$

which is a well defined object in the derived category $Ho(C(k))$ of complexes of k-modules.

Corollary 8.1 *With the notation above, there exists an isomorphism in $Ho(C(k))$*

$$\mathbb{HH}(C) \simeq \mathbb{R}\underline{Hom}(C,C)(Id,Id),$$

where Id is the identity of C, considered as an object of the dg-category $\mathbb{R}\underline{Hom}(C,C)$. In particular, one has

$$\mathbb{HH}^i(C) \simeq H^i(\mathbb{R}\underline{Hom}(C,C)(Id,Id)).$$

Proof: Using Thm. 6.1, one has

$$\mathbb{R}\underline{Hom}(C,C)(Id,Id) \simeq Int(C\otimes^{\mathbb{L}}C^{op} - Mod_{\mathbb{U}}^{rqr}).$$

Furthermore, through this identification the identity morphism of C goes to the bi-module C itself. This implies the result by the definition of Hochschild cohomology. □

An important consequence of Cor. 8.1 is the following Morita invariance of Hochschild cohomology.

Corollary 8.2 *With the notation above, there exists an isomorphism in $Ho(C(k))$*

$$\mathbb{HH}(C) \simeq \mathbb{HH}(\widehat{C}).$$

Proof: Indeed, the identity of $\widehat{C}$ is clearly continuous, and thus by Thm. 7.2 (1) one has

$$\mathbb{HH}(\widehat{C}) \simeq \mathbb{R}\underline{Hom}(\widehat{C},\widehat{C})(Id,Id) \simeq \mathbb{R}\underline{Hom}(C,\widehat{C})(\underline{h},\underline{h}),$$

where $\underline{h} : C \longrightarrow \widehat{C}$ is the Yoneda embedding. As the morphism $\underline{h}$ is quasi-fully faithful, Cor. 6.6 implies that the morphism

$$\underline{h}^* : \mathbb{R}\underline{Hom}(C,\widehat{C})(\underline{h},\underline{h}) \longrightarrow \mathbb{R}\underline{Hom}(C,C)(Id,Id)$$

is a quasi-isomorphism. Cor. 8.1 implies the result. □

Corollary 8.3 *With the notation above one has isomorphisms of groups*

$$\pi_i(Map(C,C),Id) \simeq \mathbb{HH}^{1-i}(C)$$

for any $i > 1$. For $i = 1$, one has an isomorphism of groups

$$\pi_1(Map(C,C),Id) \simeq \mathbb{HH}^0(C)^* = Aut_{Ho(C\otimes^{\mathbb{L}}C^{op}-Mod_{\mathbb{U}})}(C).$$

Proof: This follows immediately from Thm. 4.2, the well-known relations between mapping spaces and classifying spaces of model categories (see e.g. [HAGII, Cor. A.0.4]) and the formula

$$H^{-i}(\mathbb{R}\underline{Hom}_{C\otimes^{\mathbb{L}}C^{op}}(C,C)) \simeq \pi_i(Map_{C\otimes^{\mathbb{L}}C^{op}-Mod_{\mathbb{U}}}(C,C)).$$

□

Let $|dg-Cat|$ be the nerve of the category of quasi-equivalences in $dg-Cat_{\mathbb{U}}$. Using the usual relations between mapping spaces in model category and nerve of categories of equivalences (see e.g. [HAGII, Appendix A]) one finds the following consequence.

Corollary 8.4 *For a $\mathbb{U}$-small dg-category C, one has*

$$\pi_i(|dg-Cat|,C) \simeq \mathbb{HH}^{2-i}(C) \qquad \forall\, i > 2.$$

Moreover, one has

$$\pi_2(|dg-Cat|,C) \simeq \mathbb{HH}^0(C)^*.$$

Remark 8.5 The above corollary only gives an interpretation of negative Hochschild cohomology groups. The positive part of the Hochschild cohomology can also be interpreted in terms of deformation theory of dg-categories as done for example in [HAGII, §8.5].

For a ($\mathbb{U}$-small) dg-algebra A, one can define the derived Picard group $RPic(A)$ of A, as done for example in [Ro-Zi, Ke2, Ye]. Using our notations and definitions, the group $RPic(A)$ can be defined in the following way. To simplify notations let us assume that the underlying complex of A is cofibrant, and we will consider A as a dg-category with a unique object which we denote by BA. Note that the category $(A\otimes A^{op})-Mod_{\mathbb{U}}$, of $A\otimes A^{op}$-dg-modules, is also the category $(BA\otimes BA^{op})-Mod_{\mathbb{U}}$. This category can be endowed with the following monoidal structure. For X and Y two $(A \otimes A^{op})$-dg-modules, we can form the internal tensor product $X \otimes_A Y \in (A \otimes A^{op}) - Mod_{\mathbb{U}}$ as the coequalizer of the two natural morphisms

$$(X \otimes A \otimes Y) \rightrightarrows X \otimes Y.$$

This endows the model category $(A \otimes A^{op}) - Mod_{\mathbb{U}}$ with a structure of monoidal model category (see for example [K-T] where the simplicial analog is considered). Deriving this monoidal structure provides a monoidal category $(Ho((A \otimes A^{op}) - Mod_{\mathbb{U}}), \otimes^{\mathbb{L}}_A)$. By definition, the group $RPic(A)$ is the group of isomorphism classes of objects in $Ho((A \otimes A^{op}) - Mod_{\mathbb{U}})$ which are invertible for the monoidal structure $\otimes^{\mathbb{L}}_A$.

Corollary 8.6 *There is a group isomorphism*

$$RPic(A) \simeq \pi_1(|dg - Cat_{\mathbb{V}}|, \widehat{BA}).$$

Proof: This easily follows from the formula

$$\pi_1(|dg - Cat_{\mathbb{V}}|, \widehat{C}) \simeq Aut_{Ho(dg-Cat)}(\widehat{C})$$

and Cor. 7.6. □

8.2 Localization and quotient of dg-categories

Let C be a $\mathbb{U}$-small dg category, and S be a set of morphisms in $[C]$. For any $\mathbb{U}$-small dg-category D, we consider $Map_S(C, D)$ the sub-simplicial set of $Map(C, D)$ being the union of all connected components corresponding to morphisms $f : C \longrightarrow D$ in $Ho(dg - Cat)$ such that $[f] : [C] \longrightarrow [D]$ sends S to isomorphisms in $[D]$.

Corollary 8.7 *The $Ho(SSet_{\mathbb{U}})$-enriched functor*

$$Map_S(C, -) : Ho(dg - Cat_{\mathbb{U}}) \longrightarrow Ho(SSet_{\mathbb{U}})$$

is co-represented by an object $L_S(C) \in Ho(dg - Cat_{\mathbb{U}})$.

Proof: Let I_k be the dg-category with two objects 0 and 1, and freely generated by a unique morphism $0 \to 1$. Using theorem 4.2 one easily sees that $Map(I_k, C)$ can be identified with the nerve of the category $(C^{op} - Mod_{\mathbb{U}})^I_{rqr}$, of morphisms between quasi-representable C^{op}-modules. Using the dg-Yoneda lemma one sees that $[I_k, C]$ is in a natural bijection with isomorphism classes of morphisms in $[C]$. In particular, the set S can be classified by a morphism in $Ho(dg - Cat_{\mathbb{U}})$

$$S : \coprod_{f \in S} I_k \longrightarrow C.$$

We consider the natural morphism $I_k \longrightarrow \mathbf{1}$, and we define $L_S C$ to be the homotopy push-out

$$\begin{array}{ccc} \coprod_{f \in S} I_k & \longrightarrow & C \\ \downarrow & & \downarrow \\ \coprod_{f \in S} \mathbf{1} & \longrightarrow & L_S C. \end{array}$$

For any D one has a homotopy pull-back diagram

$$\begin{array}{ccc} Map(L_S C, D) & \longrightarrow & \prod_{f\in S} Map(\mathbf{1}, D) \\ \downarrow & & \downarrow \\ Map(C, D) & \longrightarrow & \prod_{f\in S} Map(I_k, D). \end{array}$$

Therefore, in order to see that $L_S C$ has the correct universal property, it is enough to check that $Map(\mathbf{1}, D) \longrightarrow Map(I_k, D)$ induces an injection on π_0, a bijection on π_i for $i > 0$, and that its image in $[I_k, D]$ consists of all morphisms in $[D]$ which are isomorphisms. Using theorem 4.2 once again we see that this follows from the following very general fact: if M is a model category, then the Quillen adjunction $Mor(M) \leftrightarrows M$ (where $Mor(M)$ is the model category of morphisms in M), sending a morphism in M to its target, induces a fully faithful functor $Ho(M) \longrightarrow Ho(Mor(M))$, whose essential image consists of all equivalences in M. $\square$.

Corollary 8.8 *Let $C \in dg-Cat_{\mathbb{U}}$ be a dg-category and S a set of morphisms in $[C]$. Then, the natural morphism $C \longrightarrow L_S C$ induces for any $D \in dg-Cat_{\mathbb{U}}$ a quasi-fully faithful morphism*

$$\mathbb{R}\underline{Hom}(L_S C, D) \longrightarrow \mathbb{R}\underline{Hom}(C, D),$$

whose quasi-essential image consists of all morphisms $C \to D$ in $Ho(dg-Cat)$ sending S to isomorphisms in $[D]$.

Proof: This follows formally from Cor. 8.7, Thm. 6.1 and Lem. 2.4. $\square$

One important example of application of the localization construction is the existence of a good theory of quotients of dg-categories. For this, let C be a $\mathbb{U}$-small dg-category, and $\{X_i\}_{i\in I}$ be a sub-set of objects in C. We assume that $[C]$ has a zero object 0. One consider S the set of morphisms in $[C]$ consisting of all $X_i \to 0$. The dg-category $L_S C$ is then denoted by $C/ < X_i >$, and is called the quotient of C by the sub-set of objects $\{X_i\}_{i\in I}$. This terminology is justified by the fact that for any dg-category D with a zero object, the morphism

$$l^* : \mathbb{R}\underline{Hom}(C/ < X_i >, D) \longrightarrow \mathbb{R}\underline{Hom}(C, D)$$

is quasi-fully faithful, and its image consists of all morphisms $f : C \longrightarrow D$ such that for all $i \in I$ $[f(X_i)] \simeq 0$ in $[D]$.

8.3 Maps between dg-categories of quasi-coherent complexes

We now pass to our last application describing maps between dg-categories of quasi-coherent complexes on k-schemes. For this, let X be a quasi-compact and separated scheme over $Spec\, k$. We consider $QCoh(X)$ the category of $\mathbb{U}$-small quasi-coherent sheaves on X. As this is a Grothendieck category we know that there exists a $\mathbb{U}$-cofibrantly generated model category $C(QCoh(X))$ of (unbounded) complexes of quasi-coherent sheaves on X (the cofibrations being the monomorphisms and the equivalences being the quasi-isomorphisms, see e.g. [Ho2]). It is easy to check that the natural $C(k)_{\mathbb{U}}$-enrichment of $C(QCoh(X))$ makes it into a $C(k)_{\mathbb{U}}$-model category, and thus as explained in §3 we can construct a $\mathbb{V}$-small dg-category $Int(C(QCoh(X))$. This dg-category will be denoted by $L_{qcoh}(X)$. Note that $[L_{qcoh}(X)]$ is naturally equivalent to the (unbounded) derived category of quasi-coherent sheaves $D_{qcoh}(X)$, and will be identified with it.

We need to recall that an object E in $L_{qcoh}(X)$ is homotopically finitely presented, or perfect in the sense of §7, if and only if it is a compact object of $D_{qcoh}(X)$, and thus if and only if it is a perfect complex on X (see for example [B-V]). We will use this fact implicitly in the sequel.

Theorem 8.9 *Let X and Y be two quasi-compact and separated schemes over k, and assume that one of them is flat over $Spec\, k$. Then, there exists an isomorphism in $Ho(dg - Cat_{\mathbb{V}})$*

$$\mathbb{R}\underline{Hom}_c(L_{qcoh}(X), L_{qcoh}(Y)) \simeq L_{qcoh}(X \times_k Y).$$

Proof: We start noticing that the model categories $C(QCoh(X))$ and $C(QCoh(Y))$ are stable, proper, cofibrantly generated, and admit a compact generator (see [B-V]). Therefore, they satisfy the conditions of the main theorem of [S-S], and thus one can find two objects E_X and E_Y in $L_{qcoh}(X)$ and $L_{qcoh}(Y)$, and two Quillen equivalences

$$C(QCoh(X)) \leftrightarrows A_X^{op} - Mod_{\mathbb{U}} \qquad C(QCoh(Y)) \leftrightarrows A_Y^{op} - Mod_{\mathbb{U}}$$

where A_X (resp. A_Y) is the full sub-dg-category of $L_{qcoh}(X)$ (resp. of $L_{qcoh}(Y)$) consisting of E_X (resp. E_Y) only (in other words, A_X is the dg-category with a unique object and $\mathbb{R}\underline{End}(E_X)$ as endomorphism dg-algebra). In the following we will write A_X for both, the dg-category and the corresponding dg-algebra $\mathbb{R}\underline{End}(E_X)$ (and the same with A_Y). These Quillen equivalences are $C(k)$-enriched Quillen equivalences, and with a bit of care one can check that they provide natural isomorphisms in $Ho(dg - Cat_{\mathbb{V}})$

$$L_{qcoh}(X) \simeq \widehat{A_X} \qquad L_{qcoh}(Y) \simeq \widehat{A_Y}.$$

Lemma 8.10 *There exists an isomorphism in $Ho(dg - Cat_{\mathbb{V}})$*

$$\widehat{A_Y} \simeq \widehat{A_Y^{op}}.$$

Proof: By the general theory of [S-S] it is enough to show that the triangulated category $D_{qcoh}(Y) \simeq [L_{qcoh}(Y)]$ possesses a compact generator F_Y such that the dg-algebra $\mathbb{R}\underline{End}(F_Y)$ is naturally equivalent to $\mathbb{R}\underline{End}(E_Y)^{op}$. For this we take $F_Y = E_Y^{\vee}$ to be the dual perfect complex of E_Y. Let $< F_Y >$ be the smallest thick triangulated sub-category of $D_{parf}(Y)$ containing F_Y. We let $\phi : D_{parf}(Y) \longrightarrow D_{parf}(Y)^{op}$ be the involution sending a perfect complex E to its dual $E^{\vee}$. Then, clearly $\phi(< F_Y >) =< E_Y >= D_{parf}(Y)$. This shows that F_Y classically generates $D_{parf}(Y)$, and thus by [B-V, Thm. 2.1.2] that F_Y is a compact generator of $D_{qcoh}(Y)$. $\Box$

Lemma 8.11 *There exists an isomorphism in $Ho(dg - Cat_{\mathbb{V}})$*

$$\widehat{A_X \otimes_k^{\mathbb{L}} A_Y} \simeq L_{qcoh}(X \times_k Y).$$

Proof: This follows from the fact that the external product $E_X \boxtimes E_Y$ is a compact generator of $D_{qcoh}(X \times_k Y)$, as explained in [B-V, Lem. 3.4.1]. Indeed, flat base change induces a natural quasi-isomorphism of dg-algebras (one uses here that either X or Y is flat over k)

$$\mathbb{R}\underline{End}(E_X \boxtimes E_Y) \simeq \mathbb{R}\underline{End}(E_X) \otimes^{\mathbb{L}}_k \mathbb{R}\underline{End}(E_Y) \simeq A_X \otimes^{\mathbb{L}}_k A_Y.$$

□

We are now ready to prove theorem 8.9. Indeed, using theorem 7.2 one finds

$$\mathbb{R}\underline{Hom}_c(L_{qcoh}(X), L_{qcoh}(Y)) \simeq \mathbb{R}\underline{Hom}_c(\widehat{A_X}, \widehat{A_Y}) \simeq \mathbb{R}\underline{Hom}(A_X, \widehat{A_Y}).$$

Lemma 8.10 and the universal properties of internal Hom's give an isomorphism

$$\mathbb{R}\underline{Hom}(A_X, \widehat{A_Y}) \simeq \mathbb{R}\underline{Hom}(A_X, \widehat{A_Y^{op}}) \simeq \widehat{A_X \otimes^{\mathbb{L}}_k A_Y}.$$

Finally lemma 8.11 implies the theorem. □

Corollary 8.12 *Under the same conditions as in Thm. 8.9, there exists a bijection between $[L_{qcoh}(X), L_{qcoh}(Y)]_c$, the sub-set of $[L_{qcoh}(X), L_{qcoh}(Y)]$ consisting of continuous morphisms, and the isomorphism classes of objects in the derived category $D_{qcoh}(X \times_k Y)$.*

Proof: Readily follows from theorem 8.9 and the fact that $[L_{qcoh}(X \times_k Y)] \simeq D_{qcoh}(X \times_k Y)$. □

Tracking back the construction of the equivalence in theorem 8.9 one sees that the bijection of corollary 8.12 can be described as follows. Let $E \in D_{qcoh}(X \times_k Y)$ be an object, and let us consider the two projections

$$p_X : X \times_k Y \longrightarrow X \qquad p_Y : X \times_k Y \longrightarrow Y.$$

We consider the functor

$$\phi_E : D_{qcoh}(X) \longrightarrow D_{qcoh}(Y)$$

defined by

$$\phi_E(F) := \mathbb{R}(p_Y)_*(\mathbb{L}p_X^*(F) \otimes^{\mathbb{L}} E),$$

for any $F \in D_{qcoh}(X)$. Then, the functor ϕ_E is the natural functor induced by the morphism $L_{qcoh}(X) \longrightarrow L_{qcoh}(Y)$ in $Ho(dg-Cat)$, corresponding to E via the bijection of Cor. 8.12.

Corollary 8.13 *Let X be a quasi-compact and separated scheme, flat over $Spec\, k$. Then, one has*

$$\pi_1(Map(L_{qcoh}(X), L_{qcoh}(X), Id)) \simeq \mathcal{O}_X(X)^*$$

$$\pi_i(Map(L_{qcoh}(X), L_{qcoh}(X), Id)) \simeq \mathbb{HH}^{1-i}(A_X) \simeq 0 \;\forall\, i > 1.$$

Proof: Indeed theorem 8.9, theorem 4.2, corollary 4.10 and corollary 6.4 give

$$Map(L_{qcoh}(X), L_{qcoh}(X)) \simeq Map(*, L_{qcoh}(X \times_k X)).$$

Furthermore, the identity on the right is clearly sent to the diagonal Δ_X in $L_{qcoh}(X \times_k X)$. Therefore, one finds for any $i > 1$

$$\pi_i(Map(L_{qcoh}(X), L_{qcoh}(X)), Id)) \simeq \pi_i(Map(*, L_{qcoh}(X \times_k X)), \Delta_X) \simeq$$

$$H^{1-i}(L_{qcoh}(X \times_k X)(\Delta_X, \Delta_X)) \simeq Ext^{1-i}_{X \times_k X}(\Delta(X), \Delta(X)) \simeq 0.$$

For $i = 1$, one has

$$\pi_1(Map(L_{qcoh}(X), L_{qcoh}(X)), Id)) \simeq \pi_1(Map(*, L_{qcoh}(X \times_k X)), \Delta_X) \simeq Aut_{D_{qcoh}(X \times_k X)}(\Delta_X) \simeq \mathcal{O}_X(X)^*.$$

□

Corollary 8.13 combined with the usual relations between mapping spaces and nerves of categories of equivalences also has the following important consequence.

Corollary 8.14 *Let X be a quasi-compact and separated scheme, flat over k. Then, one has*

$$\pi_i(|dg - Cat|, L_{qcoh}(X)) \simeq 0 \qquad \forall\, i > 2.$$

In particular, the sub-simplicial set of $|dg - Cat|$ corresponding to dg-categories of the form $L_{qcoh}(X)$, for X a quasi-compact and separated scheme flat over k, is a 2-truncated simplicial set.

We finish by a refined version of theorem 8.9 involving only perfect complexes instead of all quasi-coherent complexes. For this, we will denote by $L_{parf}(X)$ the full sub-dg-category of $L_{qcoh}(X)$ consisting of all perfect complexes.

Theorem 8.15 *Let X and Y be two smooth and proper schemes over $Spec\, k$. Then, there exists an isomorphism in $Ho(dg - Cat_{\mathbb{V}})$*

$$\mathbb{R}\underline{Hom}(L_{parf}(X), L_{parf}(Y)) \simeq L_{parf}(X \times_k Y).$$

Proof: The triangulated category $D_{qcoh}(X)$ being generated by its compact objects, one sees that the Yoneda embedding

$$L_{qcoh}(X) \longrightarrow \widehat{L_{parf}(X)}$$

is an isomorphism in $Ho(dg{-}Cat_{\mathbb{V}})$. Using our Thm. 7.2 we see that $\mathbb{R}\underline{Hom}(L_{parf}(X), L_{parf}(Y))$ can be identified, up to quasi-equivalence, with the full sub-dg-category of $\mathbb{R}\underline{Hom}_c(L_{qcoh}(X), L_{qcoh}(Y))$ consisting of all morphisms $L_{qcoh}(X) \longrightarrow L_{qcoh}(Y)$ which preserve perfect complexes. Using Thm. 8.9, we see that $\mathbb{R}\underline{Hom}(L_{parf}(X), L_{parf}(Y))$ is quasi-equivalent to the full sub-dg-category of $L_{qcoh}(X \times_k Y)$ consisting of objects E such that for any perfect complex F on X, the complex

$\mathbb{R}(p_Y)_*(p_X^*(F)\otimes^{\mathbb{L}} E)$ is perfect on Y. To finish the proof we thus need to show that an object $E\in D_{qcoh}(X\times_k Y)$ is perfect if and only if the functor

$$\Phi_E := \mathbb{R}(p_Y)_*(p_X^*(-)\otimes^{\mathbb{L}} E) : D_{qcoh}(X) \longrightarrow D_{qcoh}(Y)$$

preserves perfect objects. Clearly, as X is flat and proper over $Spec\, k$, Φ_E preserves perfect complexes if E is itself perfect.

Conversely, let E be an object in $D_{qcoh}(X\times_k Y)$ such that Φ_E preserves perfect complexes.

Lemma 8.16 *Let Z be a smooth and proper scheme over $Spec\, k$, and $E\in D_{qcoh}(Z)$. If for any perfect complex F on Z, the complex of k-modules $\mathbb{R}\underline{Hom}(F,E)$ is perfect, then E is perfect on Z.*

Proof of the lemma: We let A_Z be a dg-algebra over k such that $L_{qcoh}(Z)$ is quasi-equivalent to $\widehat{A_Z}$ (with the same abuse of notations that A_Z also means the dg-category with a unique object and A_Z as endomorphism dg-algebra). As Z is flat and proper over $Spec\, k$, the underlying complex of k-modules of A_Z is perfect. Furthermore, as Z is smooth, the diagonal $\Delta : Z \hookrightarrow Z\times_k Z$ is a local complete intersection morphism, and thus $\Delta_*(\mathcal{O}_Z)$ is a perfect complex on Z. Equivalently, the $A_Z\otimes_k^{\mathbb{L}} A_Z^{op}$-dg-module A_Z is perfect, or equivalently lies in the smallest sub-dg-category of $\widehat{A_Z^{op}\otimes_k^{\mathbb{L}} A_Z}$ containing $A_Z\otimes^{\mathbb{L}} A_Z^{op}$ and which is stable by retracts, homotopy push-outs and the loop functor (or the shift functor).

We now apply our theorem 7.2 in order to translate this last fact in terms of dg-categories of morphisms. Let $F:\widehat{A_Z}\longrightarrow \widehat{A_Z}$ be the morphism of dg-categories sending an A_Z^{op}-dg-module M to the free A_Z^{op}-dg-module

$$F(M) := \underline{M}\otimes^{\mathbb{L}} A_Z^{op},$$

where $\underline{M}$ is the underlying complex of k-modules of M. By what we have seen, the identity morphism lies in the smallest sub-dg-category of $\mathbb{R}\underline{Hom}_c(\widehat{A_Z},\widehat{A_Z})$ containing the object F and which is stable by retracts, homotopy push-outs and the loop functor. Evaluating the identity of the dg-category $\widehat{A_Z}$ at an object M, we get that the object $M\in\widehat{A_Z}$ lies in the smallest sub-dg-category of $\widehat{A_Z}$ containing $\underline{M}\otimes^{\mathbb{L}} A_Z^{op}$ and stable by retracts, homotopy push-outs and by the loop functor. Now, by our hypothesis the object E corresponds to $M\in\widehat{A_Z}$ such that $\underline{M}$ is a perfect complex of k-modules. Therefore, M itself belongs to the smallest sub-dg-category of $\widehat{A_Z}$ containing A_Z^{op} and which stable by retracts, homotopy push-outs and the loop functor. By definition of being perfect, this implies that $M\in\widehat{A_Z}_{pe}$, and thus that E is a perfect complex on Z. $\Box$

Let now E_X and E_Y be compact generators of $D_{qcoh}(X)$ and $D_{qcoh}(Y)$. Then, by the projection formula one has

$$\mathbb{R}(p_Y)_*((E_X^\vee\boxtimes E_Y^\vee)\otimes^{\mathbb{L}} E)\simeq \mathbb{R}(p_Y)_*(p_X^*(E_X^\vee)\otimes^{\mathbb{L}} E)\otimes^{\mathbb{L}} E_Y^\vee$$

which is perfect on Y. This implies in particular that

$$\mathbb{R}\underline{Hom}(E_X\boxtimes E_Y, E)\simeq \mathbb{R}\Gamma(Y,\mathbb{R}(p_Y)_*((E_X^\vee\boxtimes E_Y^\vee)\otimes^{\mathbb{L}} E)$$

is a perfect complex of k-modules. As the perfect complex $E_X \boxtimes E_Y$ is a generator on $D_{qcoh}(X \times_k Y)$, one sees that for any perfect complex F on $X \times_k Y$, the complex of k-modules $\mathbb{R}\underline{Hom}(F, E)$ is perfect. The lemma 8.16 implies that E is perfect on $X \times_k Y$, which finishes the proof of the theorem. □

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