Wave packets on Riemannian manifolds

Jean-Marc Bouclet
Institut de Mathématiques de Toulouse

November 24, 2015

Wave packets: a review of basic facts

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

$$\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}} \exp\left(\mathrm{i}\zeta \cdot (x - z) - \frac{|x-z|^2}{2} \right), \qquad x \in \mathbb{R}^n$$

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

$$\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}} \exp\left(\mathrm{i}\zeta \cdot (x - z) - \frac{|x - z|^2}{2}\right), \qquad x \in \mathbb{R}^n$$

localized around (or centered at) z in space,

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

$$\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}} \exp\left(\mathrm{i}\zeta \cdot (x - z) - \frac{|x - z|^2}{2} \right), \qquad x \in \mathbb{R}^n$$

localized around (or centered at) z in space, and near ζ in momentum

$$(\mathcal{F}\psi_{z,\zeta})(\xi) = \pi^{-\frac{n}{4}} \exp\left(-\mathrm{i}z \cdot \xi - \frac{|\xi - \zeta|^2}{2} \right)$$

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

$$\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}} \exp\left(i\zeta \cdot (x-z) - \frac{|x-z|^2}{2} \right), \qquad x \in \mathbb{R}^n$$

localized around (or centered at) z in space, and near ζ in momentum

$$(\mathcal{F}\psi_{z,\zeta})(\xi) = \pi^{-\frac{n}{4}} \exp\left(-iz \cdot \xi - \frac{|\xi-\zeta|^2}{2} \right)$$

Think of (z,ζ) as a point in phase space $T^*\mathbb{R}^n$

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

$$\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}} \exp\left(\mathrm{i}\zeta \cdot (x - z) - \frac{|x - z|^2}{2}\right), \qquad x \in \mathbb{R}^n$$

localized around (or centered at) z in space, and near ζ in momentum

$$(\mathcal{F}\psi_{z,\zeta})(\xi) = \pi^{-\frac{n}{4}} \exp\left(-\mathrm{i}z \cdot \xi - \frac{|\xi - \zeta|^2}{2}\right)$$

Think of (z, ζ) as a point in phase space $T^*\mathbb{R}^n$

We call $\psi_{z,\zeta}$ a **Gaussian wave packet centered at** (z, ζ)

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

$$\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}} \exp\left(\mathrm{i}\zeta \cdot (x - z) - \frac{|x-z|^2}{2} \right), \qquad x \in \mathbb{R}^n$$

localized around (or centered at) z in space, and near ζ in momentum

$$(\mathcal{F}\psi_{z,\zeta})(\xi) = \pi^{-\frac{n}{4}} \exp\left(-\mathrm{i}z \cdot \xi - \frac{|\xi - \zeta|^2}{2} \right)$$

Think of (z, ζ) as a point in phase space $T^*\mathbb{R}^n$

We call $\psi_{z,\zeta}$ a **Gaussian wave packet centered at** (z, ζ)

Main interests (for us) :

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

$$\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}} \exp\left(\mathrm{i}\zeta \cdot (x - z) - \frac{|x - z|^2}{2} \right), \qquad x \in \mathbb{R}^n$$

localized around (or centered at) z in space, and near ζ in momentum

$$(\mathcal{F}\psi_{z,\zeta})(\xi) = \pi^{-\frac{n}{4}} \exp\left(-\mathrm{i}z \cdot \xi - \frac{|\xi - \zeta|^2}{2} \right)$$

Think of (z, ζ) as a point in phase space $T^*\mathbb{R}^n$

We call $\psi_{z,\zeta}$ a **Gaussian wave packet centered at** (z, ζ)

Main interests (for us) :

1. One can write "waves" (i.e. functions) as superposition of wave packets

Wave packets: a review of basic facts

Canonical example: gaussian wave packet

$$\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}} \exp\left(\mathrm{i}\zeta \cdot (x - z) - \frac{|x-z|^2}{2} \right), \qquad x \in \mathbb{R}^n$$

localized around (or centered at) z in space, and near ζ in momentum

$$(\mathcal{F}\psi_{z,\zeta})(\xi) = \pi^{-\frac{n}{4}} \exp\left(-\mathrm{i}z \cdot \xi - \frac{|\xi - \zeta|^2}{2} \right)$$

Think of (z, ζ) as a point in phase space $T^*\mathbb{R}^n$

We call $\psi_{z,\zeta}$ a **Gaussian wave packet centered at** (z, ζ)

Main interests (for us) :

1. One can write "waves" (i.e. functions) as superposition of wave packets
2. The evolution of a wave packet under a Schrödinger flow can be described rather explicitly (in a suitable regime)

Wave packets: a review of basic facts

1. **Wave packet decomposition**

Wave packets: a review of basic facts

1. Wave packet decomposition

Define the **Bargmann transform** of a function u by

$$Bu(z,\zeta) = \int_{\mathbb{R}^n} \overline{\psi_{z,\zeta}(x)} u(x) dx$$

Wave packets: a review of basic facts

1. Wave packet decomposition

Define the **Bargmann transform** of a function u by

$$Bu(z,\zeta) = \int_{\mathbb{R}^n} \overline{\psi_{z,\zeta}(x)}u(x)dx$$

Then, one has the **inversion formula**

$$u = (2\pi)^{-n} B^* Bu$$

Wave packets: a review of basic facts

1. Wave packet decomposition

Define the **Bargmann transform** of a function u by

$$Bu(z,\zeta) = \int_{\mathbb{R}^n} \overline{\psi_{z,\zeta}(x)}u(x)dx$$

Then, one has the **inversion formula**

$$u = (2\pi)^{-n} B^* Bu$$

In other words

$$u(x) = (2\pi)^{-n} \int\int_{T^*\mathbb{R}^n} (Bu)(z,\zeta)\psi_{z,\zeta}(x)dzd\zeta$$

is a decomposition of u as a (continuous) sum of wave packets

Wave packets: a review of basic facts

2. Evolution of wave packets under the Schrödinger equation

Wave packets: a review of basic facts

2. Evolution of wave packets under the Schrödinger equation

For quadratic potentials, one has exact formulas.

Wave packets: a review of basic facts

2. Evolution of wave packets under the Schrödinger equation

For quadratic potentials, one has exact formulas. Set

$$p_\nu(x,\xi) = \frac{|\xi|^2}{2} + \nu\frac{|x|^2}{2}, \qquad H_\nu = -\frac{\Delta}{2} + \nu\frac{|x|^2}{2}, \qquad \nu = 0, +1, -1$$

Wave packets: a review of basic facts

2. Evolution of wave packets under the Schrödinger equation

For quadratic potentials, one has exact formulas. Set

$$p_\nu(x,\xi) = \frac{|\xi|^2}{2} + \nu\frac{|x|^2}{2}, \qquad H_\nu = -\frac{\Delta}{2} + \nu\frac{|x|^2}{2}, \qquad \nu = 0, +1, -1$$

Then

$$e^{-\mathrm{i}tH_\nu}\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}}\gamma_\nu^t \exp \mathrm{i}\left(S_\nu^t + \zeta_\nu^t\cdot(x - z_\nu^t) + \frac{\Gamma_\nu^t}{2}(x - z_\nu^t)\cdot(x - z_\nu^t)\right)$$

Wave packets: a review of basic facts

2. Evolution of wave packets under the Schrödinger equation

For quadratic potentials, one has exact formulas. Set

$$p_\nu(x,\xi) = \frac{|\xi|^2}{2} + \nu\frac{|x|^2}{2}, \qquad H_\nu = -\frac{\Delta}{2} + \nu\frac{|x|^2}{2}, \qquad \nu = 0, +1, -1$$

Then

$$e^{-\mathrm{i}tH_\nu}\psi_{z,\zeta}(x) = \pi^{-\frac{n}{4}}\gamma_\nu^t \exp \mathrm{i}\left(S_\nu^t + \zeta_\nu^t\cdot(x-z_\nu^t) + \frac{\Gamma_\nu^t}{2}(x-z_\nu^t)\cdot(x-z_\nu^t)\right)$$

where

$$(z_\nu^t,\zeta_\nu^t) = \Phi_{p_\nu}^t(z,\zeta), \qquad S_\nu^t = \int_0^t \dot z_\nu^s\cdot\zeta_\nu^s - p_\nu(z_\nu^s,\zeta_\nu^s)ds$$

2. Evolution of wave packets under the Schrödinger equation

For quadratic potentials, one has exact formulas. Set

$$p_\nu(x,\xi)=\frac{|\xi|^2}{2}+\nu\frac{|x|^2}{2},\qquad H_\nu=-\frac{\Delta}{2}+\nu\frac{|x|^2}{2},\qquad \nu=0,+1,-1$$

Then

$$e^{-\mathrm{i}tH_\nu}\psi_{z,\zeta}(x)=\pi^{-\frac{n}{4}}\gamma_\nu^t\exp\mathrm{i}\left(S_\nu^t+\zeta_\nu^t\cdot(x-z_\nu^t)+\frac{\Gamma_\nu^t}{2}(x-z_\nu^t)\cdot(x-z_\nu^t)\right)$$

where

$$(z_\nu^t,\zeta_\nu^t)=\Phi_{p_\nu}^t(z,\zeta),\qquad S_\nu^t=\int_0^t \dot z_\nu^s\cdot\zeta_\nu^s-p_\nu(z_\nu^s,\zeta_\nu^s)ds$$

and γ_ν^t, Γ_ν^t are given in term of the differential of flow $\Phi_{p_\nu}^t$,

$$D\Phi_{p_\nu}^t(z,\zeta)=\begin{pmatrix}A_\nu^t & B_\nu^t\\ C_\nu^t & D_\nu^t\end{pmatrix},$$

by

$$\Gamma_\nu^t=(C_\nu^t+\mathrm{i}D_\nu^t)(A_\nu^t+\mathrm{i}B_\nu^t)^{-1},\qquad \gamma_\nu^t=\det(A_\nu^t+\mathrm{i}B_\nu^t)^{-1/2}.$$

Wave packets: a review of basic facts

Explicitly, we obtain

$$\Gamma_0^t = \frac{t+\mathrm{i}}{1+t^2} I_n, \qquad \gamma_0^t = (1+\mathrm{i}t)^{-\frac{n}{2}}$$

$$\Gamma_1^t = \mathrm{i} I_n, \qquad \gamma_1^t = (\cos t + \mathrm{i}\sin t)^{-\frac{n}{2}}$$

$$\Gamma_{-1}^t = \frac{\sinh(2t)+\mathrm{i}}{\cosh(2t)} I_n, \qquad \gamma_{-1}^t = (\cosh t + \mathrm{i}\sinh t)^{-\frac{n}{2}}$$

Wave packets: a review of basic facts

Explicitly, we obtain

$$\begin{aligned} \Gamma_0^t &= \frac{t+\mathrm{i}}{1+t^2} I_n, & \gamma_0^t = (1+\mathrm{i}t)^{-\frac{n}{2}} \\ \Gamma_1^t &= \mathrm{i} I_n, & \gamma_1^t = (\cos t + \mathrm{i}\sin t)^{-\frac{n}{2}} \\ \Gamma_{-1}^t &= \frac{\sinh(2t)+\mathrm{i}}{\cosh(2t)} I_n, & \gamma_{-1}^t = (\cosh t + \mathrm{i}\sinh t)^{-\frac{n}{2}} \end{aligned}$$

This allows in particular to read the profile and spreading of the packets:

$$\begin{aligned} |e^{\mathrm{i}tH_0}\psi_{z,\zeta}(x)| &= \frac{1}{(\pi(1+t^2))^{\frac{n}{4}}} \exp\left(-\frac{|x-z_0^t|^2}{2(1+t^2)}\right) \\ |e^{\mathrm{i}tH_1}\psi_{z,\zeta}(x)| &= \frac{1}{\pi^{\frac{n}{4}}} \exp\left(-\frac{|x-z_1^t|^2}{2}\right) \\ |e^{\mathrm{i}tH_{-1}}\psi_{z,\zeta}(x)| &= \frac{1}{(\pi\cosh(2t))^{\frac{n}{4}}} \exp\left(-\frac{|x-z_{-1}^t|^2}{2\cosh(2t)}\right) \end{aligned}$$

Wave packets for semiclassical Schrödinger operators

From now on, we use a **semiclassical** normalization

$$\psi^h_{z,\zeta}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{\mathrm{i}}{h}\zeta \cdot (x-z) - \frac{|x-z|^2}{2h}\right)$$

Wave packets for semiclassical Schrödinger operators

From now on, we use a **semiclassical** normalization

$$\psi^h_{z,\zeta}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{\mathrm{i}}{h}\zeta\cdot(x-z) - \frac{|x-z|^2}{2h}\right)$$

$\Longrightarrow$ Localization around z on a scale $h^{1/2}$

Wave packets for semiclassical Schrödinger operators

From now on, we use a **semiclassical** normalization

$$\psi^h_{z,\zeta}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{\mathrm{i}}{h}\zeta \cdot (x-z) - \frac{|x-z|^2}{2h}\right)$$

$\Longrightarrow$ Localization around z on a scale $h^{1/2}$
Consider a semiclassical Schrödinger operator on $\mathbb{R}^n$

$$H(h) = -\frac{h^2\Delta}{2} + V(x), \qquad p(x,\xi) = \frac{|\xi|^2}{2} + V(x),$$

with $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$.

Wave packets for semiclassical Schrödinger operators

From now on, we use a **semiclassical** normalization

$$\psi^h_{z,\zeta}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{\mathrm{i}}{h}\zeta \cdot (x-z) - \frac{|x-z|^2}{2h}\right)$$

$\Longrightarrow$ Localization around z on a scale $h^{1/2}$
Consider a semiclassical Schrödinger operator on $\mathbb{R}^n$

$$H(h) = -\frac{h^2\Delta}{2} + V(x), \qquad p(x,\xi) = \frac{|\xi|^2}{2} + V(x),$$

with $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Denote

$$(z^t, \zeta^t) = \Phi^t_p(z,\zeta), \qquad \begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} := D\Phi^t_p(z,\zeta)$$

Wave packets for semiclassical Schrödinger operators

From now on, we use a **semiclassical** normalization

$$\psi^h_{z,\zeta}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{\mathrm{i}}{h}\zeta\cdot(x-z) - \frac{|x-z|^2}{2h}\right)$$

$\Longrightarrow$ Localization around z on a scale $h^{1/2}$
Consider a semiclassical Schrödinger operator on $\mathbb{R}^n$

$$H(h) = -\frac{h^2\Delta}{2} + V(x), \qquad p(x,\xi) = \frac{|\xi|^2}{2} + V(x),$$

with $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Denote

$$(z^t, \zeta^t) = \Phi^t_p(z,\zeta), \qquad \begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} := D\Phi^t_p(z,\zeta)$$

and

$$S^t = \int_0^t \dot{z}^s \cdot \zeta^s - p(z^s,\zeta^s)\,ds$$

Wave packets for semiclassical Schrödinger operators

From now on, we use a **semiclassical** normalization

$$\psi_{z,\zeta}^{h}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{\mathrm{i}}{h}\zeta \cdot (x - z) - \frac{|x - z|^2}{2h}\right)$$

$\Longrightarrow$ Localization around z on a scale $h^{1/2}$
Consider a semiclassical Schrödinger operator on $\mathbb{R}^n$

$$H(h) = -\frac{h^2 \Delta}{2} + V(x), \qquad p(x, \xi) = \frac{|\xi|^2}{2} + V(x),$$

with $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Denote

$$(z^t, \zeta^t) = \Phi_p^t(z, \zeta), \qquad \begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} := D\Phi_p^t(z, \zeta)$$

and

$$S^t = \int_0^t \dot{z}^s \cdot \zeta^s - p(z^s, \zeta^s) ds$$

Proposition [action of the symplectic group on the Siegel half space]
$A^t + \mathrm{i}B^t$ is invertible and

$$\Gamma^t := (C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1}$$

is symmetric complex, with positive definite imaginary part

Wave packets for semiclassical Schrödinger operators

Theorem (Hagedorn-Joye, Combescure-Robert) *In the limit $h \to 0$, and under general conditions on V,*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(x)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x) \exp \frac{\mathrm{i}}{h}\left(S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t)\cdot(x - z^t)\right)$$

Wave packets for semiclassical Schrödinger operators

Theorem (Hagedorn-Joye, Combescure-Robert) *In the limit $h \to 0$, and under general conditions on V,*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(x)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x) \exp \frac{\mathrm{i}}{h}\left(S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t)\cdot(x - z^t)\right)$$

for times $|t| \leq C_0 |\ln h|$ (C_0 dynamical constant). Here $\gamma^t = \det(A_t + \mathrm{i}B_t)^{-1/2}$.

Wave packets for semiclassical Schrödinger operators

Theorem (Hagedorn-Joye, Combescure-Robert) *In the limit $h \to 0$, and under general conditions on V,*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(x)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x)\exp\frac{\mathrm{i}}{h}\left(S^t + \zeta^t\cdot(x-z^t) + \frac{\Gamma^t}{2}(x-z^t)\cdot(x-z^t)\right)$$

for times $|t| \leq C_0|\ln h|$ (C_0 dynamical constant). Here $\gamma^t = \det(A_t + \mathrm{i}B_t)^{-1/2}$. The amplitude is of the form

$$\mathcal{A}^h_t(x) \sim 1 + \sum_{j\geq 1} h^{\frac{j}{2}} A_j\left(z,\zeta,t,\frac{x-z^t}{h^{\frac{1}{2}}}\right)$$

with $A_j(z,\zeta,t,X)$ polynomial of degree $\leq 3j$ in X, with coeff. depending on the classical trajectory $t \mapsto (z^t,\zeta^t)$ and the Taylor expansion of V at z^t

Wave packets for semiclassical Schrödinger operators

Theorem (Hagedorn-Joye, Combescure-Robert) *In the limit $h \to 0$, and under general conditions on V,*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(x)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x) \exp\frac{\mathrm{i}}{h}\left(S^t + \zeta^t\cdot(x-z^t) + \frac{\Gamma^t}{2}(x-z^t)\cdot(x-z^t)\right)$$

for times $|t| \le C_0|\ln h|$ (C_0 dynamical constant). Here $\gamma^t = det(A_t + \mathrm{i}B_t)^{-1/2}$. The amplitude is of the form

$$\mathcal{A}^h_t(x) \sim 1 + \sum_{j\ge 1} h^{\frac{j}{2}} A_j\left(z,\zeta,t,\frac{x-z^t}{h^{\frac{1}{2}}}\right)$$

with $A_j(z,\zeta,t,X)$ polynomial of degree $\le 3j$ in X, with coeff. depending on the classical trajectory $t\mapsto(z^t,\zeta^t)$ and the Taylor expansion of V at z^t

Rem. The polynomial growth of the amplitude in $(x-z^t)/h^{\frac{1}{2}}$ is beaten by the exponential decay of the exponential since $\mathrm{Im}(\Gamma^t)$ is positive definite

Wave packets for semiclassical Schrödinger operators

Theorem (Hagedorn-Joye, Combescure-Robert) *In the limit $h \to 0$, and under general conditions on V,*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(x)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x) \exp\frac{\mathrm{i}}{h}\left(S^t + \zeta^t\cdot(x-z^t) + \frac{\Gamma^t}{2}(x-z^t)\cdot(x-z^t)\right)$$

for times $|t| \leq C_0|\ln h|$ (C_0 dynamical constant). Here $\gamma^t = \det(A_t + \mathrm{i}B_t)^{-1/2}$. The amplitude is of the form

$$\mathcal{A}^h_t(x) \sim 1 + \sum_{j\geq 1} h^{\frac{j}{2}} A_j\left(z, \zeta, t, \frac{x-z^t}{h^{\frac{1}{2}}}\right)$$

with $A_j(z,\zeta,t,X)$ polynomial of degree $\leq 3j$ in X, with coeff. depending on the classical trajectory $t \mapsto (z^t,\zeta^t)$ and the Taylor expansion of V at z^t

Rem. The polynomial growth of the amplitude in $(x-z^t)/h^{\frac{1}{2}}$ is beaten by the exponential decay of the exponential since $\mathrm{Im}(\Gamma^t)$ is positive definite
$\Longrightarrow$ Concentration near the classical trajectory,

Wave packets for semiclassical Schrödinger operators

Theorem (Hagedorn-Joye, Combescure-Robert) *In the limit $h \to 0$, and under general conditions on V,*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(x)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x) \exp\frac{\mathrm{i}}{h}\left(S^t + \zeta^t\cdot(x-z^t) + \frac{\Gamma^t}{2}(x-z^t)\cdot(x-z^t)\right)$$

for times $|t| \leq C_0|\ln h|$ (C_0 dynamical constant). Here $\gamma^t = \det(A_t + \mathrm{i}B_t)^{-1/2}$. The amplitude is of the form

$$\mathcal{A}^h_t(x) \sim 1 + \sum_{j\geq 1} h^{\frac{j}{2}} A_j\left(z,\zeta,t,\frac{x-z^t}{h^{\frac{1}{2}}}\right)$$

with $A_j(z,\zeta,t,X)$ polynomial of degree $\leq 3j$ in X, with coeff. depending on the classical trajectory $t \mapsto (z^t,\zeta^t)$ and the Taylor expansion of V at z^t

Rem. The polynomial growth of the amplitude in $(x-z^t)/h^{\frac{1}{2}}$ is beaten by the exponential decay of the exponential since $\mathrm{Im}(\Gamma^t)$ is positive definite
$\Longrightarrow$ Concentration near the classical trajectory, at least as long as $\mathrm{Im}(\Gamma^t) \gg h$

Wave packets in semiclassical analysis

Sketch of proof.

Lemma *The matrix* Γ^t *satisfies the Ricatti equation*

$$\dot{\Gamma}^t = -V^{(2)}(z^t) - (\Gamma^t)^2, \qquad \Gamma^0 = \mathrm{i} I_n,$$

Wave packets in semiclassical analysis

Sketch of proof.

Lemma *The matrix* Γ^t *satisfies the Ricatti equation*

$$\dot{\Gamma}^t = -V^{(2)}(z^t) - (\Gamma^t)^2, \qquad \Gamma^0 = \mathrm{i} I_n,$$

and the function γ^t *satisfies*

$$\dot{\gamma}^t = -\frac{\mathrm{tr}(\Gamma^t)}{2}\gamma^t.$$

Wave packets in semiclassical analysis

Sketch of proof.

Lemma *The matrix* Γ^t *satisfies the Ricatti equation*

$$\dot{\Gamma}^t = -V^{(2)}(z^t) - (\Gamma^t)^2, \qquad \Gamma^0 = \mathrm{i} I_n,$$

and the function γ^t *satisfies*

$$\dot{\gamma}^t = -\frac{\mathrm{tr}(\Gamma^t)}{2}\gamma^t.$$

Set

$$\varphi := S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t) \cdot (x - z^t).$$

Sketch of proof.

Lemma *The matrix* Γ^t *satisfies the Ricatti equation*

$$\dot{\Gamma}^t = -V^{(2)}(z^t) - (\Gamma^t)^2, \qquad \Gamma^0 = \mathrm{i} I_n,$$

and the function γ^t *satisfies*

$$\dot{\gamma}^t = -\frac{\mathrm{tr}(\Gamma^t)}{2}\gamma^t.$$

Set

$$\varphi := S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t) \cdot (x - z^t).$$

Then

$H(h)\gamma^t e^{\frac{\mathrm{i}}{h}\varphi}$

Sketch of proof.

Lemma *The matrix* Γ^t *satisfies the Ricatti equation*

$$\dot{\Gamma}^t = -V^{(2)}(z^t) - (\Gamma^t)^2, \qquad \Gamma^0 = \mathrm{i} I_n,$$

and the function γ^t *satisfies*

$$\dot{\gamma}^t = -\frac{\mathrm{tr}(\Gamma^t)}{2}\gamma^t.$$

Set

$$\varphi := S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t) \cdot (x - z^t).$$

Then

$$H(h)\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \quad = \quad \left[\left(\dot{\varphi} + \frac{\nabla_x\varphi \cdot \nabla_x\varphi}{2} + V(x)\right) - \mathrm{i}h\left(\frac{\dot{\gamma^t}}{\gamma^t} + \frac{\Delta\varphi}{2}\right)\right]\gamma^t e^{\frac{\mathrm{i}}{h}\varphi}$$

Sketch of proof.

Lemma *The matrix* Γ^t *satisfies the Ricatti equation*

$$\dot{\Gamma}^t = -V^{(2)}(z^t) - (\Gamma^t)^2, \qquad \Gamma^0 = \mathrm{i} I_n,$$

and the function γ^t *satisfies*

$$\dot{\gamma}^t = -\frac{\mathrm{tr}(\Gamma^t)}{2}\gamma^t.$$

Set

$$\varphi := S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t)\cdot(x - z^t).$$

Then

$$\begin{aligned} H(h)\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} &= \left[\left(\dot{\varphi} + \frac{\nabla_x\varphi\cdot\nabla_x\varphi}{2} + V(x)\right) - \mathrm{i}h\left(\frac{\dot{\gamma}^t}{\gamma^t} + \frac{\Delta\varphi}{2}\right)\right]\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \\ &= \left[V(x) - V(z^t) - V^{(1)}(z^t)\cdot(x - z^t) - \frac{V^{(2)}(z^t)}{2}(x - z^t)\cdot(x - z^t)\right]\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \end{aligned}$$

Sketch of proof.

Lemma *The matrix* Γ^t *satisfies the Ricatti equation*

$$\dot{\Gamma}^t = -V^{(2)}(z^t) - (\Gamma^t)^2, \qquad \Gamma^0 = \mathrm{i} I_n,$$

and the function γ^t *satisfies*

$$\dot{\gamma}^t = -\frac{\mathrm{tr}(\Gamma^t)}{2}\gamma^t.$$

Set

$$\varphi := S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t)\cdot(x - z^t).$$

Then

$$\begin{aligned} H(h)\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} &= \left[\left(\dot{\varphi} + \frac{\nabla_x\varphi\cdot\nabla_x\varphi}{2} + V(x)\right) - \mathrm{i}h\left(\frac{\dot{\gamma}^t}{\gamma^t} + \frac{\Delta\varphi}{2}\right)\right]\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \\ &= \left[V(x) - V(z^t) - V^{(1)}(z^t)\cdot(x - z^t) - \frac{V^{(2)}(z^t)}{2}(x - z^t)\cdot(x - z^t)\right]\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \\ &= O\left(|x - z^t|^3\right)\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \end{aligned}$$

Sketch of proof.

Lemma *The matrix* Γ^t *satisfies the Ricatti equation*

$$\dot{\Gamma}^t = -V^{(2)}(z^t) - (\Gamma^t)^2, \qquad \Gamma^0 = \mathrm{i}I_n,$$

and the function γ^t *satisfies*

$$\dot{\gamma}^t = -\frac{\mathrm{tr}(\Gamma^t)}{2}\gamma^t.$$

Set

$$\varphi := S^t + \zeta^t \cdot (x - z^t) + \frac{\Gamma^t}{2}(x - z^t) \cdot (x - z^t).$$

Then

$$\begin{aligned} H(h)\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} &= \left[\left(\dot{\varphi} + \frac{\nabla_x\varphi \cdot \nabla_x\varphi}{2} + V(x)\right) - \mathrm{i}h\left(\frac{\dot{\gamma}^t}{\gamma^t} + \frac{\Delta\varphi}{2}\right)\right]\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \\ &= \left[V(x) - V(z^t) - V^{(1)}(z^t)\cdot(x - z^t) - \frac{V^{(2)}(z^t)}{2}(x - z^t)\cdot(x - z^t)\right]\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \\ &= O\left(|x - z^t|^3\right)\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \\ &= h^{3/2} O\left(\frac{|x - z^t|^3}{h^{3/2}}\right)\gamma^t e^{\frac{\mathrm{i}}{h}\varphi} \end{aligned}$$

Wave packets on Riemannian manifolds

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...).

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols).

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit,

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates and given for finite times

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates and given for finite times

Motivations and interests:

1. Consider more than the propagation along a *single* trajectory $\Rightarrow$ vary (z, ζ)

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates and given for finite times

Motivations and interests:

1. Consider more than the propagation along a *single* trajectory $\Rightarrow$ vary (z, ζ)
2. Get an (at most as possible) intrinsinc description of wave packets propagation

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates and given for finite times

Motivations and interests:

1. Consider more than the propagation along a *single* trajectory $\Rightarrow$ vary (z, ζ)
2. Get an (at most as possible) intrinsinc description of wave packets propagation
3. Get (relatively) explicit approximation of $e^{\mathrm{i}tH(h)/h}$ as a single integral

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates and given for finite times

Motivations and interests:

1. Consider more than the propagation along a *single* trajectory $\Rightarrow$ vary (z, ζ)
2. Get an (at most as possible) intrinsinc description of wave packets propagation
3. Get (relatively) explicit approximation of $e^{\mathrm{i}tH(h)/h}$ as a single integral, without need to go to the universal cover

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates and given for finite times

Motivations and interests:

1. Consider more than the propagation along a *single* trajectory $\Rightarrow$ vary (z, ζ)
2. Get an (at most as possible) intrinsinc description of wave packets propagation
3. Get (relatively) explicit approximation of $e^{\mathrm{i}tH(h)/h}$ as a single integral, without need to go to the universal cover, up to $|t| \leq C_0 |\log h|$

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates and given for finite times

Motivations and interests:

1. Consider more than the propagation along a *single* trajectory $\Rightarrow$ vary (z, ζ)
2. Get an (at most as possible) intrinsinc description of wave packets propagation
3. Get (relatively) explicit approximation of $e^{\mathrm{i}tH(h)/h}$ as a single integral, without need to go to the universal cover, up to $|t| \leq C_0 |\log h|$
4. See e.g. quite explicitly the effect of (negative) curvature

Wave packets on Riemannian manifolds

Goal: to emulate the construction on $\mathbb{R}^n$
Previous related works:

- **Construction of quasimodes:** by propagating a *single* wave packet along a *closed* geodesic (Babich-Lazutkin, Ralston, Paul-Uribe, Nonnenmacher-Eswarathasan...). Allows to use Fermi coordinates.
- **More general propagation results:** Paul-Uribe, Guillemin-Uribe-Wang: qualitative description of wave packets and their evolutions (for Hamiltonians with non homogeneous symbols). General but not so explicit, using local coordinates and given for finite times

Motivations and interests:

1. Consider more than the propagation along a *single* trajectory $\Rightarrow$ vary (z, ζ)
2. Get an (at most as possible) intrinsinc description of wave packets propagation
3. Get (relatively) explicit approximation of $e^{\mathrm{i}tH(h)/h}$ as a single integral, without need to go to the universal cover, up to $|t| \leq C_0 |\log h|$
4. See e.g. quite explicitly the effect of (negative) curvature
5. ...

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry**

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M
3. complete (for simplicity)

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M
3. complete (for simplicity)

Example. Any closed Riemannian manifold

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M
3. complete (for simplicity)

Example. Any closed Riemannian manifold

Lemma [Inverse exponential map close to the diagonal of $M \times M$] *If $d_g(z, m) < r_0$, there is a unique $W_z^m \in T_z M$ such that*

$$m = \exp_z\left(W_z^m\right).$$

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M
3. complete (for simplicity)

Example. Any closed Riemannian manifold

Lemma [Inverse exponential map close to the diagonal of $M \times M$] *If $d_g(z, m) < r_0$, there is a unique $W_z^m \in T_z M$ such that*

$$m = \exp_z \left(W_z^m \right).$$

For fixed m, $z \mapsto W_z^m$ is a vector field

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M
3. complete (for simplicity)

Example. Any closed Riemannian manifold

Lemma [Inverse exponential map close to the diagonal of $M \times M$] *If $d_g(z, m) < r_0$, there is a unique $W_z^m \in T_z M$ such that*

$$m = \exp_z \left(W_z^m\right).$$

For fixed m, $z \mapsto W_z^m$ is a vector field and one can expand its covariant derivative

$$\nabla W_z^m \sim -I + \frac{1}{3} R_z\left(., W_z^m\right) W_z^m + \frac{1}{12} (\nabla R)_z (W_z^m; ., W_z^m) W_z^m + \cdots$$

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M
3. complete (for simplicity)

Example. Any closed Riemannian manifold

Lemma [Inverse exponential map close to the diagonal of $M \times M$] *If $d_g(z, m) < r_0$, there is a unique $W_z^m \in T_z M$ such that*

$$m = \exp_z \left(W_z^m\right).$$

For fixed m, $z \mapsto W_z^m$ is a vector field and one can expand its covariant derivative

$$\nabla W_z^m \sim -I + \frac{1}{3} R_z \left(., W_z^m\right) W_z^m + \frac{1}{12} (\nabla R)_z (W_z^m; ., W_z^m) W_z^m + \cdots$$

All tensors in this expansion are bounded

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M
3. complete (for simplicity)

Example. Any closed Riemannian manifold

Lemma [Inverse exponential map close to the diagonal of $M \times M$] *If $d_g(z, m) < r_0$, there is a unique $W_z^m \in T_z M$ such that*

$$m = \exp_z \left(W_z^m \right).$$

For fixed m, $z \mapsto W_z^m$ is a vector field and one can expand its covariant derivative

$$\nabla W_z^m \sim -I + \frac{1}{3} R_z \left(., W_z^m\right) W_z^m + \frac{1}{12} (\nabla R)_z (W_z^m; ., W_z^m) W_z^m + \cdots$$

All tensors in this expansion are bounded (similar result for higher covariant derivatives)

Wave packets on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold with **bounded geometry** i.e.

1. injectivity radius bounded from below by $r_0 > 0$
2. all covariant derivatives of the Riemann curvature tensor bounded on M
3. complete (for simplicity)

Example. Any closed Riemannian manifold

Lemma [Inverse exponential map close to the diagonal of $M \times M$] *If $d_g(z, m) < r_0$, there is a unique $W_z^m \in T_z M$ such that*

$$m = \exp_z \left(W_z^m \right).$$

For fixed m, $z \mapsto W_z^m$ is a vector field and one can expand its covariant derivative

$$\nabla W_z^m \sim -I + \frac{1}{3} R_z \left(., W_z^m\right) W_z^m + \frac{1}{12} (\nabla R)_z (W_z^m; ., W_z^m) W_z^m + \cdots$$

All tensors in this expansion are bounded (similar result for higher covariant derivatives)

Rem: on $\mathbb{R}^n$, $W_z^m = m - z$.

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Proposition. *Let U be a coordinate patch, with coordinates $y_1, \ldots, y_n$.*

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Proposition. *Let U be a coordinate patch, with coordinates $y_1, \ldots, y_n$. Along each trajectory starting at $(z, \zeta) \in T^*U$, one can define intrinsincally*

$$\Gamma^t : T_{z^t} M^{\mathbb{C}} \to T_{z^t} M^{\mathbb{C}}, \qquad \textit{where} \ \ T_{z^t} M^{\mathbb{C}} = T_{z^t} M \otimes \mathbb{C}$$

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Proposition. *Let U be a coordinate patch, with coordinates $y_1, \ldots, y_n$. Along each trajectory starting at $(z, \zeta) \in T^*U$, one can define intrinsincally*

$$\Gamma^t : T_{z^t} M^{\mathbb{C}} \to T_{z^t} M^{\mathbb{C}}, \qquad \textit{where} \;\; T_{z^t} M^{\mathbb{C}} = T_{z^t} M \otimes \mathbb{C}$$

(i.e. Γ^t is a complex tensor along the curve $t \mapsto z^t$)

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Proposition. *Let U be a coordinate patch, with coordinates $y_1, \ldots, y_n$. Along each trajectory starting at $(z, \zeta) \in T^*U$, one can define intrinsincally*

$$\Gamma^t : T_{z^t} M^{\mathbb{C}} \to T_{z^t} M^{\mathbb{C}}, \qquad \textit{where } \; T_{z^t} M^{\mathbb{C}} = T_{z^t} M \otimes \mathbb{C}$$

(i.e. Γ^t is a complex tensor along the curve $t \mapsto z^t$) which is

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Proposition. *Let U be a coordinate patch, with coordinates $y_1, \ldots, y_n$. Along each trajectory starting at $(z, \zeta) \in T^*U$, one can define intrinsincally*

$$\Gamma^t : T_{z^t} M^{\mathbb{C}} \to T_{z^t} M^{\mathbb{C}}, \qquad \textit{where } \; T_{z^t} M^{\mathbb{C}} = T_{z^t} M \otimes \mathbb{C}$$

(i.e. Γ^t is a complex tensor along the curve $t \mapsto z^t$) which is **symmetric**

$$\langle \Gamma^t X, Y \rangle_{z^t} = \langle X, \Gamma^t Y \rangle_{z^t}, \qquad X, Y \in T_{z^t} M$$

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Proposition. *Let U be a coordinate patch, with coordinates $y_1, \ldots, y_n$. Along each trajectory starting at $(z, \zeta) \in T^*U$, one can define intrinsincally*

$$\Gamma^t : T_{z^t} M^{\mathbb{C}} \to T_{z^t} M^{\mathbb{C}}, \qquad \text{where } \; T_{z^t} M^{\mathbb{C}} = T_{z^t} M \otimes \mathbb{C}$$

(i.e. Γ^t is a complex tensor along the curve $t \mapsto z^t$) which is **symmetric**

$$\left\langle \Gamma^t X, Y \right\rangle_{z^t} = \left\langle X, \Gamma^t Y \right\rangle_{z^t}, \qquad X, Y \in T_{z^t} M$$

has **positive definite imaginary part**

$$\operatorname{Im} \left\langle \Gamma^t X, X \right\rangle_{z^t} > 0, \qquad X \neq 0, \; X \in T_{z^t} M$$

Wave packets on Riemannian manifolds

Consider $V \in C^\infty(M, \mathbb{R})$ and

$$H(h) := -h^2 \frac{\Delta_g}{2} + V$$

$$(z^t, \zeta^t) = \Phi^t(z, \zeta), \qquad \text{Hamiltonian flow of } \frac{|\xi|_m^2}{2} + V(m)$$

Proposition. *Let U be a coordinate patch, with coordinates $y_1, \ldots, y_n$. Along each trajectory starting at $(z, \zeta) \in T^*U$, one can define intrinsincally*

$$\Gamma^t : T_{z^t} M^{\mathbb{C}} \to T_{z^t} M^{\mathbb{C}}, \qquad \textit{where} \quad T_{z^t} M^{\mathbb{C}} = T_{z^t} M \otimes \mathbb{C}$$

(i.e. Γ^t is a complex tensor along the curve $t \mapsto z^t$) which is **symmetric**

$$\langle \Gamma^t X, Y \rangle_{z^t} = \langle X, \Gamma^t Y \rangle_{z^t}, \qquad X, Y \in T_{z^t} M$$

has **positive definite imaginary part**

$$\mathrm{Im} \langle \Gamma^t X, X \rangle_{z^t} > 0, \qquad X \neq 0, \ X \in T_{z^t} M$$

and satisfies the **Ricatti equation**

$$\nabla_{\dot{z}^t} \Gamma^t = -\mathrm{Hess}(V)_{z^t} - R_{z^t}(., \dot{z}^t)\dot{z}^t - (\Gamma^t)^2$$

where R_{z^t} is the Riemann tensor at z^t

Wave packets on Riemannian manifolds

Proof.
To construct Γ^t on $\mathbb{R}^n$, we have used the natural identifications

$$T_{(z,\zeta)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n, \qquad T_{(z^t,\zeta^t)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$$

Wave packets on Riemannian manifolds

Proof.
To construct Γ^t on $\mathbb{R}^n$, we have used the natural identifications

$$T_{(z,\zeta)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n, \qquad T_{(z^t,\zeta^t)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$$

How to proceed on a manifold ?

Wave packets on Riemannian manifolds

Proof.
To construct Γ^t on $\mathbb{R}^n$, we have used the natural identifications

$$T_{(z,\zeta)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n, \qquad T_{(z^t,\zeta^t)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$$

How to proceed on a manifold ?

1. At starting points (z,ζ) with $z \in U$, we split

$$T_{(z,\zeta)}(T^*M) \approx \mathbb{R}^n_y \oplus \mathbb{R}^n_\eta$$

using the (symplectic) coordinates $(y_1,\dots,y_n,\eta_1,\dots,\eta_n)$ on T^*U

Wave packets on Riemannian manifolds

Proof.
To construct Γ^t on $\mathbb{R}^n$, we have used the natural identifications

$$T_{(z,\zeta)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n, \qquad T_{(z^t,\zeta^t)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$$

How to proceed on a manifold ?

1. At starting points (z, ζ) with $z \in U$, we split

$$T_{(z,\zeta)}(T^*M) \approx \mathbb{R}^n_y \oplus \mathbb{R}^n_\eta$$

   using the (symplectic) coordinates $(y_1, \ldots, y_n, \eta_1, \ldots, \eta_n)$ on T^*U
2. At points (z^t, ζ^t), we use the (global) identification $\mathcal{I}_g : T^*M \to TM$

Wave packets on Riemannian manifolds

Proof.
To construct Γ^t on $\mathbb{R}^n$, we have used the natural identifications

$$T_{(z,\zeta)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n, \qquad T_{(z^t,\zeta^t)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$$

How to proceed on a manifold ?

1. At starting points (z,ζ) with $z \in U$, we split

$$T_{(z,\zeta)}(T^*M) \approx \mathbb{R}^n_y \oplus \mathbb{R}^n_\eta$$

   using the (symplectic) coordinates $(y_1,\dots,y_n,\eta_1,\dots,\eta_n)$ on T^*U
2. At points (z^t,ζ^t), we use the (global) identification $\mathcal{I}_g : T^*M \to TM$

$$\mathcal{I}_g(z^t,\zeta^t) = (z^t,\dot{z}^t)$$

Wave packets on Riemannian manifolds

Proof.
To construct Γ^t on $\mathbb{R}^n$, we have used the natural identifications

$$T_{(z,\zeta)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n, \qquad T_{(z^t,\zeta^t)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$$

How to proceed on a manifold ?

1. At starting points (z, ζ) with $z \in U$, we split

$$T_{(z,\zeta)}(T^*M) \approx \mathbb{R}^n_y \oplus \mathbb{R}^n_\eta$$

   using the (symplectic) coordinates $(y_1, \dots, y_n, \eta_1, \dots, \eta_n)$ on T^*U
2. At points (z^t, ζ^t), we use the (global) identification $\mathcal{I}_g : T^*M \to TM$

$$\mathcal{I}_g(z^t, \zeta^t) = (z^t, \dot{z}^t)$$

   and split along **horizontal** and **vertical** spaces

$$T_{(z^t,\dot{z}^t)}(\mathcal{I}_g T^*M) = \mathcal{H}_{(z^t,\dot{z}^t)} \oplus \mathcal{V}_{(z^t,\dot{z}^t)}$$

Wave packets on Riemannian manifolds

Proof.
To construct Γ^t on $\mathbb{R}^n$, we have used the natural identifications

$$T_{(z,\zeta)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n, \qquad T_{(z^t,\zeta^t)}(T^*\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}^n$$

How to proceed on a manifold ?

1. At starting points (z, ζ) with $z \in U$, we split

$$T_{(z,\zeta)}(T^*M) \approx \mathbb{R}^n_y \oplus \mathbb{R}^n_\eta$$

   using the (symplectic) coordinates $(y_1, \ldots, y_n, \eta_1, \ldots, \eta_n)$ on T^*U
2. At points (z^t, ζ^t), we use the (global) identification $\mathcal{I}_g : T^*M \to TM$

$$\mathcal{I}_g(z^t, \zeta^t) = (z^t, \dot{z}^t)$$

   and split along **horizontal** and **vertical** spaces

$$T_{(z^t,\dot{z}^t)}(\mathcal{I}_g T^*M) = \mathcal{H}_{(z^t,\dot{z}^t)} \oplus \mathcal{V}_{(z^t,\dot{z}^t)}$$

This gives a natural block decomposition

$$d\left(\mathcal{I}_g \circ \Phi^t\right) = \begin{pmatrix} \mathcal{L}_A & \mathcal{L}_B \\ \mathcal{L}_C & \mathcal{L}_D \end{pmatrix} : \mathbb{R}^n_y \oplus \mathbb{R}^n_\eta \to \mathcal{H}_{(z^t,\dot{z}^t)} \oplus \mathcal{V}_{(z^t,\dot{z}^t)}$$

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)}$$

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to T_{z^t}M^{\mathbb{C}}$$

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D)(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1, \ldots, x_n)$ near z^t, the matrix of Γ^t reads

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1, \ldots, x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

Proof (continued). One *can* then define

$$\left(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\right)\left(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\right)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1, \dots, x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

with

$$G^{-1} = (g^{ij}(x^t)),$$

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t} M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)} \to T_{z^t} M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1, \dots, x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

with

$$G^{-1} = (g^{ij}(x^t)), \qquad \Sigma^t_{ij} = \sum_{k,l} g_{kl}(x^t)\Gamma^l_{ij}(x^t)\dot x^t_k, \qquad x^t = x(z^t)$$

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1, \ldots, x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

with

$$G^{-1} = (g^{ij}(x^t)), \qquad \Sigma^t_{ij} = \sum_{k,l} g_{kl}(x^t)\Gamma^l_{ij}(x^t)\dot x^t_k, \qquad x^t = x(z^t)$$

and

$$\begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} = \begin{pmatrix} \partial x^t/\partial y & \partial x^t/\partial \eta \\ \partial \xi^t/\partial y & \partial \xi^t/\partial \eta \end{pmatrix}$$

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1,\dots,x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

with

$$G^{-1} = (g^{ij}(x^t)), \qquad \Sigma^t_{ij} = \sum_{k,l} g_{kl}(x^t)\Gamma^l_{ij}(x^t)\dot x^t_k, \qquad x^t = x(z^t)$$

and

$$\begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} = \begin{pmatrix} \partial x^t/\partial y & \partial x^t/\partial\eta \\ \partial\xi^t/\partial y & \partial\xi^t/\partial\eta \end{pmatrix}$$

$\Longrightarrow$ Symmetry of Γ^t,

Proof (continued). One *can* then define

$$\left(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\right)\left(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\right)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot{z}^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot{z}^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1, \ldots, x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

with

$$G^{-1} = (g^{ij}(x^t)), \qquad \Sigma^t_{ij} = \sum_{k,l} g_{kl}(x^t)\Gamma^l_{ij}(x^t)\dot{x}^t_k, \qquad x^t = x(z^t)$$

and

$$\begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} = \begin{pmatrix} \partial x^t/\partial y & \partial x^t/\partial \eta \\ \partial \xi^t/\partial y & \partial \xi^t/\partial \eta \end{pmatrix}$$

$\Longrightarrow$ Symmetry of Γ^t, positivity of $\mathrm{Im}(\Gamma^t)$

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1,\dots,x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

with

$$G^{-1} = (g^{ij}(x^t)), \qquad \Sigma^t_{ij} = \sum_{k,l} g_{kl}(x^t)\Gamma^l_{ij}(x^t)\dot x^t_k, \qquad x^t = x(z^t)$$

and

$$\begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} = \begin{pmatrix} \partial x^t/\partial y & \partial x^t/\partial \eta \\ \partial \xi^t/\partial y & \partial \xi^t/\partial \eta \end{pmatrix}$$

$\Longrightarrow$ Symmetry of Γ^t, positivity of $\mathrm{Im}(\Gamma^t)$ + Ricatti equation by direct computation #

Wave packets on Riemannian manifolds

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1,\ldots,x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

with

$$G^{-1} = (g^{ij}(x^t)), \qquad \Sigma^t_{ij} = \sum_{k,l} g_{kl}(x^t)\Gamma^l_{ij}(x^t)\dot x^t_k, \qquad x^t = x(z^t)$$

and

$$\begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} = \begin{pmatrix} \partial x^t/\partial y & \partial x^t/\partial \eta \\ \partial \xi^t/\partial y & \partial \xi^t/\partial \eta \end{pmatrix}$$

$\Longrightarrow$ Symmetry of Γ^t, positivity of $\mathrm{Im}(\Gamma^t)$ + Ricatti equation by direct computation #

Rem. If $(\tilde y_1,\ldots,\tilde y_n)$ are other coordinates on U, the matrix of Γ^t is changed into

$$G^{-1}\big(\tilde C^t + \tilde D^t Z\big)\big(\tilde A^t + \tilde B^t Z\big)^{-1} - G^{-1}\Sigma^t,$$

Proof (continued). One *can* then define

$$\big(\mathcal{L}_C + \mathrm{i}\mathcal{L}_D\big)\big(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B\big)^{-1} : \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)} \to \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)}$$

and then define Γ^t by composition with the natural isomorphisms

$$T_{z^t}M^{\mathbb{C}} \to \mathcal{H}^{\mathbb{C}}_{(z^t,\dot z^t)}, \qquad \mathcal{V}^{\mathbb{C}}_{(z^t,\dot z^t)} \to T_{z^t}M^{\mathbb{C}}$$

More concretely, using local coordinates $(x_1,\dots,x_n)$ near z^t, the matrix of Γ^t reads

$$G^{-1}(C^t + \mathrm{i}D^t)(A^t + \mathrm{i}B^t)^{-1} - G^{-1}\Sigma^t$$

with

$$G^{-1} = (g^{ij}(x^t)), \qquad \Sigma^t_{ij} = \sum_{k,l} g_{kl}(x^t)\Gamma^l_{ij}(x^t)\dot x^t_k, \qquad x^t = x(z^t)$$

and

$$\begin{pmatrix} A^t & B^t \\ C^t & D^t \end{pmatrix} = \begin{pmatrix} \partial x^t/\partial y & \partial x^t/\partial \eta \\ \partial \xi^t/\partial y & \partial \xi^t/\partial \eta \end{pmatrix}$$

$\Longrightarrow$ Symmetry of Γ^t, positivity of $\mathrm{Im}(\Gamma^t)$ + Ricatti equation by direct computation #

Rem. If $(\tilde y_1,\dots,\tilde y_n)$ are other coordinates on U, the matrix of Γ^t is changed into

$$G^{-1}\big(\tilde C^t + \tilde D^t Z\big)\big(\tilde A^t + \tilde B^t Z\big)^{-1} - G^{-1}\Sigma^t, \qquad Z = \left(\frac{\partial \tilde\eta}{\partial y} + \mathrm{i}\frac{\partial \tilde\eta}{\partial \eta}\right)\left(\frac{\partial \tilde y}{\partial y} + \mathrm{i}\frac{\partial \tilde y}{\partial \eta}\right)^{-1}$$

Wave packets on Riemannian manifolds

Definition of gaussian wave packets

Wave packets on Riemannian manifolds

Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$, *equal to* 1 *near* 0.

Wave packets on Riemannian manifolds

Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$, *equal to* 1 *near* 0.

$$\Psi^h_{z,\zeta}(m) := (\pi h)^{-\frac{n}{4}} \gamma^0 \exp \frac{\mathrm{i}}{h} \left(\zeta \cdot W_z^m + \frac{1}{2} \langle \Gamma^0 W_z^m, W_z^m \rangle_z \right) \rho\left(d_g(z,m)\right),$$

for $m \in M$ *and* $(z,\zeta) \in T^*U$ *(i.e.* $\zeta \in T_z^* U$*)*

$$\gamma^0 = det\big(g_{jk}(y(z))\big)^{-\frac{1}{4}}$$

Wave packets on Riemannian manifolds

Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$*, equal to* 1 *near* 0*.*

$$\Psi^h_{z,\zeta}(m) := (\pi h)^{-\frac{n}{4}} \gamma^0 \exp \frac{\mathrm{i}}{h} \left(\zeta \cdot W_z^m + \frac{1}{2} \langle \Gamma^0 W_z^m, W_z^m \rangle_z \right) \rho\left(d_g(z,m)\right),$$

for $m \in M$ *and* $(z,\zeta) \in T^*U$ *(i.e.* $\zeta \in T_z^*U$*)*

$$\gamma^0 = det\big(g_{jk}(y(z))\big)^{-\frac{1}{4}}$$

Rem. $\Psi^h_{z,\zeta}(m) = 0$ if $d_g(z,m) \geq r_0$.

Wave packets on Riemannian manifolds

Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$, *equal to* 1 *near* 0.

$$\Psi^h_{z,\zeta}(m) := (\pi h)^{-\frac{n}{4}} \gamma^0 \exp \frac{\mathrm{i}}{h} \left(\zeta \cdot W_z^m + \frac{1}{2} \langle \Gamma^0 W_z^m, W_z^m \rangle_z \right) \rho\left(d_g(z,m)\right),$$

for $m \in M$ *and* $(z, \zeta) \in T^*U$ *(i.e.* $\zeta \in T_z^* U$*)*

$$\gamma^0 = det\big(g_{jk}(y(z))\big)^{-\frac{1}{4}}$$

Rem. $\Psi^h_{z,\zeta}(m) = 0$ if $d_g(z,m) \geq r_0$.

Proposition [Wave packet decomposition - Approximate Bargmann transform]

Wave packets on Riemannian manifolds

Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$, *equal to* 1 *near* 0.

$$\Psi^h_{z,\zeta}(m) := (\pi h)^{-\frac{n}{4}} \gamma^0 \exp \frac{\mathrm{i}}{h} \left(\zeta \cdot W^m_z + \frac{1}{2} \langle \Gamma^0 W^m_z, W^m_z \rangle_z \right) \rho\left(d_g(z,m)\right),$$

for $m \in M$ *and* $(z,\zeta) \in T^*U$ *(i.e.* $\zeta \in T^*_z U$*)*

$$\gamma^0 = det\big(g_{jk}(y(z))\big)^{-\frac{1}{4}}$$

Rem. $\Psi^h_{z,\zeta}(m) = 0$ if $d_g(z,m) \geq r_0$.

Proposition [Wave packet decomposition - Approximate Bargmann transform] *Set*

$$B_h u(z,\zeta) := \left\langle \Psi^h_{z,\zeta}, u \right\rangle_{L^2(M)}, \qquad u \in C_0^\infty(U)$$

Wave packets on Riemannian manifolds

Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$*, equal to* 1 *near* 0*.*

$$\Psi^h_{z,\zeta}(m) := (\pi h)^{-\frac{n}{4}} \gamma^0 \exp \frac{\mathrm{i}}{h} \left(\zeta \cdot W^m_z + \frac{1}{2} \langle \Gamma^0 W^m_z, W^m_z \rangle_z \right) \rho\left(d_g(z,m)\right),$$

for $m \in M$ *and* $(z,\zeta) \in T^*U$ *(i.e.* $\zeta \in T^*_z U$*)*

$$\gamma^0 = det\big(g_{jk}(y(z))\big)^{-\frac{1}{4}}$$

Rem. $\Psi^h_{z,\zeta}(m) = 0$ if $d_g(z,m) \geq r_0$.

Proposition [Wave packet decomposition - Approximate Bargmann transform] *Set*

$$B_h u(z,\zeta) := \left\langle \Psi^h_{z,\zeta}, u \right\rangle_{L^2(M)}, \qquad u \in C_0^\infty(U)$$

Then

$$(2\pi h)^{-n} B_h^* B_h u = a(h) u$$

Wave packets on Riemannian manifolds

Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$, *equal to* 1 *near* 0.

$$\Psi^h_{z,\zeta}(m) := (\pi h)^{-\frac{n}{4}} \gamma^0 \exp \frac{\mathrm{i}}{h} \left(\zeta \cdot W^m_z + \frac{1}{2} \langle \Gamma^0 W^m_z, W^m_z \rangle_z \right) \rho\left(d_g(z,m)\right),$$

for $m \in M$ *and* $(z,\zeta) \in T^*U$ *(i.e.* $\zeta \in T^*_z U$*)*

$$\gamma^0 = det\big(g_{jk}(y(z))\big)^{-\frac{1}{4}}$$

Rem. $\Psi^h_{z,\zeta}(m) = 0$ if $d_g(z,m) \geq r_0$.

Proposition [Wave packet decomposition - Approximate Bargmann transform] *Set*

$$B_h u(z,\zeta) := \left\langle \Psi^h_{z,\zeta}, u \right\rangle_{L^2(M)}, \qquad u \in C_0^\infty(U)$$

Then

$$(2\pi h)^{-n} B_h^* B_h u = a(h) u = \left(1 + h^{\frac{1}{2}} a_1 + h^1 a_2 + \cdots \right) u$$

with $a(h), a_1, a_2, \ldots \in C^\infty$

Wave packets on Riemannian manifolds

Definition of gaussian wave packets *Let* $\rho \in C_0^\infty(-r_0, r_0)$*, equal to* 1 *near* 0.

$$\Psi^h_{z,\zeta}(m) := (\pi h)^{-\frac{n}{4}} \gamma^0 \exp \frac{\mathrm{i}}{h} \left(\zeta \cdot W^m_z + \frac{1}{2} \langle \Gamma^0 W^m_z, W^m_z \rangle_z \right) \rho \left(d_g(z, m) \right),$$

for $m \in M$ *and* $(z, \zeta) \in T^*U$ *(i.e.* $\zeta \in T^*_z U$*)*

$$\gamma^0 = det\big(g_{jk}(y(z))\big)^{-\frac{1}{4}}$$

Rem. $\Psi^h_{z,\zeta}(m) = 0$ if $d_g(z, m) \geq r_0$.

Proposition [Wave packet decomposition - Approximate Bargmann transform] *Set*

$$B_h u(z, \zeta) := \left\langle \Psi^h_{z,\zeta}, u \right\rangle_{L^2(M)}, \qquad u \in C_0^\infty(U)$$

Then

$$(2\pi h)^{-n} B_h^* B_h u = a(h) u = \left(1 + h^{\frac{1}{2}} a_1 + h^1 a_2 + \cdots \right) u$$

with $a(h)$*,* $a_1, a_2, \ldots \in C^\infty$*, i.e.*

$$(2\pi h)^{-n} \int \int_{T^*U} B_h u(z, \zeta) \Psi^h_{z,\zeta} dz d\zeta = a(h) u$$

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets]

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit* $h \to 0$*, and under general conditions on* V *(e.g. all covariant derivatives bounded),*

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit* $h \to 0$*, and under general conditions on* V *(e.g. all covariant derivatives bounded),*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(m)$$

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit $h \to 0$, and under general conditions on V (e.g. all covariant derivatives bounded),*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(m)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x) \exp\frac{\mathrm{i}}{h}\left(S^t + \zeta^t \cdot W^m_{z^t} + \frac{1}{2}\left\langle \Gamma^t W^m_{z^t}, W^m_{z^t}\right\rangle_{z^t}\right)\rho\big(d_g(z_t, m)\big)$$

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit $h \to 0$, and under general conditions on V (e.g. all covariant derivatives bounded),*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(m)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t\mathcal{A}^h_t(x)\exp\frac{\mathrm{i}}{h}\left(S^t+\zeta^t\cdot W^m_{z^t}+\frac{1}{2}\left\langle\Gamma^t W^m_{z^t},W^m_{z^t}\right\rangle_{z^t}\right)\rho\big(d_g(z_t,m)\big)$$

with

$$\gamma^t = det(g_{jk}(x^t))^{-1/4}det(A^t+\mathrm{i}B^t)^{-1/2}$$

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit $h \to 0$, and under general conditions on V (e.g. all covariant derivatives bounded),*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(m)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x)\exp\frac{\mathrm{i}}{h}\left(S^t+\zeta^t\cdot W^m_{z^t}+\frac{1}{2}\langle\Gamma^t W^m_{z^t}, W^m_{z^t}\rangle_{z^t}\right)\rho\big(d_g(z_t,m)\big)$$

with

$$\gamma^t = det(g_{jk}(x^t))^{-1/4}det(A^t+\mathrm{i}B^t)^{-1/2}$$

and an amplitude of the form

$$\mathcal{A}^h_t(x)\sim 1+\sum_{j\geq 1}h^{\frac{j}{2}}T_j\left(t,z^t,\zeta^t,\frac{W^m_{z^t}}{h^{\frac{1}{2}}}\right)$$

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit $h \to 0$, and under general conditions on V (e.g. all covariant derivatives bounded),*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(m)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x)\exp\frac{\mathrm{i}}{h}\left(S^t+\zeta^t\cdot W^m_{z^t}+\frac{1}{2}\left\langle\Gamma^t W^m_{z^t},W^m_{z^t}\right\rangle_{z^t}\right)\rho\big(d_g(z_t,m)\big)$$

with

$$\gamma^t = det(g_{jk}(x^t))^{-1/4}det(A^t+\mathrm{i}B^t)^{-1/2}$$

and an amplitude of the form

$$\mathcal{A}^h_t(x)\sim 1+\sum_{j\geq 1}h^{\frac{j}{2}}T_j\left(t,z^t,\zeta^t,\frac{W^m_{z^t}}{h^{\frac{1}{2}}}\right)$$

for times $|t|\leq C_0|\ln h|$

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit $h \to 0$, and under general conditions on V (e.g. all covariant derivatives bounded),*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(m)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x)\exp\frac{\mathrm{i}}{h}\left(S^t+\zeta^t\cdot W^m_{z^t}+\frac{1}{2}\left\langle\Gamma^t W^m_{z^t},W^m_{z^t}\right\rangle_{z^t}\right)\rho\big(d_g(z_t,m)\big)$$

with

$$\gamma^t = det(g_{jk}(x^t))^{-1/4}det(A^t+\mathrm{i}B^t)^{-1/2}$$

and an amplitude of the form

$$\mathcal{A}^h_t(x)\sim 1+\sum_{j\geq 1}h^{\frac{j}{2}}\,T_j\left(t,z^t,\zeta^t,\frac{W^m_{z^t}}{h^{\frac{1}{2}}}\right)$$

for times $|t|\leq C_0|\ln h|$ with $T_j(t,z^t,\zeta^t,.)$ polynomial (i.e. sum of tensors) of degree at most $3j$,

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit $h \to 0$, and under general conditions on V (e.g. all covariant derivatives bounded),*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(m)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t\mathcal{A}^h_t(x)\exp\frac{\mathrm{i}}{h}\left(S^t+\zeta^t\cdot W^m_{z^t}+\frac{1}{2}\langle\Gamma^t W^m_{z^t},W^m_{z^t}\rangle_{z^t}\right)\rho(d_g(z_t,m))$$

with

$$\gamma^t = det(g_{jk}(x^t))^{-1/4}det(A^t+\mathrm{i}B^t)^{-1/2}$$

and an amplitude of the form

$$\mathcal{A}^h_t(x)\sim 1+\sum_{j\geq 1}h^{\frac{j}{2}}T_j\left(t,z^t,\zeta^t,\frac{W^m_{z^t}}{h^{\frac{1}{2}}}\right)$$

for times $|t|\leq C_0|\ln h|$ with $T_j(t,z^t,\zeta^t,.)$ polynomial (i.e. sum of tensors) of degree at most $3j$, depending on the classical trajectory and the Taylor expansions of V and $W^m_{.}$ at z^t.

Wave packets on Riemannian manifolds

Theorem [Propagation of gaussian wave packets] *In the limit $h \to 0$, and under general conditions on V (e.g. all covariant derivatives bounded),*

$$e^{-\mathrm{i}\frac{t}{h}H(h)}\psi^h_{z,\zeta}(m)$$

is well approximated by

$$(\pi h)^{-\frac{n}{4}}\gamma^t \mathcal{A}^h_t(x) \exp\frac{\mathrm{i}}{h}\left(S^t + \zeta^t \cdot W^m_{z^t} + \frac{1}{2}\langle \Gamma^t W^m_{z^t}, W^m_{z^t}\rangle_{z^t}\right)\rho(d_g(z_t, m))$$

with

$$\gamma^t = det(g_{jk}(x^t))^{-1/4} det(A^t + \mathrm{i}B^t)^{-1/2}$$

and an amplitude of the form

$$\mathcal{A}^h_t(x) \sim 1 + \sum_{j\geq 1} h^{\frac{j}{2}} T_j\left(t, z^t, \zeta^t, \frac{W^m_{z^t}}{h^{\frac{1}{2}}}\right)$$

for times $|t| \leq C_0|\ln h|$ with $T_j(t, z^t, \zeta^t, .)$ polynomial (i.e. sum of tensors) of degree at most $3j$, depending on the classical trajectory and the Taylor expansions of V and $W^m_{\cdot}$ at z^t.

Remark on the proof: The transport equations

Wave packets on Riemannian manifolds

Remark on the proof: The transport equations are of the form

$$(\nabla_{\dot{z}^t} T)(\underbrace{.,\dots,.}_{k\ \ factors}) + \underbrace{T[\Gamma^t\cdot,\dots] + \cdots + T[\dots,\Gamma^t\cdot]}_{k\ \ terms} = F[.,\dots,.]$$

Remark on the proof: The transport equations are of the form

$$(\nabla_{\dot{z}^t} T)(\underbrace{.,\dots,.}_{k \ \ factors}) + \underbrace{T[\Gamma^t\cdot,\dots] + \cdots + T[\dots,\Gamma^t\cdot]}_{k \ \ terms} = F[.,\dots,.]$$

which turns out to be equivalent to

$$\frac{d}{dt}\left(T[E_t\cdot,\dots,E_t\cdot]\right) = F[E_t\cdot,\dots,E_t\cdot]$$

with $E_t := d\pi(\mathcal{L}_A + \mathrm{i}\mathcal{L}_B) : \mathbb{C}^n \to T_{z^t}M \otimes \mathbb{C}$ ($d\pi =$ projection from the horizontal space at $(z^t, \dot{z}^t)$ to the tangent space at z^t)

$\Longrightarrow$ Control on the exponential growth in time of $T_j(t, z^t, \zeta^t, .)$.

Wave packets on Riemannian manifolds

Theorem [Propagator approximation]

Wave packets on Riemannian manifolds

Theorem [Propagator approximation] *If* A_h *is a pseudodifferential operator supported in* U*,*

Wave packets on Riemannian manifolds

Theorem [Propagator approximation] *If* A_h *is a pseudodifferential operator supported in* U*, with principal symbol* χ*,*

Wave packets on Riemannian manifolds

Theorem [Propagator approximation] *If A_h is a pseudodifferential operator supported in U, with principal symbol χ, then (the kernel of) $e^{-\mathrm{i}\frac{t}{h}H(h)}A_h$*

Wave packets on Riemannian manifolds

Theorem [Propagator approximation] *If A_h is a pseudodifferential operator supported in U, with principal symbol χ, then (the kernel of) $e^{-\mathrm{i}\frac{t}{h}H(h)}A_h$ is well approximated by*

$$K_t^h(m, m') = h^{-\frac{3n}{2}} \int\int_{T^*U} b_h(t, z, \zeta, m, m') \exp \frac{\mathrm{i}}{h} F(t, z, \zeta, m, m') dz d\zeta$$

for times $|t| \leq C_0 |\log h|$.

Wave packets on Riemannian manifolds

Theorem [Propagator approximation] *If A_h is a pseudodifferential operator supported in U, with principal symbol χ, then (the kernel of) $e^{-\mathrm{i}\frac{t}{h}H(h)}A_h$ is well approximated by*

$$K_t^h(m,m') = h^{-\frac{3n}{2}} \int\int_{T^*U} b_h(t,z,\zeta,m,m') \exp\frac{\mathrm{i}}{h}F(t,z,\zeta,m,m')dzd\zeta$$

for times $|t| \leq C_0|\log h|$. The **phase** *reads*

$$F = S^t_{(z,\zeta)} + \zeta^t \cdot W^m_{z^t} + \frac{1}{2}\left\langle \Gamma^t_{(z,\zeta)} W^m_{z^t}, W^m_{z^t}\right\rangle_{z^t} - \zeta \cdot W^{m'}_z + \frac{1}{2}\left\langle \widetilde{\Gamma^0_{(z,\zeta)}} W^{m'}_z, W^{m'}_z\right\rangle_z$$

Wave packets on Riemannian manifolds

Theorem [Propagator approximation] *If A_h is a pseudodifferential operator supported in U, with principal symbol χ, then (the kernel of) $e^{-\mathrm{i}\frac{t}{h}H(h)}A_h$ is well approximated by*

$$K_t^h(m,m') = h^{-\frac{3n}{2}} \int\int_{T^*U} b_h(t,z,\zeta,m,m') \exp\frac{\mathrm{i}}{h}F(t,z,\zeta,m,m')dzd\zeta$$

for times $|t| \leq C_0|\log h|$. The **phase** *reads*

$$F = S^t_{(z,\zeta)} + \zeta^t \cdot W^m_{z^t} + \frac{1}{2}\left\langle \Gamma^t_{(z,\zeta)} W^m_{z^t}, W^m_{z^t}\right\rangle_{z^t} - \zeta \cdot W^{m'}_z + \frac{1}{2}\left\langle \widetilde{\Gamma^0_{(z,\zeta)}} W^{m'}_z, W^{m'}_z\right\rangle_z$$

where

$$\left\langle \widetilde{\Gamma^0_{(z,\zeta)}} W^{m'}_z, W^{m'}_z\right\rangle_z = -\mathrm{Re}\left\langle \Gamma^0_{(z,\zeta)} W^{m'}_z, W^{m'}_z\right\rangle_z + \mathrm{i}\ \mathrm{Im}\ \left\langle \Gamma^0_{(z,\zeta)} W^{m'}_z, W^{m'}_z\right\rangle_z$$

Wave packets on Riemannian manifolds

Theorem [Propagator approximation] *If A_h is a pseudodifferential operator supported in U, with principal symbol χ, then (the kernel of) $e^{-\mathrm{i}\frac{t}{h}H(h)}A_h$ is well approximated by*

$$K_t^h(m,m') = h^{-\frac{3n}{2}} \int\int_{T^*U} b_h(t,z,\zeta,m,m') \exp\frac{\mathrm{i}}{h}F(t,z,\zeta,m,m')dzd\zeta$$

for times $|t| \leq C_0|\log h|$. The **phase** *reads*

$$F = S^t_{(z,\zeta)} + \zeta^t \cdot W^m_{z^t} + \frac{1}{2}\left\langle \Gamma^t_{(z,\zeta)} W^m_{z^t}, W^m_{z^t}\right\rangle_{z^t} - \zeta\cdot W^{m'}_z + \frac{1}{2}\left\langle \widetilde{\Gamma^0_{(z,\zeta)}} W^{m'}_z, W^{m'}_z\right\rangle_z$$

where

$$\left\langle \widetilde{\Gamma^0_{(z,\zeta)}} W^{m'}_z, W^{m'}_z\right\rangle_z = -\mathrm{Re}\left\langle \Gamma^0_{(z,\zeta)} W^{m'}_z, W^{m'}_z\right\rangle_z + \mathrm{i}\ \mathrm{Im}\ \left\langle \Gamma^0_{(z,\zeta)} W^{m'}_z, W^{m'}_z\right\rangle_z$$

The **amplitude** *$b_h(t,z,\zeta,m,m')$ reads $b_0(t,z,\zeta,m,m') + O_t(h^{1/2})$,*

$$b_0 = \det\left((g_{jk}(x^t))^{1/2}(A^t + \mathrm{i}B^t)\right)^{-\frac{1}{2}}\det\left(g_{jk}(y)\right)^{-\frac{1}{4}}\chi(z,\zeta)\rho\left(d_g(z,m')\right)\rho\left(d_g(z^t,m)\right)$$

Wave packets on Riemannian manifolds

Theorem [Propagator approximation] *If A_h is a pseudodifferential operator supported in U, with principal symbol χ, then (the kernel of) $e^{-\mathrm{i}\frac{t}{h}H(h)}A_h$ is well approximated by*

$$K_t^h(m,m') = h^{-\frac{3n}{2}} \int\int_{T^*U} b_h(t,z,\zeta,m,m') \exp \frac{\mathrm{i}}{h} F(t,z,\zeta,m,m') dz d\zeta$$

for times $|t| \leq C_0 |\log h|$. The **phase** *reads*

$$F = S^t_{(z,\zeta)} + \zeta^t \cdot W^m_{z^t} + \frac{1}{2}\left\langle \Gamma^t_{(z,\zeta)} W^m_{z^t}, W^m_{z^t} \right\rangle_{z^t} - \zeta \cdot W^{m'}_z + \frac{1}{2}\left\langle \widetilde{\Gamma^0_{(z,\zeta)}} W^{m'}_z, W^{m'}_z \right\rangle_z$$

where

$$\left\langle \widetilde{\Gamma^0_{(z,\zeta)}} W^{m'}_z, W^{m'}_z \right\rangle_z = -\mathrm{Re} \left\langle \Gamma^0_{(z,\zeta)} W^{m'}_z, W^{m'}_z \right\rangle_z + \mathrm{i}\ \mathrm{Im} \left\langle \Gamma^0_{(z,\zeta)} W^{m'}_z, W^{m'}_z \right\rangle_z$$

The **amplitude** *$b_h(t,z,\zeta,m,m')$ reads $b_0(t,z,\zeta,m,m') + O_t(h^{1/2})$,*

$$b_0 = \det\left((g_{jk}(x^t))^{1/2}(A^t + \mathrm{i}B^t)\right)^{-\frac{1}{2}} \det\left(g_{jk}(y)\right)^{-\frac{1}{4}} \chi(z,\zeta)\rho\left(d_g(z,m')\right)\rho\left(d_g(z^t,m)\right)$$

Proof:

$$e^{-\mathrm{i}\frac{t}{h}H(h)}A_h u = (2\pi h)^{-n} \int\int_{T^*U} e^{-\mathrm{i}\frac{t}{h}H(h)} \Psi^h_{z,\zeta} \left\langle A_h^* a_h^{-1} \Psi^h_{z,\zeta}, u \right\rangle_{L^2(M)} dz d\zeta$$

Thank you for your attention