Low frequency resolvent estimates on asymptotically flat manifolds

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The setup

We consider an **asymptotically conical manifold** $(\mathcal{M}^n, G)$, ie for $\mathcal{K} \Subset \mathcal{M}$ and some $\mathcal{S}$ closed manifold, we have a diffeomorphism

$$\kappa : \mathcal{M} \setminus \mathcal{K} \to (R, \infty) \times \mathcal{S}$$

such that

$$G = \kappa^* \left(A(r)dr^2 + 2rB(r)dr + r^2 H(r) \right)$$

where $A(r)$ is a function (on $\mathcal{S}$), $B(r)$ a 1-form and $H(r)$ Riemannian metric, all depending smoothly on r, such that for some $\rho > 0$,

$$||\partial_r^j (A(r) - 1)||_0 + ||\partial_r^j B(r)||_1 + ||\partial_r^j (H(r) - H_0)||_2 \lesssim r^{-j-\rho},$$

where H_0 is a fixed metric on $\mathcal{S}$. This means $G \approx dr^2 + r^2 H_0$ close to infinity.

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Examples

1. $(\mathbb{R}^n, G_0)$, $G_0 =$ Euclidean metric
2. $(\mathbb{R}^n, G)$, G **long range** perturbation of G_0, ie

$$\left|\partial_x^\alpha \big(G(x) - G_0\big)\right| \lesssim (1 + |x|)^{-\rho - |\alpha|}$$

3. $(\mathcal{M}, G)$ **scattering manifold**, ie if $\mathcal{M}$ can be smoothly compactified as a manifold $\overline{\mathcal{M}}$ with boundary $\partial\overline{\mathcal{M}} = \mathcal{S}$, with boundary defining function x ($\mathcal{S} = \{x = 0\}$), and close to $x = 0$ (= infinity)

$$G = \frac{dx^2}{x^4} + \frac{h(x)}{x^2},$$

$h(.) =$ family of metrics on $\mathcal{S}$ smooth w.r.t. x up to $x = 0$. Then take $r = 1/x$ and $H(r) = h(1/r)$.

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The limiting absorption principle (LAP)

Set $R(z) = (-\Delta_G - z)^{-1}$. The LAP at energy $\lambda \in \mathbb{R}$ is the existence of

$$R_s(\lambda \pm i0) := \lim_{\varepsilon \to 0+} \langle r \rangle^{-s} R(\lambda \pm i\varepsilon) \langle r \rangle^{-s}$$

for some suitable $s > 0$ or (slightly) more simply

$$\sup_{\varepsilon > 0} \left|\left| \langle r \rangle^{-s} R(\lambda + i\varepsilon) \langle r \rangle^{-s} \right|\right| < \infty$$

More generally, one can consider

$$R_s^{(k)}(\lambda \pm i0) = (k!)^{-1} \lim_{\varepsilon \to 0+} \langle r \rangle^{-s} (-\Delta_G - \lambda \mp i\varepsilon)^{-1-k} \langle r \rangle^{-s}$$

The LAP is related to the spectral resolution E_λ of Δ_G, via

$$\frac{dE_\lambda}{d\lambda} = \frac{1}{2i\pi} \left(R(\lambda - i0) - R(\lambda + i0) \right)$$

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- The LAP is a well known consequence of the Mourre theory (+ Jensen-Mourre-Perry).
- Problem: getting estimates $R_s(\lambda \pm i0)$ as $\lambda \to \infty$, high frequency regime, and $\lambda \to 0^+$, low frequency regime
- High frequency (= semiclassical) estimates depend on the geodesic flow
  1. in general: $\mathcal{O}(e^{C\lambda^{1/2}})$
  2. non trapping: $\mathcal{O}(\lambda^{-1/2})$
  3. "weak" trapping: at least $\mathcal{O}(\lambda^{-1/2}\log\lambda)$, or $\mathcal{O}(\lambda^{\sigma})$...
- Low frequency estimates do not depend on the geodesic flow, but rather use global homogeneous Hardy-Poincaré or Sobolev inequalities

$$||\langle r\rangle^{-1}u||_{L^2} \lesssim ||\nabla_G u||_{L^2}, \qquad ||u||_{L^{2*}} \lesssim ||\nabla_G u||_{L^2},$$

  where $2^* = 2n/(n-2)$ for $n \geq 3$ (cf assumptions to get long time gaussian heat kernel estimates)

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where $2^* = 2n/(n-2)$ for $n \geq 3$ (cf assumptions to get long time gaussian heat kernel estimates)

Connection with time dependent problems

If B is a bounded operator

$$\int_{\mathbb{R}} ||Be^{it\Delta_G}u_0||^2 dt \leq 2\pi \left(\sup_{\substack{\lambda\in\mathbb{R},\\ \varepsilon>0}} ||BR(\lambda+i\varepsilon)B^*|| \right) ||u_0||^2$$

Using $B = \langle r \rangle^{-s}\phi(h^2\Delta_G)$, with $\phi \in C_0^\infty(\mathbb{R}\setminus 0)$ and semiclassical resolvent estimates

$$\sup_{\lambda\sim h^{-2}} ||\langle r\rangle^{-s}(-\Delta_G - \lambda \pm i0)^{-1}\langle r\rangle^{-s}|| \leq C_s h l(h), \qquad s > 1/2$$

we get, *eg* with $l(h) = h^{-l}$, a *local smoothing effect*

$$\int_{\mathbb{R}} ||\langle r\rangle^{-s}\phi(h^2\Delta_G)e^{it\Delta_G}u_0||^2_{H^{\frac{1-l}{2}}} dt \leq C_s ||u_0||^2_{L^2}.$$

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Connection with time dependent problems (continued)

By almost orthogonality, we can sum over $h = 2^{-k}$, $k \geq 0$, and get

$$\int_{\mathbb{R}} ||\langle r \rangle^{-s} (1 - \Phi)(\Delta_G) e^{it\Delta_G} u_0||^2_{H^{\frac{1-l}{2}}} dt \leq C_\Phi ||u_0||^2_{L^2},$$

for some (actually *all*) $\Phi \in C_0^\infty(\mathbb{R})$, $\Phi \equiv 1$ near 0. If we want to *remove this spectral cutoff*, we only get that for all T

$$\int_{-T}^{T} ||\langle r \rangle^{-s} e^{it\Delta_G} u_0||^2_{H^{1/2-l}} dt \leq C_T ||u_0||^2_{L^2},$$

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The result

Theorem 1 (B + Royer) Let $(\mathcal{M}, G)$ be an asymptotically conical manifold of dimension $n \geq 3$.

1. There exists $C > 0$ such that, for $|\mathrm{Re}(z)| \leq 1$,

$$||\langle r \rangle^{-1}(-\Delta_G - z)^{-1}\langle r \rangle^{-1}|| \leq C.$$

2. For all $s \in (0, 1/2)$, there exists $C_s > 0$ such that, for $0 < |\mathrm{Re}(z)| \leq 1$,

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3. Fix $[E_1, E_2] \Subset (0, \infty)$. For all integer $k \geq 1$, there exists C_k such that, for all $\epsilon \in (0, 1]$ and all ζ s.t. $\mathrm{Re}(\zeta) \in [E_1, E_2]$,

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Comments

1. The weight $\langle r \rangle^{-1}$ is sharp and improves on previous results by B and Bony-Häfner (on $\mathbb{R}^n$). Maybe contained implicitly in Guillarmou-Hassell for scattering manifolds.
2. In higher dimensions, one has better estimates. Morevoer when $n = 3$ and $(\mathcal{S}, H_0) = (S^2, can)$, one can take $s = 1/2$.
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Connection with Strichartz estimates

We want to know if global Strichartz estimates for $u(t) = e^{it\Delta_G} u_0$ hold, ie

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$$\phi(\epsilon^{-2}\Delta_G)u(t) = \chi(\epsilon r)\phi(\epsilon^{-2}\Delta_G)u(t) + (1-\chi)(\epsilon r)\phi(\epsilon^{-2}\Delta_G)u(t)$$

1. The first term is treated by L^2 estimates (Sobolev + LAP)

$$\begin{aligned} ||\chi(\epsilon r)\phi(\epsilon^{-2}\Delta_G)u(t)||_{L^{2*}} &\lesssim ||\nabla_G \chi(\epsilon r)\phi(\epsilon^{-2}\Delta_G)u(t)||_{L^2} \\ &\lesssim \epsilon ||\langle \epsilon r\rangle^{-1}\phi(\epsilon^{-2}\Delta_G)u(t)||_{L^2} \\ &\lesssim ||\langle r\rangle^{-1}\phi(\epsilon^{-2}\Delta_G)u(t)||_{L^2} \end{aligned}$$

2. $(1-\chi)(\epsilon r)\phi(\epsilon^{-2}\Delta_G)$ is a (micro)localization where $r \gtrsim \epsilon^{-1}$ and $|\xi| \sim \epsilon \Rightarrow$ outside of the 'uncertainty region' $\Rightarrow$ one can use microlocal techniques (rescaled pseudodifferential and Fourier integral operators). Here, the 'type 3 estimates' are very useful.

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Global Strichartz estimates

Let

$$u(t) = e^{it\Delta_G} u_0.$$

Theorem 2 (B + Mizutani - in progress) Let $(\mathcal{M}, G)$ be an asymptotically conical manifold of dimension $n \geq 3$. Assume we have polynomial resolvent estimates at high frequency

$$||\langle r\rangle^{-s}(-\Delta_G - \lambda - i0)\langle r\rangle^{-s}|| \leq C\lambda^{\sigma}, \qquad \lambda \gg 1,$$

for some $s > 0$ and $\sigma \in \mathbb{R}$. Then

1. There exists $\chi \in C_0^\infty(\mathcal{M})$ equal to 1 on a large enough compact set such that

$$||(1-\chi)u||_{L^2(\mathbb{R};L^{2^*}(\mathcal{M}))} \lesssim ||u_0||_{L^2(\mathcal{M})}.$$

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Proof of Theorem 1 (item 1)

Lemma One can choose $\kappa : \mathcal{M} \setminus \mathcal{K} \to (R, \infty) \times \mathcal{S}$ (or equivalently the radial coordinates r near infinity) such that

$$d\mathrm{vol}_G = \kappa^* \big(r^{n-1} dr d\mathrm{vol}_{H_0} \big).$$

Consequence: Outside a compact set, a good model for $(\mathcal{M}, d\mathrm{vol}_G)$ is $(\mathcal{M}_0, r^{n-1} dr d\mathrm{vol}_{H_0})$ with $\mathcal{M}_0 = (0, \infty) \times \mathcal{S}$, and

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$$(r\partial_r)^k K_{ij} \text{ small for all } k \geq 0. \qquad (S)$$

Then

$$(P - \lambda - i\varepsilon)^{-1} = \lambda^{-1} e^{i \ln \lambda^{1/2} A}(P_\lambda - 1 - i\mu)e^{-i \ln \lambda^{1/2} A}$$

where P_λ is the rescaled operator obtained by rescaling $r \mapsto r/\lambda^{1/2}$ in the K_{ij}, scaling under which (S) is invariant.
Remark: all theses λ dependent operators are selfadjoint with respect to $r^{n-1} dr d\mathrm{vol}_{H_0}$.

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