Propagation estimates for the Schrödinger equation

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Workshop on Harmonic Analysis and Spectral Theory
Consider a differential operator in divergence form, on $\mathbb{R}^d$, $d \geq 3$,

$$P = -\text{div} \left( G(x) \nabla \right),$$

with $G(x)$ a real, positive definite matrix, such that,

$$c \leq G(x) \leq C, \quad x \in \mathbb{R}^d,$$

for some $C, c > 0$.

Under weak regularity assumptions on $G$, $P$ has a selfadjoint realization on $L^2(\mathbb{R}^d)$ and one may define its resolvent

$$R(z) = (P - z)^{-1} : \text{Dom}(P) \to L^2(\mathbb{R}^d), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

which is bounded on $L^2$:

$$\|R(z)\|_{L^2 \to L^2} \leq |\text{Im}(z)|^{-1}.$$

One may (and will) more generally consider powers of the resolvent

$$R(z)^k = (P - z)^{-k} = \partial_z^{k-1}(P - z)^{-1}/(k - 1)!$$
In this talk, we are interested in the limit $\text{Im}(z) \to 0$ of (powers of) the resolvent.

1. If $\text{Re}(z) < 0$: no problem! $R(z)$ is bounded on $L^2$ since

$$\text{Re} \left( u, (P - z)u \right)_{L^2} \geq c \| \nabla u \|^2_{L^2} - \text{Re}(z) \| u \|^2_{L^2},$$

hence

$$\| R(z)f \|_{L^2} \leq -\frac{1}{\text{Re}(z)} \| f \|_{L^2}.$$

2. If $\text{Re}(z) = 0$. The situation is more difficult but, under very general conditions one may define

$$P^{-1} = \int_0^{+\infty} e^{-tP} dt : L^{2d}(\mathbb{R}^d) \to L^{2d-2}(\mathbb{R}^d).$$

One uses heat kernel bounds

$$0 \leq \left[ e^{-tP} \right] (x, y) \lesssim t^{-\frac{d}{2}} e^{-c \frac{|x-y|^2}{t}}, \quad t > 0,$$

which imply

$$\left[ P^{-1} \right] (x, y) \lesssim |x - y|^{2-d},$$

and then concludes with Hardy-Littlewood-Sobolev inequality.
If \( \text{Re}(z) > 0 \) ?

One needs much stronger assumptions on \( G \). Here we will assume that, for some \( \rho > 0 \),

\[
|\partial^\alpha (G(x) - I_d) | \lesssim \langle x \rangle^{-\rho - |\alpha|}.
\]

This is a flatness assumption at infinity: \( P \) is a long range perturbation of \(-\Delta\) (short range \( \leftrightarrow \) \( \rho > 1 \)).

The spectrum of \( P \) is then the half line \([0, +\infty)\).

**Absence of embedded eigenvalues** Any \( u \in L^2 \) such that

\[
P u = \lambda u,
\]

for some \( \lambda \geq 0 \), is identically 0. (Most general proof by Koch-Tataru '06; previous results by Froese-Herbst-Hoffmann-Ostenhoff and Cotta-Ramuniso-Krüger-Schrader)
Consider the *generator of dilations* (on $L^2$)

$$A = \frac{x \cdot \nabla + \nabla \cdot x}{2i},$$

i.e., the selfadjoint generator of the unitary group

$$e^{i\tau A} \varphi(x) = e^{\tau \frac{d}{2}} \varphi(e^{\tau} x).$$

One controls the behavior of the resolvent as $\text{Im}(z) \to 0$ as follows.

**Jensen-Mourre-Perry weighted estimates** For any $I \Subset (0, +\infty)$ and any $k \geq 1$,

$$\sup_{\text{Re}(z) \in I} \|(A + i)^{-k} R(z)^k (A - i)^{-k}\|_{L^2 \to L^2} < \infty.$$ 

Furthermore, the limits

$$R(\lambda \pm i0)^k = \lim_{\epsilon \to 0^+} R(\lambda \pm i\epsilon)^k, \quad \lambda > 0,$$

exist (in weighted spaces) and

$$R(\lambda \pm i0)^k = \partial^{k-1}_\lambda R(\lambda \pm i0)/(k-1)!.$$

Here the weights $(A \pm i)^{-1}$ may be replaced by $\langle x \rangle^{-1}$. 
Consider the time dependent Schrödinger equation

\[ i\partial_t u - Pu = 0, \quad u|_{t=0} = u_0 \in L^2, \]

ie \( u(t) = e^{-itP}u_0 \). By the Spectral Theorem

\[ e^{-itP} = \int e^{-it\lambda}dE_\lambda, \]

where the spectral measure can be (formally) written as

\[ 2i\pi \frac{dE_\lambda}{d\lambda} = (P - \lambda - i0)^{-1} - (P - \lambda + i0)^{-1}. \]

Thus, by (formal) integrations by parts

\[ t^k e^{-itP} = c_k \int_\mathbb{R} e^{-it\lambda} \left( (P - \lambda - i0)^{-k-1} - (P - \lambda + i0)^{-k-1} \right) d\lambda. \]

**Conclusion.** If the R.H.S. is bounded in \( t \), then we get a time decay for \( e^{-itP} \).

**Problem.** To justify the integrations by parts, we need to know the behaviour of \( (P - \lambda \pm i0)^{-k-1} \) at the thresholds: \( \lambda \to 0, \lambda \to +\infty \).
Behavior of the resolvent as $\lambda \to \infty$ Under the non trapping condition, one has for all $k \geq 1$,

$$\|\langle x \rangle^{-k}(P - \lambda \pm i0)^{-k}\langle x \rangle^{-k}\|_{L^2 \to L^2} \lesssim \lambda^{-\frac{k}{2}}, \quad \lambda \to \infty.$$  

From such well known estimates and the integrations by parts trick, one gets spectrally localized estimates of the form

$$\|\langle x \rangle^{-k}e^{-itP}(1 - \varphi)(P)\langle x \rangle^{-k}\|_{L^2 \to L^2} \leq C_{\varphi, k}\langle t \rangle^{-k},$$

if $\varphi \in C^\infty_0(\mathbb{R})$ satisfies $\varphi \equiv 1$ near 0, and $k \geq 0$.

To avoid the spectral cutoffs, we need to study the regime $\lambda \to 0$.  


Results

Let $N(d)$ be the largest even integer $< \frac{d}{2} + 1$.

**Theorem 1** If $\nu > \frac{d}{2} + N(d)$, then, as $|\lambda| \to 0$

$$||\langle x \rangle^{-\nu} (P - \lambda \pm i0)^{-N(d)} \langle x \rangle^{-\nu}||_{L^2 \to L^2} \lesssim \begin{cases} |\lambda|^{-1/2} & \text{if } d \equiv 3 \text{ mod } 4, \\ |\lambda|^{-\varepsilon} \text{ for any } \varepsilon & \text{if } d \equiv 0 \text{ mod } 4 \\ 1 & \text{otherwise}. \end{cases}$$

**Theorem 2** Under the non trapping condition,

$$||\langle x \rangle^{-\nu} e^{-itP} \langle x \rangle^{-\nu}||_{L^2 \to L^2} \lesssim \langle t \rangle^{1-N(d)}.$$
Main steps of the proof

Assume for simplicity that $G - I_d$ is small everywhere.

1 - Scaling

$$P - \lambda - i\epsilon = \lambda e^{i\tau A} \left( P_\lambda - 1 - i\mu \right) e^{-i\tau A}$$

with $\mu = \epsilon/\lambda$,

$$P_\lambda = -\text{div} \left( G_\lambda(x) \nabla \right), \quad G_\lambda(x) = G \left( \frac{x}{\lambda^{1/2}} \right),$$

and $\tau$ such that

$$\left( e^{-i\tau A} \varphi \right)(x) = \lambda^{-d/4} \varphi(x/\lambda^{1/2}).$$

Interest: prove estimates for the resolvent of $P_\lambda$ near energy 1 (i.e. away from the 0 threshold).

Problem: behavior of the coefficients of $P_\lambda$ as $\lambda \to 0$ (the condition (1) for $G_\lambda$ is not uniform with respect to $\lambda$).
2- Jensen-Mourre-Perry estimates. We obtain, for any $k \in \mathbb{N}$,

$$
\sup_{\mu \in \mathbb{R} \setminus \{0\}, \lambda > 0} \|(A + i)^{-k}(P_\lambda - 1 - i\mu)^{-k}(A - i)^{-k}\|_{L^2 \to L^2} < \infty.
$$

These estimates rely on the positive commutator estimate

$$
i[P_\lambda, A] = -\text{div} (2G_\lambda(x) - (x \cdot \nabla G_\lambda)(x)) \nabla \geq -\Delta,
$$

if $\|G_\lambda - I_d\|_\infty + \|x \cdot \nabla G_\lambda\|_\infty = \|G - I_d\|_\infty + \|x \cdot \nabla G\|_\infty$ is small enough, and on the fact that higher commutators

$$\text{ad}_A^k(P_\lambda) = [A, \text{ad}_A^{k-1}(P_\lambda)] \quad \text{ad}_A^0(P_\lambda) = P_\lambda,$$

are bounded from $H^{-1}$ to $H^1$.

The uniformity of the bounds w.r.t. $\lambda$ is simply due to the fact that we only need to control the scale invariant norms

$$
\|(x \cdot \nabla)^j G_\lambda\|_\infty = \|(x \cdot \nabla)^j G\|_\infty.
$$
3- **Elliptic estimates.** Let $N = N(d)$. We show that we can improve $L^2$ bounds into

$$\sup_{\mu \in \mathbb{R} \setminus \{0\}, \lambda > 0} \| (hA + i)^{-N} (P_\lambda - 1 - i\mu)^{-N} (hA - i)^{-N} \|_{H^{-N} \rightarrow H^N} < \infty.$$ 

for some fixed $h > 0$ small enough.

1. Choose $h$ small to guarantee that $(hA \pm i)^{-1}$ is bounded on $H^{\pm N}$.

2. Pick $\phi \in C_0^\infty(0, \infty)$, $\phi \equiv 1$ near 1. Then

$$ (P_\lambda - z)^{-N} = \phi(P_\lambda)(P_\lambda - z)^{-N} \phi(P_\lambda) + (1 - \phi^2(P_\lambda))(P_\lambda - z)^{-N} = I + II. $$

By the Spectral Theorem,

$$ II = (P_\lambda + 1)^{-N/2} B_\lambda(z)(P_\lambda + 1)^{-N/2}, $$

with $B_\lambda(z)$ bounded in $L^2$ uniformly w.r.t. $\lambda > 0$ and $\text{Re}(z) = 1.$
Lemma. If the scale invariant norms $\|\partial^\alpha (G - I_d)\|_{L^d/|x|}$ are small enough for $|\alpha| < d/2$, then

$$\sup_{\lambda > 0} \|(P_{\lambda} + 1)^{-N/2}\|_{H^{-N} \to L^2} \lesssim 1.$$ 

3. By setting

$$K^-_{\lambda} = (hA - i)^N \phi(P_{\lambda}), \quad K^+_{\lambda} = (K^-_{\lambda})^*,$$

observe that

$$I = K^+_{\lambda} (hA + i)^{-N} (P_{\lambda} - 1 - i\mu)^{-N} (hA - i)^{-N} K^-_{\lambda}.$$

Lemma. If the scale invariant norms

$$\|(x \cdot \nabla)^j \partial^\alpha (G - I_d)\|_{L^d/|x|}, \quad |\alpha| < \frac{d}{2}, \quad j \leq N(d),$$

are small enough, then

$$\sup_{\lambda > 0} \|K^-_{\lambda} (hA - i)^{-N}\|_{H^{-N} \to L^2} < \infty.$$
Conclusion: Sobolev imbeddings. We obtain: for some \( h > 0 \),

\[
\sup_{\lambda > 0, \mu \in \mathbb{R}\{0\}} \| (hA + i)^{-N}(P\lambda - 1 - i\mu)^{-N}(hA - i)^{-N} \|_{L^p \to L^{p'}} =: C_N < \infty,
\]

with \( N = N(d) \) and

\[
p = \frac{2d}{d + 2s} \quad \text{with} \quad s = \begin{cases} \frac{d}{2} & \text{if } d \equiv 3 \mod 4, \\ \text{any } s < \frac{d}{2} & \text{if } d \equiv 0 \mod 4, \\ N & \text{otherwise}. \end{cases}
\]

But

\[
(P - \lambda - i\epsilon)^{-N} = \lambda^{-N} e^{i\tau A} (P\lambda - 1 - i\mu)^{-N} e^{-i\tau A}
\]

and

\[
\| e^{i\tau A} \|_{L^p' \to L^{p'}} = e^{\tau \left( \frac{d}{2} - \frac{d}{p'} \right)} = \lambda^\frac{d}{4} \left(1 - \frac{2}{p'} \right) = \lambda^\frac{s}{2},
\]

thus

\[
\| (hA + i)^{-N}(P - \lambda - i\epsilon)^{-N}(hA - i)^{-N} \|_{L^p \to L^{p'}} \leq C_N \lambda^{-N+s}.
\]