

# On the scattering theory of asymptotically flat manifolds and Strichartz inequalities

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## Purpose of the talk

- ▶ Take the question of **Strichartz inequalities** (for the Schrödinger equation) on **asymptotically flat manifolds** as a case study to review some related **scattering estimates** (resolvent estimates, time decay, smoothing estimates), either for comparison or because they are crucial inputs in the proofs of Strichartz inequalities
- ▶ Present some recent results (joint with H. Mizutani) on Strichartz inequalities on asymptotically flat manifolds

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$$\|e^{it\Delta} u_0\|_{L^2(K)} \lesssim_K \|e^{it\Delta} u_0\|_{L^q(\mathbb{R}^n)}, \quad K \in \mathbb{R}^n.$$

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Rem. This correspondence  $\lambda \rightarrow t$  also allows to convert resolvent estimates into time decay/propagation estimates (smoothness of  $R_0(\lambda \pm i0) \leftrightarrow$  decay of  $e^{itP}$ )

# Strichartz inequalities vs smoothing effect for a wave packet

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## Smoothing effect (local in time)

$$|\langle D \rangle^s e^{i\frac{t}{2}\Delta} G_{z,\zeta,h}(x)|$$



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## Smoothing effect (local in time)

$$\begin{aligned} |\langle D \rangle^s e^{i\frac{t}{2}\Delta} G_{z,\zeta,h}(x)| &\sim \langle \zeta/h \rangle^s \frac{\pi^{-\frac{n}{4}}}{(h\langle t/h \rangle^2)^{\frac{n}{4}}} \exp\left(-\frac{|x-z-(t/h)\zeta|^2}{2h\langle t/h \rangle^2}\right) \quad h \rightarrow 0, \\ &= \langle \zeta/h \rangle^s G_{z,\zeta,h}^t(x). \end{aligned}$$

We assume that  $\zeta \neq 0$ , say  $|\zeta| = 1$  and then, by possibly rotating the axis, that  $\zeta = (1, 0, \dots, 0)$ . Then

$$\|\langle x \rangle^{-\nu} \langle \zeta/h \rangle^s G_{z,\zeta,h}^t\|_{L_x^2}^2 = c_n \langle \zeta/h \rangle^{2s} \langle t/h \rangle^{-n} \int \langle h^{\frac{1}{2}} y + z + t\zeta/h \rangle^{-2\nu} \exp\left(-\frac{y^2}{\langle t/h \rangle^2}\right) dy$$

If we further integrate in time on  $[-T, T]_t$ ,

$$c_n h \langle \zeta/h \rangle^{2s} \int_{-T/h}^{T/h} \langle \tau \rangle^{-n} \int \langle h^{\frac{1}{2}} y + z + \tau\zeta \rangle^{-2\nu} \exp\left(-\frac{y^2}{\langle \tau \rangle^2}\right) dy d\tau$$

which is bounded by

$$c_n h \langle 1/h \rangle^{2s} \int_{-T/h}^{T/h} \int \langle h^{\frac{1}{2}} Y_1 \langle \tau \rangle + z_1 + \tau \rangle^{-2\nu} \exp(-Y^2) dY d\tau$$

**Remark.** Up to the term  $Y_1 \langle \tau \rangle$ , there is no more contribution of the spreading  $\langle \tau \rangle$ .

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**Remark.** Up to the term  $Y_1\langle \tau \rangle$ , there is no more contribution of the spreading  $\langle \tau \rangle$ . Here, the main role will be played the translation by  $(t/h)\zeta = \tau\zeta$ .

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**Scattering inequalities** turn out to play a crucial role in this problem.

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$$G_0 = dx_1^2 + \cdots + dx_n^2 = \sum_{j,k} G_{jk} dx_j dx_k, \quad G_0 := (G_{jk}) = I.$$

The geodesic flow  $\phi^t : \mathbb{R}^n \times \mathbb{R}^n (= T^*\mathbb{R}^n) \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  is given by

$$\phi^t(x, \xi) = (x + 2t\xi, \xi) =: (x^t, \xi^t),$$

it solves the **Hamilton equations**

$$\dot{x}^t = (\partial_\xi p)(x^t, \xi^t), \quad \dot{\xi}^t = -(\partial_x p)(x^t, \xi^t)$$

where

$$p(x, \xi) = |\xi|^2 = \xi \cdot G_0^{-1} \xi$$

is the (principal) symbol of  $-\Delta = D_1^2 + \cdots + D_n^2$  with  $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$

- ▶ Perturbed model:  $\mathbb{R}^n$ , equipped with a metric  $\sum_{j,k} G_{jk}(x) dx_j dx_k$  such that

$$G(x) - I \rightarrow 0 \text{ as } x \rightarrow \infty, \quad G(x) := (G_{jk}(x))$$

more precisely,  $\partial^\alpha (G_{jk}(x) - \delta_{jk}) = O(\langle x \rangle^{-\mu - |\alpha|})$  for some  $\mu > 0$ . The geodesic flow is defined analogously with

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$$-\Delta_G = - \sum_{j,k} G^{jk}(x) \partial_{x_j} \partial_{x_k} + \sum_{j,k,\ell} G^{jk}(x) \Gamma_{jk}^\ell(x) \partial_{x_\ell}$$

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**Question:** behavior of  $R(\lambda \pm i0)$  and (2) as  $\lambda \rightarrow \infty$  (high energy) and  $\lambda \rightarrow 0$  (low energy) ?

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$$\left\| \langle r \rangle^{-\nu_1} R(\lambda \pm i0) \langle r \rangle^{-\nu_2} \right\|_{L^2(M) \rightarrow L^2(M)} \lesssim 1$$

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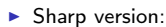
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$$\left\| \langle r \rangle^{-\nu_1} R(\lambda \pm i0) \langle r \rangle^{-\nu_2} \right\|_{L^2(M) \rightarrow L^2(M)} \lesssim 1$$

[Bony-Hafner]



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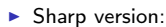
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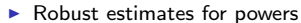
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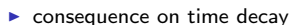
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$$\| \langle \lambda^{\frac{1}{2}} r \rangle^{-k} (\lambda^{-1} P - 1 \pm i0)^{-k} \langle \lambda^{\frac{1}{2}} r \rangle^{-k} \|_{L^2(M) \rightarrow L^2(M)} \lesssim 1$$

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$$\| \langle \lambda^{\frac{1}{2}} r \rangle^{-k} \varphi(\lambda^{-1} P) e^{-itP} \langle \lambda^{\frac{1}{2}} r \rangle^{-k} \|_{L^2(M) \rightarrow L^2(M)} \lesssim \langle \lambda t \rangle^{1-k}$$

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## Intuition (non trapping case):

- ▶ Inside a compact set  $K$ , combine

$$\|\mathbf{1}_K e^{i \cdot P} u_0\|_{L^2([-T, T], L^{2^*})} \lesssim_T \|u_0\|_{H^{1/2}(M)} \quad \text{and} \quad \|\mathbf{1}_K e^{i \cdot P} v_0\|_{L^2([-T, T], H^{1/2})} \lesssim_T \|v_0\|_{L^2}$$

- ▶ Outside a compact set: use that the geometry is close to a nice model (...)

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**Few about global in time estimates** (partially due to the low energy analysis)

- ▶ Tataru , Tataru-Marzuola-Metcalf: asymptotically euclidean case, allow relatively weak trapping at infinity
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**Theorem 4 (nonlinear scattering)** *Under the assumptions of Theorem 3, the  $L^2$  critical equation*

$$i\partial_t u - Pu = \sigma |u|^{\frac{4}{n}} u, \quad u|_{t=0} = u_0, \quad \sigma = \pm 1,$$

*with  $\|u_0\|_{L^2} \ll 1$ , has a unique solution in (a subspace of)  $C(\mathbb{R}, L^2) \cap L^{2+\frac{4}{n}}(\mathbb{R} \times M)$  and*

$$\|u(t) - e^{-itP} u_\pm\|_{L^2(M)} \rightarrow 0, \quad t \rightarrow \pm\infty.$$

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**Low frequency localization in the uncertainty region:**



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**Low frequency localization in the uncertainty region:** in the regime  $\lambda = \epsilon^2 \rightarrow 0$ , how to prove

$$\int_{\mathbb{R}} \|\chi(\epsilon r) f(P/\epsilon^2) e^{itP} u_0\|_{L^2(\mathbb{R}; L^{2^*})}^2 dt \leq C \|f(P/\epsilon^2) u_0\|_{L^2}^2$$

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où  $\tilde{f}, \tilde{\tilde{f}} \in C_0^\infty(0, +\infty)$ .

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où  $\tilde{f}, \tilde{\tilde{f}} \in C_0^\infty(0, +\infty)$ . One concludes by mean of an optimally weighted resolvent inequality [B-Royer, 2015]

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**Low frequency localization in the uncertainty region:** in the regime  $\lambda = \epsilon^2 \rightarrow 0$ , how to prove

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with  $C$  independent of  $\lambda$  (and  $u_0$ )

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**Rem.** For the localization,  $(1 - \chi(\epsilon r)) f(P/\epsilon^2)$ , one has “ $|\xi| \sim \epsilon$ ” and “ $|x| \gtrsim \epsilon^{-1}$ ”  $\Rightarrow$  no problem of uncertainty principle to use microlocal techniques



## Rest of the proof

At infinity: split  $f(P/\lambda)e^{itP}$  into sums of

$$T_\lambda(t) = L_\lambda f(P/\lambda)e^{itP}$$

with suitable localization operators  $L_\lambda$ , and show

$$\|T_\lambda(t)\|_{L^2 \rightarrow L^2} \lesssim 1, \quad \|T_\lambda(t)T_\lambda(s)\|_{L^1 \rightarrow L^\infty} \lesssim |t-s|^{-\frac{n}{2}}$$

by writing

$$T_\lambda(t)T_\lambda(s) = \text{approximation} + \text{remainder}$$

- ▶ the “approximation” is explicit enough operator to bound sharply its integral kernel by  $|t-s|^{-\frac{n}{2}}$  (dispersion bound)
- ▶ the remainder is a remainder term in a Duhamel formula in which we combine  $L^2$  time decay/propagation estimates (for the time decay) and Sobolev estimates (to replace  $L^2 \rightarrow L^2$  by  $L^1 \rightarrow L^\infty$ ) to derive dispersion bounds.

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