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MONTE CARLO METHODS FOR SENSITIVITY ANALYSIS OF POISSON-DRIVEN STOCHASTIC SYSTEMS, AND APPLICATIONS

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Abstract

We extend a result due to Zazanis [34] on the analyticity of the expectation of suitable functionals of homogeneous Poisson processes, with respect to the intensity of the process. As our main result, we provide Monte Carlo estimators for the derivatives. We apply our results to stochastic models which are of interest in stochastic geometry and insurance.

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1. Introduction

Let N be an independently marked homogeneous Poisson process (IMHPP) with points in $\mathbb{R}^d$ and marks with distribution Q taking values on some complete separable metric space $\mathbb{M}$. Under the probability measure P_λ, the intensity of the Poisson point process is $\lambda > 0$. Moreover, let $\varphi(N)$ be a real valued functional of the process and E_λ the expectation under P_λ. The function $\lambda \mapsto E_\lambda[\varphi(N)]$ is known to be smooth in λ under several and different assumptions.

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Zazanis [34] focuses on functionals depending only on the configuration, up to a finite stopping time, of a homogeneous Poisson process on the half-line. For this class of functionals he proves that the function $\lambda \mapsto E_\lambda[\varphi(N)]$ is analytic under a specific moment condition on the functional, and a light-tailed assumption on the stopping time. However, he does not provide an explicit expression for the derivatives.

For one dimensional IMHPP, Baccelli, Hasenfuss and Schmidt [5] provide sufficient conditions for the m-differentiability of $E_\lambda[\varphi(N)]$, with respect to λ, in a neighborhood of the origin, and closed form expressions for the derivatives in terms of multiple integrals. However, their method does not address the question of analyticity, and their set of conditions is different from ours.

A more general framework is considered by Molchanov and Zuyev [25]. Let N be a (not necessarily homogeneous) Poisson process on a locally compact separable metric space, with intensity measure Λ. Moreover, let $\varphi(N)$ be a suitable functional of the process. They study the analyticity of the expectation $E_\Lambda[\varphi(N)]$, with respect to Λ. Particularly, they prove that, under some assumptions on φ, the function $\Lambda \mapsto E_\Lambda[\varphi(N)]$ is analytic on the cone of positive measures.

For Poisson processes with a finite intensity measure Λ, a relevant work is also that one of Albeverio, Kondratiev and Röckner [1], where it is proved that the expectation $E_\Lambda[\varphi(N)]$ is analytic with respect to a perturbation of Λ by a semi-group.

In this paper we basically rely on Zazanis' paper [34] for the analyticity of $\lambda \mapsto E_\lambda[\varphi(N)]$, where N is an IMHPP on $\mathbb{R}^d \times \mathbb{M}$. As our main result, we derive explicit formulas for all the derivatives of $\lambda \mapsto E_\lambda[\varphi(N)]$. These formulas provide Monte Carlo methods for sensitivity analysis of suitable Poisson driven stochastic systems, with respect to the intensity of the process.

There are several motivations for being interested in sensitivity analysis: the main reasons are the applications to optimization and control of complex systems occurring, for instance, in stochastic geometry and insurance. Sensitivity analysis was introduced by Ho and Cao [16], and has been addressed by many authors (see, for instance, the book by Glassermann [14] and the references cited therein). There are mainly three ways to handle this problem: the infinitesimal perturbation analysis (IPA), the likelihood ratio method (LRM), and the rare perturbation analysis (RPA). We refer the reader to L'Ecuyer [19] and Suri and Zazanis [32] for more insight into the IPA method, and to Reimann and Weiss [30] for more details on the LRM. It is worthwhile to mention the work by Decreusefond [11] where, using Malliavin calculus, it is shown that IPA, RPA and LRM can be seen as a part of the stochastic calculus of variations. Decreusefond's paper main achievement is that he may, potentially, consider discrete-event systems more general than Poisson processes.

As already mentioned, we derive explicit formulas for all the derivatives of $\lambda \mapsto E_\lambda[\varphi(N)]$. For this we use the RPA method. Suppose we wish to compute the derivative $\frac{d}{d\lambda}E_\lambda[\varphi(N)]$. We distinguish two different RPA methods: the virtual and the phantom. The virtual RPA method may be attributed to Reiman and Simon [29], and has been revisited by Baccelli and Brémaud [4]. Following the ideas of these articles, we evaluate the limit

$$\lim_{\Delta\lambda\to 0} \frac{E_{\lambda+\Delta\lambda}[\varphi(N)] - E_\lambda[\varphi(N)]}{\Delta\lambda}.$$

The key idea is to use the superposition property of IMHPPs to generate an IMHPP of intensity $\lambda + \Delta\lambda$ from a small perturbation of an IMHPP of intensity λ. By a coupling argument, an IMHPP of intensity $\lambda + \Delta\lambda$ is generated from the superposition of two independent IMHPPs of respective intensity λ and $\Delta\lambda$. The phantom RPA method was introduced by Brémaud and Vazquez-Abad [7]. Following the approach in this paper, we compute the limit

$$\lim_{\Delta\lambda\to 0} \frac{E_\lambda[\varphi(N)] - E_{\lambda-\Delta\lambda}[\varphi(N)]}{\Delta\lambda}.$$

The idea is to use the thinning property of IMHPPs to generate an IMHPP of intensity $\lambda - \Delta\lambda$: similarly to the previous case, this process is generated from a small perturbation of an IMHPP of intensity λ by a coupling argument. We generalize this approach to compute the n-th order derivatives $\frac{d^n}{d\lambda^n}E_\lambda[\varphi(N)]$.

Our results can be applied to suitable functionals of random sets arising in stochastic geometry. Furthermore, by using importance sampling and large deviations techniques we show that our results can be applied to ruin probabilities of risk processes with Poisson arrivals and delayed or un-delayed claims. In the case of classical risk processes (un-delayed claims) we provide an asymptotically optimal Monte Carlo estimator for the first order derivative of the ruin probability.

The paper is organized as follows. In Section 2 we fix the notation and extend Zazanis' result about analyticity of functionals of homogeneous Poisson processes. In

Section 3 we state our results about n-th order derivatives of functionals of homogeneous Poisson processes. In Section 4 we prove the results given in Section 3. Finally, in Section 5 we apply our results to stochastic models which are of interest in stochastic geometry and insurance.

2. Preliminaries

2.1. Notation

Let $d \geq 1$ be an integer, $\mathbb{M}$ a complete separable metric space, and $\mathcal{N}$ the space of all counting measures on $\mathbb{R}^d \times \mathbb{M}$, defined on the Borel σ-field $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{M})$, such that each measure $\mu \in \mathcal{N}$ is simple and locally finite, that is: $\mu(\{(x,z)\})$ is equal to 0 or 1 for each $(x,z) \in \mathbb{R}^d \times \mathbb{M}$, and μ is finite on each set of the form $B \times \mathbb{M}$, where B is a bounded Borel set. We endow the space $\mathcal{N}$ with its usual topology (see, for instance, the book by Daley and Vere-Jones [10] for the details). Any measure in $\mathcal{N}$ can be represented as

$$\mu = \sum_{n=1}^{\mu(\mathbb{R}^d \times \mathbb{M})} \delta_{(x_n,z_n)} = \sum_{n=1}^{\mu(\mathbb{R}^d \times \mathbb{M})} \delta_{\mathbf{x}_n}, \tag{1}$$

where $(x_n, z_n) = \mathbf{x}_n \in \mathbb{R}^d \times \mathbb{M}$ and $\delta_{(x,z)}$, $(x,z) \in \mathbb{R}^d \times \mathbb{M}$, is the Dirac's measure on $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{M})$: for any $B \in \mathcal{B}(\mathbb{R}^d)$ and $M \in \mathcal{B}(\mathbb{M})$, $\delta_{(x,z)}(B \times M)$ is equal to 1 if $(x,z) \in B \times M$ and equal to 0 otherwise. The elements of the sequence $\{z_n\}_{n\geq 1} \subseteq \mathbb{M}$ are called marks. The symbol $\mathrm{supp}(\mu)$ will denote the support of the counting measure $\mu \in \mathcal{N}$, that is, if μ is given by (1), $\mathrm{supp}(\mu) = \{(x_n, z_n)_{n\geq 1}\}$.

Let B_r be the closed ball centered in 0 with radius r, and $B(x,r) = x + B_r$ the closed ball centered at x with radius r. If K is a compact set, throughout this paper we denote by B_K the smallest closed ball centered in 0 which contains K.

Let $\mu = \sum_{n\geq 1} \delta_{\mathbf{x}_n} = \sum_{n\geq 1} \delta_{(x_n,z_n)}$ be in $\mathcal{N}$. For $B \in \mathcal{B}(\mathbb{R}^d)$ and $M \in \mathcal{B}(\mathbb{M})$, we set

$$\int_{B\times M} \psi(\mathbf{x})\mu(d\mathbf{x}) = \int_{B\times M} \psi(x,z)\mu(dx \times dz) = \sum_{n\geq 1} \psi(x_n,z_n)\mathbb{1}\{(x_n,z_n) \in B \times M\},$$

for any measurable functional $\psi : \mathbb{R}^d \times \mathbb{M} \to \mathbb{R}$ such that the sum is well defined.

Let $B \subseteq \mathbb{R}^d$ be a Borel set. Throughout this work, we denote the set of points of $\mathrm{supp}(\mu)$ in $B \times \mathbb{M}$ by $\mu_{|B}$ and the number of points of μ in $B \times \mathbb{M}$ by μ_B. With an abuse of notation, $|B|$ denotes the Lebesgue measure of B, and for real numbers $x \in \mathbb{R}$ the symbol $|x|$ denotes the usual absolute value.

For any measurable functional $\varphi : \mathcal{N} \to \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^d \times \mathbb{M}$, we define the increments:

$$D^+_{\mathbf{x}} \varphi(\mu) = \varphi(\mu + \delta_{\mathbf{x}}) - \varphi(\mu) \quad \text{and} \quad D^-_{\mathbf{x}} \varphi(\mu) = \varphi(\mu) - \varphi(\mu - \delta_{\mathbf{x}}),$$

where $D^-_{\mathbf{x}} \varphi(\mu)$ is properly defined only if $\mathbf{x} \in \mathrm{supp}(\mu)$. Similarly, if $\mu' \in \mathcal{N}$,

$$D^+_{\mu'} \varphi(\mu) = \varphi(\mu + \mu') - \varphi(\mu) \quad \text{and} \quad D^-_{\mu'} \varphi(\mu) = \varphi(\mu) - \varphi(\mu - \mu'),$$

where $D^-_{\mu} \varphi(\mu)$ is properly defined only if $\mathrm{supp}(\mu') \subseteq \mathrm{supp}(\mu)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A (simple and locally finite) marked point process on $\mathbb{R}^d$ with marks in $\mathbb{M}$ is a measurable mapping from Ω to $\mathcal{N}$. Throughout the paper we fix a marked point process N on $\mathbb{R}^d$ with marks in $\mathbb{M}$. For a Borel set $B \subseteq \mathbb{R}^d$, define the σ-field on Ω:

$$\mathcal{F}_B = \sigma\{N(C \times M) : C \in \mathcal{B}(\mathbb{R}^d), C \subseteq B, M \in \mathcal{B}(\mathbb{M})\}.$$

Let $\mathbb{F}$ and $\mathbb{K}$ denote, respectively, the family of closed and compact sets of $\mathbb{R}^d$. We endow these families with their standard topology (see Matheron [22] and Stoyan, Kendall and Mecke [31]). Let $S : \mathcal{N} \to \mathbb{F}$ be a measurable mapping. We say that S is a stopping set if $S(N)$ is a measurable mapping from Ω to $\mathbb{K}$ such that $\{S(N) \subseteq K\} \in \mathcal{F}_K$ for each $K \in \mathbb{K}$. The stopping σ-field is the following collection on Ω:

$$\mathcal{F}_S = \sigma \left\{ F \in \bigvee_{K \in \mathbb{K}} \mathcal{F}_K : F \cap \{S(N) \subseteq K\} \in \mathcal{F}_K \text{ for all } K \in \mathbb{K} \right\}.$$

For details and properties of stopping sets and stopping σ-fields, we refer to Zuyev [36].

All the random elements considered in this work are defined on the measurable space $(\Omega, \mathcal{F})$. We endow such space with the family of probability measures $\{P_\lambda\}_{\lambda>0}$ such that, under P_λ, the marked point process

$$N = \sum_{n \geq 1} \delta_{(X_n, Z_n)} = \sum_{n \geq 1} \delta_{\mathbf{X}_n}$$

is an IMHPP of intensity $\lambda > 0$, that is: the ground point process $\{X_n\}_{n\geq 1}$ is a homogeneous Poisson process with intensity λ, the random marks $\{Z_n\}_{n\geq 1}$ are independent and identically distributed (iid) with law Q, and the sequences $\{X_n\}_{n\geq 1}$

and $\{Z_n\}_{n\geq 1}$ are independent. We denote by E_λ the expectation associated to P_λ. Note that N is actually a Poisson point process on $\mathbb{R}^d \times \mathbb{M}$ with intensity measure $\lambda\Lambda$, where

$$\Lambda(dx \times dz) = dxQ(dz)$$

is the product measure on $\mathbb{R}^d \times \mathbb{M}$ of the Lebesgue measure and Q.

Although in Subsection 2.2 and Section 3 we assume that $\{X_n\}_{n\geq 1}$ is a homogeneous Poisson process on $\Sigma = \mathbb{R}^d$, the results therein still hold if Σ is a Borel subset of $\mathbb{R}^d$, and $\{X_n\}_{n\geq 1}$ is the restriction on Σ of a homogeneous Poisson process on $\mathbb{R}^d$ (for instance, note that in Subsection 5.2 we apply the results in Subsection 2.2 and Section 3 to stochastic models where $\{X_n\}_{n\geq 1}$ are the points of a homogeneous Poisson process on $[0,\infty)$).

2.2. Analyticity of functionals of independently marked homogeneous Poisson processes

Our analysis is based on a result, due to Zazanis [34], which can be extended to the context of stopping sets as follows. Let φ be a measurable functional from $\mathcal{N}$ to $\mathbb{R}$, $f(\lambda) = E_\lambda[\varphi(N)]$, $f^{(n)}(\lambda) = \frac{d^n f(\lambda)}{d\lambda^n}$, and $[a,b)$ an interval of the positive half-line. We consider the following conditions:

$$\text{There exists a stopping set } S \text{ such that } \varphi(N) \text{ is } \mathcal{F}_S\text{-measurable.} \tag{2}$$

$$\text{For any } \lambda \in [a,b) \text{ there exists } \gamma = \gamma(\lambda) > 1 \text{ such that} \tag{3}$$

$$E_\lambda[|\varphi(N)|^\gamma] < \infty.$$

$$\text{For any } \lambda \in [a,b) \text{ there exists } s = s(\lambda) > 0 \text{ such that} \tag{4}$$

$$E_\lambda[\exp(s|B_{S(N)}|)] < \infty.$$

It holds:

Theorem 1. *Assume* (2), (3), *and* (4), *then* $f(\cdot)$ *is analytic on* $[a,b)$ *that is, for a fixed* $x_0 \in [a,b)$, *we have*

$$f(x) = \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \quad x \in [a,b).$$

In its paper Zazanis considers a homogeneous Poisson process N on the half-line, and stopping sets of the form $S(N) = [0, T(N)]$, where $T(N)$ is a stopping time with respect to the natural filtration of the Poisson process. Moreover, he assumes the stronger condition:

$$\text{For any } \lambda \in [a, b),\ E_\lambda[\varphi^4(N)] < \infty$$

in place of (3).

To prove Theorem 1 we need the following lemmas.

Lemma 1. *Under assumptions* (2) *and* (4)*, for all* $\lambda \in [a, b)$*, there exists* $s' = s'(\lambda) > 0$ *such that* $E_\lambda[\exp(s' N_{S(N)})] < \infty$.

Lemma 2. *Under assumptions of Theorem 1, we have that*

$$E_\lambda\left[|\varphi(N)|\left(\frac{\lambda+\varepsilon}{\lambda}\right)^{N_{S(N)}}\exp(\varepsilon|B_{S(N)}|)\right] < \infty,\ \ \varepsilon \in \left(0, \min\left\{\frac{s(\gamma-1)}{2\gamma}, \frac{s'\lambda(\gamma-1)}{2\gamma}\right\}\right).$$

Here γ *is given by assumption* (3)*,* s *by* (4)*, and* s' *is determined by Lemma 1.*

Proof of Lemma 1. For ease of notation, throughout this proof we write $S = S(N)$. Let s be given by assumption (4), and set $C = E_\lambda[\exp(s|S|)]$ and $\delta > e^2\lambda$. For $k \geq 0$, let r_k be such that $|B_{r_k}| = k/\delta$ (that is $r_k = (k/(\delta\pi_d))^{1/d}$, where π_d is the volume of the ball B_1). We notice that

$$P_\lambda(N_S > k) \leq P_\lambda(|B_S| > k/\delta) + P_\lambda(N_{B_{r_k}} > k), \quad \text{for all } k \geq 0. \tag{5}$$

By a standard large deviation estimate for the Poisson distribution (see, for instance, the book by Penrose [26], Lemma 1.2) we have, for all $k \geq e^2\lambda$,

$$P_\lambda(N_{B_{r_k}} > k) \leq \exp\left(-\frac{k}{2}\log\left(\frac{k}{\lambda|B_{r_k}|}\right)\right) = \exp\left(-\frac{k}{2}\log\left(\frac{\delta}{\lambda}\right)\right). \tag{6}$$

Therefore, by (5), (6) and Markov inequality, it follows that, for all $k \geq e^2\lambda$,

$$P_\lambda(N_S > k) \leq C\exp\left(-\frac{s}{\delta}k\right) + \exp\left(-\frac{k}{2}\log\left(\frac{\delta}{\lambda}\right)\right).$$

Finally, we easily deduce that, for $0 < s' < \min\{s/\delta, \frac{1}{2}\log(\delta/\lambda)\}$, $E_\lambda[\exp(s' N_S)] = 1 + (e^{s'} - 1)\sum_{k\geq 0}\exp(s'k)P_\lambda(N_S > k) < \infty$. $\square$

Proof of Lemma 2. As above we set $S = S(N)$. The proof is similar to the proof of Lemma 2 in Zazanis [34]. Following the proof of Lemma 2 in [34] and using Hölder inequality (in place of the first application of Cauchy-Schwarz inequality) and then Cauchy-Schwarz inequality, we have that

$$
\begin{aligned}
&E_\lambda\left[|\varphi(N)|\left(\frac{\lambda+\varepsilon}{\lambda}\right)^{N_S}\exp(\varepsilon|B_S|)\right]\\
&\le \left(E_\lambda\left[|\varphi(N)|^\gamma\right]\right)^{\frac{1}{\gamma}}\left(E_\lambda\left[\left(\frac{\lambda+\varepsilon}{\lambda}\right)^{\frac{\gamma}{\gamma-1}N_S}\exp\left(\varepsilon\frac{\gamma}{\gamma-1}|B_S|\right)\right]\right)^{\frac{\gamma-1}{\gamma}}\\
&\le \left(E_\lambda\left[|\varphi(N)|^\gamma\right]\right)^{\frac{1}{\gamma}}\left(E_\lambda\left[\left(\frac{\lambda+\varepsilon}{\lambda}\right)^{\frac{2\gamma}{\gamma-1}N_S}\right]E_\lambda\left[\exp\left(\varepsilon\frac{2\gamma}{\gamma-1}|B_S|\right)\right]\right)^{\frac{\gamma-1}{2\gamma}}.
\end{aligned}
$$

The claim follows by Lemma 1 and assumptions (3) and (4). □

Proof of Theorem 1. Zazanis' result can be extended as stated by Theorem 1. We briefly outline the main changes in the proof: follow the proofs of Theorem 4 and Corollary 2 in Zazanis [34] replacing Lemma 2 in [34] with Lemma 2 and the so-called Cameron-Martin-Girsanov change of measure by the change of measure

$$\frac{dP_{\lambda,S}}{dP_{a,S}}=\left(\frac{\lambda}{a}\right)^{N_{S(N)}}\exp(-|S(N)|(\lambda-a)), \tag{7}$$

where $P_{\lambda,S}$ denotes the restriction of P_λ to $\mathcal{F}_S$ (note that $P_\lambda \ll P_a$ on the stopping σ-field $\mathcal{F}_S$ with density (7) due to the results in Zuyev [36]). □

Remark 1. A function $g:\mathbb{R}\to[0,\infty)$ is said absolutely monotonic in $[a,b)$ if it has derivatives of all orders that satisfy $g^{(k)}(x)\ge 0$ for all $x\in(a,b)$, $k\ge 0$. Consider a nonnegative functional $\varphi(\mu,\lambda)$ on $\mathcal{N}\times\mathbb{R}^+$, which depends explicitely on λ in such a way that, for each $\mu\in\mathcal{N}$, the function $\lambda\mapsto\varphi(\mu,\lambda)$ is absolutely monotonic in $[a,b)$. If moreover conditions (2), (3) and (4) are satisfied with $\varphi(\mu,\lambda)$ in place of $\varphi(\mu)$, then $f(\lambda)=E_\lambda[\varphi(N,\lambda)]$ is analytic on $[a,b)$. The proof is similar to that one of Theorem 1. In particular, note that the absolute monotonicity of $\lambda\mapsto\varphi(N,\lambda)$ implies the absolute monotonicity of

$$\lambda\mapsto\varphi(N,\lambda)\left(\frac{\lambda}{a}\right)^{N_{S(N)}}\exp(-|S(N)|(\lambda-a))$$

in $[a,b)$. Indeed, similarly to [34], one can prove that the function

$$\lambda\mapsto\left(\frac{\lambda}{a}\right)^{N_{S(N)}}\exp(-|S(N)|(\lambda-a))$$

is absolutely monotonic. The claim follows using that the product of two absolutely monotonic functions is an absolutely monotonic function.

3. Rare perturbation analysis

Sensitivity analysis is concerned with evaluating derivatives of cost functions, with respect to parameters of interest. It plays a central role in identifying the most significant system parameters. In this section, we give Monte Carlo methods to estimate the derivatives of the cost function $f(\lambda) = E_\lambda[\varphi(N)]$. An application of importance could be the use of such gradient estimates in stochastic gradient algorithms to find the optimal value λ_0 that minimizes the cost function.

3.1. Monotone mappings

The following notion of monotonicity is crucial in this work.

Definition 1. Let S be a measurable mapping from $\mathcal{N}$ to $\mathbb{F}$. We say that S is nonincreasing (nondecreasing) if, for any μ_1, μ_2 in $\mathcal{N}$, the inclusion $\mathrm{supp}(\mu_1) \subseteq \mathrm{supp}(\mu_2)$ implies $S(\mu_1) \supseteq S(\mu_2)$ $(S(\mu_1) \subseteq S(\mu_2))$. The mapping S is said monotone if it is nonincreasing or nondecreasing.

We give a couple of examples as a guide to intuition. Let $\mu = \{x_n\}_{n\geq 1}$ be a locally finite counting measure on $[0,\infty)$. Define the functional $\varphi(\mu) = 1 \wedge x_1$, where x_1 is the first point of μ on $[0,\infty)$, and the measurable mapping $S(\mu) = [0, x_1]$. Then S is nonincreasing but it is not nondecreasing. Instead, if we define $S(\mu) = [0,1]$, then S is nonincreasing and nondecreasing.

3.2. First order derivative

In this subsection we state the result concerning the first order derivative of $f(\cdot)$. Its proof is given in Section 4.

Theorem 2. *Under assumptions of Theorem 1, with γ in* (3) *such that $\gamma > 2$, if moreover the mapping S is monotone then, for all $\lambda \in [a,b)$,*

$$\begin{aligned} f'(\lambda) &= E_\lambda\Big[\,|S(N)|D^+_{\mathbf{X}}\varphi(N)\,\Big] && (8)\\ &= E_\lambda\Big[\,\frac{N_{S(N)}}{\lambda}D^-_{\mathbf{X}'}\varphi(N)\,\Big], && (9)\end{aligned}$$

where $\mathbf{X} = (\xi,\zeta)$ and $\mathbf{X}' = (\xi',\zeta')$ are random variables on $\mathbb{R}^d \times \mathbb{M}$. Given $S(N)$, ξ is uniformly distributed on $S(N)$; ζ is independent of N and ξ and has law Q. Given the collection of points $N_{|S(N)}$, $\mathbf{X}' = (\xi',\zeta')$ is uniformly distributed on the collection.

The closed form formulas provided by equations (8) and (9) both give a Monte Carlo method to simulate the derivative of $f(\lambda)$. We note also that if we consider an IMHPP on $(-\infty,0]$ with marks in $(0,\infty)$, and the assumptions of Theorem 2, our formulas of the first order derivative coincide with the corresponding formula in Baccelli, Hasenfuss and Schmidt [5] (see formula (10) therein with $k=1$). This easily follows by equality (9) and the forthcoming equalities (39), (40). A similar remark holds for the n-th order derivatives (see Theorem 3 below).

3.3. Higher order derivatives

We now generalize Theorem 2, stating the result for the n-th order derivatives $f^{(n)}(\lambda)$. The details of the proof are given in Section 4.

Let φ be a measurable functional from $\mathcal{N}$ to $\mathbb{R}$. As in Reiman and Simon [29] and Blaszczyszyn [6], for $\mu\in\mathcal{N}$, $n\geq 1$ and $\mathbf{x}_i=(x_i,z_i)\in\mathbb{R}^d\times\mathbb{M}$, $i=1,\dots,n$, define

$$\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu)=\varepsilon(\mathbf{x}_1,\dots,\mathbf{x}_n)\sum_{k=0}^{n}(-1)^{n-k}\sum_{\pi\in\{\binom{n}{k}\}}\varphi\left(\mu+\sum_{i\in\pi}\delta_{\mathbf{x}_i}\right), \tag{10}$$

where $\varepsilon(\mathbf{x}_1,\dots,\mathbf{x}_n)=\mathbb{1}(\{x_1,\dots,x_n \text{ are distinct}\})$ and $\{\binom{n}{k}\}$ denotes the collection of all subsets with cardinality k of $\{1,\dots,n\}$. We shall consider also the functionals

$$\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu)=\varepsilon(\mathbf{x}_1,\dots,\mathbf{x}_n)\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}\left(\mu-\sum_{i=1}^{n}\delta_{\mathbf{x}_i}\right), \tag{11}$$

which are properly defined only if $\mathbf{x}_1,\dots,\mathbf{x}_n\in\operatorname{supp}(\mu)$. Note that $\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}$ (and therefore $\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_n}$) is invariant by permutations in the sense that for any permutation σ of $\{1,...,n\}$ $\varphi_{\mathbf{x}_{\sigma(1)},\dots,\mathbf{x}_{\sigma(n)}}(\mu)=\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu)$. Furthermore, as can be easily seen reasoning by induction on $n\geq 1$, we have that if $\varepsilon(\mathbf{x}_1,\cdots,\mathbf{x}_{n+1})=1$, then

$$\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_{n+1}}(\mu)=\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}\left(\mu+\delta_{\mathbf{x}_{n+1}}\right)-\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu)=D^+_{\mathbf{x}_{n+1}}\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu) \tag{12}$$

and

$$\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_{n+1}}(\mu)=\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu)-\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_n}\left(\mu-\delta_{\mathbf{x}_{n+1}}\right)=D^-_{\mathbf{x}_{n+1}}\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu). \tag{13}$$

In particular, $\varphi_{\mathbf{x}_1}(\mu)=D^+_{\mathbf{x}_1}\varphi(\mu)$ and $\varphi^{\mathbf{x}_1}(\mu)=D^-_{\mathbf{x}_1}\varphi(\mu)$. In the following theorem we use the standard convention that the sum over an empty set is zero and $k!/(k-n)!=0$ for $n>k$. It holds

Theorem 3. *Under assumptions of Theorem 2, for all $\lambda \in [a,b)$ and $n \geq 1$,*

$$\begin{aligned}
f^{(n)}(\lambda) &= E_\lambda[|S(N)|^n \varphi_{\mathbf{X}_1,\dots,\mathbf{X}_n}(N)] && (14)\\
&= E_\lambda\left[\left(\frac{N_{S(N)}}{\lambda}\right)^n \varphi^{\mathbf{X}'_1,\dots,\mathbf{X}'_n}(N)\right] && (15)\\
&= E_\lambda\left[\frac{N_{S(N)}!}{(N_{S(N)}-n)!\lambda^n}\varphi^{\mathbf{X}''_1,\dots,\mathbf{X}''_n}(N)\right], && (16)
\end{aligned}$$

where, for $1 \leq i \leq n$, $\mathbf{X}_i = (\xi_i, \zeta_i)$, $\mathbf{X}'_i = (\xi'_i, \zeta'_i)$ and $\mathbf{X}''_i = (\xi''_i, \zeta''_i)$ are random variables on $\mathbb{R}^d \times \mathbb{M}$. Given $S(N)$, $(\xi_i)_{1\leq i\leq n}$ are independent and uniformly distributed on $S(N)$, and independent of N; $(\zeta_i)_{1\leq i\leq n}$ are independent, independent of N and $(\xi_i)_{1\leq i\leq n}$, and with law Q. Given the collection of points $N_{|S(N)}$, $(\mathbf{X}'_i)_{1\leq i\leq n}$ are independent and uniformly distributed on the collection; $\{\mathbf{X}''_1,\dots,\mathbf{X}''_n\}$ is uniformly distributed on the set of subsets of n distinct points of $N_{|S(N)}$.

Note that equation (16) implies that $f^{(n)}(\lambda) = 0$ if $N_{S(N)} < n$ with probability one.

Putting together Theorems 1 and 3, we obtain the following corollary:

Corollary 1. *Under assumptions of Theorem 2 and notation of Theorem 3, for all $\lambda \in [a,b)$,*

$$f(\lambda) = E_a\left[\sum_{n=0}^{N_{S(N)}} \left(\frac{\lambda - a}{a}\right)^n \frac{N^n_{S(N)}}{n!}\varphi^{\mathbf{X}'_1,\cdots,\mathbf{X}'_n}(N)\right].$$

4. Proofs of Theorems 2 and 3

4.1. Integrability lemmas

In the core of the proof of Theorems 2 and 3, we use the integrability of some functionals. In this subsection we prove such integrability results.

We start with a simple continuity result.

Lemma 3. *Under assumptions of Theorem 1, for all $\alpha \in [0, \gamma(\lambda))$, the function $\lambda' \mapsto E_{\lambda'}[|\varphi(N)|^\alpha]$ is defined in an open neighborhood of λ and is continuous at λ.*

Proof. Throughout this proof we set $S = S(N)$. The conclusion is trivial for $\alpha = 0$. Assume $\alpha > 0$, we prove that

$$\lim_{\varepsilon \to 0^-} E_{\lambda+\varepsilon}[|\varphi(N)|^\alpha] = E_\lambda[|\varphi(N)|^\alpha]. \tag{17}$$

A similar argument can be used to prove the same limit as $\varepsilon \to 0^+$. Let $s = s(\lambda) > 0$ be given by assumption (4), $\beta = \gamma/\alpha > 1$ and $\varepsilon \in (\min\{-\lambda, -s(\beta-1)/\beta\}, 0)$. By the Cameron-Martin-Girsanov change of measure (7) it follows

$$E_{\lambda+\varepsilon}[|\varphi(N)|^\alpha] = E_\lambda \left[|\varphi(N)|^\alpha \left(\frac{\lambda+\varepsilon}{\lambda} \right)^{N_S} \exp(-\varepsilon|S|) \right].$$

By the choice of ε, we have that

$$|\varphi(N)|^\alpha \left(\frac{\lambda+\varepsilon}{\lambda} \right)^{N_S} \exp(-\varepsilon|S|) \leq |\varphi(N)|^\alpha \exp(-\varepsilon|S|) \leq |\varphi(N)|^\alpha \exp(s(\beta-1)|S|/\beta)$$

Now, Hölder inequality and assumptions (3) and (4) give

$$E_\lambda[|\varphi(N)|^\alpha \exp(s(\beta-1)|S|/\beta)] \leq E_\lambda\left[|\varphi(N)|^\gamma\right]^{\frac{\alpha}{\gamma}} E_\lambda\left[\exp(s|S|)\right]^{\frac{\beta-1}{\beta}} < \infty.$$

The limit (17) is then a consequence of Lebesgue's dominated convergence theorem. □

For any $\mu \in \mathcal{N}$, $n \geq 1$ and $\mathbf{x}_i = (x_i, z_i) \in \mathbb{R}^d \times \mathbb{M}$, $i = 1, \dots, n$, define the functionals

$$\psi(\mu) = \int_{(\mathbb{R}^d \times \mathbb{M})^n} |\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu)| \, \Lambda(d\mathbf{x}_1) \dots \Lambda(d\mathbf{x}_n), \tag{18}$$

and (with the convention that the sum over an empty set is zero)

$$\chi(\mu) = \sum_{\{\mathbf{x}_1,\dots,\mathbf{x}_n\} \subset \mathrm{supp}(\mu)} |\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu)|, \tag{19}$$

where the sum is taken on sets of n distinct points of μ. It holds:

Lemma 4. *Under assumptions of Theorem 1, if moreover the mapping S is nonincreasing then, for all $\lambda \in [a,b)$ and $\alpha \in [1,\gamma)$, $E_\lambda[\psi(N)^\alpha] < \infty$.*

Lemma 5. *Under assumptions of Theorem 1, if moreover the mapping S is nondecreasing then, for all $\lambda \in [a,b)$ and $\alpha \in [1,\gamma)$, $E_\lambda[\chi(N)^\alpha] < \infty$.*

Proof of Lemma 4. For ease of notation, set $P = P_\lambda$ and $E = E_\lambda$. Let $q > 1$ be such that $q\alpha \leq \gamma$ and $p > 1$ such that $1/p + 1/q = 1$. Moreover, let $\widehat{N} = \sum_{n\geq 1} \delta_{(\widehat{X}_n, \widehat{Z}_n)}$ be an IMHPP with intensity $\Delta\lambda$, such that $\widehat{Z}_1$ has law Q and $\widehat{N}$ is independent of N. Here $\Delta\lambda$ is chosen so that $\lambda + \Delta\lambda < b$ and $E[\exp(2p\alpha\Delta\lambda|S(N)|)] < \infty$. Reasoning by induction on $n \geq 1$ we have that condition (2) and the monotonicity of S imply

$$\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}(N) = 0 \quad \text{for any } (x_1,\dots,x_n) \notin S(N)^n. \tag{20}$$

Indeed, for $n=1$, the $\mathcal{F}_S$-measurability of $\varphi(N)$ and the inclusion $S(N+\delta_{\mathbf{x}})\subseteq S(N)$ for all $\mathbf{x}=(x,z)\in\mathbb{R}^d\times\mathbb{M}$, imply $\varphi(N+\delta_{\mathbf{x}})=\varphi(N)$, for each $\mathbf{x}=(x,z)\in(\mathbb{R}^d\setminus S(N))\times\mathbb{M}$. The general case is proved similarly. Therefore

$$\psi(N)=\int_{(S(N)\times\mathbb{M})^n}|\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}(N)|\,\Lambda(d\mathbf{x}_1)\dots\Lambda(d\mathbf{x}_n).$$

By the superposition property of Poisson processes, $N+\widehat{N}$ is an IMHPP with intensity $\lambda+\Delta\lambda$. It follows

$$\begin{aligned}
\psi(N) &\le |S(N)|^n\sum_{k=0}^n\binom{n}{k}|S(N)|^{-k}\int_{(S(N)\times\mathbb{M})^k}\Big|\,\varphi\left(N+\sum_{i=1}^k\delta_{\mathbf{x}_i}\right)\Big|\,\Lambda(d\mathbf{x}_1)\dots\Lambda(d\mathbf{x}_k)\\
&\le |S(N)|^n\sum_{k=0}^n\binom{n}{k}E[|\varphi(N+\widehat{N})|\,|\,\widehat{N}_{S(N)}=k,N]\\
&\le |S(N)|^nE[|\varphi(N+\widehat{N})|\,|\,N]\sum_{k=0}^n\binom{n}{k}P(\widehat{N}_{S(N)}=k\,|\,N)^{-1}\\
&\le |S(N)|^nE[|\varphi(N+\widehat{N})|\,|\,N]\sum_{k=0}^n\binom{n}{k}\frac{k!}{(\Delta\lambda|S(N)|)^k}\exp(\Delta\lambda|S(N)|)\\
&= \frac{n!}{(\Delta\lambda)^n}\exp(\Delta\lambda|S(N)|)E[|\varphi(N+\widehat{N})|\,|\,N]\sum_{k=0}^n\frac{(\Delta\lambda|S(N)|)^{n-k}}{(n-k)!}\\
&\le \frac{n!}{(\Delta\lambda)^n}\exp(2\Delta\lambda|S(N)|)E[|\varphi(N+\widehat{N})|\,|\,N].
\end{aligned}$$

Using Jensen and Hölder inequalities we deduce that

$$\begin{aligned}
E[\psi(N)^\alpha] &\le \left(\frac{n!}{(\Delta\lambda)^n}\right)^\alpha E[\exp(2\alpha\Delta\lambda|S(N)|)\left(E[|\varphi(N+\widehat{N})|\,|\,N]\right)^\alpha]\\
&\le \left(\frac{n!}{(\Delta\lambda)^n}\right)^\alpha E[\exp(2\alpha\Delta\lambda|S(N)|)E[|\varphi(N+\widehat{N})|^\alpha\,|\,N]]\\
&\le \left(\frac{n!}{(\Delta\lambda)^n}\right)^\alpha E[\exp(2p\alpha\Delta\lambda|S(N)|)]^{1/p}E[|\varphi(N+\widehat{N})|^{q\alpha}]^{1/q}<\infty.
\end{aligned}$$

□

Proof of Lemma 5. Set $P=P_\lambda$, $E=E_\lambda$, and let $N^{\otimes n}$ (respectively, $N^{\otimes n}_{|S(N)}$) be the set of the n-tuples of n distinct points of N (respectively, $N_{|S(N)}$). Let $p,q>1$ be such that $\alpha q\le\gamma$ and $1/p+1/q=1$. Let $\{\beta_n\}_{n\ge1}$ be an iid sequence of Bernoulli random variables, independent of N and defined by

$$P(\beta_n=0)=1-P(\beta_n=1)=\Delta\lambda/\lambda.$$

Consider the thinned IMHPP of intensity $\lambda - \Delta\lambda$ given by $\widetilde{N} = \sum_{n\geq 1} \beta_n \delta_{(X_n, Z_n)}$. Let $s > 0$ be such that $E[\exp(sN_{S(N)})] < \infty$ (see Lemma 1). Here we choose $\Delta\lambda$ in such a way that $2p\alpha \log(\lambda/(\lambda - \Delta\lambda)) < s$. Reasoning by induction on $n \geq 1$ it can be proved that condition (2) and the monotonicity of S imply

$$\varphi^{\mathbf{X}_1,\dots,\mathbf{X}_n}(N) = 0 \quad \text{for } \mathbf{X}_1,\dots,\mathbf{X}_n \in \mathrm{supp}(N)\colon\ X_1,\dots,X_n \notin S(N). \tag{21}$$

Indeed, for $n = 1$, the $\mathcal{F}_S$-measurability of $\varphi(N)$ and the inclusion $S(N - \delta_{\mathbf{X}}) \subseteq S(N)$ for all $\mathbf{X} = (X, Z) \in \mathrm{supp}(N)$, imply $\varphi(N - \delta_{\mathbf{X}}) = \varphi(N)$, for each $\mathbf{X} \in \mathrm{supp}(N)$ such that $X \in \mathbb{R}^d \setminus S(N)$. The general case is proved similarly. Therefore

$$\begin{aligned}
\chi(N) &= \sum_{\{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*\}\subset \mathrm{supp}(N)} |\varphi^{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*}(N)| && (22)\\
&= \frac{1}{n!} \sum_{(\mathbf{X}_1^*\dots,\mathbf{X}_n^*)\in N^{\otimes n}} |\varphi^{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*}(N)| && (23)\\
&= \frac{1}{n!} \sum_{(\mathbf{X}_1^*,\dots,\mathbf{X}_n^*)\in N^{\otimes n}_{|S(N)}} |\varphi^{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*}(N)| && (24)\\
&= \frac{1}{n!} E\left[(N_{S(N)})^n |\varphi^{\mathbf{X}_1',\dots,\mathbf{X}_n'}(N)| \,|\, N\right], && (25)
\end{aligned}$$

where the equality in (23) follows from the invariance by permutations of $\varphi^{\mathbf{x}_1\dots,\mathbf{x}_n}(\mu)$, the equality in (24) follows by (21), and the equality in (25) follows by the definition of $(\mathbf{X}_i')_{1\leq i\leq n}$ (see the statement of Theorem 3). If $N_{S(N)} < n$ then $\chi(N) = 0$. On the other hand, if $N_{S(N)} \geq n$ we deduce that

$$\begin{aligned}
\chi(N) &\leq E\left[(N_{S(N)})^n |\varphi^{\mathbf{X}_1',\dots,\mathbf{X}_n'}(N)| \,|\, N\right]\\
&\leq (N_{S(N)})^n E\left[\sum_{k=0}^n \sum_{\{i_1,\dots,i_k\}\in\{\binom{n}{k}\}} \left|\, \varphi\left(N - \sum_{j=1}^k \delta_{\mathbf{X}_{i_j}'}\right)\right| \,\Big|\, N\right]\\
&= (N_{S(N)})^n \sum_{k=0}^n \binom{n}{k} E[|\varphi(\widetilde{N})| \,|\, N, N_{S(N)} - \widetilde{N}_{S(N)} = k]\\
&\leq (N_{S(N)})^n E[|\varphi(\widetilde{N})| \,|\, N] \sum_{k=0}^n \binom{n}{k} P(N_{S(N)} - \widetilde{N}_{S(N)} = k \,|\, N)^{-1}\\
&= (N_{S(N)})^n E[|\varphi(\widetilde{N})| \,|\, N] \sum_{k=0}^n \binom{n}{k} \frac{k!(N_{S(N)} - k)!}{N_{S(N)}!} \left(\frac{\Delta\lambda}{\lambda}\right)^{-k} \left(1 - \frac{\Delta\lambda}{\lambda}\right)^{k - N_{S(N)}}\\
&\leq K (N_{S(N)})^n \left(\frac{\lambda}{\lambda - \Delta\lambda}\right)^{N_{S(N)}} E[|\varphi(\widetilde{N})| \,|\, N],
\end{aligned}$$

where $K = \left(\frac{\lambda}{\Delta\lambda}\right)^n \sum_{k=0}^n k!\binom{n}{k}$. Finally, using Jensen, Hölder and Cauchy-Schwartz inequalities we get

$$\begin{aligned}
E[\chi(N)^\alpha] &\leq K^\alpha E\left[(N_{S(N)})^{\alpha n}\left(\frac{\lambda}{\lambda-\Delta\lambda}\right)^{\alpha N_{S(N)}} |\varphi(\widetilde{N})|^\alpha\right] \\
&\leq K^\alpha E\left[(N_{S(N)})^{p\alpha n}\left(\frac{\lambda}{\lambda-\Delta\lambda}\right)^{p\alpha N_{S(N)}}\right]^{1/p} E[|\varphi(\widetilde{N})|^{q\alpha}]^{1/q} \\
&\leq K^\alpha E[(N_{S(N)})^{2p\alpha n}]^{1/(2p)} E\left[\left(\frac{\lambda}{\lambda-\Delta\lambda}\right)^{2p\alpha N_{S(N)}}\right]^{1/(2p)} E[|\varphi(\widetilde{N})|^{q\alpha}]^{1/q} < \infty.
\end{aligned}$$

□

4.2. Case of nonincreasing mappings

In this subsection, we prove the closed form formulas given by equations (8) and (14) in the case of nonincreasing mappings S. More precisely, the following propositions hold:

Proposition 1. *Under assumptions of Theorem 2, if moreover the mapping S is nonincreasing then, for all $\lambda \in [a,b)$, equation (8) holds.*

Proposition 2. *Under assumptions of Theorem 2, if moreover the mapping S is nonincreasing then, for all $\lambda \in [a,b)$ and $n \geq 1$, equation (14) holds.*

We start proving Proposition 1. The proof is based on the virtual Rare Perturbation method considered in Baccelli and Brémaud [4].

Proof of Proposition 1. For ease of notation, we set $P = P_\lambda$ and $E = E_\lambda$. A straightforward computation gives

$$\begin{aligned}
E\left[\,|D^+_{\mathbf{X}}\varphi(N)| \;\middle|\; N\,\right] &= \frac{1}{|S(N)|}\int_{S(N)\times\mathbb{M}} |D^+_{\mathbf{x}}\varphi(N)|\Lambda(d\mathbf{x}) \\
&= \frac{1}{|S(N)|}\int_{\mathbb{R}^d\times\mathbb{M}} |D^+_{\mathbf{x}}\varphi(N)|\Lambda(d\mathbf{x}), \quad a.s., \qquad (26)
\end{aligned}$$

where the latter equality follows by condition (2) and by the assumption that S is nonincreasing. Indeed, as in the proof of Lemma 4, the $\mathcal{F}_S$-measurability of $\varphi(N)$ and the inclusion $S(N+\delta_{\mathbf{x}}) \subseteq S(N)$ for each $\mathbf{x} \in \mathbb{R}^d \times \mathbb{M}$, imply $\varphi(N+\delta_{\mathbf{x}}) = \varphi(N)$ for each $\mathbf{x} = (x,z) \in (\mathbb{R}^d \setminus S(N)) \times \mathbb{M}$. Thus, the integrability of the random variable $|S(N)|D^+_{\mathbf{X}}\varphi(N)$ follows by Lemma 4. Now, as in Lemma 4, let $\widehat{N} = \sum_{n\geq 1}\delta_{(\widehat{X}_n,\widehat{Z}_n)}$ be an IMHPP with intensity $\Delta\lambda$, such that $\widehat{Z}_1$ has law Q and $\widehat{N}$ is independent of N. By the superposition property of Poisson processes, $N+\widehat{N}$ is an IMHPP with intensity $\lambda+\Delta\lambda$. Here we choose $\Delta\lambda$ small enough so that $\lambda+\Delta\lambda<b$. Due to the monotonicity of S, we have $S(N+\widehat{N})\subseteq S(N)$, and so the $\mathcal{F}_S$-measurability of $\varphi(N)$ yields:

$$\varphi(N)=\varphi(N_{|S(N)})\quad\text{and}\quad\varphi(N+\widehat{N})=\varphi((N+\widehat{N})_{|S(N)}).\tag{27}$$

We then notice that

$$\begin{aligned}\frac{f(\lambda+\Delta\lambda)-f(\lambda)}{\Delta\lambda}&=E[D^+_{\widehat{N}}\varphi(N)]/\Delta\lambda\\&=\frac{1}{\Delta\lambda}E\left[\sum_{k\geq 1}\mathbb{1}(\widehat{N}_{S(N)}=k)D^+_{\widehat{N}_{|S(N)}}\varphi(N)\right]\\&=\frac{1}{\Delta\lambda}E[\mathbb{1}(\widehat{N}_{S(N)}=1)D^+_{\widehat{N}_{|S(N)}}\varphi(N)]\qquad(28)\\&\qquad+\frac{1}{\Delta\lambda}E[\mathbb{1}(\widehat{N}_{S(N)}\geq 2)D^+_{\widehat{N}_{|S(N)}}\varphi(N)],\qquad(29)\end{aligned}$$

where the second equality follows noticing that by (27) on $\{\widehat{N}_{S(N)}=0\}$ we have $\varphi(N)=\varphi(N+\widehat{N})$. Fix $\alpha\in(2,\gamma(\lambda))$, by Lemma 3 the function $\lambda'\mapsto E[|\varphi(N)|^\alpha]$ is continuous at λ. Therefore, there exists a positive constant $C>0$ such that $E[|\varphi(N)|^\alpha]<C^\alpha$ and $E[|\varphi(N+\widehat{N})|^\alpha]<C^\alpha$ for $\Delta\lambda$ small enough. Using Hölder and Minkowski inequalities we have

$$\begin{aligned}\Big|\,E[\mathbb{1}(\widehat{N}_{S(N)}\geq 2)D^+_{\widehat{N}_{|S(N)}}\varphi(N)]\,\Big|&\leq\Big(\,P(\widehat{N}_{S(N)}\geq 2)\,\Big)^{1-1/\alpha}\Big(\,E[|\varphi(N+\widehat{N}_{|S(N)})-\varphi(N)|^\alpha]\,\Big)^{1/\alpha}\\&\leq 2C\,\left(\,E\left[\sum_{k\geq 2}\frac{(\Delta\lambda)^k|S(N)|^k}{k!}e^{-\Delta\lambda|S(N)|}\right]\,\right)^{1-1/\alpha}\\&\leq 2C(\Delta\lambda)^{2(1-1/\alpha)}(E[|S(N)|^2])^{1-1/\alpha}.\qquad(30)\end{aligned}$$

By assumption (4) we have $E[|S(N)|^2]<\infty$. Therefore, by inequality (30) it follows that the term in (29) goes to zero, as $\Delta\lambda\to 0$. Since $\widehat{N}$ is independent of N it follows

$$\begin{aligned}E[\mathbb{1}(\widehat{N}_{S(N)}=1)D^+_{\widehat{N}_{|S(N)}}\varphi(N)]&=E[E[\mathbb{1}(\widehat{N}_{S(N)}=1)D^+_{\widehat{N}_{|S(N)}}\varphi(N)\,|\,N]]\\&=E[\Delta\lambda|S(N)|e^{-\Delta\lambda|S(N)|}E[D^+_{\widehat{N}_{|S(N)}}\varphi(N)\,|\,N,\,\widehat{N}_{S(N)}=1]]\\&=E[\Delta\lambda|S(N)|e^{-\Delta\lambda|S(N)|}E[D^+_{\mathbf{X}}\varphi(N)\,|\,N]]\\&=\Delta\lambda E[|S(N)|e^{-\Delta\lambda|S(N)|}D^+_{\mathbf{X}}\varphi(N)].\end{aligned}$$

Thus by the dominated convergence theorem the term in (28) converges to $E[|S(N)|D^+_{\mathbf{X}}\varphi(N)]$, as $\Delta\lambda \to 0$. □

Proof of Proposition 2. Set $E = E_\lambda$, and note that by (20) we have

$$\psi(N) = E[|S(N)|^n|\varphi_{\mathbf{X}_1,\dots,\mathbf{X}_n}(N)| \,|\, N].$$

Thus the integrability of $|S(N)|^n\varphi_{\mathbf{X}_1,\dots,\mathbf{X}_n}(N)$ for any $n \geq 1$ follows by Lemma 4. We prove formula (14) by induction on $n \geq 1$. As already shown it holds for $n = 1$. Let $\widetilde{\psi}$ be the functional defined as ψ without the absolute value. By (2) and (20) it follows that $\widetilde{\psi}(N)$ is $\mathcal{F}_S$-measurable. Assume the inductive hypothesis $f^{(n)}(\lambda) = E[\widetilde{\psi}(N)]$ for $n > 1$. Fix $\alpha \in (2, \gamma)$, by Lemma 4 we have $E[|\widetilde{\psi}(N)|^\alpha] < \infty$. Define the random variable $\mathbf{X}_{n+1} = (\xi_{n+1}, \zeta_{n+1})$ with values on $\mathbb{R}^d \times \mathbb{M}$ as follows: given $S(N)$, ξ_{n+1} is uniformly distributed on $S(N)$, and is independent of $N, \mathbf{X}_1, \dots, \mathbf{X}_n$; ζ_{n+1} has law Q and is independent of N, $\mathbf{X}_1, \dots, \mathbf{X}_n$ and ξ_{n+1}. By Proposition 1 we get

$$f^{(n+1)}(\lambda) = E[|S(N)|D^+_{\mathbf{X}_{n+1}}\widetilde{\psi}(N)].$$

The conclusion follows noticing that by (20) and (12) we have

$$\begin{aligned} E[|S(N)|D^+_{\mathbf{X}_{n+1}}\widetilde{\psi}(N)] &= \int_{\mathbb{R}^d\times\mathbb{M}} E[D^+_{\mathbf{x}}\widetilde{\psi}(N)]\Lambda(d\mathbf{x}) \\ &= \int_{(\mathbb{R}^d\times\mathbb{M})^{n+1}} E[\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_{n+1}}(N)]\,\Lambda(d\mathbf{x}_1)\cdots\Lambda(d\mathbf{x}_{n+1}) \\ &= E[|S(N)|^{n+1}\varphi_{\mathbf{X}_1,\dots,\mathbf{X}_{n+1}}(N)]. \end{aligned}$$

□

4.3. Case of nondecreasing mappings

In this subsection, we prove the closed form formulas given by equations (9) and (15) in the case of nondecreasing mappings S. More precisely, the following propositions hold:

Proposition 3. *Under assumptions of Theorem 2, if moreover the mapping S is nondecreasing then, for all $\lambda \in [a, b)$, $f'(\lambda)$ equals the term in (9).*

Proposition 4. *Under assumptions of Theorem 2, if moreover the mapping S is nondecreasing then, for all $\lambda \in [a, b)$ and $n \geq 1$, $f^{(n)}(\lambda)$ equals the term in (15).*

We first prove Proposition 3. For this we use the so-called phantom Rare Perturbation method introduced in Brémaud and Vazquez-Abad [7].

Proof of Proposition 3. Set $P = P_\lambda$ and $E = E_\lambda$. As in the proof of Lemma 5, the $\mathcal{F}_S$-measurability of $\varphi(N)$ and the inclusion $S(N - \delta_{\mathbf{X}}) \subseteq S(N)$ for all $\mathbf{X} = (X, Z) \in \mathrm{supp}(N)$, imply $\varphi(N - \delta_{\mathbf{X}}) = \varphi(N)$, for each $\mathbf{X} \in \mathrm{supp}(N)$ such that $X \in \mathbb{R}^d \setminus S(N)$. Therefore

$$\int_{S(N)\times\mathbb{M}} |D^-_{\mathbf{x}}\varphi(N)|N(d\mathbf{x}) = \int_{\mathbb{R}^d\times\mathbb{M}} |D^-_{\mathbf{x}}\varphi(N)|N(d\mathbf{x}). \tag{31}$$

Thus, the integrability of the random variable $\int_{S(N)\times\mathbb{M}} D^-_{\mathbf{x}}\varphi(N)N(d\mathbf{x})$ follows from Lemma 5. Now note that

$$\begin{aligned} E\left[\frac{N_{S(N)}}{\lambda} D^-_{\mathbf{X}'}\varphi(N)\right] &= E\left[E\left[\frac{N_{S(N)}}{\lambda} D^-_{\mathbf{X}'}\varphi(N) \;\Big|\; N_{|S(N)}\right]\right] \\ &= E\left[\frac{1}{\lambda}\int_{S(N)\times\mathbb{M}} D^-_{\mathbf{x}}\varphi(N)N(d\mathbf{x})\right]. \end{aligned} \tag{32}$$

We finally show

$$f'(\lambda) = E\left[\frac{1}{\lambda}\int_{S(N)\times\mathbb{M}} D^-_{\mathbf{x}}\varphi(N)N(d\mathbf{x})\right].$$

Let $\{\beta_n\}_{n\geq1}$ be the sequence of Bernoulli random variables defined in the proof of Lemma 5. Consider the thinned IMHPP of intensity $\lambda-\Delta\lambda$ given by $\widetilde{N} = \sum_{n\geq1}\beta_n\delta_{(X_n,Z_n)}$. By condition (2) and the monotonicity of S it follows that $\varphi(N) = \varphi(N_{|S(N)})$ and $\varphi(\widetilde{N}) = \varphi(\widetilde{N}_{|S(N)})$. By the independence of $\{\beta_n\}_{n\geq1}$ and N we have, for $0 \leq k \leq N_{S(N)}$,

$$P(N_{S(N)} - \widetilde{N}_{S(N)} = k \,|\, N) = \binom{N_{S(N)}}{k}\left(\frac{\Delta\lambda}{\lambda}\right)^k\left(1 - \frac{\Delta\lambda}{\lambda}\right)^{N_{S(N)}-k}.$$

This equation implies

$$\begin{aligned} &E[\mathbb{1}(N_{S(N)} - \widetilde{N}_{S(N)} = k)(\varphi(N) - \varphi(\widetilde{N}))] = \\ &E\left[\binom{N_{S(N)}}{k}\left(\frac{\Delta\lambda}{\lambda}\right)^k\left(1 - \frac{\Delta\lambda}{\lambda}\right)^{N_{S(N)}-k} E[\varphi(N) - \varphi(\widetilde{N}) \,|\, N, N_{S(N)} - \widetilde{N}_{S(N)} = k]\right]. \end{aligned}$$

Since $\varphi(N) = \varphi(N_{|S(N)})$ and $\varphi(\widetilde{N}) = \varphi(\widetilde{N}_{|S(N)})$, we have $E[\mathbb{1}(N_{S(N)} - \widetilde{N}_{S(N)} = 0)(\varphi(N) - \varphi(\widetilde{N}))] = 0$. Therefore,

$$\begin{aligned}
\frac{f(\lambda) - f(\lambda - \Delta\lambda)}{\Delta\lambda} &= \frac{E[\varphi(N) - \varphi(\widetilde{N})]}{\Delta\lambda} \\
&= \frac{1}{\Delta\lambda} E\left[\sum_{k\geq 1} \mathbb{1}(N_{S(N)} - \widetilde{N}_{S(N)} = k)(\varphi(N) - \varphi(\widetilde{N}))\right] \\
&= \frac{1}{\Delta\lambda} E\left[N_{S(N)} \left(\frac{\Delta\lambda}{\lambda}\right)\left(1 - \frac{\Delta\lambda}{\lambda}\right)^{N_{S(N)}-1} E[\varphi(N) - \varphi(\widetilde{N})|N, N_{S(N)} - \widetilde{N}_{S(N)} = 1]\right] \qquad (33) \\
&\quad + \frac{1}{\Delta\lambda} E\left[\mathbb{1}(N_{S(N)} - \widetilde{N}_{S(N)} \geq 2)(\varphi(N) - \varphi(\widetilde{N}))\right]. \qquad (34)
\end{aligned}$$

We note that, given N and the event $\{N_{S(N)} - \widetilde{N}_{S(N)} = 1\}$, the law of the random variable $\varphi(\widetilde{N})$ is equal to the law of $\varphi(N - \delta_{\mathbf{X}'})$. Thus,

$$E[\varphi(N) - \varphi(\widetilde{N}) \,|\, N,\, N_{S(N)} - \widetilde{N}_{S(N)} = 1] = \frac{1}{N_{S(N)}} \int_{S(N)\times\mathbb{M}} (\varphi(N) - \varphi(N - \delta_{\mathbf{x}}))N(d\mathbf{x}). \qquad (35)$$

By the dominated convergence theorem and (35) it follows that, as $\Delta\lambda \to 0$, the term in (33) goes to

$$E\left[\frac{1}{\lambda}\int_{S(N)\times\mathbb{M}} D^-_{\mathbf{x}}\varphi(N)N(d\mathbf{x})\right].$$

The proof of the proposition is complete if we prove that the term in (34) goes to zero as $\Delta\lambda \to 0$. Fix $\alpha \in (2, \gamma(\lambda))$, by Lemma 3 the function $\lambda' \to E[|\varphi(N)|^\alpha]$ is continuous at λ. Therefore, there exists a positive constant $C > 0$ such that $E[|\varphi(N)|^\alpha] < C^\alpha$ and $E[|\varphi(\widetilde{N})|^\alpha] < C^\alpha$ for $\Delta\lambda$ small enough. Using Hölder and Minkowski inequalities we have

$$\begin{aligned}
&\left| E\left[\mathbb{1}(N_{S(N)} - \widetilde{N}_{S(N)} \geq 2)(\varphi(N) - \varphi(\widetilde{N}))\right] \right| \\
&\leq \left(P(N_{S(N)} - \widetilde{N}_{S(N)} \geq 2)\right)^{1-1/\alpha} \left(E[|\varphi(N) - \varphi(\widetilde{N})|^\alpha]\right)^{1/\alpha} \\
&\leq 2C\left(E\left[\sum_{k=2}^{N_{S(N)}} \binom{N_{S(N)}}{k}\left(\frac{\Delta\lambda}{\lambda}\right)^k\left(1 - \frac{\Delta\lambda}{\lambda}\right)^{N_{S(N)}-k}\right]\right)^{1-1/\alpha}. \qquad (36)
\end{aligned}$$

As can be easily checked, for any $n \geq 2$ and $p \in (0,1)$,

$$\sum_{m=2}^{n} \binom{n}{m} p^m (1-p)^{n-m} \leq \frac{1}{2}n^2p^2. \qquad (37)$$

Thus by (36) and (37) the absolute value of the term in (34) can be bounded from above by

$$2C\left(1/2\left(\frac{\Delta\lambda}{\lambda}\right)^2 E[N_{S(N)}^2]\right)^{1-1/\alpha}\Big/\Delta\lambda,$$

and this quantity goes to zero as $\Delta\lambda \to 0$, since $E[N_{S(N)}^2] < \infty$ by Lemma 1. □

Proof of Proposition 4. Set again $E = E_\lambda$. By (22)-(25) we get

$$E\left[\left(\frac{N_{S(N)}}{\lambda}\right)^n |\varphi^{\mathbf{X}'_1,\ldots,\mathbf{X}'_n}(N)| \;\Big|\; N\right] = n!\chi(N)/\lambda^n.$$

Thus, the integrability of $\left(\frac{N_{S(N)}}{\lambda}\right)^n \varphi^{\mathbf{X}'_1,\ldots,\mathbf{X}'_n}(N)$ for any $n \geq 1$ follows by Lemma 5. We prove formula (15) by induction on $n \geq 1$. By Proposition 3, it holds for $n = 1$. Let $\widetilde{\chi}$ be the functional defined by

$$\widetilde{\chi}(\mu) = \frac{n!}{\lambda^n} \sum_{\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}\subset \operatorname{supp}(\mu)} \varphi^{\mathbf{x}_1,\ldots,\mathbf{x}_n}(\mu).$$

Let $N_{|S(N)}^{\otimes n}$ denote the set of the n-tuples of n distinct points of $N_{|S(N)}$. Since

$$\widetilde{\chi}(N) = \frac{1}{\lambda^n} \sum_{(\mathbf{X}_1^*,\ldots,\mathbf{X}_n^*)\in N_{|S(N)}^{\otimes n}} \varphi^{\mathbf{X}_1^*,\ldots,\mathbf{X}_n^*}(N)$$

(see (22)-(24)) we have that $\widetilde{\chi}(N)$ is $\mathcal{F}_S$-measurable. Moreover, for each $n \geq 1$,

$$\widetilde{\chi}(N) = E\left[\left(\frac{N_{S(N)}}{\lambda}\right)^n \varphi^{\mathbf{X}'_1,\ldots,\mathbf{X}'_n}(N) \;\Big|\; N\right] \tag{38}$$

(see (24)-(25)). Assume the inductive hypothesis $f^{(n)}(\lambda) = E[\widetilde{\chi}(N)]$ for $n > 1$. Fix $\alpha \in (2,\gamma)$, by Lemma 5 we have $E[|\widetilde{\chi}(N)|^\alpha] < \infty$. Let $\mathbf{X}'_{n+1}$ be a random variable on $\mathbb{R}^d \times \mathbb{M}$ such that, given $N_{|S(N)}$, $\mathbf{X}'_{n+1}$ is independent of $(\mathbf{X}'_i)_{1\leq i\leq n}$, and uniformly distributed on the collection $N_{|S(N)}$. By Proposition 3 we get

$$f^{(n+1)}(\lambda) = E\left[\frac{N_{S(N)}}{\lambda} D^-_{\mathbf{X}'_{n+1}} \widetilde{\chi}(N)\right].$$

The conclusion follows noticing that by (38) and (13) we have

$$\begin{aligned}
&E[D^-_{\mathbf{X}'_{n+1}}\widetilde{\chi}(N)\,|\,N] = \\
&E\left[\left(\frac{N_{S(N)}}{\lambda}\right)^n \left(\varphi^{\mathbf{X}'_1,\ldots,\mathbf{X}'_n}(N) - \varphi^{\mathbf{X}'_1,\ldots,\mathbf{X}'_n}(N - \delta_{\mathbf{X}'_{n+1}})\right) \;\Big|\; N\right] = \\
&E\left[\left(\frac{N_{S(N)}}{\lambda}\right)^n \varphi^{\mathbf{X}'_1,\ldots,\mathbf{X}'_{n+1}}(N) \;\Big|\; N\right].
\end{aligned}$$

□

4.4. Proof of Theorem 2

For ease of notation we set again $E = E_\lambda$. In view of Propositions 1 and 3, it is sufficient to show that

$$E[|S(N)|D^+_{\mathbf{X}}\varphi(N)] = E\left[\frac{N_{S(N)}}{\lambda}D^-_{\mathbf{X}'}\varphi(N)\right].$$

Arguing as for (26) we have

$$E[D^+_{\mathbf{X}}\varphi(N)\,|\,N] = \frac{1}{|S(N)|}\int_{\mathbb{R}^d\times\mathbb{M}} D^+_{\mathbf{x}}\varphi(N)\Lambda(d\mathbf{x}) \quad a.s..$$

Therefore,

$$E[|S(N)|D^+_{\mathbf{X}}\varphi(N)] = \int_{\mathbb{R}^d\times\mathbb{M}} E[D^+_{\mathbf{x}}\varphi(N)]\Lambda(d\mathbf{x}). \tag{39}$$

On the other hand, using the same argument as for (31) and the Slivnyak-Mecke theorem (see, for instance, Daley and Vere-Jones [10]) we get

$$\begin{aligned} E\left[\int_{S(N)\times\mathbb{M}} D^-_{\mathbf{x}}\varphi(N)N(d\mathbf{x})\right] &= E\left[\int_{\mathbb{R}^d\times\mathbb{M}} D^-_{\mathbf{x}}\varphi(N)N(d\mathbf{x})\right] \\ &= \lambda\int_{\mathbb{R}^d\times\mathbb{M}} E[D^+_{\mathbf{x}}\varphi(N)]\Lambda(d\mathbf{x}). \end{aligned} \tag{40}$$

The conclusion follows by equalities (39), (40) and equation (32), which does not depend on the monotonicity of S.

4.5. Proof of Theorem 3

As usual set $E = E_\lambda$. Let $\widetilde{\psi}$ and $\widetilde{\chi}$ be the functionals defined in the proofs of Propositions 2 and Proposition 4, respectively. Equations (14)-(15) will follow if we prove

$$E[\widetilde{\psi}(N)] = E[\widetilde{\chi}(N)]. \tag{41}$$

Indeed, by the proof of Proposition 2, if S is nonincreasing we have $f^{(n)}(\lambda) = E[\widetilde{\psi}(N)]$, and by the proof of Proposition 4, if S is nondecreasing we have $f^{(n)}(\lambda) = E[\widetilde{\chi}(N)]$. Equality (41) follows since by the extended Slivnyak-Campbell theorem (see Møller and Waagepetersen [24]) and the invariance by permutation of $\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_n}(\mu)$ we have

$$\begin{aligned}
E[\widetilde{\chi}(N)] &= \frac{n!}{\lambda^n} E\left[\sum_{\{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*\}\subset \mathrm{supp}(N)} \varphi^{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*}(N)\right] \\
&= \int_{(\mathbb{R}^d\times\mathbb{M})^n} E\left[\varphi^{\mathbf{x}_1,\dots,\mathbf{x}_n}\left(N+\sum_{i=1}^n \delta_{\mathbf{x}_i}\right)\right] \Lambda(d\mathbf{x}_1)\dots\Lambda(d\mathbf{x}_n) \\
&= \int_{(\mathbb{R}^d\times\mathbb{M})^n} E[\varphi_{\mathbf{x}_1,\dots,\mathbf{x}_n}(N)]\,\Lambda(d\mathbf{x}_1)\dots\Lambda(d\mathbf{x}_n) \\
&= E[\widetilde{\psi}(N)],
\end{aligned}$$

where we have used (11).

It remains to show equality (16). To this end, we write:

$$\begin{aligned}
\widetilde{\chi}(N) &= \frac{n!}{\lambda^n} \sum_{\{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*\}\subset N_{|S(N)}} \varphi^{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*}(N) \\
&= \frac{N_{S(N)}!}{\lambda^n (N_{S(N)}-n)!} \binom{N_{S(N)}}{n}^{-1} \sum_{\{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*\}\subset N_{|S(N)}} \varphi^{\mathbf{X}_1^*,\dots,\mathbf{X}_n^*}(N) \\
&= \frac{N_{S(N)}!}{\lambda^n (N_{S(N)}-n)!} E[\varphi^{\mathbf{X}_1'',\dots,\mathbf{X}_n''}(N)\,|\,N],
\end{aligned}$$

where the latter equality follows from the invariance by permutations of $\varphi^{\mathbf{x}_1,\cdots,\mathbf{x}_n}(\mu)$ and the fact that $\varphi^{\mathbf{x}_1,\cdots,\mathbf{x}_n}(\mu) = 0$ if $\varepsilon(\mathbf{x}_1,\dots,\mathbf{x}_n) = 0$. □

5. Applications

5.1. Stochastic geometry

Stabilizing functionals are widely used in stochastic geometry. This class of functionals was first introduced by Lee [20] and further developed by Penrose and Yukich (see, for instance, [27] and [28]). Assumption (2) is closely related to assuming φ stabilizing. The main difference is that in (2) we require that S is a stopping set. Thus, stochastic geometry is a natural field of application of Theorems 1, 2 and 3. In the next two paragraphs, we develop two examples of application in this field.

5.1.1. *Cluster in subcritical continuum percolation.* Let $N = \sum_{n\geq 1}\delta_{(X_n,Z_n)}$ be an IMHPP on $\mathbb{R}^d$ of intensity λ, with marks in $[0,r]$, $r>0$. Consider the Boolean model

$$\Xi = \bigcup_{n\geq 1} B(X_n, Z_n).$$

Continuum percolation deals with the existence of an infinite connected component in Ξ. It is well-known that there exists a critical value of λ, say $\lambda_c > 0$, such that if $\lambda < \lambda_c$, a.s. there are not infinite connected components in Ξ, and if $\lambda > \lambda_c$, a.s. there is a unique infinite connected component in Ξ (see Meester and Roy [23] as a general reference on continuum percolation).

Define by $W(N)$ the connected component (or cluster) of Ξ containing the origin (note that $W(N)$ is possibly empty) and, with a little abuse of notation, by $N_{|W(N)}$ the restriction of the random measure N to the cluster $W(N)$ (this is indeed an abuse of notation since usually throughout this paper $\mu_{|B}$ denotes the set of points of μ on $B\times\mathbb{M}$). If $\lambda < \lambda_c$, then $W(N)$ is a.s. a compact set. We define $\varphi(N) = \mathbb{1}(N_{|W(N)} \in A)$, for some measurable set $A \subseteq \mathcal{N}$. It is of general interest to analyze the function

$$f(\lambda) = E_\lambda[\varphi(N)], \quad \lambda \in (0, \lambda_c).$$

In this paragraph, we provide a continuous analog of the Russo's formula for Poisson point fields, see Zuyev [35]. More precisely, for $\mu \in \mathcal{N}$, define the sets of pivotal points of A by

$$\begin{aligned}\mathcal{P}^+(\mu) = \{\mathbf{x} \in (\mathbb{R}^d \times [0,r])\backslash\mathrm{supp}(\mu) : \varphi(\mu+\delta_{\mathbf{x}}) = 1, \varphi(\mu) = 0\} \\ \cup \{\mathbf{x} \in \mathrm{supp}(\mu) : \varphi(\mu) = 1, \varphi(\mu-\delta_{\mathbf{x}}) = 0\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}^-(\mu) = \{\mathbf{x} \in (\mathbb{R}^d \times [0,r])\backslash\mathrm{supp}(\mu) : \varphi(\mu+\delta_{\mathbf{x}}) = 0, \varphi(\mu) = 1\} \\ \cup \{\mathbf{x} \in \mathrm{supp}(\mu) : \varphi(\mu) = 0, \varphi(\mu-\delta_{\mathbf{x}}) = 1\}.\end{aligned}$$

It holds:

Theorem 4. *The function $f(\lambda) = P_\lambda(N_{|W(N)} \in A)$ is analytic on the interval $(0, \lambda_c)$, and*

$$f'(\lambda) = E_\lambda[\Lambda(\mathcal{P}^+(N)) - \Lambda(\mathcal{P}^-(N))] = \frac{1}{\lambda}E_\lambda[N_{\mathcal{P}^+(N)} - N_{\mathcal{P}^-(N)}] \quad \textit{for } 0 < \lambda < \lambda_c.$$

The proof of Theorem 4 is based on Theorems 1, 2. The main difficulty in applying these theorems to $f(\lambda)$ is that W is not a stopping set. This difficulty can be circumvented as follows. The Minkowski addition is defined by

$$A \oplus B = \{a + b : a \in A, b \in B\}, \quad A, B \in \mathbb{K}$$

(see Matheron [22] for a complete treatment of the Minkowski operations). Next Lemma 6 provides a stopping set S which satisfies conditions (2) and (4).

Lemma 6. *Define the random compact set* $S(N) = W(N) \oplus B_r$. *Then*
(i) S *is a stopping set such that* $W(N)$ *(and therefore* $\varphi(N)$*) is* $\mathcal{F}_S$*-measurable.*
(ii) For each $\lambda > 0$ *there exists* $s = s(\lambda) > 0$ *such that* $E_\lambda[\exp(s|B_{S(N)}|)] < \infty$.

Proof. We first prove that S is a stopping set. Note first that if $\mathbf{X}_k = (X_k, Z_k) \in \mathrm{supp}(N_{|W(N)})$ then either X_k is at distance at most $Z_k + Z_n$ from at least one other point X_n with $\mathbf{X}_n = (X_n, Z_n) \in \mathrm{supp}(N_{|W(N)})$ or X_k is at distance at most Z_k from the origin. Note that $W(N) = \bigoplus_{\mathbf{X}_n \in \mathrm{supp}(N_{|W(N)})}(X_n + B_{Z_n})$, and therefore by the definition of $S(N)$ we have that $S(N) = \left(\bigoplus_{\mathbf{X}_n \in \mathrm{supp}(N_{|W(N)})}(X_n + B_{Z_n})\right) \oplus B_r$. Letting $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^d$, we deduce that, for any $K \in \mathbb{K}$,

$$\begin{aligned}\{S(N) \subseteq K\}^c = \{\exists\, y \in K^c,\, &\mathbf{X}_1, \cdots, \mathbf{X}_n \in \mathrm{supp}(N) : \\ &\|X_1\| \le Z_1, \|X_{i+1} - X_i\| \le Z_i + Z_{i+1}, 1 \le i \le n-1,\, \|X_n - y\| \le Z_n + r\}.\end{aligned}$$

Now, set in the above expression $m = \min\{k \in \{1, \ldots, n+1\} : X_k \in K^c\}$, with the convention $X_{n+1} = y$. Since by assumption, $Z_{i+1} \le r$, $\|X_{i+1} - X_i\| \le Z_i + Z_{i+1}$ implies $\|X_{i+1} - X_i\| \le Z_i + r$, hence, the event $\{S(N) \subseteq K\}^c$ can be rewritten as:

$$\begin{aligned}\{S(N) \subseteq K\}^c = \{\exists\, &\mathbf{X}_1, \cdots, \mathbf{X}_m \in \mathrm{supp}(N) : X_1, \cdots, X_m \in K \text{ and} \\ &\|X_1\| \le Z_1, \|X_i - X_{i+1}\| \le Z_i + Z_{i+1}, 1 \le i \le m-1, (X_m + B_{Z_m + r}) \cap K^c \ne \emptyset\}.\end{aligned}$$

It follows that $\{S(N) \subseteq K\}^c \in \mathcal{F}_K$, and thus S is a stopping set. Now, by construction $N_{|W(N)}$ is $\mathcal{F}_S$-measurable. We deduce that $W(N)$ is also $\mathcal{F}_S$-measurable and (i) is proved. It remains to prove (ii). Define $r_{W(N)} = \inf\{r \ge 0 : W(N) \subseteq B_r\}$, then $B_{S(N)} = B_{r_{W(N)}+r}$. Thus (ii) is a consequence of the exponential decrease of the subcritical cluster, see Section 3.7 and Lemma 3.3 of Meester and Roy [23]. □

Proof of Theorem 4. Clearly, the functional φ satisfies condition (3). Moreover, by Lemma 6 conditions (2) and (4) are satisfied with $S(N) = W(N) \oplus B_r$. Thus, by Theorem 1 the function $f(\cdot)$ is analytic on $(0, \lambda_c)$. Note that S is nondecreasing, thus using formula (8) of Theorem 2, it follows that

$$\begin{aligned}
f'(\lambda) &= E_\lambda[|S(N)|(\mathbb{1}(\mathbf{X} \in \mathcal{P}^+(N)) - \mathbb{1}(\mathbf{X} \in \mathcal{P}^-(N)))] \\
&= E_\lambda[|S(N)|E_\lambda[\mathbb{1}(\mathbf{X} \in \mathcal{P}^+(N)) - \mathbb{1}(\mathbf{X} \in \mathcal{P}^-(N)) \,|\, N]] \\
&= E_\lambda\left[|S(N)|\frac{1}{|S(N)|}\int_{S(N)\times[0,r]}(\mathbb{1}(\mathbf{x} \in \mathcal{P}^+(N)) - \mathbb{1}(\mathbf{x} \in \mathcal{P}^-(N)))\Lambda(d\mathbf{x})\right] \\
&= E_\lambda[\Lambda(\mathcal{P}^+(N)) - \Lambda(\mathcal{P}^-(N))],
\end{aligned}$$

where $\mathbf{X} = (\xi, \zeta)$ is defined in Theorem 2. In particular, the first equality above follows since, given $S(N)$, ξ is uniformly distributed on $S(N)$, and therefore $\mathbf{X} \notin \mathrm{supp}(N)$ a.s.. This proves the first equality of the claim. The second equality of the claim can be proved similarly, using formula (9) of Theorem 2. □

5.1.2. *Typical cell of Poisson-Voronoi tessellation* Let $A \subset \mathbb{R}^d$ be a locally finite point set, $\mu_A = \sum_{a\in A}\delta_a$, and $\|\cdot\|$ the Euclidean norm of $\mathbb{R}^d$. The Voronoi's cell with respect to A with nucleus $y \in A$ is by definition

$$C(y, \mu_A) = \{x \in \mathbb{R}^d : \|x - y\| \leq \|x - a\| \,\forall\, a \in A\}.$$

Let $N = \sum_{n\geq 1}\delta_{X_n}$ be a Poisson point process on $\mathbb{R}^d$ of intensity λ. The Poisson-Voronoi cell with nucleus X_k is by definition the random convex set $C(X_k, N)$ (see, for instance, Stoyan, Kendall and Mecke [31]). Let $\mathcal{K}_0^d$ be the set of convex bodies of $\mathbb{R}^d$ containing the origin, equipped with the Hausdorff metric and the related Borel σ-field. Moreover, let N_0 be the point process obtained by N adding a point at the origin. The typical Poisson-Voronoi cell is defined by $C(0, N_0)$. This cell is called typical since by Slivnyak's theorem (see, for instance, Stoyan, Kendall and Mecke [31]), $P_\lambda^0(C(0, N) \in A) = P_\lambda(C(0, N_0) \in A)$, where P_λ^0 is the Palm version of P_λ and A is a Borel set of $\mathcal{K}_0^d$.

Let ϕ be a measurable functional from $\mathcal{K}_0^d$ to $\mathbb{R}$. Define $\varphi(N) = \phi(C(0, N_0))$ and

$$f(\lambda) = E_\lambda[\phi(C(0, N_0))] = E_\lambda[\varphi(N)].$$

The Voronoi flower $V(N)$ is the union of the closed balls that have the origin and d points of N on their boundary, and no points of N inside. It is known that the centers of the balls which form $V(N)$ are the vertices of the typical Poisson-Voronoi cell. Then

V is a stopping set and $\varphi(N) = \phi(C(0, N_0))$ is $\mathcal{F}_V$-measurable (see, for instance, Zuyev [36]). It is also known that $B_{V(N)}$ satisfies (4) for all $\lambda > 0$, indeed by Lemma 1 and Remark 5 in Foss and Zuyev [13] it follows that, for all $\lambda > 0$,

$$P_\lambda(|B_{V(N)}| > 2^{-d}t) < e^{-c_d t} \quad \text{for each } t > 0,$$

for some positive constant c_d depending only on the dimension d. Furthermore, it can be easily realized that the mapping V is monotone nonincreasing. Hence our results can be applied to $f(\lambda)$ provided that $E_\lambda[|\phi(C(0, N_0))|^\gamma] < \infty$ for some $\gamma = \gamma(\lambda) > 2$.

Note that this latter condition holds if $\varphi(N) = \phi(C(0, N_0)) = \mathbb{1}(C(0, N_0) \in A)$, for some measurable set $A \subset \mathcal{K}_0^d$. In particular, in this case the following analog of Theorem 4 holds. For each $\mu \in \mathcal{N}$ such that $0 \in \operatorname{supp}(\mu)$, consider the sets of pivotal points of the measurable set $A \subset \mathcal{K}_0^d$:

$$\begin{aligned}\mathcal{P}^+(\mu) = \{x \in \mathbb{R}^d \backslash \operatorname{supp}(\mu) : C(0, \mu + \delta_x) \in A, C(0, \mu) \notin A\} \\ \cup \{x \in \operatorname{supp}(\mu) : C(0, \mu) \in A, C(0, \mu - \delta_x) \notin A\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}^-(\mu) = \{x \in \mathbb{R}^d \backslash \operatorname{supp}(\mu) : C(0, \mu + \delta_x) \notin A, C(0, \mu) \in A\} \\ \cup \{x \in \operatorname{supp}(\mu) : C(0, \mu) \notin A, C(0, \mu - \delta_x) \in A\}.\end{aligned}$$

It holds:

Theorem 5. *The function* $f(\lambda) = P_\lambda(C(0, N_0) \in A)$ *is analytic on* $(0, \infty)$ *and*

$$f'(\lambda) = E_\lambda[|\mathcal{P}^+(N)| - |\mathcal{P}^-(N)|] = \frac{1}{\lambda} E_\lambda[N_{\mathcal{P}^+(N)} - N_{\mathcal{P}^-(N)}], \quad \lambda > 0.$$

The proof is similar to that one of Theorem 4 and therefore omitted.

5.2. Insurance

In this subsection we apply our results to risk processes described in terms of Poisson shot noise and compound Poisson processes. The former have been introduced in Klüppelberg and Mikosch [17, 18] to model delayed claims, the latter correspond to the classical Cramér-Lundberg model (see, for instance, Asmussen [2]). The main results of this subsection are Theorems 6 and 7. Under suitable light-tailed conditions on the claims, they provide, respectively, closed form formulas for the n-th order derivative of the ruin probability of risk processes with delayed (and un-delayed) claims, and an efficient Monte Carlo estimator for the first order derivative of the ruin probability of the classical Cramér-Lundberg model. The estimator proposed in the Paragraph 5.2.2 is alternative to that one of Asmussen and Rubinstein [3] (see Remark 2).

Now we briefly recall the notion of asymptotically optimal estimator, which will be considered in this subsection. Let $z(u)$ be a positive function such that $z(u) \to 0$, as $u \to \infty$. To get an asymptotically efficient estimator of $z(u)$ one looks for an unbiased estimator $\hat{r}_u$ of $z(u)$ whose relative error is asymptotically bounded. In the following we focus on a weaker concept of efficiency. We say that $\hat{r}_u$ is asymptotically optimal (as $u \to \infty$) if

$$\liminf_{u\to\infty} \frac{\log \sqrt{E[\hat{r}_u^2]}}{\log z(u)} \geq 1$$

(see Asmussen [2] and Asmussen and Rubinstein [3]).

All the random variables considered in this subsection are defined on a measurable space $(\Omega, \mathcal{F})$. Here we consider marked point processes on $[0,\infty)$ with marks in $[0,\infty)$. We endow $(\Omega, \mathcal{F})$ with the family of probability measures $\{P_\lambda\}_{\lambda>0}$ such that, under P_λ, $X_1 < X_2 < \dots$ are the points of a homogeneous Poisson process on $[0,\infty)$ with intensity $\lambda > 0$, and $\{Z_n\}_{n\geq 1}$ are iid nonnegative random variables with distribution Q, and independent of the Poisson process. We denote by N the IMHPP $\sum_{n\geq 1} \delta_{(X_n,Z_n)}$, by N_t the number of points of N on $[0,t]\times[0,\infty)$, by $N_{|t}$ the set of points of supp(N) on $[0,t]\times[0,\infty)$, and by E_λ the expectation with respect to P_λ.

5.2.1. *Derivatives of the ruin probability of risk processes with delayed claims.* Consider the following risk model. Let $u - Y(t)$ be the surplus of the insurance portfolio described by the shot noise process with drift:

$$Y(t) = \sum_{n\geq 1} H(t - X_n, Z_n)\mathbb{1}_{(0,t]}(X_n) - ct, \quad t \geq 0.$$

Here $u > 0$ is the initial capital, $c > 0$ is the premium density (which is assumed to be constant), and $H : \mathbb{R}\times[0,\infty) \to [0,\infty)$ is a nondecreasing continuous function such that $H(t,z) = 0$ for $t \leq 0$. Throughout this paragraph we assume $H(\infty, z) = z$ and $P_\lambda(Z_1 > 0) > 0$. Since the law of Z_1 under P_λ does not depend on λ, from now on, for a measurable function g, we set $E_\lambda[g(Z_1)] = E[g(Z_1)]$.

Note that the function H models the delay in claim settlement in the sense that the insurance company honors a claim at time X_n paying the quantity $H(t - X_n, Z_n)$ at time t. The associated ruin probability is defined by the quantity

$$f_u(\lambda) = P_\lambda\left(T_u(N) < \infty\right), \qquad u \geq 0,$$

where

$$T_u(N) = \inf\{t \geq 0 : Y(t) > u\}, \qquad T_u(N) = \infty \quad \text{if } \{...\} = \emptyset$$

is the ruin time. Brémaud [8] proved that under the following assumptions:

$$\kappa(\theta) = E[e^{\theta Z_1}] < \infty \quad \text{for all } \theta \text{ in a neighborhood of 0, say } (0, \eta) \text{ with } \eta \leq \infty \tag{42}$$

and

$$c > \lambda E[Z_1], \tag{43}$$

it holds

$$f_u(\lambda) \leq e^{-wu} \qquad \text{for all } u \geq 0 \tag{44}$$

and

$$\lim_{u\to\infty} \frac{1}{u} \log f_u(\lambda) = -w, \tag{45}$$

where w (called Lundberg parameter) is the unique positive zero of the function

$$\Lambda(\theta) = \lambda(\kappa(\theta) - 1) - c\theta.$$

(note that this function Λ should not be confused with the intensity measure Λ considered in the previous sections. In the remaining part of the paper, the symbol Λ will not be used anymore to denote the intensity measure). Thus, under (42) and (43), the event $\{T_u(N) < \infty\}$ is rare as $u \to \infty$ and this yields problems if we want to estimate $f_u(\lambda)$ by an efficient Monte Carlo simulation (we refer the reader to Buclew [9] for an introduction to rare event simulation). Such difficulties can be overcome using importance sampling. Define the stochastic process

$$C(t) = \sum_{n\geq 1} Z_n \mathbb{1}_{(0,t]}(X_n),$$

and consider the family of laws $\{P_\lambda^\theta\}_{\theta:\kappa(\theta)<\infty}$ defined as follows: the probability measure P_λ^θ is absolutely continuous with respect to the original law P_λ on the σ-field

$\mathcal{F}_{[0,t]}$, for each $t \geq 0$, and the corresponding density is

$$\ell_t^{P_\lambda^\theta, P_\lambda} = \frac{e^{\theta C(t)}}{E_\lambda[e^{\theta C(t)}]} = \exp\left\{\theta C(t) - \lambda t(\kappa(\theta) - 1)\right\}. \tag{46}$$

We point out (see, for instance, Asmussen [2]) that, under P_λ^θ, the process $\{X_n\}_{n\geq 1}$ is a homogeneous Poisson process with intensity $\lambda\kappa(\theta)$, independent of the sequence $\{Z_n\}_{n\geq 1}$ of iid random variables, whose common law Q^θ is absolutely continuous with respect to their common law Q under P_λ, with density $\frac{dQ^\theta}{dQ}(z) = \frac{e^{\theta z}}{\kappa(\theta)}$.

Throughout this subsection we denote by E_λ^θ the expectation under P_λ^θ. Furthermore, since the law of Z_1 under P_λ^θ does not depend on λ, for a measurable function g, we set $E_\lambda^\theta[g(Z_1)] = E^\theta[g(Z_1)]$.

The following result can be found in [33].

Proposition 5. *Assume* (42) *and* (43)*, then* $P_\lambda^w(T_u(N) < \infty) = 1$ *for all* $u > 0$ *and, under* P_λ^w*,*

$$\hat{r}_u(N) = \ell_{T_u(N)}^{P_\lambda, P_\lambda^w}$$

is an asymptotically optimal estimator of $f_u(\lambda)$*.*

Let $\mathbf{x}_n = (x_n, z_n) \in (0,\infty) \times (0,\infty)$, $n \geq 1$. For each locally finite counting measure $\mu = \sum_{n\geq 1} \delta_{\mathbf{x}_n}$, define the functionals

$$\varphi_\theta(\mu, \lambda) = \exp\left\{-\theta \sum_{n\geq 1} z_n \mathbb{1}_{(0,T_u(\mu)]}(x_n) + \lambda(\kappa(\theta) - 1)T_u(\mu)\right\}, \quad 0 \leq \theta < w \tag{47}$$

and

$$\varphi_w(\mu) = \exp\left\{-w\left(\sum_{n\geq 1} z_n \mathbb{1}_{(0,T_u(\mu)]}(x_n) - cT_u(\mu)\right)\right\}. \tag{48}$$

Moreover, we consider the functionals $\varphi_{w,\mathbf{x}_1,\ldots,\mathbf{x}_n}(\mu)$ and $\varphi_w^{\mathbf{x}_1,\ldots,\mathbf{x}_n}(\mu)$, which are defined, respectively, by (10) and (11) with φ_w in place of φ.

The following theorem provides closed form expressions for the n-th order derivatives of the ruin probability. As usual, we use the standard convention that the sum over an empty set is zero and $k!/(k-n)! = 0$ for $n > k$.

Theorem 6. *Under assumptions of Proposition 5, we have that for a fixed* $u > 0$*, the* *function $f_u(\cdot)$ is analytic in a neighborhood of λ, and for all $n \geq 1$:*

$$\begin{aligned}
f_u^{(n)}(\lambda) - (\kappa(w)-1)^n E_\lambda^w[(T_u(N))^n \varphi_w(N)] &= E_\lambda^w[(T_u(N))^n \varphi_{w,\mathbf{X}_1,\ldots,\mathbf{X}_n}(N)] \\
&= E_\lambda^w\left[\left(\frac{N_{T_u(N)}}{\lambda}\right)^n \varphi_w^{\mathbf{X}_1',\ldots,\mathbf{X}_n'}(N)\right] \\
&= E_\lambda^w\left[\frac{N_{T_u(N)}!}{(N_{T_u(N)}-n)!\lambda^n}\varphi_w^{\mathbf{X}_1'',\ldots,\mathbf{X}_n''}(N)\right],
\end{aligned}$$

where, for $1 \leq i \leq n$, $\mathbf{X}_i = (\xi_i, \zeta_i)$, $\mathbf{X}_i' = (\xi_i', \zeta_i')$, $\mathbf{X}_i'' = (\xi_i'', \zeta_i'')$ are random variables on $(0,\infty)\times(0,\infty)$. Given $T_u(N)$, $(\xi_i)_{1\leq i\leq n}$ are independent and uniformly distributed on $[0, T_u(N)]$, and independent of N; $(\zeta_i)_{1\leq i\leq n}$ are independent, independent of N and $(\xi_i)_{1\leq i\leq n}$, and with law Q^w. Given the collection of points $N_{|T_u(N)}$, $(\mathbf{X}_i')_{1\leq i\leq n}$ are independent and uniformly distributed on the collection; $\{\mathbf{X}_1'', \ldots, \mathbf{X}_n''\}$ is uniformly distributed on the set of subsets of n distinct points of $N_{|T_u(N)}$.

To prove Theorem 6 we need the following Lemmas 7 and 8. Here we consider the notion of large deviation principle for which we refer the reader to the book by Dembo and Zeitouni [12].

Lemma 7. *Assume (42), if moreover the function $\theta \mapsto \kappa(\theta)$, $\theta \in (0,\eta)$, is steep, namely $\lim_{n\to\infty} \kappa'(\theta_n) = \infty$ whenever $\{\theta_n\}$ is a sequence converging to η, we have that the stochastic process $\{Y(t)/t\}_{t>0}$ satisfies a large deviation principle with rate function $\Lambda^*(x) = \sup_{\theta\in\mathbb{R}}(\theta x - \Lambda(\theta))$.*

The proof of Lemma 7 can be found in Macci, Stabile and Torrisi [21] (see Proposition 3.1 therein).

Lemma 8. *Under assumptions of Proposition 5, we have that for a fixed $u > 0$, and $\overline{\theta} \in (0, w]$ such that $\lambda\kappa'(\overline{\theta}) - c > 0$, there exists $s = s(\lambda) > 0$ such that $E_\lambda^{\overline{\theta}}[\exp(sT_u(N))] < \infty$.*

Proof of Lemma 8. In this proof we write T_u in place of $T_u(N)$. Since

$$\kappa_{\overline{\theta}}(\alpha) = E^{\overline{\theta}}[e^{\alpha Z_1}] = \frac{\kappa(\alpha + \overline{\theta})}{\kappa(\overline{\theta})},$$

by the assumptions it follows that: $\kappa_{\overline{\theta}}(\alpha) < \infty$ for $\alpha \in (0, \eta - \overline{\theta})$ and the function $\alpha \mapsto \kappa_{\overline{\theta}}(\alpha)$ is steep. Therefore, by Lemma 7 the stochastic process $\{Y(t)/t\}_{t>0}$ satisfies a large deviation principle with respect to $P_\lambda^{\overline{\theta}}$ with rate function $\Lambda_{\overline{\theta}}^*(x) = \sup_{\theta\in\mathbb{R}}(\theta x -$

$\Lambda_{\overline{\theta}}(\theta))$, where $\Lambda_{\overline{\theta}}(\theta) = \lambda\kappa(\overline{\theta})(\kappa_{\overline{\theta}}(\theta) - 1) - c\theta$. Since $\lambda\kappa'(\overline{\theta}) - c > 0$ by assumption, we can choose $\beta \in (0, \kappa'(\overline{\theta}))$ such that $\gamma = \lambda\beta - c > 0$. By the large deviation principle of $\{Y(t)/t\}_{t>0}$ with respect to $P_\lambda^{\overline{\theta}}$, and the regularity properties of the rate function $\Lambda^*_{\overline{\theta}}(\cdot)$ we have

$$\lim_{t\to\infty} \frac{1}{t} \log P_\lambda^{\overline{\theta}}(Y(t)/t < \gamma) = -\Lambda^*_{\overline{\theta}}(\gamma). \tag{49}$$

Moreover, for any $u > 0$, we have that there exists $\bar{t}_1 = \bar{t}_1(u, \gamma)$ such that

$$P_\lambda^{\overline{\theta}}(T_u > t) \le P_\lambda^{\overline{\theta}}(Y(t) < u) \le P_\lambda^{\overline{\theta}}(Y(t)/t < \gamma), \quad \text{for all } t \ge \bar{t}_1. \tag{50}$$

Therefore, by (49) and (50) it follows that, for any $\varepsilon, u > 0$ there exists $\bar{t} = \bar{t}(\varepsilon, u, \gamma)$ such that

$$P_\lambda^{\overline{\theta}}(T_u > t) < e^{-(\Lambda^*_{\overline{\theta}}(\gamma)-\varepsilon)t}, \quad \text{for all } t \ge \bar{t}. \tag{51}$$

Now, take $0 < s < \Lambda^*_{\overline{\theta}}(\gamma) - \varepsilon$. The conclusion follows noticing that by (51) we have:

$$\begin{aligned} E_\lambda^{\overline{\theta}}[\exp(sT_u)] &= 1 + s\int_0^\infty e^{st} P_\lambda^{\overline{\theta}}(T_u > t)dt \\ &\le e^{s\bar{t}} + s\int_{\bar{t}}^\infty e^{-[(\Lambda^*_{\overline{\theta}}(\gamma)-\varepsilon)-s]t}dt < \infty. \end{aligned}$$

□

Proof of Theorem 6. We start noticing that by the properties of the function $\theta \mapsto \Lambda(\theta)$, $\theta \in (0, \eta)$, there exists a strictly increasing sequence $\{\theta_k\}$ converging to w such that $\theta_k \in (0, w)$ and $\lambda\kappa'(\theta_k) - c > 0$. By the implicit function theorem the function $\lambda \mapsto w(\lambda)$ is continuous. Therefore, for each k there exists a neighborhood of λ, say $I_k = (\lambda - \varepsilon_k, \lambda + \varepsilon_k)$, such that for all $\lambda' \in I_k$ we have $\theta_k < w(\lambda')$ and $\lambda'\kappa'(\theta_k) - c > 0$. We note that since $\theta_k \in (0, w)$ is such that $\lambda\kappa'(\theta_k) - c > 0$ it holds $P_\lambda^{\theta_k}(T_u(N) < \infty) = 1$ (see Lemma 3.2 in [33] for details). Therefore,

$$f_u(\lambda) = E_\lambda[\mathbb{1}(T_u(N) < \infty)] = E_\lambda^{\theta_k}[\varphi_{\theta_k}(N, \lambda)].$$

Note that $[0, T_u]$ is a stopping set. Furthermore, the functional $\varphi_{\theta_k}(N, \lambda)$ is $\mathcal{F}_{[0,T_u]}$-measurable and absolutely monotonic in λ (see Remark 1 for the definition of absolutely monotonic function). Note also that $\Lambda(\theta) \le 0$ for each $\theta \in [0, w]$. Therefore, by the definition of $T_u(N)$ and the assumption $H(t, z) \nearrow z$ as $t \nearrow \infty$, we have that, for each u,

$$\varphi_\theta(N, \lambda) \le e^{-\theta u} \quad \text{and} \quad \varphi_w(N) \le e^{-wu}.$$

In particular, this implies that the functional $\varphi_{\theta_k}(N,\lambda)$ is bounded. Therefore, by Lemma 8 and Remark 1 it follows that $f_u(\cdot)$ is analytic on I_k. Consider the functions:

$$F_k(x,y) = E_x^{\theta_k}[\varphi_{\theta_k}(N,y)], \quad (x,y) \in I_k \times I_k.$$

Using obvious notation, we shall show later that:

$$\partial_x^n F_k(x,y) = E_x^{\theta_k}[(T_u(N))^n \varphi_{\theta_k,\mathbf{x}_1,\ldots,\mathbf{x}_n}(N,y)], \quad n \geq 1 \tag{52}$$

and

$$\partial_y^n F_k(x,y) = (\kappa(\theta_k)-1)^n E_x^{\theta_k}[(T_u(N))^n \varphi_{\theta_k}(N,y)], \quad n \geq 1. \tag{53}$$

Therefore,

$$f_u^{(n)}(\lambda) - (\kappa(\theta_k)-1)^n E_\lambda^{\theta_k}[(T_u(N))^n \varphi_{\theta_k}(N,\lambda)] = E_\lambda^{\theta_k}[(T_u(N))^n \varphi_{\theta_k,\mathbf{x}_1,\ldots,\mathbf{x}_n}(N,\lambda)], \quad n \geq 1.$$

We now show that

$$\lim_{k\to\infty} E_\lambda^{\theta_k}[(T_u(N))^n \varphi_{\theta_k,\mathbf{x}_1,\ldots,\mathbf{x}_n}(N,\lambda)] = E_\lambda^{w}[(T_u(N))^n \varphi_{w,\mathbf{x}_1,\ldots,\mathbf{x}_n}(N)]. \tag{54}$$

Using the exponential tilting we have that

$$\begin{aligned}
&|E_\lambda^{\theta_k}[(T_u(N))^n \varphi_{\theta_k,\mathbf{x}_1,\ldots,\mathbf{x}_n}(N,\lambda)] - E_\lambda^{w}[(T_u(N))^n \varphi_{w,\mathbf{x}_1,\ldots,\mathbf{x}_n}(N)]| \leq \\
&E_\lambda^{w}[(T_u(N))^n | \exp\{-(w-\theta_k)(C(T_u(N)) - cT_u(N)) - \Lambda(\theta_k)T_u(N)\}\varphi_{\theta_k,\mathbf{x}_1,\ldots,\mathbf{x}_n}(N,\lambda) \\
&\qquad - \varphi_{w,\mathbf{x}_1,\ldots,\mathbf{x}_n}(N)|].
\end{aligned} \tag{55}$$

Note that the argument of the mean in (55) is less than or equal to $(2T_u(N))^n(\exp\{-\Lambda(\theta_k)T_u(N)\}+1)$. By Lemma 8 there exists $s = s(\lambda) > 0$ such that $E_\lambda^w[e^{sT_u(N)}] < \infty$. Fix $\varepsilon \in (0,s)$ and choose $\bar{k} = \bar{k}(\varepsilon)$ such that for all $k > \bar{k}$ it holds $0 < -\Lambda(\theta_k) < \varepsilon < s$ (a such $\bar{k}$ exists since $\lim_{k\to\infty} \Lambda(\theta_k) = \Lambda(w) = 0$). Then, for all $k > \bar{k}$, the argument of the mean in (55) is less than or equal to

$$(2T_u(N))^n(\exp\{\varepsilon T_u(N)\}+1),$$

which is integrable under P_λ^w since $E_\lambda^w[e^{sT_u(N)}] < \infty$. Thus (54) follows by the dominated convergence theorem. The limit

$$\lim_{k\to\infty} E_\lambda^{\theta_k}[(T_u(N))^n \varphi_{\theta_k}(N,\lambda)] = E_\lambda^{w}[(T_u(N))^n \varphi_{w}(N)]$$

can be proved similarly. This shows the first equality in the statement. The other equalities can be proved as in Proposition 4 and Theorem 3.

It remains to show that $F_k(\cdot,\cdot)$ has partial derivatives (52) and (53). Equality (52) follows by Proposition 2. Indeed, by Lemma 8, for each $x \in I_k$, there exists $s = s(x) > 0$ such that $E_x^{\theta_k}[e^{sT_u(N)}] < \infty$, and the mapping $S_u := [0, T_u]$ is nonincreasing. We now show (53) with $n = 1$. The general case follows along similar lines, reasoning by induction. If we justify the interchange between the sign of limit and the sign of mean in the expression:

$$\lim_{h\to 0^+} \frac{E_x^{\theta_k}[\varphi_{\theta_k}(N, y+h) - \varphi_{\theta_k}(N, y)]}{h},$$

then the right-hand derivative equals the right-hand side of (53). In fact we can pass the limit into the sign of expectation in that a straightforward computation gives:

$$\left|\frac{\varphi_{\theta_k}(N, y+h) - \varphi_{\theta_k}(N, y)}{h}\right| \leq (\kappa(\theta_k) - 1)T_u(N)\exp\{h(\kappa(\theta_k) - 1)T_u(N)\}.$$

Here again by Lemma 8 the right-hand side of the above inequality is integrable under $P_x^{\theta_k}$, and therefore we can apply the dominated convergence theorem. Similarly, one can show that the left-hand derivative equals the right-hand side of (53). This concludes the proof. $\square$

5.2.2. *Classical risk processes: an efficient Monte Carlo algorithm for the first order derivative of the ruin probability* .

The classical risk model is defined by the surplus $u - Y(t)$ of the insurance portfolio described by the compound process with drift:

$$Y(t) = \sum_{n\geq 1} Z_n \mathbb{1}_{(0,t]}(X_n) - ct.$$

The interpretation of the quantities in the above formulas is exactly as in the previous paragraph. Moreover, we consider the same statistical assumptions and the same notation (clearly, the ruin probability $f_u(\lambda)$ and the ruin time $T_u(N)$ are now defined with respect to the classical risk process). For the Cramér-Lundberg model it is well-known that assuming (42) and (43) it holds

$$\lim_{u\to\infty} \frac{e^{wu}}{u} f'_u(\lambda) = \frac{c^2 w(c - \lambda E[Z_1])}{\lambda(\lambda\kappa'(w) - c)^2}, \tag{56}$$

where w is the unique positive zero of the function $\Lambda(\cdot)$ (see Proposition 9.4 in Asmussen [2]). Moreover, note that in the case of classical risk processes, the corresponding

Theorem 6 can be proved along similar lines, and provides the following unbiased estimator of $f_u'(\lambda)$ under P_λ^w:

$$\hat{s}_u(N) = (\kappa(w) - 1)T_u(N)\varphi_w(N) + T_u(N)(\varphi_w(N + \delta_{\mathbf{X}}) - \varphi_w(N)).$$

Here, $\mathbf{X} = (\xi, \zeta)$ is a random variable on $(0, \infty) \times (0, \infty)$; given $T_u(N)$, ξ is uniformly distributed on $[0, T_u(N)]$, and independent of N; ζ is independent of N and ξ, and has law Q^w.

The following theorem holds.

Theorem 7. *Assume* (42) *and* (43). *Then* $\hat{s}_u(N)$ *is an asymptotically optimal estimator of* $f_u'(\lambda)$, *as* $u \to \infty$, *under the law* P_λ^w.

Proof. We only need to prove that

$$\liminf_{u\to\infty} \frac{\log \sqrt{E_\lambda^w[(\hat{s}_u(N))^2]}}{\log f_u'(\lambda)} \geq 1. \tag{57}$$

For any (locally finite) counting measure μ on $(0, \infty) \times (0, \infty)$ and $u > 0$, we have that $\varphi_w(\mu) \leq e^{-wu}$, thus

$$|\hat{s}_u(N)| \leq (\kappa(w) + 1)e^{-wu}T_u(N),$$

and therefore

$$E_\lambda^w[(\hat{s}_u(N))^2] \leq (\kappa(w) + 1)^2 e^{-2wu} E_\lambda^w[T_u(N)^2]. \tag{58}$$

Denote by $\{\tilde{X}_i\}$ the inter-arrivals of the Poisson process $\{X_i\}$. Under P_λ^w, $\sum_{i=1}^n (Z_i - c\tilde{X}_i)$ is a random walk with positive drift, indeed

$$E_\lambda^w[Z_1 - c\tilde{X}_1] = (\lambda\kappa'(w) - c)/\lambda\kappa(w) > 0.$$

Furthermore, $T_u(N)$ is the hitting time of this random walk. Therefore, by the results in Gut [15] it follows that

$$E_\lambda^w[T_u(N)^2] = O(u^2), \quad \text{as } u \to \infty. \tag{59}$$

Finally, (57) follows by (58), (59) and relation (56). □

While it is tempting to conjecture that a similar optimality result holds for risk processes with delay in claim settlement, we do not have a proof of this claim.

Remark 2. Note that, under assumptions of Theorem 7, Asmussen and Rubinstein [3] (see also Asmussen [2]) proved that

$$\hat{\sigma}_u(N) = \left(N_{T_u(N)}/\lambda - T_u(N)\right) e^{-w(C(T_u(N)) - cT_u(N))}$$

is asymptotically optimal for $f'_u(\lambda)$, under P^w_λ. The estimator $\hat{s}_u(N)$ is alternative to $\hat{\sigma}_u(N)$.

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