Spatial Capacity of Multiple Access Wireless Networks

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***Abstract*— We study the capacity of multiple access networks both on uplink and downlink. In our model each user requires a given signal to interference plus noise ratio (SINR) and the capacity region is obtained as a solution of a power allocation problem. In this paper, we emphasize on the differences between uplink and downlink. The mathematical analysis of the capacity region is led in the framework of ergodic point processes and we exhibit the links between the geometry of the network and its capacity region. On the downlink we pay attention to various network architectures and levels of cooperation between base stations: macrodiversity, load balancing and traditional cellular networks.**

***Index Terms*— spatial capacity, CDMA networks, macrodiversity, spatial point processes, power control, spectral radius.**

I. Introduction

THIS paper deals with the capacity of wireless multiple access networks. Primarily, it covers the analysis of CDMA networks (Code Division Multiple Access) in *macrodiversity*. In a network in macrodiversity the base stations are fully coordinated and they jointly code (for downlink) or decode (for uplink) the emitted signals. Macrodiversity networks supersedes the traditional *cellular* architecture of wireless networks where each user is attached to a unique base station based on its location. As an intermediate architecture, there are *load balanced* networks, where each user is attached to a unique base station but this allocation depends on the whole configuration of the network. Computing the load capacity of such networks is an important issue of wireless communications. This problem relies on finding a power allocation satisfying all users in the network.

The problem of power control and load constraints in CDMA networks has drawn much attention. However, most authors are only considering CDMA networks without macrodiversity. On the downlink in the seminal papers of Gilhousen et al. [1] and Zander [2], [3], the authors rely the solution of the power control problem to a condition of the type:

$$\rho(T) < 1, \tag{1}$$

where T is a square non negative matrix depending on the channel state and $\rho(T)$ denote the spectral radius of T. Baccelli et al. [4], [5] have developed a probabilistic geometric model to analyze the feasibility condition given by Equation (1). The users and the base stations are instances of spatial point processes and the authors compute the probability that a base station satisfies the SINR ratio requirement of each users in its cell. In this paper, we extend the geometric model of Baccelli et al. to networks in macrodiversity.

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On the uplink, Hanly [6], [7] has solved the power control problem for finite networks in macrodiversity. The solution of the power control problem reduces to a condition of the type:

$$\sum_{i=1}^{M} h_i < N, \tag{2}$$

where N and M are the numbers of base stations and users respectively and h_i is the SINR requirement of the i^{th} user. In the present paper, we generalize the work of Hanly to infinite networks where users and base stations are instance of ergodic point processes.

The feasibility conditions given by Equations (1) and (2) can be understood as a condition of the type: "$\rho < 1$" where ρ is the *load* of the network. In this paper, we compute the value of the load both on uplink (denoted by $\rho_\uparrow$), and downlink (denoted by $\rho_\downarrow$) in a probabilistic setting where the channel condition depends on the relative positions of users and base stations. This modelling contribution will enable to understand better what is the impact of the geometry of the network in its capacity. On the uplink, if the mean SINR requirement is denoted by h, the mean number of users (respectively base stations) per surface unit is λ_u (resp. λ_s) we will obtain (Theorem 2):

$$\rho_\uparrow = h\frac{\lambda_s}{\lambda_u}, \tag{3}$$

$h\lambda_s$ is the mean SINR requirement per surface unit, so that the geometrical term of the uplink load reduces to $1/\lambda_u$.

On the downlink for a network with N base stations and M users, we will prove, for networks either cellular, load balanced or in macrodiversity, that the downlink load is asymptotically equal, as M grows large, to:

$$\rho_\downarrow \sim hM\gamma, \tag{4}$$

(Theorem 4) where γ is explicitly computed and depends on the relative position of the base stations.

Both Equations (3) and (4) show that the load may be decomposed as a mean SINR requirement per surface unit ($h\lambda_s$ on the uplink, hM on the downlink) and a geometric term ($1/\lambda_u$ on the uplink and γ on the downlink). This decoupling between mean SINR and geometry is of prime interest: given a required level of user quality of service, we can design a network architecture.

On the downlink, another consequence of our results is the comparison between the various possible levels of cooperation between base stations: macrodiversity, load balancing and cellular networks. We will hint that the main improvement between a fixed cell network and a macrodiversity network seems to be in the flexibility into affecting each user to a specific base station and not on the possibility to share a user between several base stations (Theorem 3). In other words, load balancing is as efficient as user sharing (i.e. macrodiversity). We will prove that the constant γ appearing in Equation (4) is the same for a network in macrodiversity, load balanced or for an optimal cellular network. On the contrary, as already known, for the uplink, macrodiversity has a much larger impact and appears as a major improvement compared to traditional cellular network structure.

In this paper, we are primarily concerned by the impact of the locations of users and base stations on the capacity of the network. To this end, somewhat artificially, the path gain between a user located at x and a base station located at y will be set as $L(x, y)$. Averaging over the channel conditions we will often assume that the path gain depends only on the distance between x and y. This assumption is not meant to be realistic, however it captures the spatial features of wireless networks.

The remainder of this paper is organized as follows. Section II is devoted to the macrodiversity on the uplink. In §II-A, we introduce our model, its key features are the spatial location of base stations and users, and the SINR requirements of each user. In §II-B we extend Hanly's Theorem to ergodic spatial point processes of users and base stations and establish Equation (3).

In Section III, we analyze the downlink. We present the model in §III-A and establish a necessary and sufficient condition for the feasibility of the power control problem in §III-B. §III-C gives a characterization of the optimal power allocation. This characterization establishes a bound on the increase of capacity brought by macrodiversity in a network. In §III-D, we pay attention to the limit downlink load as the number of users grows large and we establish Equation (4). At last, in §III-E we extend our results to infinite networks and prove a negative result for the feasibility of power control problem when the point process of users is a Poisson point process.

II. Uplink

A. Model description

We consider a network consisting of M users and N base stations. The users are located at points $\{X_i\}_{1\leq i\leq M} \in \mathbb{R}^2$ and the base station at points $\{Y_j\}_{1\leq j\leq N} \in \mathbb{R}^2$. We denote by $U(x, y)$ the channel gain from y to x, $x, y \in \mathbb{R}^2$. $|U(x, y)|$ represents the path loss due to shadowing, fading and distance attenuation effects.

In an uplink multiple access network in macrodiversity, each user sends independently from the other a signal and the base stations are jointly decoding the received signals. This kind of channel is known as multi-receiver networks (see Hanly and Whiting [8]). A base station j receives a signal equal to the sum of all the signals sent by the users plus an external white Gaussian noise. Let $w = (w_j)_{1\leq j\leq N}$ denote the power of the noise vector, $U = (U(X_i, Y_j))_{1\leq i\leq M,1\leq j\leq N}$, the channel matrix. The user i sends a signal s_i. Let $s = (s_i)_{1\leq i\leq M}$ be the vector of the signal sent by users. Then mathematically the signals received by the base stations is a $\mathbb{R}^{M\times 1}$ vector:

$$v = U's + w.$$

We set the channel bandwidth to ΔHz and we suppose that user i requires a rate R_i in bits per second. Let $S_i = E(|s_i|^2)$ and $\eta_j = E(|w_j|^2)$ denote the powers of the signals. If the users are sending their signals independently, it is known (see [8]) that the rate vector $(R_1, ..., R_M)$ is achievable if and only if there exists $S \in \mathbb{R}_+^M$ such that:

$$\forall i, \quad R_i \leq \Delta \log_2(1 + \sum_{j=1}^{N} \frac{S_i|U(X_i, Y_j)|^2}{\eta_j + \sum_{m\neq i} S_m|U(X_m, Y_j)|^2}).$$

We restrain ourselves to the sufficient condition:

$$\forall i, \quad R_i \leq \Delta \log_2(1 + \sum_{j=1}^{N} \frac{S_i|U(X_i, Y_j)|^2}{\eta_j + \sum_{m=1}^{M} S_m|U(X_m, Y_j)|^2}).$$

This last condition is only sufficient but when M is large it is expected not to be far from being necessary. Let $L(X_i, Y_j) = |U(X_i, Y_j)|^2$ denote the attenuation function. Thus, feasibility of a given rate vector is equivalent to a minimal requirement on the signal to interference ratio:

$$\forall i, \quad h_i \leq \sum_j \frac{L(X_i, Y_j)S_i}{\eta_j + \sum_m L(X_m, Y_j)S_m}, \tag{5}$$

where $h_i = 2^{R_i/\Delta} - 1$, with an abuse of language h_i will be called the SINR requirement of user i. The power allocation problem is stated as follows, for a given vector of bit rates $(R_i)_i$ does there exists a power vector $(S_i)_i$ such that the set inequalities (5) is satisfied. The following theorem solves the power allocation problem:

Theorem 1 (Hanly): Suppose that for all i, j, $L(X_i, Y_j) > 0$ *and* $\eta_j > 0$.

Then, there exists a solution of (5) if and only if

$$\sum_{i=1}^{M} h_i < N.$$

This theorem is surprising , since the feasibility condition does not rely on the geometry of the network (i.e. the coefficients $L(X_i, Y_j)$).

B. Stochastic Model

In this paragraph, we generalize the work done by Hanly in [6] for stochastic infinite networks. This generalization proves that Hanly's Theorem is not due to the finiteness of the network but is intrinsic to uplink communications in macrodiversity.

We follow the probabilistic setting of [4]. The set of users is a marked point process $\Pi_u = \{(X_i, h_i)\}_i$, where X_i is the location of the user i and h_i is its SINR requirement. We model similarly the base stations by a point processes on $\mathbb{R}^2$: $\Pi_s = \{(Y_j, \eta_j)\}_j$, η_j is the noise power. We can suppose $\eta_j > 0$ and $h_i > 0$ for all i, j. Moreover, Π_u and Π_s are supposed to be a stationary and ergodic marked point processes. We denote by λ_u (resp. λ_s) the intensity of Π_u (resp. Π_s) which are assumed to be finite. The Palm probability of the process Π_u (resp. Π_s) is denoted by P_u^0 (resp. P_s^0), (for an introduction to Palm probability, refer to Daley and Vere-Jones [9]). We assume that $E_u^0(h_0) < \infty$. We remind that $E_u^0(h_0)$ may be understood as the mean SINR requirement of typical user. At last, we consider a radial positive attenuation function, that is: $L(x,y) = l(|x-y|)$.

In infinite networks, the power control problem is still given by the set of inequalities (5). A SINR vector $(h_i)_{i\geq 0} > 0$ is feasible, if there exists a power allocation $(S_i)_{i\geq 0}$ such that the set of inequalities (5) is satisfied.

Following Hanly [6], we introduce:

$$G : \begin{cases} {\mathbb{R}_+}^{\mathbb{N}} & \to \quad {\mathbb{R}_+}^{\mathbb{N}} \\ (S_i)_i & \mapsto \quad \left(h_i (\sum_j \frac{L(X_i,Y_j)}{\eta_j + \sum_{m=0}^{\infty} S_m L(X_m,Y_j)})^{-1} \right)_i \end{cases}$$

The power allocation problem is equivalent to finding $S \in {\mathbb{R}_+}^{\mathbb{N}}$ such that, component-wise: $G(S) \leq S$.

Lemma 1: With the foregoing assumptions, there exists a power allocation satisfying (5) with probability 0 *or* 1.

Proof: The event $\{\text{Equation (5) has a solution}\} = \{\text{there exists } S, \text{ such that } G(S) \leq S\}$ is invariant under a translation on $\mathbb{R}^2$ since the value $G(S)$ does not change if we translate simultaneously all users and all base stations. Thus, by ergodicity, this event has probability 0 or 1. ∎

We define the *uplink load* by:

$$\rho_{\uparrow} = \frac{\lambda_u}{\lambda_s} E_u^0(h_0). \tag{6}$$

The following result is a natural extension of Theorem 1.

Theorem 2: We assume that $E_s^0(\eta_0^{-1}) < +\infty$ and that one of the two following conditions holds:

- *$x \mapsto xl(x)$ is in $L^1(\mathbb{R})$ and $x \mapsto xl(x)$ is non-increasing,*
- *or, there exists $\beta > 1$ such that $x \mapsto x^{\beta} l(x)$ is integrable.*

then

- *If $\rho_{\uparrow} > 1$, then (5) has almost surely no solution.*
- *If $\rho_{\uparrow} < 1$, then (5) admits almost surely a solution.*

An analogy can be made between this theorem and the stability of G/G/s queues. The intensity of user arrival is λ_u, $\lambda_u E_u^0(h_0)$ is the mean SINR requirement per surface unit and λ_s plays the role of the number of service booths per surface unit. As for $G/G/s$ queues, the limit case $\frac{\lambda_u}{\lambda_s} E^0(h_0) = 1$ is harder and the power allocation problem is not solved for these networks.

As for finite networks, The feasibility condition depends only on the bit rates requirement and the density of users and base stations in the network.

The technical hypothesis on $l(x)$ is used to ensure a rapid decay of the tail of the shot-noise process $\sum_i l(|X_i|)$. It covers a usual model for the attenuation function: $l(x) \sim x^{-\alpha}$, $\alpha > 2$. The assumption $E_s^0(\eta_0^{-1}) < +\infty$ simplifies the proof of the sufficient condition. The result should hold for weaker assumptions.

The proof of Theorem 2 can be found in Appendix. The main idea is to follow the lines of the original proof of Theorem 1 and use ergodicity to ensure convergence and some uniform bounds on shot noise processes.

III. DOWNLINK

A. Model Description

We consider the same network as in the previous section, with the same notations. There are M users and N base stations. In a downlink multiple access network in macrodiversity, the base stations are jointly coding a signal for each user and users are decoding independently. This kind of channel is known as multiple input multiple output (MIMO) broadcast channel (see in particular Caire and Shamai [10], Goldsmith, Jindal and Vishwanath [11]). A user i receives a signal equal to the sum of all the signals sent by the base stations plus an external white Gaussian noise. As above, $w = (w_i)_i$ denote the noise vector, $U = (U(X_i,Y_j))_{i,j}$, the channel matrix and $U_i = (U(X_i,Y_j))_j$ the channel vector to i . The base station j sends a signal s_{ij} to the user i. Let $s_i = (s_{ij})_j$ be the vector of the signal sent to i. Then the signals received by users is a vector of size N equal to

$$u = U \sum_{i=1}^{M} s_i + w,$$

Let Γ_i be the covariance matrix of $(s_{ij})_{1\leq j\leq N}$ and $\eta_i = E(|w_i|^2)$ the power of the noise at i. User i requires a rate R_i in bits per second. If we make the assumption, that for all j, for all $m \neq i$, the signals s_{mj} are regarded as noise by the base stations in the coding of signal s_i, the gaussian channel capacity theorem (refer to Cover and Thomas [12]) implies that the rate vector $R = (R_1, ..., R_M)$ is achievable if:

$$\forall i, \quad R_i \leq \Delta \log_2 (1 + \frac{U_i^* \Gamma_i U_i}{\eta_i + U_i^* \sum_{m\neq i} \Gamma_m U_i}). \tag{7}$$

(This last condition is only sufficient and it is not necessary.) In this work, we only consider achievable rates satisfying in Equation (7) in the case where Γ_i is diagonal: the base stations are sending uncorrelated signals to each user. This is a natural assumption for an efficient coding. We note $S_{ij} = \Gamma_i(j,j)$ and $l_{ij} = L(X_i,Y_j) = |U(X_i,Y_j)|^2$ the attenuation function. Thus the rate vector $R = (R_1, ..., R_M)$ is achievable if there exists a power allocation (S_{ij}) such that:

$$\forall i, \quad R_i \leq \Delta \log_2 (1 + \frac{\sum_j L(X_i,Y_j) S_{ij}}{\eta_i + \sum_j L(X_i,Y_j) \sum_{m\neq i} S_{mj}}),$$

thus, letting $h_i = 1 - 2^{-R_i/\Delta}$, feasibility of a given rate vector is equivalent to the existence a power allocation (S_{ij}) such that:

$$\forall i, \quad h_i \leq \frac{\sum_j L(X_i,Y_j) S_{ij}}{\eta_i + \sum_j L(X_i,Y_j) \sum_m S_{mj}}. \tag{8}$$

The set of inequalities (8) is our macrodiversity model for multiple access downlink networks. Note that the definition of the SINR $h_i = 1 - 2^{-R_i/\Delta}$ is not consistent with the definition of h_i on the uplink (that is $h_i = 2^{R_i/\Delta} - 1$). However since they will play exactly the same role, we use the same notation for these two scalars, in the limit Δ large, they are equivalent.

B. Power Allocation Algebras

In this section, we study the power allocation problem (8), following Baccelli, Blaszczyszyn and Tournois [4].

We introduce the set of stochastic matrices:

$$\mathcal{A} = \{A = (a_{ij}) \in \mathbb{R}^{M\times N}, A \geq 0, \forall i \quad \sum_j a_{ij} = 1\}.$$

A matrix A in $\mathcal{A}$ will be called an allocation matrix.

The following obvious lemma restates Equation (8).

Lemma 2: An power allocation $(S_{ij})_{1\leq i\leq M, 1\leq j\leq N}$ is a solution of (8) if and only if there exists a non-negative matrix $A \in \mathcal{A}$ such that:

$$\forall i, j \quad a_{ij} h_i \leq \frac{L(X_i, Y_j) S_{ij}}{\eta_i + \sum_j L(X_i, Y_j) \sum_m S_{mj}}. \tag{9}$$

For a fixed $A = (a_{ij})$, the restatement given by Equation (9) reduces our problem to a power allocation problem without macrodiversity as it is addressed in [4]. Our $M \times N$ macrodiversity network is equivalent to a $MN \times N$ fixed cell network: each user X_i is subdivided into N independent users $(X_i^j)_{1\leq j\leq N}$, X_i^j is affiliated to base station j and has SINR requirement of $a_{ij} h_i$. We define the linear mapping:

$$\mathcal{T} : \left\{ \begin{array}{lll} \mathcal{A} & \rightarrow & \mathbb{R}^{N\times N} \\ A & \mapsto & T = (\sum_i a_{ij} h_i \frac{l_{ik}}{l_{ij}})_{1\leq j,k\leq N} \end{array} \right. .$$

Let $\rho(T)$ denotes the spectral radius of the square matrix T. We then have the following necessary and sufficient condition:

Proposition 1: Let,

$$\rho_\downarrow = \min_{A\in\mathcal{A}} \rho(\mathcal{T}(A)). \tag{10}$$

Equation (9) has a solution if and only if $\rho_\downarrow < 1$.

$\rho_\downarrow$ is the *downlink load* of the network. Since $\rho(T + \tilde{T}) \leq \rho(T) + \rho(\tilde{T})$, $\rho_\downarrow$ is computed as an optimization of a convex function over a convex set.

Proof: Note that $\rho_\downarrow = \rho(\mathcal{T}(A^*))$, $A^* \in \mathcal{A}$. This proposition is a consequence of Propositions 3.1 to 3.3 of [4] in the finite dimensional case. For the reader convenience, we sketch the main idea. Consider the allocation matrix A^*. The base station j guarantees an individual signal to noise ratio of at least $h_i a_{ij}$ to user i. We define S_j has the total power emitted by station j: $S_j = \sum_i S_{ij}$. Let $S = (S_j)_j$ be the vector of total emitted powers, by elementary calculations that Equation (9) implies component-wise: $S \geq \mathcal{T}(A^*)S + b$, where b contains the noise of the channel. This inequality is solved by the Perron-Frobenius theory, and the existence of a non-negative vector S relies on whether or not the spectral radius of $\mathcal{T}(A^*)$ is less than one. It remains to prove that if the inequality for the total emitted powers S has a solution, then it is possible to compute the individual powers S_{ij}. ■

On the uplink, the feasibility of the power control problem did not depend on the geometry of the network. On the downlink, on the contrary, in the computation of $\rho_\downarrow$, the locations of the users is relevant.

Lemma 3: If $A \in \mathcal{A}^*$:

$$\frac{1}{N}\sum_{i=1}^{M} h_i \leq \rho(\mathcal{T}(A)) \leq \sum_{i=1}^{M} h_i. \tag{11}$$

The right hand side bound of (11) is simply obtained by removing all base stations but one in the network. This bound cannot be improved without taking into account the locations of the users (see Remark 1). We can compare the left hand side with Theorem 1. On the uplink, there is a solution to the power allocation if and only if $1/N\sum_{i=1}^M h_i < 1$. On the downlink this condition is only necessary.

Proof: For any matrix T, trace$(T) = \sum_{i=1}^M h_i = \sum_j \lambda_j$, where $(\lambda_j)_j$ are the eigenvalues of T. Since $\rho(T)$ is the largest eigenvalue, we deduce the left hand side.

It remains the right hand side of Equation (11). Consider the allocation matrix $A \in \mathcal{A}$ where the j^{th} column is 1 and all the others are set to 0. We immediately check: $\rho(\mathcal{T}(A)) = \sum_i h_i$. ■

Remark 1: There exists configurations such that the two bounds of Equation (11) are reached.

A limit configuration reaching left hand side of Equation (11). Consider a network on a line and suppose to simplify: $M = KN$, K integer. Then place the base stations Y_j at locations jr and place K users $(X_1^j, ..., X_K^j)$ at jr. Consider now the allocation $A = (a_{ij})$, a_{ij} taking value 1 if X_i is an X_m^j and 0 otherwise. We can check directly that if $L(x, y)$ goes to 0 as the distance between x and y goes to infinity, $\rho(\mathcal{T}(A))$ tends to $\frac{1}{N}\sum_{i=1}^M h_i$ as r tends toward infinity.

A configuration reaching right hand side of Equation (11). Consider, the case where all M users are at the same location. We define $l_j = L(X_i, Y_j) > 0$ and let D be the diagonal matrix whose diagonal is $(l_1, ..., l_N)$. In this case, we have $T = D^{-1}MD$, with $M_{jk} = \sum_i a_{ij} h_i$. T and M have the same spectral radius. Then notice that $M = U\mathbb{1}^t$, where U and $\mathbb{1}$ are $\mathbb{R}^N$ positive vectors and it follows that $\rho(T) = \rho(M) = \mathbb{1}^t U = \sum_i h_i$.

C. Optimal Power Allocation

In this paragraph, we state an interesting property shared by the optimal allocation matrices $A \in \mathcal{A}^* = \{A \in \mathcal{A} : \rho(\mathcal{T}(A)) = \rho_\downarrow\}$.

For the sake of simplicity, we will suppose that for all $x, y \in \mathbb{R}^2$, $L(x, y) > 0$. We can also suppose that if $T = \mathcal{T}(A)$ where $A \in \mathcal{A}^*$:

$$\forall j, k, \quad T_{jk} > 0. \tag{12}$$

Indeed, if $T_{jk} = 0$ for some k, then the j^{th} row is equal to 0. Thus, T and the sub-matrix of T obtained by removing the j^{th} row and the j^{th} column have the same spectral radius.

For $A \in cA$, we define two sets:

$$I(A) = \{i \in \{1, .., M\}, \exists a_{i,j} \in (0, 1) \text{ for some } j\},$$

$$J(A) = \{(i, j), a_{i,j} \in (0, 1)\}.$$

$I(A)$ is understood as the set of users for which two or more base stations are actively contributing to satisfy its SINR requirement. For a discrete set K, $|K|$ denotes the cardinal of K. We have the following theorem:

Theorem 3: We assume that for all integer n, for all sequences $i_1, ..., i_n$ of $\{1, ..., M\}$ and for all sequences of *distinct integers* $j_1, ..., j_n$ *of* $\{1, ..., N\}$, *we have (with* $j_{n+1} = j_1$*):*

$$\prod_{k=1}^{n} \frac{l_{i_k,j_k}}{l_{i_k,j_{k+1}}} \neq 1. \quad (13)$$

Then if $A \in \mathcal{A}^*$*:*

$$|J(A)| - |I(A)| < N. \quad (14)$$

Corollary 1: If $A \in \mathcal{A}^*$, $|I(A)| < N$.

This theorem gives an upper bound to the number of users which are really in macrodiversity, i.e. to the number of users which are receiving a signal from more than two different base stations. Provided that the assumption is satisfied, this upper bound does not depend on the geometry. This bound is also surprisingly small: on a typical wireless network, $M \gg N$, so the proportion of users in macrodiversity is small.

We denote $\tilde{\mathcal{A}} = \{A \in \mathcal{A} : \ \forall i, j \ a_{ij} \in \{0, 1\}\}$, the set of allocation matrices such that each user is affiliated to a unique base stations, the *load-balanced downlink load* is defined as:

$$\tilde{\rho}_\downarrow = \min_{A \in \tilde{\mathcal{A}}} \rho(\mathcal{T}(A)).$$

$\tilde{\rho}_\downarrow$ is the load corresponding to a network where each user is affiliated to a unique base station.

In view of Theorem 3, we may guess that $\tilde{\rho}_\downarrow / \rho_\downarrow$ is close to 1. In fact, in the special case, $N = 2$ (two base stations) we can actually show that the two minima are equal. In the §III-D, we will state that this intuition makes sense when M grows large.

Assumption (13) is not very restrictive in our context. In a probabilistic setting, it would be easily almost surely satisfied.

The proof of Theorem 3 is postponed to Appendix. It does not contain any intuition on the result. Note however that even if Theorem 3 may be surprising in view of its application, it is quite natural if $\rho_\downarrow$ is seen as the minimum of a convex function, $T \mapsto \rho(T)$, on a compact convex set, $\mathcal{A}$. With reasonable assumptions, this minimum is reached on the boundary of the set $\mathcal{A}$, that is the subset of $\tilde{\mathcal{A}}$

D. Asymptotic Load

Even for the simplest probabilistic models, the computation of $\rho_\downarrow$ is by far less easy than the computation of $\rho_\uparrow$. In this paragraph, we show however that it is possible to compute the scaling limit of $\rho_\downarrow$ when the number of users tends to infinity.

The N base stations are fixed and deployed in a bounded region $\Omega \subset \mathbb{R}^2$. We consider an ergodic sequence of users $\{X_i, h_i\}_{i \in \mathbb{Z}}$ with h_i independent of X_i, $0 < E(h_0) < \infty$, $X_i \in \Omega$ and for all measurable subset $A \subset \Omega$, $P(X_i \in A) = \int_A \lambda(x) dx$. $\lambda(x)$ is the spatial intensity (or density here) of users in Ω. As last the attenuation $L(x, y)$ is positive.

We pay attention to the load in the network when the set of users is $\{X_i, h_i^{(M)}\}_{1 \leq i \leq M}$ where $h_i^{(M)} = h_i / M$ is the scaled SINR of user i. In this paragraph, we need to explicit the dependency of the problem in M so that we define $\mathcal{A}_M = \{A = (a_{ij}) \in \mathbb{R}_+^{\mathbb{N} \times \mathbb{N}} : \text{ for } i > M \ a_{ij} = 0, \text{ for } 1 \leq i \leq M \ \sum_j a_{ij} = 1\}$, $\mathcal{A}_\infty$ is simply denoted by $\mathcal{A}$ and we introduce the linear mapping:

$$\mathcal{T}_M : \begin{cases} \mathcal{A} & \rightarrow \quad \mathbb{R}^{N \times N} \\ A & \mapsto \quad T = (\sum_i a_{ij} h_i^{(M)} \frac{l_{ik}}{l_{ij}})_{1 \leq j,k \leq N} \end{cases},$$

$\mathcal{T}_1$ is simply denoted by $\mathcal{T}$ (this is consistent with its definition in §III-B). The downlink load associated to the set of users $\{X_i, h_i^{(M)}\}_{1 \leq i \leq M}$ is by definition:

$$\rho_\downarrow^{(M)} = \min_{A \in \mathcal{A}_M} \rho(\mathcal{T}_M(A)) = \frac{1}{M} \min_{A \in \mathcal{A}_M} \rho(\mathcal{T}(A)).$$

For load balanced allocations, we define similarly, $\tilde{A}_M$ and $\tilde{\rho}_\downarrow^{(M)}$.

Lemma 4: There exists $0 < \rho_\downarrow^\infty \leq \tilde{\rho}_\downarrow^\infty$ *such that almost surely:*

$$\lim_{M \to \infty} \rho_\downarrow^{(M)} = \rho_\downarrow^\infty \ \textit{ and } \ \lim_{M \to \infty} \tilde{\rho}_\downarrow^{(M)} = \tilde{\rho}_\downarrow^\infty. \quad (15)$$

Proof: For $p < q \in \mathbb{N}$, we define $\mathcal{A}_{p,q} = \{A = (a_{ij}) \in \mathbb{R}_+^{\mathbb{N} \times \mathbb{N}} : \text{ for } i \notin \{p, \cdots, q\} \ a_{ij} = 0, \text{ for } i \in \{p, \cdots, q\} \ \sum_j a_{ij} = 1\}$, the set of allocations matrices for users indexed from p to q. Note that if $A_{1,p} \in \mathcal{A}_{1,q}$, we have $\mathcal{T}(A_{1,q}) = \mathcal{T}(A_{1,p}) + \mathcal{T}(A_{p+1,q})$, where the matrices $A_{1,p} \in \mathcal{A}_{1,p}$ and $A_{p+1,q} \in \mathcal{A}_{p+1,q}$ are obtained from $A_{1,q}$ by setting to 0 all rows not in $\{1, p\}$ and $\{p+1, q\}$ respectively. Since, $\rho(T + \tilde{T}) \leq \rho(T) + \rho(\tilde{T})$ we deduce:

$$\min_{A \in \mathcal{A}_{1,q}} \rho(\mathcal{T}(A)) \leq \min_{A \in \mathcal{A}_{1,p}} \rho(\mathcal{T}(A)) + \min_{A \in \mathcal{A}_{p+1,q}} \rho(\mathcal{T}(A)).$$

The existence of $\rho_\downarrow^\infty$ and $\tilde{\rho}_\downarrow^\infty$ follows then directly from Kingman's subadditive ergodic theorem. The positivity of $\rho_\downarrow^\infty$ is a consequence of Lemma 3. ■

Before stating the main result of this paragraph, we need a couple of definitions.

A set of measurable functions, $f_j : \mathbb{R}^2 \to \mathbb{R}$, $1 \leq j \leq N$ is said to be *singular* if there exists a measurable set A of positive Lebesgue measure and a constant C such that $f_j(x) = C f_k(x)$ for some $j \neq k$. By extension, the base stations locations is said to be singular if the set of attenuation functions $x \mapsto L(x, Y_j)$ is singular. This notion of singularity is purely technical and it is not a strong assumption in view of applications.

A tessellation is a collection of measurable sets partitioning the region Ω, we denote by $\mathcal{V} = \{V = (V_j)_{1 \leq j \leq N} : \text{ almost everywhere } \sum_j \mathbb{1}_{V_j}(x) = 1\}$ the set of tessellation composed of N sets. We identify two tessellations V and V' in $\mathcal{V}$ if for all j, $\mathbb{1}_{V_j}$ and $\mathbb{1}_{V'_j}$ are almost everywhere (a.e.) equal.

Theorem 4: If the base stations locations are non singular then

$$\rho_\downarrow^\infty = \tilde{\rho}_\downarrow^\infty = E(h_0) \gamma,$$

where

$$\gamma = \min_{V \in \mathcal{V}} \rho(\mathcal{T}'(V))$$

and $\mathcal{T}'(V)_{j,k} = \int_{V_j} \frac{L(x, Y_k)}{L(x, Y_j)} \lambda(x) dx$.

This theorem sheds a new light on the downlink load when the number of users is large. First, it strengthens the intuition that macrodiversity and load balancing lead to the same level of load in the network. Secondly, we have been able to compute explicitly the limit behavior of the asymptotic behavior of the network. As an example, a practical consequence is the following approximation for a set of M users located at (X_i) with SINR requirement (h_i), from Equation (15) we get:

$$\rho_\downarrow \sim \gamma \sum_{j=1}^{M} h_i,$$

We have completely decoupled the SINR requirement and the geometry of the network which is contained in the scalar γ.

There is a third consequence of Theorem 4. Define $V^* = (V_j^*)_j$ as the optimal tessellation (defined up to null measure sets) such that:

$$\rho(\mathcal{T}'(V^*)) = \gamma.$$

We consider a traditional cellular network architecture with associated cells V_j^* with M users (X_i) distributed according to $\lambda(x)dx$ with SINR requirement $(h_i^{(M)})$. The user i is attached to base station j if $X_i \in V_j^*$: that is the associated allocation matrix A_M^* satisfies: for $i \leq M$, $a^*_{ij\,M} = \mathbb{1}(X_i \in V_j)$. The *cellular downlink load* is equal to $\overline{\rho}_\downarrow^{(M)} = \rho(\mathcal{T}_M(A_M^*))$. As the number of users M grows large, from the law of large number, $\overline{\rho}_\downarrow^{(M)}$ tends to $E(h_0)\gamma$. Therefore an optimal cellular architecture has asymptotically the same load than a network in macrodiversity.

The proof of Theorem 4 is postponed to Appendix.

Example 1: Hexagonal Grid.

$\Omega = [0,1]^2$ is seen as a torus to avoid boundary effects, and the users are uniformly distributed on Ω. We suppose that the set of base stations is located on a regular hexagonal grid of radius $R = 1/L$, with L integer. We index our base station with two indices in $\{0, \cdots, L-1\}$ and with a complex representation of $\mathbb{R}^2$, the base station (p,q) is located at $Y_{p,q} = R(p + qe^{i\frac{\pi}{3}})$. Let $\{V_j\}$ be the Voronoi Tessellation of the hexagonal network (that is, $x \in V_j$ if for all $j' \neq j$, $|x - Y_j| < |x - Y_{j'}|$). If $L(x,y) = l(|x-y|)$, then the symmetry of the network leads to

$$\gamma = \int_{V_{0,0}} \frac{I(x)}{l(|x|)} dx,$$

where $I(x) = \sum_j l(|x - Y_j|)$. This last equation has an intuitive meaning: in a symmetric network, the optimal cellular architecture is obtained by equalizing the individual load of each base station.

E. Infinite Networks

In the previous paragraph, we have computed the downlink load as the number of users grows large and the number of base stations is fixed. As on the uplink, it is an appealing idea to compute $\rho_\downarrow$ for infinite networks, that is when both the numbers of users and base stations are infinite. The power control problem is still given by the set of inequalities (8) and Lemma 2 remains obviously true. Thus, we can still follow the line of [4]. We can still define $\mathcal{A}$ and the linear mapping $\mathcal{T}$. Proposition 1 has an infinite dimensional analogue.

First, we recall some results on infinite recurrent matrices. Let us denote by $T^n = (T^n_{jk})$, the n^{th} power of T. The power series $T_{jk}(z) = \sum_n T^n_{jk} z^n$ have a common convergence radius $R(T) = \frac{1}{\rho(T)}$; $\rho(T)$ is by definition the spectral radius of T. $T_{jj}(R)$ is finite or infinite at the same time for all j, making T respectively *transient* or *recurrent*. For more refer to Seneta [13]. As a consequence of Propositions 3.1 to 3.3 of [4].

Proposition 2: Let,

$$\rho_\downarrow = \min_{A \in \mathcal{A}} \rho(\mathcal{T}(A)),$$

- *if $\rho_\downarrow < 1$ then Equation (8) has a solution ,*
- *if $\rho_\downarrow > 1$ then Equation (8) does not admit any solution,*
- *if $\rho_\downarrow = 1$ and $\rho_\downarrow = \rho(\mathcal{T}(A^*))$, Equation (8) has a solution if $\mathcal{T}(A^*)$ is transient.*

We model base stations and users by considering two point processes on $\mathbb{R}^2$: $\Pi_s = \{Y_j\}_j$ and $\Pi_u = \{(X_i, h_i, \eta_i)\}_i$, h_i and η_i are the marks of the point process. The marks are supposed identically distributed, independent and independent of the rest of the model. We suppose that the point process of users Π_u is a stationary Poisson process of intensity $\lambda_u > 0$. At last, we consider a radial attenuation function, that is: $L(x,y) = l(|x-y|)$. As usual, we can suppose $l(r) > 0$ for all $r \in \mathbb{R}^+$.

We have the following negative result:

Theorem 5: For $t \in \mathbb{R}$, let $l_t : r \mapsto l(\max(r-t, 0))$, we denote by $\|\cdot\|_\infty$ the uniform norm. If:

$$\lim_{t \to 0} \|\frac{l_t}{l}\|_\infty = 1, \tag{16}$$

then:

$$\rho_\downarrow = +\infty, \quad \textit{almost surely}.$$

Assumption (16) is used to get a continuity of the entries of $\mathcal{T}(A)$ with respect to the users' locations $\{X_i\}_{i\in\mathbb{N}}$. However, the theorem should be true for a larger class of attenuation functions.

This result asserts that whatever the intensity of base stations is, there is no solution of the power allocation problem. It implies that some admission congestion protocol must be enforced in a CDMA network on the downlink. Otherwise, as the proof of Theorem 5 shows, there will always be a local concentration of users which saturates the whole network. If we compare to Theorem 2, this result is in complete opposition with what happens on the uplink. Theorem 5 is somewhat disappointing, the stationary point process for users' location framework does not lead to a right concept of spatial load.

The proof of Theorem 5 relies on classical results on spectral radius (see [13] for details).

Lemma 5: Let T and S be non-negative matrices (possibly infinite), then:

- *if $\forall j,k$ $T_{jk} \geq S_{jk}$, then $\rho(T) \geq \rho(S)$,*
- *for all square sub-matrix $\tilde{T}$ of T, $\rho(T) \geq \rho(\tilde{T})$.*

Proof of Theorem 5. Without loss of generality we can suppose that $h_i > 0$, indeed $\sum_i \mathbb{1}(h_i > 0)\delta_{X_i,h_i,\eta_i}$ is still a poisson point process with independent marks. Let R, h be some positive real numbers and M an integer. The event $A_i = \{\Pi_u(B(X_i,R)) \geq M\} \cap \{\forall X_k \in B(X_i,R), \quad h_k > h\}$ has a positive probability, provided h small enough. Hence using the independency property of Poisson processes, $\sum_i \mathbb{1}_{A_i} = \infty$ almost surely. We consider one of these configurations.

Without loss of generality, we can also suppose $i = 1$ and $X_1 = 0$: $\forall k \in \{1...M\}$, $X_k \in B(0,R)$ and $h_k > h$.

Fix $1 > \epsilon > 0$ from Hypothesis (16), for ϵ small enough, there exists R such that:

$$\forall x \in B(0,R), \forall y \in \mathbb{R}^2, \quad |l(|x-y|) - l(|y|)| \leq l(|y|)\epsilon.$$

Hence, for all $X_i \in B(0,R)$ we easily check:

$$\Big| \frac{L(0,Y_k)}{L(0,Y_j)} - \frac{L(X_i,Y_k)}{L(X_i,Y_j)} \Big| \leq \frac{\epsilon}{1-\epsilon} \frac{L(0,Y_k)}{L(0,Y_j)}. \tag{17}$$

Let $T = \mathcal{T}(A)$, we have

$$T_{jk} \geq \tilde{T}_{jk} = h \sum_{i=1}^{M} a_{ij} \frac{L(X_i,Y_k)}{L(X_i,Y_j)},$$

and, by lemma 5, $\rho(T) \geq \rho(\tilde{T})$.

Now, if $\tilde{T}^{(N)}$ denotes the sub-matrix of $\tilde{T}$ extracted from the first N rows and N columns, from (17), we deduce:

$$\tilde{T}^{(N)}_{jk} \geq h(1 - \frac{\epsilon}{1-\epsilon}) \frac{L(0,Y_k)}{L(0,Y_j)} \sum_{i=1}^{M} a_{ij}. \tag{18}$$

Moreover, there exists N such that $\sum_{j=1}^{N} \sum_{i=1}^{M} a_{ij} \geq M(1-\epsilon)$. For such N, define, the $N \times N$ matrix, $\hat{T}^{(N)}$, with $\hat{T}^{(N)}_{jk}$ is equal to the right hand side of (18). From lemma 5, $\rho(\tilde{T}) \geq \rho(\tilde{T}^{(N)}) \geq \rho(\hat{T}^{(N)})$. Computing the spectral radius of $\hat{T}^{(N)}$, we obtain:

$$\rho(T) \geq \rho(\hat{T}^{(N)}) \geq hM(1-2\epsilon).$$

We thus have proved that $\rho(T)$ cannot be upper bounded.

APPENDIX I
PROOF OF THEOREM 2

The following lemma on shot noise processes is needed in the proof. In what follows, $|\cdot|$ is the Euclidean norm and $B(x,R)$ is the closed ball of center x and radius R.

Lemma 6: Let $\Pi = \{(X_i, Z_i)\}_i$ be a stationary marked point process on $\mathbb{R}^d \times \mathbb{R}_+$. We suppose Π has a finite intensity λ and $E^0(Z_0) < \infty$. Let $\alpha < 1$ and $x \mapsto l(x)$ a non-negative function on $\mathbb{R}$. If $x \mapsto x^{d-1}l(x)$ is integrable and $x \mapsto x^{d-1}l(x)$ is non-increasing on a neighborhood of $+\infty$, or if there exists $\epsilon > 0$ such that $x \mapsto x^{d-1+\epsilon}l(x)$ is integrable. Then, almost surely:

$$\liminf_{R\to+\infty} \sup_{x\in B(0,\alpha R)} \sum_{X_i \notin B(0,R)} Z_i l(|x - X_i|) = 0.$$

Proof: Suppose for example, $x \mapsto x^{d-1}l(x)$ is non-increasing on a neighborhood of $+\infty$ For n integer, let $C_n(R) = \{x \in \mathbb{R}^d : x \in B(0,(n+1)R) \backslash B(0,nR)\}$. We can write for all $x \in B(0,\alpha R)$:

$$\sum_{X_i \notin B(0,R)} Z_i l(|x-X_i|) \leq \sum_{n=1}^{\infty} l((n-\alpha)R) \sum_{X_i} Z_i \mathbb{1}_{X_i \in C_n(R)}.$$

If π_d denote the d-dimensional Lebesgue measure of the unit ball, from Campbell formuli, we deduce:

$$\begin{aligned}
&E \sup_{x\in B(0,\alpha R)} \sum_{X_i \notin B(0,R)} Z_i l(|x - X_i|) \\
&\leq \lambda \sum_{n=1}^{\infty} l((n-\alpha)R) \int_{\mathbb{R}^d} \int_0^{+\infty} z \mathbb{1}_{x\in C_n(R)} P^0(dz)dx \\
&\leq \lambda \sum_{n=1}^{\infty} l((n-\alpha)R) E^0(Z_0) \pi_d R^d ((n+1)^d - n^d) \\
&\leq \lambda C R E^0(Z_0) \sum_{n=1}^{\infty} l((n-\alpha)R) R^{d-1} n^{d-1},
\end{aligned}$$

where C is a constant depending on the dimension d only. From the hypothesis on $x \mapsto x^{d-1}l(x)$, we can apply the dominated convergence theorem to conclude:

$$\lim_{R\to+\infty} E \sup_{x\in B(0,\alpha R)} \sum_{X_i \notin B(0,R)} Z_i l(|x - X_i|) = 0.$$

In order to get the result in almost sure convergence, it suffices to recall that from any sequence converging in L^1, we can extract a sequence converging almost surely. We thus obtain the stated result. The case $x \mapsto x^{d-1+\epsilon}l(x)$ in $L^1(\mathbb{R})$ is similar. ■

The next lemma will be used to build a stationary solution. The proof is straightforward.

Lemma 7: With the hypothesis of Theorem 2, the mapping G as it is defined in §II-B is continuous on $G^{-1}(\mathbb{R}_*^{+\mathbb{N}})$ for the L^∞-norm: $\|S\| = \sup_{i\in\mathbb{N}} |S_i|$.

Proof of theorem 2. The idea is to follow the proof of Hanly in the finite case and use ergodicity and the uniform bound given by Lemma 6 to extend to infinite case.

Case $\rho_\uparrow > 1$.

Suppose that there exists a solution of (5) with a positive probability. From Proposition 1, this solution exists almost surely, we denote the solution by $S = (S_i)$. We have component-wise $G(S) \leq S$. Let $\underline{0} = (0)_{i\in\mathbb{N}}$, notice that almost surely for all i, $G(\underline{0})_i > 0$. The function G is monotonous component-wise: if $S \leq S'$ then $G(S) \leq G(S')$. We deduce that $G(\underline{0}) \leq G(S) \leq S$ and for all i, $G^n(\underline{0})_i$ is an increasing sequence and is upper bounded by S_i. This sequence converges toward S^*_i, which by continuity (Lemma 7) satisfies $S^* = G(S^*)$. Since G is invariant under a translation, we can define a solution (S^*_i) as a mark on Π_u. For the sake of simplicity, we drop the $''*''$ exponent in S^* and suppose directly $G(S) = S$, $S_i > 0$.

We consider the thinned point process: $\Pi_{u,t} = \sum_i \mathbb{1}_{S_i<t} \delta_{\{X_i,h_i,S_i\}}$, this marked point process is still stationary and ergodic. Let $\lambda_{u,t}$ be its intensity. The Palm probability of $\Pi_{u,t}$ is $P^0_{u,t}(\cdot) = P^0_u(\cdot | S_0 \in [0,t))$, (see Baccelli and Brémaud [14]). Let $\alpha < 1$, and, to simplify notations, let $N_R = \Pi_s(B(0,R))$, $M_R = \Pi_u(B(0,R))$ and $M_{t,R} = \Pi_{u,t}(B(0,R))$. Now, from the ergodicity of our model, almost surely:

$$\lim_{R\to+\infty} \frac{1}{N_R} \sum_{i=1}^{M_{\alpha R}} h_i = \rho_\uparrow, \text{ and } \lim_{R\to+\infty} \frac{M_{t,\alpha R}}{N_R} = \frac{\lambda_{u,t}}{\lambda_s} \alpha^2. \tag{19}$$

Let $Z_j = \eta_j^{-1}$, now, from Lemma 6, almost surely:

$$\liminf_{R\to+\infty} \sup_{X_i \in B(0,\alpha R)} \sum_{Y_j \notin B(0,R)} Z_j l(|X_i - Y_j|) = 0. \quad (20)$$

The integrability of $E_u^0(h_0)$ implies that $\lim_{t\to+\infty} E_u^0(h_0 \mathbb{1}(h_0 \geq t)) = 0$. This last limit implies thanks to ergodicity and an exchange of limit (justified by Fubini's Theorem):

$$\lim_{t\to\infty} \lim_{R\to\infty} \frac{1}{N_R} \sum_{i=1}^{M_{\alpha R}} h_i \mathbb{1}(h_i \geq t) = 0. \quad (21)$$

Then we do the following decomposition:

$$\frac{1}{N_R} \sum_{i=1}^{M_{\alpha R}} h_i = \frac{1}{N_R} \sum_{X_i \in \Pi_u^t \cap B(0,\alpha R)} h_i + \frac{1}{N_R} \sum_{i=1}^{M_{\alpha R}} h_i \mathbb{1}(h_i \geq t). \quad (22)$$

The first term of the right hand side of Equation (22), say A, is upper bounded by:

$$\begin{aligned} A &= \frac{1}{N_R} \sum_{X_i \in \Pi_u^t \cap B(0,\alpha R)} \sum_{j=1}^{\infty} \frac{S_i L(X_i, Y_j)}{\eta_j + \sum_{m=1}^{\infty} S_m L(X_m, Y_j)} \\ &\leq \frac{1}{N_R} \sum_{X_i \in \Pi_u^t \cap B(0,\alpha R)} \sum_{j=1}^{N_R} \frac{S_i L(X_i, Y_j)}{\eta_j + \sum_{m=1}^{\infty} S_m L(X_m, Y_j)} \\ &\quad + \frac{1}{N_R} \sum_{X_i \in \Pi_u^t \cap B(0,\alpha R)} \sum_{Y_j \notin B(0,R)} t Z_j l(|X_i - Y_j|) \\ &\leq 1 + t \frac{M_{\alpha R}^t}{N_R} \sup_{X_i \in B(0,\alpha R)} \sum_{Y_j \notin B(0,R)} Z_j l(|X_i - Y_j|). \end{aligned}$$

We can compute the $\liminf_{R\to\infty}$ of Equation (22) on both side and then let t tends to infinity. From Equation (19), the left hand side of the previous inequality converges to $\rho_\uparrow$ whereas from Equations ((19), (20) and (21), the right hand side is bounded by 1 (by letting t tends to infinity). Thus $\rho_\uparrow \leq 1$ is a necessary condition of the feasibility of the power allocation problem.

Case $\rho_\uparrow < 1$ and $h_i < h$ for all i.

The central argument of Hanly is a change of variables and an application of Brouwer's fixed point theorem (see Goebel and Kirk [15]). Hanly defines:

$$g : \begin{cases} \bigotimes_{i\in\mathbb{N}} (h_i, +\infty] & \to \quad \mathbb{R}^{+\mathbb{N}} \\ (t_i)_{i\in\mathbb{N}} & \mapsto \quad (\frac{h_i}{t_i - h_i})_i \end{cases},$$

and

$$f_i : \begin{cases} \mathbb{R}^{+\mathbb{N}} & \to \quad \mathbb{R}^+ \\ (S_m)_{m\in\mathbb{N}} & \mapsto \quad \sum_{j=1}^{\infty} \frac{(S_i+1) L(X_i,Y_j)}{\eta_j + \sum_{m=1}^{\infty} S_m L(X_m,Y_j)} \end{cases}.$$

Let $\epsilon > 0$ and define:

$$\phi^\epsilon : \begin{cases} \bigotimes_{i\in\mathbb{N}} [h_i(1+\epsilon), \frac{1}{\epsilon}] & \to \quad \bigotimes_{i\in\mathbb{N}} [h_i(1+\epsilon), \frac{1}{\epsilon}] \\ (t_i)_{i\in\mathbb{N}} & \mapsto \quad (\Phi_i^\epsilon(t))_{i\in\mathbb{N}} \end{cases},$$

where:

$$\Phi_i^\epsilon(t) = \begin{cases} f_i \circ g_i(t_i) \quad if & f_i \circ g(t) \in [h_i(1+\epsilon), \frac{1}{\epsilon}] \\ h_i(1+\epsilon) & f_i \circ g(t) < h_i(1+\epsilon) \\ \frac{1}{\epsilon} & f_i \circ g(t) > \frac{1}{\epsilon} \end{cases}$$

From Assumption $E_s^0(\eta_0^{-1}) < +\infty$ and Lemma 6, it is easy to see that f_i is continuous on $\bigotimes_{i\in\mathbb{N}} [\frac{h_i}{\epsilon^{-1} - h_i}, \frac{1}{\epsilon}]$ for the L^∞-norm. Thus, ϕ^ϵ is a continuous map. $\bigotimes_{i\in\mathbb{N}} [h_i(1+\epsilon), \frac{1}{\epsilon}]$ is a compact convex set and hence by Brouwer's fixed point theorem: there exists t^ϵ such that $\phi^\epsilon(t^\epsilon) = t^\epsilon$. We will first show that we can extract a converging sequence from t^ϵ.

We consider the thinned point process: $\Pi_s^{q,w} = \sum_j \mathbb{1}_{\eta_j > w} \mathbb{1}_{\sum_i l(|X_i - Y_j|) < q} \delta_{\{Y_j, \eta_j\}}$, this point process is still stationary and ergodic. Let $\lambda_s^{q,w}$ be its intensity. Since, $\sum_j l(|X_i - Y_j|)$ is almost surely finite for all j and $\eta_j > w$, for q large and w small, we still have:

$$\frac{\lambda_u}{\lambda_s^{q,w}} E_u^0(h_0) > 1,$$

thus we can suppose directly that $\sum_i l(|X_i - Y_j|) < q$ and $\eta_j > w$ for all j.

Let $a > h$ large enough to guarantee: $\frac{h}{a-h} \frac{q}{w} < a$ and suppose $t_i^\epsilon \geq a$. Then $S_i^\epsilon = (g(t^\epsilon))_i \leq \frac{h_i}{a - h_i}$. Hence $a \leq t_i^\epsilon \leq f_i(S^\epsilon) \leq \frac{h_i}{a_i - h_i} \sum_j \frac{L(X_i,Y_j)}{\eta_j} < a$. Thus, we have proved: for all i, $t_i^\epsilon \in [h_i, a]$. We thus can extract a sequence t^ϵ converging toward $t \in \bigotimes_{i\in\mathbb{N}} [h_i, a]$. We now want to show that $\lim_{\epsilon\to 0} g(t^\epsilon)$ exists. To do so, we prove that for all i, there exists ϵ_i such that for all $\epsilon < \epsilon_i$, t_i^ϵ satisfies: $t_i^\epsilon > h_i(1+\epsilon_i)$.

Suppose that for some i, for all $\eta > 0$, there exists $\epsilon < \eta$ such that: $t_i^\epsilon = h_i(1+\epsilon)$. We consider a sequence of such ϵ. Let $S_m^\epsilon = (g(t^\epsilon))_m$ and $I_j^\epsilon = \sum_m S_m^\epsilon L(X_m, Y_j)$, the interference at base station j. We have $I_j^\epsilon \geq \epsilon^{-1} L(X_i, Y_j)$, thus for all j: $\lim_{\epsilon\to 0} I_j^\epsilon = +\infty$. Since $t_k^\epsilon = \max(\sum_j \frac{(S_k^\epsilon + 1) L(X_k, Y_j)}{\eta_j + I_j^\epsilon}, h_k(1+\epsilon))$, by a dominated convergence argument we deduce that S_k^ϵ cannot be bounded, hence for all k:

$$\lim_{\epsilon\to 0} t_k^\epsilon = h_k.$$

Since $\rho_\uparrow < 1$, there exists $\alpha > 1$ such that:

$$\frac{\lambda_u \alpha^2}{\lambda_s} E_u^0(h_0) < 1.$$

Thus, ergodicity implies:

$$\lim_{\epsilon\to 0} \lim_R \frac{1}{N_R} \sum_{i=1}^{M_{\alpha R}} t_i^\epsilon = \frac{\lambda_u \alpha^2}{\lambda_s} E_u^0(h_0) < 1. \quad (23)$$

Since t^ϵ is a fixed point, we have for $\epsilon < a^{-1}$:

$$\frac{1}{N_R} \sum_{i=1}^{M_{\alpha R}} t_i^\epsilon = \frac{1}{N_R} \sum_{i=1}^{M_{\alpha R}} \phi_i^\epsilon(t^\epsilon) \geq \frac{1}{N_R} \sum_{i=1}^{M_{\alpha R}} f_i \circ g(t^\epsilon), \quad (24)$$

We write:

$$\begin{aligned}
&\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i\circ g(t^\epsilon)\\
&\geq \frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}}\sum_{j=1}^{N_R}\frac{L(X_i,Y_j)(S_i^\epsilon+1)}{\eta_j+I_j^\epsilon}\\
&\geq \frac{1}{N_R}\sum_{j=1}^{N_R}\frac{I_j^\epsilon+\sum_i L(X_i,Y_j)}{\eta_j+I_j^\epsilon}\\
&\quad -\frac{1}{N_R}\sum_{j=1}^{N_R}\sum_{X_i\notin B(0,\alpha R)}\frac{L(X_i,Y_j)(\epsilon^{-1}+1)}{w}\\
&\geq \frac{1}{N_R}\sum_{j=1}^{N_R}\frac{I_j^\epsilon+\sum_i L(X_i,Y_j)}{\eta_j+I_j^\epsilon}\\
&\quad -\sup_{y\in B(0,R)}\sum_{X_i\notin B(0,\alpha R)}\frac{L(X_i,y)(\epsilon^{-1}+1)}{w}.
\end{aligned}$$

Now, by letting R tends toward infinity, using Lemma 6, we obtain:

$$\liminf_{R\to+\infty}\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i\circ g(t^\epsilon)\geq \lim_R\frac{1}{N_R}\sum_{j=1}^{N_R}\frac{I_j^\epsilon+\sum_i L(X_i,Y_j)}{\eta_j+I_j^\epsilon},$$

We can apply the ergodic theorem for point processes (see [9]):

$$\liminf_{R\to+\infty}\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i\circ g(t^\epsilon)\geq E_s^0(\frac{I_0^\epsilon}{\eta_0+I_0^\epsilon})+E_s^0(\frac{\sum_i L(X_i,0)}{\eta_0+I_0^\epsilon}),$$

letting ϵ tends toward 0 and using the dominated convergence theorem, we conclude that:

$$\lim_{\epsilon\to0}\liminf_{R\to+\infty}\frac{1}{N_R}\sum_{i=1}^{M_{\alpha R}} f_i\circ g(t^\epsilon)\geq 1.$$

This last inequality together with (24) contradicts (23). Thus we cannot have $t_i^\epsilon=h_i(1+\epsilon)$ an infinite number of times. We have proved that for $\epsilon<\epsilon_i$, $t_i^\epsilon>h_i(1+\epsilon_i)$. Since $g_i(t)=\frac{h_i}{t-h_i}$ is a continuous map on $[h_i(1+\epsilon_i),a]$, we can define: $S_i^*=g_i(t_i)=\lim_{\epsilon\to0}g_i(t_i^\epsilon)$. From the continuity of f_i:

$$f_i(S^*)=h_i\frac{S_i^*+1}{S_i^*},$$

which is equivalent to:

$$h_i=\sum_{j=1}^{\infty}\frac{S_i^*L(X_i,Y_j)}{\eta_j+\sum_{m=1}^\infty S_m^*L(X_m,Y_j)}.$$

This concludes the proof of the theorem when $h_i<h$ for all i.

Case $\rho_\uparrow<1$, general case.

Let $h>0$, we consider a new user point process: $\Pi_u'=\sum_i\lceil\frac{h_i}{h}\rceil\delta_{\{X_i,h\lceil\frac{h_i}{h}\rceil^{-1}\}}$. Since, by hypothesis, the marked point process $\{(X_i,h_i)\}$ is ergodic, Π_u' is a stationary ergodic marked point process, its marks: $h\lceil\frac{h_i}{h}\rceil^{-1}$ are upper bounded by h. Moreover, if we find a power allocation satisfying (5) for Π_u', by additivity of (5), we have found a solution of (5) for Π_u. A direct computation shows that $\lambda_u'\leq\lambda_u(\frac{E_u^0(h_0)}{h}+P_u^0(h_0\geq h))$. Hence for h large enough, $\frac{\lambda_u'}{\lambda_s}E_{u'}^0(h_0)<1$. This conclude the proof in the general case.

Appendix II
Proof of Theorem 3

In the following, $\|\cdot\|$ is any given norm on $\mathbb{R}^{N\times N}$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^N$. I is the identity matrix in $\mathbb{R}^{N\times N}$. Two lemmas are necessary before turning to the proof. The first lemma is simply an expansion of order 1 of $T\mapsto det(xI-T)$ in the neighborhood of T.

Lemma 8: Let $\Phi_T(x)$ be the characteristic polynomial of T and $Adj(T)$ its adjoint; for all $H\in\mathbb{R}^{N\times N}$ we have:

$$\Phi_{T+H}(x)=\Phi_T(x)+\sum_{j,k}H_{jk}Adj(xI-T)_{jk}+o(\|H\|). \quad (25)$$

For $T\in\mathcal{T}(\mathcal{A})$, we define: $\mathcal{H}_T=\{H\in\mathbb{R}^{N\times N}:T+H\in\mathcal{T}(\mathcal{A})\}$.

Lemma 9: If $T\in\mathcal{T}(\mathcal{A}^*)$ then on a neighborhood $\mathcal{V}_0$ of the null matrix:

$$\forall H\in\mathcal{H}_T\cap\mathcal{V}_0,\quad \langle Hv_T,w_T\rangle\leq0, \quad (26)$$

where, v_T and w_T are respectively the left and right eigenvectors of T associated to eigenvalue $\rho(T)$.

Proof: From Equation (12), T is primitive, hence (from Seneta [13]): $Adj(\rho(T)I-T)=\Phi_T'(\rho(T))w_Tv_T'$ and $\Phi_T'(\rho(T))>0$. For $x=\rho(T)$, Equation (25) reduces to:

$$\Phi_{T+H}(\rho(T))=\Phi_T'(\rho(T))\langle Hv_T,w_T\rangle+o(\|H\|). \quad (27)$$

If $T\in\mathcal{T}(\mathcal{A}^*)$, then $\rho(T+H)\geq\rho(T)$ for all $H\in\mathcal{H}^T$. This implies $\Phi_{T+H}(\rho(T))\leq0$ for H sufficiently small. (26) follows from (27) and $\Phi_T'(\rho(T))>0$. ■

We can now prove Theorem 3.

Proof of Theorem 3.

Let $A=(a_{ij})\in\mathcal{A}^*$ and $T=\mathcal{T}(A)$. w and v are the right and left eigenvectors of T associated to $\rho(T)$. For each $i_0\in I$, we can find $j_1\neq j_2$ such that $a_{i_0,j_1}>0$ and $a_{i_0,j_1}>0$, we define the matrice A^ϵ by:

$(A^\epsilon)_{ij}=A_{ij}+\epsilon\delta_{i,i_0}\delta_{j,j_1}-\epsilon\delta_{i,i_0}\delta_{j,j_2}$ (δ is the Kronecker symbol).

For $\epsilon>0$ small enough A^ϵ and $A^{-\epsilon}$ are in $\mathcal{A}$, hence $H=\mathcal{T}(A^\epsilon)-T$ and $-H=\mathcal{T}(A^{-\epsilon})-T$ are both in $\mathcal{H}_T$. We can apply Lemma 9 and it follows:

$$\begin{aligned}
0 &= \langle Hv,w\rangle\\
&= (\sum_k l_{i_0k}v_k)(\frac{w_{j_1}}{l_{i_0j_1}}-\frac{w_{j_2}}{l_{i_0j_2}})
\end{aligned}$$

The last equality implies, since $l_{i_0k}>0$ and $v_k>0$:

$$\frac{w_{j_1}}{l_{i_0j_1}}=\frac{w_{j_2}}{l_{i_0j_2}} \quad (28)$$

The end of the proof relies on a simple argument on graphs. Let $I=I(A)$ and $J=J(A)$, without loss of generality, we can suppose $I=\{1,...,|I|\}$. Let $J_i=\{j,(i,j)\in J\}$.

We now define the embedded non-oriented graphs $\mathcal{G}_i$ on the set $\{1,..,N\}$ of base stations. We put an edge in $\mathcal{G}_i$ between j_1,j_2 if there exists an integer $i_0\leq i$ such that j_1 and j_2 are in J_{i_0}. From what precedes, this implies (28).

Similarly we define the graph $\mathcal{J}_i$ by putting an edge between j_1 and j_2 if j_1 and j_2 are in J_i. By construction, we have $\cup_{i=1}^l\mathcal{J}_i=\mathcal{G}_l$.

We now remark that Assumption (13) together with Equation (28) implies that if there is a path leading from j_1 to j_2 in $\mathcal{G}_i$, there cannot be any edge between j_1 and j_2 in $\mathcal{J}_{i+1}$. In other words, a set of connected nodes in $\mathcal{G}_i$ and a set of connected nodes in $\mathcal{J}_{i+1}$ cannot have more than one common node.

Let N_i be the number of non-isolated nodes in $\mathcal{G}_i$ and $n_c(i)$ be the number of connected components in $\mathcal{G}_i$ not reduced to an isolated node. We obtain:

$$N_1 = |J_1|.$$

The constraint on our embedded graphs implies that adding the edges of $\mathcal{J}_{i+1}$ to $\mathcal{G}_i$ can either merge two distinct connected components of $\mathcal{G}_i$, increase a connected component or add a new connected component. In these three possible cases, the following formula is satisfied:

$$N_{i+1} = N_i + |J_{i+1}| + n_c(i+1) - n_c(i) - 1,$$

at last, by summing this last equation from 1 to $|I|-1$, we obtain

$$|J| - |I| \leq N - n_c(|I|),$$

which in turn implies Equation (14). Since $|J_i| \geq 2$, $|J| \geq 2|I|$ and the corollary follows.

Appendix III
Proof of Theorem 4

Let $V = (V_j)_j$ a tessellation in $\mathcal{V}$ and $A^{(M)} \in \mathcal{A}_M$ the allocation matrix corresponding to the cellular network with cells $(V_j)_j$: for $i \leq M$, $a_{ij}{}^{(M)} = \mathbb{1}(X_i \in V_i)$. By ergodicity, for all j,k a.s. we have:

$$\lim_{M\to\infty} \mathcal{T}(A^{(M)})_{jk} = E(h_0) \int_{V_j} \frac{L(x,Y_k)}{L(x,Y_j)} \lambda(x)dx.$$

The spectral radius is a continuous function of the entries of the matrix. Hence, taking the infimum over $\mathcal{V}$, we thus deduce:

$$\rho_{\downarrow}^{\infty} \leq \tilde{\rho}_{\downarrow}^{\infty} \leq E(h_0)\gamma.$$

It remains to prove that $E(h_0)\gamma \leq \rho_{\downarrow}^{\infty}$. To this end, we define the following set of measurable functions:

$$\mathcal{F} = \{f = (f_j)_{1\leq j\leq N} : f_j : \Omega \to \mathbb{R}_+, \text{ a.e. } \sum_j f_j(x) = 1\}.$$

$\mathcal{F}$ is the convex hull of the set of tessellations.

Let $A^{(M)} = (a_{ij}^{(M)})$ be a sequence of allocation matrices such that $\rho_{\downarrow}^{(M)} = \rho(\mathcal{T}(A^{(M)}))/M$. We define the empirical allocation measure $\mu_j^{(M)}$ as,

$$\mu_j^{(M)} = \frac{1}{M}\sum_{i=1}^{M} a_{ij}^{(M)} \delta_{X_i}.$$

For each j, the sequence $\{\mu_j^{(M)}\}_M$ is tight, so that we may extract a converging subsequence to a limit measure μ_j (for the weak convergence of measures). Notice that:

$$\sum_{j=1}^{N} \mu_j^{(M)} = \frac{1}{M}\sum_{i=1}^{M} \delta_{X_i},$$

letting M tends to infinity, we get:

$$\sum_{j=1}^{N} \mu_j = \ell_\lambda,$$

with $\ell_\lambda(A) = \int_A \lambda(x)dx$. In particular μ_j is absolutely continuous with respect to ℓ_λ. Let f_j^* be the Radon-Nikodym derivative of μ_j with respect to ℓ_λ. $\int_\Omega \lambda(x)dx = 1$ implies that $f^* = (f_j^*) \in \mathcal{F}$. If $h(x) = \sum_i \mathbb{1}(x = X_i)h_i$, the entry (j,k) of the matrix $\mathcal{T}(A^{(M)})/M$ is equal to:

$$\int h(x)\frac{L(x,Y_k)}{L(x,Y_j)}\mu^{(M)}(dx)$$

The spectral radius is a continuous function of the entries of the matrix (remember that the size of $\mathcal{T}(A^{(M)})$ is fixed to $N \times N$, so no continuity problem may occur). We obtain:

$$\rho_{\downarrow}^{\infty} = E(h_0)\rho(\mathcal{T}'(f^*)).$$

where

$$\mathcal{T}'(f)_{j,k} = \int \frac{L(x,Y_k)}{L(x,Y_j)} f_j(x)\lambda(x)dx.$$

(Assume first that h_i takes a finite number of distinct values and then extend to the general case).

It remains to prove that $\rho(\mathcal{T}'(f^*)) = \gamma$. First note that by definition of $\rho_{\downarrow}^{\infty}$:

$$\rho(\mathcal{T}'(f^*)) = \min_{f\in\mathcal{F}} \rho(\mathcal{T}'(f)). \tag{29}$$

So that $\rho(\mathcal{T}'(f^*))$ is the minimum of a convex function over a compact convex set. The last step is the following Lemma:

Lemma 10: If the base stations locations are not singular then

$$\gamma = \min_{V\in\mathcal{V}} \rho(\mathcal{T}'(V)) = \min_{f\in\mathcal{F}} \rho(\mathcal{T}'(f))$$

This lemma is a continuous analog of Theorem 3.

Proof: We consider the $f^* \in \mathcal{F}$ given by Equation (29). Let $E = f_1^*(]0,1[)^{-1} \cap f_2^*(]0,1[)^{-1}$. In this proof, ℓ will denote the Lebesgue measure. We need to show that $\ell(E) = 0$. Suppose instead that $\ell(E) > 0$, we can suppose without loss of generality that $\ell(E) < +\infty$. For ϵ_0 small enough, there exists $E' \subset E$ with $\ell(E') > 0$ such that for all $x \in E'$, $\min(f_1(x), f_2(x)) > \epsilon$ and $\max(f_1(x), f_2(x)) < 1-\epsilon$. Let $A \subset E'$ and let $f_1^\epsilon(x) = f_1(x) + \epsilon\mathbb{1}_A(x)$, $f_2^\epsilon(x) = f_2(x) - \epsilon\mathbb{1}_A(x)$ and $f_j^\epsilon(x) = f_j(x)$ for $j \notin \{1,2\}$. If $0 < \epsilon < \epsilon_0$, f^ϵ and $f^{-\epsilon}$ are in $\mathcal{F}$.

Let $T = \rho(\mathcal{T}'(V^*))$ and w and v are the right and left eigenvectors of $\rho(T) = \gamma$. We can apply Lemma 9 to $H = \mathcal{T}'(f^\epsilon)$ and $-H = \mathcal{T}'(f^{-\epsilon})$, we deduce that:

$$\begin{aligned} 0 &= \langle Hv, w\rangle \\ &= \epsilon\int_A (\sum_k L(x,Y_k)v_k)(\frac{w_1}{L(x,Y_1)} - \frac{w_2}{L(x,Y_2)})\lambda(x)dx. \end{aligned}$$

The last equality implies,

$$w_1 \int_A \frac{1}{L(x,Y_1)}dx = w_2 \int_A \frac{1}{L(x,Y_2)}dx. \tag{30}$$

Thus, for all A included E', such that $\ell(A) > 0$:

$$\frac{1}{\ell(A)}\int_A \frac{1}{L(x,Y_1)}dx - \frac{w_1}{w_2}\frac{1}{\ell(A)}\int_A \frac{1}{L(x,Y_1)}dx = 0.$$

We can apply Theorem 1.40 of [16] and conclude that a.e. in E':

$$L(x, Y_1) = \frac{w_1}{w_2} L(x, Y_2).$$

This contradicts our hypothesis the non singularity assumption. Therefore $\ell(E) = 0$. We have also proved that the minimum is uniquely reached (up to null measure sets). ■

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