The dead leaves model : a general tessellation modeling occlusion

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Abstract

In this article, we study a particular example of general random tessellation, the *dead leaves* model. This model, first studied by the Mathematical Morphology school, is defined as a sequential superimposition of random closed sets, and provides the natural tool to study the occlusion phenomenon, essential ingredient in the formation of visual images. We generalize results from G. Matheron, and in particular we compute the probability for n compact sets to be included in *visible parts*. This result characterizes the distribution of the boundary of the dead leaves tessellation.

Keywords: General Tessellation — Dead Leaves — Typical Cell — Image Modeling.

AMS Classification: 60D05 (Primary) ; 60G55 ; 52A22 ; 68U10 (Secondary)

1 Introduction

The dead leaves model has been introduced by G. Matheron in [18]. This model results from sequential superimposition of random sets. As such, it provides the natural tool for studying the non-linear *occlusion* phenomena, of great importance in image modeling and processing. However, to the best of our knowledge, this model has not been systematically investigated, and even its mere definition lacks some precision. Our purpose in this paper is twofold: first to provide a rigorous definition of the model as a random tessellation, second to give new proofs or extensions of Matheron's results in the framework of Palm calculus.

A first motivation to study this model comes from applications. Amongst existing stochastic models for natural images, the dead leaves is the only one whose definition agrees with their physical formation. Several recent studies have demonstrated the ability of specific dead leaves models to reproduce most known statistics of natural images, see [23], [1], [16]. The model has also been proposed as a tool to resample random fields for texture synthesis, see [10]. Other examples of application come from material sciences, see [14] and [8].

As a second motivation, let us stress that the dead leaves model provides non-trivial examples of general random tessellations, in the sense that their cells are general closed sets. In particular, they are not necessarily polygonal, connected or convex, as it is the case for the most popular tessellation models, such a Poisson flats, Voronoi or Delaunay tessellations. Note that non-convex and non-polygonal cells are encountered in the case of Johnson-Mehl tessellation (see e.g. [26]), but that there are relatively few such examples. Therefore, there are few studies of "general" tessellations, even though classical formulae originally proved in the convex and polygonal case have been shown to hold in more general contexts, see [25], [28] and [6].

In Section 2 we first recall some facts on random closed sets and slightly reformulate [21] and [25] to define random tessellations and typical cell distributions. In Section 3 we define the dead leaves model as a random tessellation obtained from an initial Poisson process, and give some of its elementary properties. Then, in Section 4, we generalize results from G. Matheron. In order to do so in a rigorous way, we make use of point processes theory through the systematic use of Palm calculus. We first give the probability for n compact sets to be included in n different visible parts, a result which completely characterizes the distribution of the boundary of our model as a random closed set. Then we compute the distribution of "objects" that remain completely visible. Eventually, we reobtain in the Palm calculus framework a nice result from G. Matheron giving the length distribution of the intersection of objects with a line of fixed direction, stating in particular that its expectation is divided by two as a result of occlusion.

Previous work. The dead leaves model was introduced in [18], an internal note written in an informal style, but containing all basic ideas. The model is defined as the superimposition of infinitesimal boolean models, and formula for the probability of a compact set to be included in a visible part and for the distribution of completely visible parts, among other things, are derived. Most of these definitions and results are stated in the book by J. Serra [24]. D. Jeulin further studied this model in [13], still with the same infinitesimal formalism, and gave an explicit formula for the joint probability of two compact sets to be included in visible parts. In [12] he generalizes the model to the case of random functions and extend to this setting formulae for the distribution of visible parts and for inclusion probabilities. R. Cowan and A. Tsang, in a very interesting paper [5], make use of mean value formulae for tessellations to derive the expectations of various quantities such as the number of connected components of visible parts or the length of their boundaries per surface unit.

2 Basic definitions

2.1 Closed Sets and Tessellations

Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}$ be respectively the sets of all closed, open and compact sets of $\mathbb{R}^d$, $d \geq 1$. Let us denote for any $A \subset \mathbb{R}^d$,

$$\mathcal{F}^A = \{F \in \mathcal{F} \,:\, F \cap A = \emptyset\} \quad \text{and} \quad \mathcal{F}_A = \{F \in \mathcal{F} \,:\, F \cap A \neq \emptyset\}.$$

The Borel σ-field $\mathcal{B}_{\mathcal{F}}$ on $\mathcal{F}$ is generated by the basis of open sets $\{\mathcal{F}^K, K \in \mathcal{K}; \mathcal{F}_G, G \in \mathcal{G}\}$. Borel sets are defined on $\mathcal{G}$ and $\mathcal{K}$ in a way similar to those of $\mathcal{F}$, see [19]. A random closed set (RACS) of $\mathbb{R}^d$ is a measurable function from a probability space $(\Omega, \mathcal{S}, P)$ into $(\mathcal{F}, \mathcal{B}_{\mathcal{F}})$. For any sets A and B, we will denote

$$A \ominus B = \{x \in \mathbb{R}^d \,:\, x + \check{B} \subset A\} \quad \text{and} \quad A \oplus B = \{x + y \,:\, x \in A, y \in B\},$$

where $\check{B} = \{-x, x \in B\}$. $A \ominus \check{B}$ is called the erosion of A by B, and $A \oplus \check{B}$ the dilation of A by B. Measurability properties of these operators are established in [19].

A σ-finite measure on $\mathcal{F}' := \mathcal{F}\backslash\{\emptyset\}$ (endowed with its Borel σ-algebra $\mathcal{B}_{\mathcal{F}'}$) is a measure taking finite values on $\mathcal{F}_K$, for all $K \in \mathcal{K}$, see [19]. We denote by $\mathcal{N}_{\mathcal{F}'}$ the set of σ-finite counting measures on $(\mathcal{F}', \mathcal{B}_{\mathcal{F}'})$. For all $M \in \mathcal{N}_{\mathcal{F}'}$, we write $M = \sum_i \delta_{F_i}$, where δ_{F_i} is the unit mass measure at point F_i. The *boundary* of M is defined as $\partial M = \bigcup_i \partial F_i$, where ∂F_i denotes the topological boundary of F_i. A point process on $\mathcal{F}'$ is a measurable function from a probabilistic space to $(\mathcal{N}_{\mathcal{F}'}, \mathcal{B}_{\mathcal{N}_{\mathcal{F}'}})$, where $\mathcal{B}_{\mathcal{N}_{\mathcal{F}'}}$ is the usual σ-field on $\mathcal{N}_{\mathcal{F}'}$, see e.g. [7].

Following Stoyan [25], a tessellation of $\mathbb{R}^d$ is defined as follows.

Definition 1 *Let $T = \sum_i \delta_{F_i} \in \mathcal{N}_{\mathcal{F}'}$. We say that T is a tessellation of $\mathbb{R}^d$ if*

(i) $\bigcup_i F_i = \mathbb{R}^d$.

(ii) for all $i \neq j$, $\mathrm{Int}F_i \cap F_j = \emptyset$, where $\mathrm{Int}F$ denotes the interior of F,

or equivalently if $\{(\mathrm{Int}F_i)_i, \partial T\}$ is a partition of $\mathbb{R}^d$.

Note that $T \in \mathcal{N}_{\mathcal{F}'}$ implies that the number of cells F_is hitting a compact set is finite. This condition is added in the original definition in [25], where the F_is are marks of a point process $N = \sum_i \delta_{x_i}$ on $\mathbb{R}^d$, where x_i is called the *centroid* of F_i. The centroids are unimportant for the definition of a tessellation but they are quite useful for defining *the typical cell distribution* as we will recall below.

Let $\mathcal{T}$ be the set of all tessellations in $\mathcal{N}_{\mathcal{F}'}$. Expressing assertions (i) and (ii) as limits of the elementary sets operations $(F, F') \mapsto F \cup F'$, $(F, F') \to F \cap F'$ and $F \to \partial F$, whose measurability may be found in [19, Section 1-2], one easily gets that $\mathcal{T} \in \mathcal{B}_{\mathcal{N}_{\mathcal{F}'}}$. A random tessellation of $\mathbb{R}^d$ is then defined as a point process T on $\mathcal{F}'$, such that $T \in \mathcal{T}$ almost surely (a.s.). Classical examples of random tessellations (see the references in [26, Chapter 10] and [22]) include Poisson hyperplanes processes, Delaunay, Voronoi and Johnson-Mehl tessellations. A standard approach (see e.g. [2], [4], [20], [21] or [26]), which applies in these examples, is to define ∂T directly as a RACS without considering the underlying random tessellation. However, it is not always possible to recover the F_i's from ∂T (they may not be connected, see [6] and Remark 10 below for a precise example).

2.2 Typical Cell distribution

In [21] a typical cell is defined by using the Palm distribution of a simple marked point process $N = \sum_i \delta_{x_i, F_i}$ of points in $\mathbb{R}^n$ with marks in $\mathcal{F}'$, stationary with respect to shifts $N \mapsto \sum \delta_{x_i - x, F_i - x}$, $x \in \mathbb{R}^d$. More precisely, let us denote by μ the intensity of N, which we assume to be finite, and by $\mathbb{P}_N^0$ its Palm distribution. Let x_0 be the point nearest to the origin and F_0 be its corresponding cell. Then the *typical cell distribution* is defined on the σ-field $\mathcal{I}$ of all translation-invariant events in $\mathcal{B}_{\mathcal{F}'}$ by $\chi \mapsto \mathbb{P}_N^0(F_0 \in \chi)$, $\chi \in \mathcal{I}$. A result in [21], proven in the case of tessellations whose cells are bounded polytopes, can be easily extended as follows.

Proposition 2 *Let B be a Borel set in $\mathbb{R}^d$ such that*

$$0 < \nu(F_i \oplus B) < +\infty \quad \textit{for all } i \textit{ a.s.}, \tag{1}$$

where ν is the Lebesgue measure on $\mathbb{R}^n$. Then $\mu = \mathbb{E}\sum_i \frac{\mathbb{1}(0\in F_i\oplus B)}{\nu(F_i\oplus B)}$ and

$$\mathbb{P}_N^0(F_0 \in \chi) = \frac{1}{\mu}\mathbb{E}\sum_i \frac{\mathbb{1}(0 \in F_i \oplus B)\mathbb{1}(F_i \in \chi)}{\nu(F_i \oplus B)}, \quad \chi \in \mathcal{I}.$$

When starting from a stationary point process $M = \sum_i \delta_{F_i}$ on $\mathcal{F}'$, a marked point process N can be obtained by constructing points $x_i = \Delta(F_i)$, where Δ is such that $\Delta(F_i - x) = \Delta(F_i) - x$. Classical examples for Δ include the set-centroid, the median point or the extremal point in a given direction. Observe that, under Condition (1), it is always possible to define such a set-centroid by taking for each coordinate the median of the marginal measure of ν restricted to $F_i \oplus B$; for instance, the first coordinate is then defined as the smallest x such that $\nu((F_i \oplus B) \cap (-\infty, x] \times \mathbb{R}^{d-1}) \geq \nu(F_i \oplus B)/2$. As noticed by [21], the typical cell distribution should not depend on the choice of the x_is, which is insured by Proposition 2 provided that one can find a Borel set B for which (1) is fulfilled. This will be the case for the dead leaves model considered below.

In order to define the typical cell of a tessellation, assume that

$$\left\{ \begin{array}{l} 0 < \nu(F_i) < \infty \\ \nu(\partial F_i) = 0 \end{array} \right. \qquad \text{for all } i \text{ a.s.} \tag{2}$$

Note that the first condition above is Condition (1) with $B = \{0\}$. The second condition enables to define, almost everywhere, $F_{\{x\}}$ as the cell to which the point x belongs. By stationarity of N, $F_{\{0\}}$ is defined a.s. Applying Proposition 2, we then get

$$\mu = \mathbb{E}\frac{1}{\nu(F_{\{0\}})} \quad \text{and} \quad \mathbb{P}_N^0(F_0 \in \chi) := \frac{1}{\mu}\mathbb{E}\frac{\mathbb{1}(F_{\{0\}} \in \chi)}{\nu(F_{\{0\}})}, \quad \chi \in \mathcal{I}. \tag{3}$$

We thus obtain the formula of the typical cell distribution derived in [20], [21] (when the F_i's are bounded polytopes) and [4] (when the F_i's are uniformly bounded polytopes).

We end this section with a limit theorem. Let $B_n = B(0, r_n)$ be the ball centered at 0 of radius r_n where $r_n \to \infty$. Let $(A_n)_{n\in\mathbb{N}}$ be any increasing sequence of compact convex sets such that for all n, $B_n \subset A_n$. The individual ergodic theorem (Proposition 10.2.II of [7]) easily yields the following.

Proposition 3 *If N is ergodic and satisfies (2), then, for all $\chi \in \mathcal{I}$,*

$$\lim_n \frac{\sum_i \mathbb{1}(F_i \in \chi)\frac{\nu(F_i\cap A_n)}{\nu(F_i)}}{\sum_i \frac{\nu(F_i\cap A_n)}{\nu(F_i)}} = \mathbb{P}_N^0(F_0 \in \chi) \quad a.s. \tag{4}$$

Equation (4) is a weighted average, where each F_i has a weight equal to its proportion included in A_n. From a statistical point of view, (4) can be used for deriving a strongly consistent estimator of $\mathbb{P}_N^0(F_0 \in \chi)$ for a given $\chi \in \mathcal{I}$. Under stronger hypothesis on the cells, there may be different sequences having the same limit as in (4). For example, if the cells are uniformly bounded (as in [4]), Relation (4) implies, a.s.,

$$\mathbb{P}_N^0(F_0 \in \chi) = \lim_n \frac{\sum_i \mathbb{1}(F_i \in \chi)\mathbb{1}(F_i \subset A_n)}{\sum_i \mathbb{1}(F_i \subset A_n)} = \lim_n \frac{\sum_i \mathbb{1}(F_i \in \chi)\mathbb{1}(F_i \cap A_n \neq \emptyset)}{\sum_i \mathbb{1}(F_i \cap A_n \neq \emptyset)} .$$

Sufficient conditions under which these equalities hold are studied in [6].

3 The dead leaves model

3.1 Definition

The dead leaves model is obtained through sequential superimposition of random objects falling on $\mathbb{R}^d$. Let $\sum_{i\in\mathbb{N}} \delta_{x_i,t_i}$ be a homogeneous Poisson point process on the half-space $\mathbb{R}^d \times (-\infty, 0]$ with intensity one. Let P be a probability measure on $(\mathcal{F}, \mathcal{B}_{\mathcal{F}})$, and $(X_i)_{i\in\mathbb{N}}$, be i.i.d. random variables on $\mathcal{F}$ with distribution P and independent of the Poisson point process above. Equivalently, $\Phi = \sum_i \delta_{x_i,t_i,X_i}$ is a Poisson point process on $\mathbb{R}^d \times (-\infty, 0] \times \mathcal{F}$ with intensity measure $\nu(dx)dtP(dX)$.

We write $(\Omega, \mathcal{S}, \mathbb{P})$ for the probabilistic space on which Φ is defined and $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$. From now on, X will always denote a random variable on $\mathcal{F}$ with distribution P independent of all other variables, and E will denote the expectation with respect to P.

Definition 4 *For all $i \in \mathbb{N}$, the random closed set $x_i + X_i$ is called a leaf and*

$$V_i = (x_i + X_i) \setminus \left(\bigcup_{t_j \in (t_i, 0)} (x_j + \mathrm{Int} X_j) \right) \tag{5}$$

is called a visible part.

From now on we assume that X satisfies the following three conditions:

(C-1) For all $K \in \mathcal{K}$, $E\nu(X \oplus K) < +\infty$,

(C-2) There exists a ball B with strictly positive radius, such that $E\nu(X \ominus B) > 0$.

(C-3) X is a regular closed set, i.e. X is the closure of its interior, P-a.s.

Proposition 5 *We denote by M the point process on $\mathcal{F}'$ obtained by removing all sets with empty interior in the collection $\{V_i\}$, that is,*

$$M = \sum_i \mathbb{1}\{\mathrm{Int} V_i \neq \emptyset\} \, \delta_{V_i} \, . \tag{6}$$

Then M is a random tessellation of $\mathbb{R}^d$. Moreover $N = \sum_i \mathbb{1}\{\mathrm{Int} V_i \neq \emptyset\} \, \delta_{x_i, V_i}$ is stationary, mixing and has finite intensity.

Remark 6 *The condition $\mathrm{Int} V_i \neq \emptyset$ in the definitions of M and N is adopted for convenience as it eliminates visible parts with zero d-dimensional Lebesgue measure. The question arises whether $M' := \sum_i \mathbb{1}\{V_i \neq \emptyset\} \, \delta_{V_i}$ also verifies such property. For simple examples of X, it is easily shown that $M = M'$ a.s. but we do not know whether this equality is true under the general assumptions (C-1)-(C-3). In any case, because (5) implies that $\partial V_i \subset \cup_{t_j > t_i} \partial \{\mathrm{Int} V_j\}$, we always have $\partial M = \partial M'$.*

In order to prove Proposition 5 we will make use of the following two lemmas. The first one, which is easy to prove by referring to the definition of the intensity of the Poisson point process Φ, will be repeatedly needed in the sequel.

Lemma 7 *Let K be a bounded Borel set, $-\infty < s_1 < s_2 < 0$ and define*

$$\begin{aligned}\Phi_K(s_1, s_2) &:= \sum_i \mathbb{1}\{t_i \subset (s_1, s_2] \text{ and } K \subset x_i + X_i\} \ , \\ \Phi^K(s_1, s_2) &:= \sum_i \mathbb{1}\{t_i \subset (s_1, s_2] \text{ and } K \cap x_i + X_i \neq \emptyset\} \ .\end{aligned}$$

$\Phi_K(t_1, t_2)$ and $\Phi^K(t_1, t_2)$ are Poisson random variables with respective means $(t_2 - t_1)E\nu(X \ominus \check{K})$ and $(t_2 - t_1)E\nu(X \oplus \check{K})$.

Lemma 8 *If K is a Borel set of $\mathbb{R}^d$ such that $E\nu(X \ominus \check{K}) > 0$, then K is almost surely covered by some leaf $x_i + X_i$. As a consequence, any bounded set is a.s. covered by a finite number of leaves.*

Proof. Let us fix $t < 0$. Using Lemma 7, the probability $\mathbb{P}(\Phi_K(t, 0) = 0)$ that none of the leaves $x_i + X_i$ with $t < t_i < 0$ satisfies $K \subset x_i + X_i$ is $\exp(tE\nu(X \ominus \check{K}))$, which yields the first assertion. Now let B be a ball such that Condition (**C**-2) is satisfied, that is $E\nu(X \ominus B) > 0$. Since any bounded set K is covered by a finite number of balls with the same radius as B, it also follows that K is covered by $\cup_{t_i > T}(x_i + X_i)$ for some $T < 0$. □

Proof. [Proof of Proposition 5] Let us now show that, $\mathbb{P}$-a.s., $M \in \mathcal{N}_{\mathcal{F}'}$. In fact, we show that, $\mathbb{P}$-a.s., $M' := \sum_i \mathbb{1}(V_i \neq \emptyset)\delta_{V_i} \in \mathcal{N}_{\mathcal{F}'}$ (which implies $M \in \mathcal{N}_{\mathcal{F}'}$), that is, that only a finite number of visible parts V_i may intersect a given compact set K. By Lemma 8, $\mathbb{P}$-a.s., there exists a negative T such that K is covered by leaves $x_i + X_i$ satisfying $t_i > T$. It follows that the visible parts intersecting K correspond to leaves falling after time T. The number of such leaves is thus $\Phi^K(T, 0)$, which is finite $\mathbb{P}$-a.s. by Lemma 7 with Condition (**C**-1). To show that M is a random tessellation, we now verify that it satisfies Conditions (i) and (ii) of Definition 1. Let $T < 0$. Since $\cup_{t_i > T} V_i \subseteq \cup_{t_i > T}(x_i + X_i)$ and since a point in $x_i + X_i$ either belongs to V_i or to $x_j + \mathrm{Int}X_j$ for some $t_j > t_i$, we have $\cup_{t_i > T}(x_i + X_i) = \cup_{t_i > T} V_i$. Therefore by Lemma 8 we get, $\mathbb{P}$-a.s., $\cup_i V_i = \mathbb{R}^d$. We observe from Condition (C-3) that $\mathrm{Int}V_i = (x_i + \mathrm{Int}X_i) \cap \left\{\cap_{t_j > t_i}(x_j + X_j)^{\mathrm{c}}\right\}$. It follows that $\mathrm{Int}V_i = \emptyset$ if and only if $V_i \subset \cup_{t_j > t_i}(x_j + X_j) = \cup_{t_j > t_i} V_j$. Indeed, the "if" part is obvious, while the "only if" part is obtained by observing that $x_i + \mathrm{Int}X_i \subseteq \cap_{t_j > t_i}(x_j + X_j)$ implies the same inclusion for $\overline{x_i + \mathrm{Int}X_i} = x_i + X_i \supseteq V_i$. Finally, consider a realization of Φ such that $M' \in \mathcal{N}_{\mathcal{F}'}$ and $\cup_i V_i = \mathbb{R}^d$, which happens $\mathbb{P}$-a.s., as we have shown above. Pick any point $x \in \mathbb{R}^d$. Since $M' \in \mathcal{N}_{\mathcal{F}'}$, there exists

a positive and finite number of indices i such that $x \in V_i$ and hence one i such that $x \in V_i$ and $x \notin V_j$ for all $t_j > t_i$. By the above characterization, this implies $\mathrm{Int}V_i \neq \emptyset$. Hence $\cup\{V_i : \mathrm{Int}V_i \neq \emptyset\} = \mathbb{R}^d$, that is, M satisfies Condition (i) of Definition 1. Condition (ii) of Definition 1 is easily obtained from (5) and (C-3) by considering the cases $t_j > t_i$ and $t_i > t_j$ successively.
Next we show stationarity and mixing property. Define

$$\Pi : \sum_i \delta_{x_i,t_i,X_i} \mapsto \sum_i \mathbb{1}(\mathrm{Int}V_i \neq \emptyset)\delta_{x_i,V_i} . \tag{7}$$

Recall that $\mathbb{P}$ denotes the distribution of the initial (homogeneous) Poisson point process Φ, so that $\mathbb{P}_\Pi = \mathbb{P} \circ \Pi^{-1}$ is the distribution of N. Further observe that translations on the x_i's correspond to translations on the V_i's through Π. It follows that the stationarity and the mixing property of N (respect to shifts $N \to \sum \delta_{x_i - x, V_i - x}$, $x \in \mathbb{R}^d$) are inherited from Φ.
It remains to prove that the intensity μ of N is finite. For all $T < 0$, let $N_T := \sum \delta_{x_i,V_i} \mathbb{1}(t_i > T, \mathrm{Int}V_i \neq \emptyset)$. Let μ_T denote the intensity of N_T; we have $\mu_T \leq \mathbb{E}\sum \mathbb{1}(x_i \in [0,1]^n, t_i > T) \leq -T$, hence μ_T is finite. By monotone convergence, since μ_T is non-decreasing as T decreases to $-\infty$, $\mu = \lim_{T\to-\infty} \mu_T$. Below we provide a uniform upper bound for μ_T, which will thus apply to μ and conclude the proof. Using Proposition 2 with B given by (C-2), we get

$$\begin{aligned}\mu_T &= \mathbb{E}\sum_i \frac{\mathbb{1}(0 \in V_i \oplus B)}{\nu(V_i \oplus B)} \mathbb{1}(t_i > T, \mathrm{Int}V_i \neq \emptyset) \\ &\leq \nu(B)^{-1}\,\mathbb{E}\sum_i \mathbb{1}\left\{0 \in x_i + X_i \oplus B, 0 \notin \cup_{t_i > t}(x_i + \mathrm{Int}X_i \ominus B)\right\},\end{aligned}$$

where the inequality follows both from $\nu(V_i \oplus B) \geq \nu(B)$, and $V_i \oplus B \subset (x_i + X_i \oplus B) \setminus \cup_{t_i > t} (x_i + \mathrm{Int}X_i \ominus B)$, which in turn follows from (5) and standard properties of morphological operations. Now, Campbell's theorem and Slivnyak's theorem yield

$$\mu_T \leq \frac{1}{\nu(B)} \int_{[T,0]\times\mathbb{R}^d\times\mathcal{F}} \mathbb{1}(0 \in x + X \oplus B)\mathbb{P}(0 \notin \cup_{t_i > t}\{x_i + \mathrm{Int}X_i \ominus B\})dt\nu(dx)P(dX).$$

Noticing that $\cup_{t_i > t}(x_i + \mathrm{Int}X_i \ominus B)$ is a boolean model with intensity t, we thus get

$$\mu_T \leq \frac{1}{\nu(B)} E\nu(X \oplus B) \int_T^0 \exp(tE\nu(X \ominus B))dt \leq \frac{1}{\nu(B)} \frac{E\nu(X \oplus B)}{E\nu(X \ominus B)},$$

which is finite under (C-1) and (C-2). □

In the definition of M, we assume that $\sum_i \delta_{x_i,t_i}$ has intensity one. However, rescaling the x_i's is equivalent, up to a global rescaling of the model, to a rescaling of X and any order preserving modification of the t_i's is unimportant as seen from the definition.

Definition 9 *The point process M defined in Proposition 5 is called the dead leaves tessellation associated with the RACS X.*

Remark 10 *The dead leaves model clearly shows the necessity to define a tessellation through its cells, and not only its boundary. Indeed, visible parts defined by (5) are not necessarily connected, see Figure 2.*

3.2 Perfect simulation

The term "dead leaves model" originates from a more natural definition which consists in putting each new leaf *above* the previous ones and then considering the stationary distribution of this Markov process. Let K be a compact set of $\mathbb{R}^2$. A classical "coupling from the past" argument enables perfect simulation of the stationary distribution restricted to K, by putting each new leaf *below* the already fallen leaves until K is completely covered (see the illustrating web applet [15]). This elegant argument was first introduced for the dead leaves model in [27]. In Figures 1 and 2 we show simulations of the model computed this way. To visualize the model each grain is allocated a random gray level.

3.3 Regularity properties of visible parts

Some almost sure regularity results about visible parts are a consequence of the following remark. From Lemma 7, a visible part V_i is $\mathbb{P}$-a.s. equal to a leaf $x_i + X_i$ to which a *finite* number of other leaves have been removed. Now remark that if A is a closed set and B an open set, then $\partial(A \setminus B) = (\partial A \setminus B) \cup (\partial B \cap A)$. It follows that ∂V_i is a finite union of sets, each of which is included in $x_j + \partial X_j$ for some $t_j \geq t_i$ so that some regularity properties on ∂X are inherited by the ∂V_i's. Note however that possible convexity of the grain X is not inherited by the V_i's, see Figure 1.

Proposition 11 *We have $\nu(\partial M) = 0$ $\mathbb{P}$-a.s. if and only if $\nu(\partial X) = 0$ P-a.s.*

Figure 1: on the left, simulation of a dead leaves model, where the grain X_0 is a disk with constant radius. On the right, simulation of a dead leaves model, where the grain X_0 is a disk with a uniformly distributed radius.

Proof. The discussion above implies that $\nu(\partial V_i) \leq \sum_{t_j \geq t_i} \nu(\partial X_i)$ $\mathbb{P}$-a.s. Since $\partial M = \cup_i \partial V_i$, $\nu(\partial X) = 0$ P-a.s. implies $\nu(\partial M) = 0$ $\mathbb{P}$-a.s.

Now, $\nu(\partial M) = 0$ $\mathbb{P}$-a.s. implies $\nu(\partial V_i) = 0$ for all i and in particular for all cells such that $V_i = x_i + X_i$ (the so-called *relief cells* studied in the forthcoming Section 4.2). We will see in Remark 17 below that this in turn implies $\nu(\partial X) = 0$ P-a.s. □

If $\mathrm{Int} V_i \neq \emptyset$ then $\nu(V_i) > 0$. Besides, $V_i \subset x_i + X_i$ is bounded $\mathbb{P}$-a.s. by (C-1). If in addition $\nu(\partial X) = 0$ P-a.s., then we are in the framework of Section 2.2 for tessellations. When $\nu(\partial X) = 0$, one says that X is ν-regular, a property that neither implies nor is implied by (C-3). It is easy to find a set X which is ν-regular and not closed regular, for instance a set containing isolated points. To construct a closed regular set which is not ν-regular, one can proceed as follows (for $d \geq 2$). Let $\tilde{\nu}$ be the $(d-1)$-dimensional Lebesgue measure on the hyperplane $\{\mathbf{x} = (x_1, \dots, x_d) : x_1 = 1/2\}$. Then there exists a homeomorphism $h : [0,1]^d \to [0,1]^d$ such that $\nu + \tilde{\nu} = \nu \circ h$, see [9]. It follows that $X := h([0,1/2]^d)$ is not ν-regular although it is closed regular.

Figure 2: simulations of dead leaves models. Left: the grain X_0 is a rectangle with a direction uniformly distributed in $[0, \pi]$. Right: the grain is a more complicated shape, the distribution of its size being uniform.

4 Some characteristics of the dead leaves tessellation

4.1 Inclusion probabilities and boundary distribution

The main practical result from the original paper by Matheron introducing the dead leaves model [18] is concerned with a functional, defined on compact sets of the plane, equal to the probability that a given compact set is included in a visible part of the model. It is shown that, for a non-empty $K \in \mathcal{K}$,

$$\mathbb{P}(K \subset \mathrm{Int} V_i \text{ for some } i \in \mathbb{N}) = \frac{E\nu(\mathrm{Int} X \ominus \check{K})}{E\nu(X \oplus \check{K})}. \tag{8}$$

Considering simple examples of possible K's such as bipoints or segments leads to valuable geometric information on the model.

In what follows, we generalize this result by taking interest in the probability that n compact sets are included in n distinct visible parts. We define

$$Q^{(n)}(K_1, \ldots, K_n) = \mathbb{P}(K_1 \subset \mathrm{Int} V_{i_1}, \ldots, K_n \subset \mathrm{Int} V_{i_n} \text{ for some } t_{i_1} < \cdots < t_{i_n} < 0).$$

Proposition 12 *Let us denote*

$$F^{(n)}(K_1,\ldots,K_n) = E\nu(\mathrm{Int}X \ominus \check{K}_1)\prod_{j=2}^{n} E\nu\left((\mathrm{Int}X \ominus \check{K}_j)\cap(X\oplus \underline{\check{K}}_{j-1})^{C}\right), \tag{9}$$

and

$$G^{(n)}(K_1,\ldots,K_n) = \prod_{j=1}^{n} E\nu\left(X\oplus \underline{\check{K}}_j\right), \tag{10}$$

where, for all $j=1,\ldots,n$,

$$\underline{K}_j = \bigcup_{k=1}^{j} K_k. \tag{11}$$

Then

$$Q^{(n)}(K_1,\ldots,K_n) = \frac{F^{(n)}(K_1,\ldots,K_n)}{G^{(n)}(K_1,\ldots,K_n)}. \tag{12}$$

Remark 13 *Note that (**C**-2) implies $E\nu(X)>0$ and thus that $G^{(n)}(K_1,\ldots,K_n)$ does not vanish for non-empty compact sets.*

Proof. Within this proof section, we fix n non-empty compact sets $K_1,\ldots,K_n$ and we write $Q^{(n)}$ for $Q^{(n)}(K_1,\ldots,K_n)$. Summing over disjoint events we have that

$$Q^{(n)} = \mathbb{E}\left(\sum \mathbb{1}(t_{i_1}<\cdots<t_{i_n}<0)\prod_{j=1}^{n}\mathbb{1}(K_j\subset \mathrm{Int}V_{i_j})\right), \tag{13}$$

where the sum is taken over all n-tuples of points in Φ. First note that only n-tuples of distinct points may be considered in this sum and that, from the definition of visible parts in (5) and (C-3), the summand in this equation may be written as

$$\mathbb{1}(t_{i_1}<\cdots<t_{i_n}<0)\prod_{j=1}^{n}\mathbb{1}(K_j\subset(x_{i_j}+\mathrm{Int}X_{i_j}))\prod_{t_i>t_{i_j}}\mathbb{1}(K_j\cap(x_i+X_i)=\emptyset). \tag{14}$$

In the simplest case $n=1$, this amounts to say that $Q^{(1)}$ is the probability that there exists a leaf X_i such that K_1 is included in $\mathrm{Int}X_i$ and is not hit by subsequent leaves. We will now apply the Campbell Formula to compute this expectation, and therefore need the following notation. Let $\mathcal{E} := \mathbb{R}^2 \times (-\infty, 0] \times \mathcal{F}$. We write $\mathcal{N}^{(n)}$ ($\mathcal{N}$ for $n = 1$) for the space of σ-finite counting measures on $\mathcal{E}^n$. For all $n \geq 1$, we define the point process on $\mathcal{E}^n$, $\Phi^{(n)} = \sum_{i_1,\dots,i_n} \delta_{z_{i_1},\dots,z_{i_n}}$, where the sum is taken over all indices $(i_1, \dots, i_n)$ such that $z_{i_1}, \dots, z_{i_n}$ are distinct points of Φ. We define a function f from $\mathcal{E}^n \times \mathcal{N}^{(n)}$ to $\mathbb{R}$ so that (14) reads $f(\{z_{i_j}\}_{j=1}^n, \Phi^{(n)})$. Applying the refined Campbell Theorem (see [7]) to compute the expectation in (13), we get

$$Q^{(n)} = \int_{Z \in \mathcal{E}^n} \int_{\phi \in \mathcal{N}^{(n)}} f(Z, \phi)\, \mathbb{P}^Z(d\phi) \prod_{j=1}^n \mu_\Phi(d\tilde{z}_j),$$

where $Z = \{\tilde{z}_j\}_{j=1}^n$, μ_Φ is the intensity measure of Φ and $\mathbb{P}^Z$ is the Palm distribution of the process $\Phi^{(n)}$ at Z. Applying the generalized Slivnyak Theorem (see [26]) gives

$$Q^{(n)} = \int_{Z \in \mathcal{E}^n} \mathbb{E}\left[f(Z, (\Phi + \delta_{\tilde{z}_1} + \cdots + \delta_{\tilde{z}_n})^{(n)})\right] \prod_{j=1}^n \mu_\Phi(d\tilde{z}_j), \tag{15}$$

where, as usual, $\mathbb{E}$ is the expectation associated to Φ. Writing $\tilde{z}_j = (\tilde{x}_j, \tilde{t}_j, \tilde{X}_j)$ for $j = 1, \dots, n$, with $\tilde{t}_1 < \cdots < \tilde{t}_n < 0$, by definition of f, we have

$$\begin{aligned} f(Z, (\Phi + \delta_{\tilde{z}_1} + \cdots + \delta_{\tilde{z}_n})^{(n)}) = f(Z, \Phi^{(n)}) = \\ \left(\prod_{j=1}^n \mathbb{1}(K_j \subset \tilde{x}_j + \mathrm{Int}\tilde{X}_j)\right) \left(\prod_{j=2}^n \mathbb{1}(\underline{K}_{j-1} \cap (\tilde{x}_j + \tilde{X}_j) = \emptyset)\right) \\ \left(\prod_{j=1}^{n-1} \prod_{t_i \in (\tilde{t}_j, \tilde{t}_{j+1}]} \mathbb{1}(\underline{K}_j \cap (x_i + X_i) = \emptyset)\right) \prod_{t_k \in (\tilde{t}_n, 0]} \mathbb{1}(\underline{K}_n \cap (x_k + X_k) = \emptyset), \end{aligned} \tag{16}$$

with $\underline{K}_j$ as defined in (11). The expectation in (15) is computed as follows. Since Φ is a Poisson process, the last line of (16) can be written as a product of independent terms whose expectations can be computed using that, at fixed $s < t \leq 0$, and for K compact,

$$\mathbb{P}(K \cap (x_i + X_i) = \emptyset \text{ for all } t_i \in (s, t]) = \exp\left((s - t) E\nu(X \oplus \check{K})\right)$$

(see Lemma 7). Next, integrating with respect to $\mathbb{1}(\tilde{t}_1 < \cdots < \tilde{t}_n < 0)d\tilde{t}_1 \dots d\tilde{t}_n$ and using a change of variable $u_j = \tilde{t}_j - \tilde{t}_{j+1}$, for $j = 1, \dots, n-1$, we obtain

$$Q^{(n)} = \prod_{j=1}^{n} E\nu\left(X \oplus \check{\underline{K}}_j\right)^{-1}$$

$$\int_{(\mathbb{R}^2\times\mathcal{F})^n} \left(\prod_{j=1}^{n} \mathbb{1}(K_j \subset \tilde{x}_j + \mathrm{Int}\tilde{X}_j)\right) \left(\prod_{j=2}^{n} \mathbb{1}(\underline{K}_{j-1} \cap (\tilde{x}_j + \tilde{X}_j) = \emptyset)\right) \prod_{j=1}^{n}(d\tilde{x}_j P(d\tilde{X}_j)).$$

The first term of the right-hand side of the previous equation is $(G^{(n)})^{-1}$, and the term of the second line writes

$$\prod_{j=1}^{n} \left(\int_{\mathbb{R}^2\times\mathcal{F}} \mathbb{1}(K_j \subset \tilde{x} + \mathrm{Int}\tilde{X})\mathbb{1}(\underline{K}_{j-1} \cap (\tilde{x} + \tilde{X}) = \emptyset) d\tilde{x} P(d\tilde{X})\right),$$

with the convention $\underline{K}_0 = \emptyset$. Now, for two compact sets A and B, we have

$$\int \mathbb{1}(A \subset (x+\mathrm{Int}X))\mathbb{1}(B\cap(x+X) = \emptyset)\nu(dx)P(dX) = E\nu((\mathrm{Int}X\ominus\check{A})\cap(X\oplus\check{B})^c),$$

which, along with the last equations, yields $F^{(n)}$ and then (12). $\square$

For $n = 1$, we get the original result of Matheron, (8), and the case $n = 2$ was treated in [13]. Note that from the $Q^{(n)}$'s, we can compute the probability

$$\mathbb{P}(K_1 \subset \mathrm{Int}V_{i_1}, \dots, K_n \subset \mathrm{Int}V_{i_n} \text{ for some } i_1, \dots, i_n \in \mathbb{N})$$

and thus the probability for n connected compact sets $K_1, \dots, K_n$ to avoid the boundary of the dead leaves tessellation. For $n = 2$ for instance, this is

$$\mathbb{P}((K_1 \cup K_2) \cap \partial M = \emptyset) = Q^{(2)}(K_1, K_2) + Q^{(2)}(K_2, K_1) + Q^{(1)}(K_1 \cup K_2).$$

Moreover, it is easily seen that if we consider the random field obtained by independently coloring each visible part, then Proposition 12 enables to compute the finite dimensional distributions of this field. This is a useful result in the context of image modeling, see [11]. Next, we show that the knowledge of $Q^{(n)}$ for all n characterizes the distribution of ∂M in $(\mathcal{F}, \mathcal{B}_{\mathcal{F}})$.

Proposition 14 *The distribution of the boundary ∂M is uniquely determined by the functionals $Q^{(n)}$, $n \in \mathbb{N}$.*

Proof. The distribution of ∂M is characterized by its capacity functional defined for every compact set K by $\mathbb{P}(F \cap K = \emptyset)$, see [19]. Let $K \in \mathcal{K}$, let $r_n > 0$ be a sequence converging to 0, and for each n, let $\{x_i^{(n)}\}_{i=1,\dots,N_n}$ be finite sequences in K such that $K \subset C_n = \cup_i B(x_i^n, r_n)$, where $B(x, r)$ is the (closed) ball centered at x with radius r. Note that since each C_n is a finite union of connected compact sets, the knowledge of the $Q^{(i)}$, $i \in \mathbb{N}$, uniquely determines $\mathbb{P}(C_n \cap \partial M = \emptyset)$. Now since $C_n \downarrow K$, we have that $\mathcal{F}^{C_n} \uparrow \mathcal{F}^K$, and thus that $\mathbb{P}(C_n \cap \partial M = \emptyset) \uparrow \mathbb{P}(K \cap \partial M = \emptyset)$. $\square$

4.2 Typical relief cells

In this section, we take interest in the distribution of cells that remain completely visible. This problem was first addressed in [18], see also [17], [24] and [12].

Definition 15 *A cell V_i is a relief cell if $(x_i + X_i) = V_i$. Denote by $N_r = \sum_i \mathbb{1}(V_i = (x_i + X_i))\delta_{x_i,V_i}$ the point process of relief cells.*

As in the proof of Proposition 5, one can show that N_r is stationary and mixing. From Condition (C-3) if $V_i = (x_i + X_i)$ then $\mathrm{Int}V_i \neq \emptyset$. It follows that N_r is a thinning of N and since N has finite intensity, so has N_r.

Proposition 16 *The typical relief cell distribution is absolutely continuous with respect to P with Radon-Nikodym derivative $F \mapsto \left(\mu_r E\nu(\mathrm{Int}X \oplus \check{F})\right)^{-1}$, where $\mu_r := \displaystyle\int_{\mathcal{F}} \frac{P(dF)}{E\nu(\mathrm{Int}X \oplus \check{F})}$ is the intensity of N_r.*

Remark 17 *As a consequence of this Proposition, the typical relief cell distribution and the leaf distribution P are equivalent measures on $\mathcal{I}$. This remark completes the proof of the "only if" part of Proposition 11.*

Proof. N_r is a simple point process with finite intensity. We denote by $\mathbb{P}^0_{N_r}$ the Palm distribution of N_r. Writing $N_r = \sum \delta_{x_i^r, V_i^r}$, we have, for all $\chi \in \mathcal{I}$,

$$\begin{aligned}
\mathbb{P}^0_{N_r}(V_0^r \in \chi) &= \frac{1}{\mu_r}\mathbb{E}\sum_i \mathbb{1}(V_i^r \in \chi)\mathbb{1}(x_i^r \in [0,1]^2) \\
&= \frac{1}{\mu_r}\mathbb{E}\sum_i \mathbb{1}(V_i \in \chi,\, x_i \in [0,1]^2,\, (x_i + X_i) \cap \bigcup_{t_j \in (t_i, 0]} (x_j + \mathrm{Int}X_j) = \emptyset)\,.
\end{aligned}$$

From Slivnyak's theorem and Campbell's formula,

$$\begin{aligned}
\mathbb{P}^0_{N_r}(V_0^r \in \chi) &= \frac{1}{\mu_r}\int_{\mathbb{R}^2\times\mathbb{R}_-\times\chi} \mathbb{P}((x+F)\cap \bigcup_{t_j\in(t,0]}(x_j+\mathrm{Int}X_j)=\emptyset)\,\nu(dx)dtP(dF)\\
&= \frac{1}{\mu_r}\int_{\mathbb{R}_-\times\chi} \exp(tE\nu(\mathrm{Int}X\oplus\check{F}))\,dtP(dF)\\
&= \frac{1}{\mu_r}\int_{\chi}\left[E\nu(\mathrm{Int}X\oplus\check{F})\right]^{-1}\,P(dF),
\end{aligned}$$

where the second equality follows from Lemma 7. Taking $\chi=\mathcal{F}'$, we also find the announced formula for the intensity. $\square$

For example, we can compute the area distribution of a typical relief cell. For $\chi_s=\{F\in\mathcal{F}':\nu(F)>s\}$, we find $\mathbb{E}^0_{N_r}(\nu(X_0^r))=\mu_r^{-1}\int_{\mathcal{F}'}\nu(F)[E\nu(\mathrm{Int}X\oplus \check{F})]^{-1}\,P(dF)$.

Remark 18 *For $d=2$, if X is convex and isotropic a.s., we obtain the original result of Matheron by applying the Steiner Formula to compute μ_r. Let $l(K)$ denote the length of ∂K, for K convex, we have $\mu_r=E\left[(\nu(X)+\frac{2}{\pi}l(X)El(X)+E\nu(X))^{-1}\right]$.*

4.3 Cells intersected with a line

We now take interest in the intersection between the dead leaves model and a fixed line D. In this section we take $d\geq 2$ and, in addition to (C-1)-(C-3), we assume that

(C-4) $\nu(\partial X)=0$ a.s. and, for any line D', $D'\cap\partial X$ is either empty, finite or has positive $\nu_{D'}$ measure a.s.,

where $\nu_{D'}$ is the one-dimensional Lebesgue measure on D'. This assumption is for instance verified if X is a finite union of convex sets, a.s.

We will compute the Palm distribution of the point process $\partial M\cap D$ and, in the case where X is convex, prove a result from [18] in the Palm calculus framework.

Lemma 19 *$\partial M\cap D$ is a point process on D.*

Proof. Since ∂M is a locally finite union of sets ∂V_is a.s. and since, for all i, ∂V_i is included in a finite union of sets $(x_j+\partial X_j)$, it is sufficient to show, that, a.s., for any j, $(x_j+\partial X_j)\cap D$ is a finite or empty set. Let us suppose that this does not hold. By (C-4), it implies that with positive probability, there exists j such that $\nu_D(x_j + \partial X_j) > 0$. Thus $\mathbb{E}\nu_D\{\cup_j(x_j + \partial X_j)\} > 0$. Without loss of generality, we let D be the first coordinate axis. By Fubini's theorem and translation invariance, we obtain

$$\mathbb{E}\nu\left\{\bigcup_j (x_j + \partial X_j)\right\} = \int_{\mathbf{y}\in\mathbb{R}^{d-1}} \mathbb{E}\nu_{D_\mathbf{y}}\left\{\bigcup_j (x_j + \partial X_j)\right\} d\mathbf{y} > 0,$$

where, for any $\mathbf{y} = (y_2, \ldots, y_d)$, $D_\mathbf{y}$ is the line parallel to D going through the point $(0, y_2, \ldots, y_d)$. Thus, a.s., there exists j such that $\mathbb{E}\nu(\partial X_j) > 0$, which is in contradiction with (C-4). $\square$

We let $\mathbf{u}$ be a unit vector colinear to D, denote by $[0, x\mathbf{u}]$ the segment $\{\lambda x\mathbf{u},\, \lambda \in [0,1]\}$ and define, for all $x \geq 0$,

$$L(x) = \mathbb{P}([0, x\mathbf{u}] \subset \mathrm{Int}V_i \text{ for some } i \in \mathbb{N}) = Q^{(1)}([0, x\mathbf{u}]) = \frac{E\nu(\mathrm{Int}X \ominus [0, -x\mathbf{u}])}{E\nu(X \oplus [0, -x\mathbf{u}])}, \tag{17}$$

where $Q^{(1)}$ is defined above in Section 4.1 and the last equality follows from (8).

From now on we denote by $N_\ell = \sum_i \delta_{y_i}$ the simple point process defined in Lemma 19, with points in $\mathbb{R}$, write $\mathbb{P}_{N_\ell}$ for its law and $\mathbb{P}^0_{N_\ell}$ for its associated Palm distribution. We index N_ℓ such that $\{y_i\}$ is increasing and $y_0 < 0 < y_1$. The following lemma links the Palm distribution of N_ℓ to L.

Lemma 20 *Let $N_\ell = \sum_i \delta_{y_i}$ be the simple stationary point process defined above. Then $L(x)$ is absolutely continuous, has a negative right derivative $L'(0)$ at $x = 0$ and, almost everywhere,*

$$\mathbb{P}^0_{N_\ell}(y_1 > x) = \frac{L'(x)}{L'(0)}. \tag{18}$$

Proof. Observe that $L(x) = \mathbb{P}_{N_\ell}(y_1 > x)$ for all non-negative x. Let λ be the intensity of N_ℓ. The inversion formula (see for example [3]) gives, for all $x \geq 0$,

$$L(x) = \mathbb{P}_{N_\ell}(y_1 > x) = \lambda \int_x^\infty \mathbb{P}^0_{N_\ell}(y_1 > t)\, dt.$$

By derivating we obtain that $L'(x) = -\lambda \mathbb{P}^0_{N_\ell}(y_1 > x)$. Observing that $\mathbb{P}^0_{N_\ell}(y_1 = 0) = 0$, we obtain the differentiability of L at the origin and $L'(0) = -\lambda < 0$. $\square$

We end this section by considering the case of an a.s. convex X. First, we introduce the *geometric covariogram* γ_X of X, defined for $x \geq 0$ by

$$\gamma_X(x) := \nu(X \cap (x\mathbf{u} \oplus X)).$$

Note that the covariogram is usually defined on $\mathbb{R}^d$, but that here we only take interest in a half-line. Let $p_{\mathbf{u}^\perp}$ denote the orthogonal projection on the hyperplane orthogonal to $\mathbf{u}$ and $\nu_{\mathbf{u}^\perp}$ denote the $(d-1)$-dimensional Lebesgue measure on this hyperplane. If X is convex, then γ_X is a convex function on $[0, W_{\mathbf{u}})$, where $W_{\mathbf{u}}$ is the width of X in direction $\mathbf{u}$, and is identically zero outside this interval. Moreover, it is continuously differentiable on $[0, W_{\mathbf{u}})$ with derivative $\gamma'_X(x) = -\nu_{\mathbf{u}^\perp}[p_{\mathbf{u}^\perp}(X \cap (x\mathbf{u} \oplus X))] \geq -\nu_{\mathbf{u}^\perp}(p_{\mathbf{u}^\perp}(X))$, see [19]. From ($\mathbf{C}$-1) and ($\mathbf{C}$-2), we have $E\nu_{\mathbf{u}^\perp}(p_{\mathbf{u}^\perp}(X)) < \infty$. Hence, $E\gamma_X$ is absolutely continuous with derivative $E\left(\gamma'_X(x)\right)$ almost everywhere; from now on we simply write $E\gamma'_X(x)$ for $E\left(\gamma'_X(x)\right)$. Moreover $\gamma'_X(x)$ is right continuous at $x = 0$ and so is $E\gamma'_X(x)$ by dominated convergence, so that $E\gamma_X(x)$ has the right-hand derivative $E\gamma'_X(0) = -E\nu_{\mathbf{u}^\perp}(p_{\mathbf{u}^\perp}(X))$ at $x = 0$.

Definition 21 *The intercept distribution (in the direction* $\mathbf{u}$*) of* X *is defined as*

$$F_X(x) = \frac{E\gamma_X{}'(x)}{E\gamma_X{}'(0)}, \quad x \geq 0. \tag{19}$$

Remark 22 *The term intercept distribution refers to the fact that* $\gamma_X{}'(x)/\gamma_X{}'(0)$ *is the probability distribution of the length of the intersection of* X *with lines having direction* $\mathbf{u}$ *uniformly distributed among those hitting* X*, see [24].*

Proposition 23 *Let* M *be a dead leaves model associated to a RACS* X *which is convex with intercept distribution* F_X *a.s. and let* $\mathbb{P}^0_{N_\ell}$ *and* y_1 *be defined as above. Then, for all* $x \geq 0$*,*

$$\int_x^\infty \mathbb{P}^0_{N_\ell}(y_1 > t)\, dt = \frac{1}{2}(1 + Kx)^{-1} \int_x^{+\infty} F_X(t)\, dt, \tag{20}$$

where $K = -E\gamma_X{}'(0)/E\gamma_X(0)$*.*

Proof. It can be shown that, when X is convex, $\nu(X \ominus [0, -x\mathbf{u}]) = \gamma_X(x)$ and $\nu(X \oplus [0, -x\mathbf{u}]) = \gamma_X(0) + x\nu_{\mathbf{u}^\perp}(p_{\mathbf{u}^\perp}(X))$. Since $\nu_{\mathbf{u}^\perp}(p_{\mathbf{u}^\perp}(X)) = -E\gamma_X{}'(0)$, Relation (17) yields

$$L(x) = \frac{E\gamma_X(x)}{E\gamma_X(0) - xE\gamma_X{}'(0)},$$

and the result then follows from (18) and (19) through easy calculations. □

Let us finally notice that $\mathbb{P}^0_{N_\ell}(y_1 > x)$ may be seen (as in section 2.2) as the length distribution of the "typical cell" of the tessellation $D \cap M := \sum_i \mathbb{1}\{V_i \cap D \neq \emptyset\}\, \delta_{V_i \cap D}$, and thus as the intercept distribution of the typical cell of M (which is not convex). Notice also that by taking $x = 0$ in formula (20), we obtain

$$\mathbb{E}^0_{N_\ell}(y_1) = \frac{1}{2}\int_0^{+\infty} F_X(t)\, dt,$$

which says (see Remark 22) that, for a convex X, the mean intercept in any direction is divided by two as a result of occlusion.

5 Conclusion

Various generalizations of this model are possible. Non homogeneous point processes could be considered, or the independence assumption between time and objects could be broken (see [12]), enabling perspective laws to be taken into account. In the homogeneous and independent case, many open problems remain, in particular for computing typical cell properties given the distribution of the leaf X. The computation of the mean perimeter and area of typical cells, as done in [5] for the connected components of visible parts, is an interesting direction for further work.

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