Report

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Abstract

We study the convergence of some particle processes to Dawson-Watanabe superprocesses through their Radon-Nikodým derivatives convergence, as in Lalley and Zheng [5]. The processes we will look at here are branching random walks with drift, contact processes, voter models and Lotka-Volterra models.

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1 Introduction

The particle processes we study here are all birth and death particle processes, each particle can give birth to a child in a neighbouring site, or die. Our processes takes place on a lattice $\mathbf{Z}^d$. The rate at which a particle at site x gives birth or dies only depends on the number of particle at x and the number of particles in the neighbourhood of x.

Here we introduce some of the notations we use throughout this report. First we put the heuristic about the particle processes in two definitions, continuous time and discrete time. We denote by p an irreductible symmetric random walk kernel on Z^d such that $p(0) = 0$ and $\sum_x x^i x^j p(x) = \delta_{i,j}\sigma^2 < +\infty$, which denotes the way the particles are walking on the lattice. For all bounded functions ϕ, we will denote $P\phi(x) = \sum_e p(e)\phi(x+e)$.

Definition 1. A continuous time particle system with kernel p, birth rate b_t and death rate k_t is a Markov process $\xi_t : \mathbf{Z}^d \to \mathbf{N}$, where $\xi_t(x)$ represent the number of particles at x at time t.

This population increases if a neighbour produces a child at x, and decreases if a particle at x dies. Both rates $b_t(x)$ and $k_t(x)$ are function of x, t, $\xi_{t-}(x)$ and $V_{t-}(x) = \sum_e p(e)\xi_{t-}(x+e)$. We rewrite it as the following :

$$\begin{cases} \xi_t(x) \to \xi_t(x) + 1 \text{ at rate } V_t(x)b_t(x) \\ \xi_t(x) \to \xi_t(x) - 1 \text{ at rate } \xi_t(x)k_t(x). \end{cases}$$

Definition 2. A discrete time particle system with kernel p and offspring distribution Π is a Markov chain $\xi_n : \mathbf{Z}^d \to \mathbf{N}$, where $\xi_n(x)$ represent the number of particles at x at time n. The probability on $\mathbf{N}$ $\Pi_{x,n}$ is a function of x, n, $\xi_n(x)$ and $V_n(x) = \sum_e p(e)\xi_n(x+e)$. We rewrite it as the following :

$$\xi_{n+1}(x) = k \text{ with probability } \Pi_{n,x}(k).$$

Remark 1. There are some differences between the definitions of the continuous time and the discrete time models. In the first one, we give two different rates which both play on the speed at which the process evolves (doubling both b_t and k_t is the same as considering ξ_{2t}) and the number of children of one particle before its death (roughly, if b_t and k_t are constant during all the life of the particle, the number of children follow a geometric law of parameter $\dfrac{b_t(x)}{b_t(x)+k_t(x)}$, because this is the number of heads we obtain with an unfair coin before the first tail).

In the second model, we just take in account the number of children of a single particle. The variation of the speed of the process can be introduced after, in a rescaling, considering that a step of 1 at position x correspond to a jump of time $\gamma_n \approx b_n + k_n$.

We now have to define the limits we hope to find, the Dawson-Watanabe super-processes :

Definition 3. A Dawson-Watanabe super-process with branching rate $\gamma > 0$, drift $\theta_t(x)$ and diffusion coefficient σ^2 starting at X_0 is an adapted a.s-continuous $\mathcal{M}_f(\mathbb{R}^d)$-valued process X_t on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ which solve the martingale problem :

$$\forall \phi \in \mathcal{C}_b^\infty(\mathbb{R}^d), M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s(\gamma \frac{\sigma^2 \Delta\phi}{2})ds - \int_0^t X_s(\theta_s \phi)ds, \tag{1}$$

is a continuous $(\mathcal{F}_t)$-martingale, with $M_0(\phi) = 0$ and quadratic variation :

$$\langle M(\phi) \rangle_t = \gamma \int_0^t X_s(\phi^2)ds.$$

The solution and existence in law of this martingale problem is well known, let $\mathbf{P}_{X_0}^{\gamma,\theta,\sigma^2}$ the law of the solution on $\Omega_{X,D} = D([0, +\infty[, \mathcal{M}_f(\mathbb{R}^d))$.

Remark 2. The Dawson-Watanabe superprocess without drift is also called the super-Brownian motion.

In this report, we try to prove the convergence of some sequences of these particle systems, suitably rescaled, to Dawson-Watanabe processes, in the following way. We compute the Radon-Nikodým derivative of the particle system we are studying against an other one with a known convergence. The following lemma will complete the proofs.

Lemma 1. *Let X_n, X be random variables valued in a metric space E all defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and let L_n, L be non-negative real valued random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ with mean 1.*

Let $\mathbf{Q}_n, \mathbf{Q}$ be probability measures on $(\Omega, \mathcal{F})$ with Radon-Nikodym derivative L_n, L against $\mathbf{P}$, if :

$$(X_n, L_n) \underset{N \to +\infty}{\Longrightarrow} (X, L),$$

then the $\mathbf{Q}_n$-distribution of X_n converge weakly to the $\mathbf{Q}$-distribution of X.

Proof. Let $\phi \in \mathcal{C}_b(E)$, by definition, we have :

$$\mathbf{E}_{\mathbf{Q}_n}(\phi(X_n)) = \mathbf{E}_{\mathbf{P}}(L_n \phi(X_n)).$$

Since $(X_n, L_n) \underset{N \to +\infty}{\Longrightarrow} (X, L)$, we know too that for all $A > 0$,

$$\mathbf{E}((L_n \wedge A)\phi(X_n)) \underset{N \to +\infty}{\longrightarrow} \mathbf{E}((L \wedge A)\phi(X)).$$

Moreover $\mathbf{E}(L_n) = 1$, and $L_n \geq 0$, so we know that :

$$\mathbf{E}(|L_n \wedge A - L_n|) = \mathbf{E}(L_n \mathbf{1}_{\{L_n > A\}}) = 1 - \mathbf{E}(L_n \mathbf{1}_{\{L_n \leq A\}}),$$

converge to $\mathbf{E}(|L \wedge A - L|)$ when $n \to +\infty$. We can now compute :

$$\begin{aligned}
& \limsup_{n \to \infty} |\mathbf{E}(L_n \phi(X_n) - L\phi(X))| \\
\leq \ & \limsup_{n \to \infty} \|\phi\|_\infty \mathbf{E}(|(L_n \wedge A) - L_n| + |(L \wedge A) - L|) \\
\leq \ & 2\|\phi\|_\infty \mathbf{E}(|(L \wedge A) - L|).
\end{aligned}$$

And we just have to take $A \to +\infty$ to conclude. □

We now need a particle system whose rescaled limit is well known, which we will use as a reference law with all the other processes. This process will be the branching random walk. A branching random walk is the simplest process among those we have defined above. In this process, the birth and the death rate are both equal to a constant γ, in the continuous time model.

Definition 4. A continuous time branching random walk with rate γ and kernel p is a Markov process $\xi_t : \mathbf{Z}^d \to \mathbf{N}$ evolving as following :

$$\begin{cases} \xi_t(x) \to \xi_t(x) + 1 \ \text{ at rate } \ \gamma V_t(x) \\ \xi_t(x) \to \xi_t(x) - 1 \ \text{ at rate } \ \gamma \xi_t(x). \end{cases}$$

In order to prove the convergence of the particle processes we are studying to Dawson-Watanabe super-processes, we rescale them in time, space and mass, and take only care of the local density of particle near each point. More precisely, if ξ^N is a sequence of branching random walks with rate γN, we define the following sequence of $\mathcal{M}_f(\mathbb{R}^d)$-valued processes :

$$X_t^N = \frac{1}{N} \sum_x \xi_t^N(x) \delta_{\frac{x}{\sqrt{N}}}.$$

The measure X_t^N is localized on the rescaled lattice $Z_N = \dfrac{\mathbf{Z}}{\sqrt{N}}$. We shall study the convergence in law of this measure-valued process. The following theorem holds :

Theorem 1 (Dawson-Watanabe theorem). *If $X_0^N \Rightarrow \mu$, where $\mu \in \mathcal{M}_f(\mathbb{R}^d)$ then we have :*

$$X^N \Rightarrow X,$$

where X is the law of a super-Brownian motion with initial distribution μ, rate 2γ and diffusion σ^2.

The same result holds for the discrete time branching random walk.

Definition 5. A discrete time branching random walk with kernel p is a Markov chain $\xi_n : \mathbf{Z}^d \to \mathbf{N}$ evolving as follows :

$$\xi_{n+1}(x) = k \text{ with probability } e^{-V_n(x)}\frac{(V_n(x))^k}{k!}.$$

In other words, this is a discrete time particle process, where at each step, each particle dies and leaves children with the following distribution :

$$\Pi_{n,x}(k) = e^{-V_n(x)}\frac{(V_n(x))^k}{k!}.$$

Remark 3. Here we chose the model with a Poisson offspring distribution to facilitate computations, but other laws can be used, such as a geometric law, which is closer to the continuous time model, because there is no difference in the limit when $N \to +\infty$, but then the calculations becomes harder.

Let ξ^N be a sequence of discrete time branching random walks with rate 1, as in the previous part, but we consider it sped up by a factor γN, and we define :

$$X_t^N = \frac{1}{N}\sum_x \xi^N_{\lfloor \gamma N t\rfloor}(x)\delta_{\frac{x}{\sqrt{N}}}.$$

The following theorem holds :

Theorem 2 (Dawson-Watanabe theorem). *If $X_0^N \Rightarrow \mu$, where $\mu \in \mathcal{M}_f(\mathbb{R}^d)$ then we have :*

$$X^N \Rightarrow X,$$

where X is the law of a super Brownian motion with rate γ and diffusion σ^2.

Remark 4. The disappearence of a factor 2 in the rate is natural if we remember the correspondence rule between the two processes. A rate γ in both birthing and killing rate in the continuous time branching random walk is equal to a rate 2γ in the discrete time one.

2 Radon-Nikodým derivative of the particle processes

2.1 About the continuous time particle processes

2.1.1 Notations

For all of these particles systems, we give a few general definitions which will be useful later, when we will compute the Radon-Nikodým derivative of two of them.

First we will give a characterisation of a particle process through a countable set of particles with a common law which is easily computable.

Proposition 1. *Let X_t be a particle system with birth rate b_t and death rate k_t, we denote :*

- $T_0 = 0$ *and* $T_{n+1} = \inf \{t > T_n | X_t \neq X_{T_n}\}$ *;*
- $t_n = T_{n+1} - T_n$ *;*
- $x_n \in \mathbf{Z}^d$ *such as* $X_{T_{n+1}}(x_n) \neq X_{T_n}(x_n)$ *which is a.s. unique ;*
- $\delta_n = X_{T_{n+1}}(x_n) - X_{T_n}(x_n) \in \{-1, 1\}$.

Then, knowing $\mathcal{F}_{T_n}$, the joint law of (t_n, x_n, δ_n) is given by :

$$\mathbf{P}(t_n \in [t, t+dt[, x_n = x, \delta_n = 1 | \mathcal{F}_{T_n}) = \exp\left(-\int_{T_n}^{t} X_{T_n}(Pb_s + k_s)ds\right) P(X_{T_n})(x) b_{T_n+t-}(x)dt,$$

$$\mathbf{P}(t_n \in [t, t+dt[, x_n = x, \delta_n = -1 | \mathcal{F}_{T_n}) = \exp\left(-\int_{T_n}^{t} X_{T_n}(Pb_s + k_s)ds\right) X_{T_n}(x) k_{T_n+t-}(x)dt.$$

Moreover there is a continuous bijection between $X_{t\wedge T_n}$ and $(t_k, x_k, \delta_k)_{k<n}$.

Proof. We use the Markov property, which say us that :

$$\mathbf{P}(t_n \in [t, t+dt[, x_n = x, \delta_n = \delta | \mathcal{F}_{T_n}) = \mathbf{P}^{X_{T_n}}(t_0 \in [t + T_n, t + T_n + dt[, x_0 = x, \delta_0 = \delta).$$

And then, to find when the first jump occurs and what is his location and type, we use the "clock-alarm lemma".

The last affirmation is obvious. □

We now know a way to build a birth and death particle system. We just have to construct this countable set of random variables, one after the other.

2.1.2 Radon-Nikodým derivative of continuous time particle systems

Proposition 2. *Let* $\mathbf{P}$ *be the law of a particle system with birthing rate b_t and killing rate k_t, and* $\mathbf{Q}$ *the law of a modification of this process with rates $b_t(1+\alpha_t)$ and $k_t(1+\beta_t)$, where α, β are* *continuous bounded functions on $\mathbb{R}^+ \times \mathbf{Z}^d$. The Radon-Nikodým derivative of the process until time t can be written :*

$$\begin{aligned}\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t}(X) &= \exp\left\{-\int_0^t X_s(P(b_s\alpha_s)+k_s\beta_s)ds)\right\} \\ &\prod_{n|T_n<t}\left(1+\mathbf{1}_{\{\delta_n=1\}}\alpha_{T_{n+1}-}(x_n)+\mathbf{1}_{\{\delta_n=-1\}}\beta_{T_{n+1}-}(x_n)\right).\end{aligned}$$

Proof. We first use Proposition 1, to compute the Radon-Nikodým derivative L_n of $(X_{t\wedge T_n})_t$ under laws $\mathbf{P}$ and $\mathbf{Q}$:

$$\begin{aligned}L_n &= \exp\left\{\int_0^{T_n} X_s(P(b_s\alpha_s)+k_s\beta_s)ds)\right\} \\ &\times\prod_{k=0}^{n-1}\left(1+\mathbf{1}_{\{\delta_k=1\}}\alpha_{T_{k+1}-}(x_k)+\mathbf{1}_{\{\delta_k=-1\}}\beta_{T_{k+1}-}(x_k)\right).\end{aligned}$$

Now we give an extension in the following way :

Let $n_t=\inf\{n>0|T_n>t\}$, we can compute the Radon-Nikodým derivative of the process until time t, using the following :

$$\begin{aligned}\mathbf{E}\left(\left.\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t}\right|\mathcal{F}_{T_n}\right) &= \mathbf{E}\left(\left.\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t}\mathbf{1}_{\{n\le n_t\}}\right|\mathcal{F}_{T_n}\right) \\ &+ \mathbf{E}\left(\left.\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t}\mathbf{1}_{\{n>n_t\}}\right|\mathcal{F}_{T_n}\right) \\ &= L_n\mathbf{1}_{\{n\le n_t\}}+L_{n_t-1}\frac{\mathbf{Q}(T_{n_t}>t)}{\mathbf{P}(T_{n_t}>t)}\mathbf{1}_{\{n>n_t\}},\end{aligned}$$

using the fact that the information we use between the time T_{n_t-1} and t is just $T_{n_t}>t$. We now just have to let $n\to+\infty$ to find the following.

$$\begin{aligned}\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t}(X) &= \exp\left\{-\int_0^t X_s(P(b_s\alpha_s)+k_s\beta_s)ds)\right\} \\ &\prod_{n|T_n<t}\left(1+\mathbf{1}_{\{\delta_n=1\}}\alpha_{T_{n+1}-}(x_n)+\mathbf{1}_{\{\delta_n=-1\}}\beta_{T_{n+1}-}(x_n)\right).\end{aligned}$$

□

We will now write this derivative in an easier way. To do this we have to recall what is the exponential of a càdlàg martingale.

Definition 6. Let M_t be a càdlàg martingale (right-continuous with left limits). The exponential of M is the martingale, denoted $\mathcal{E}(M)$, defined by :

$$\mathcal{E}(M)_t=\exp\left(M_t-\frac{1}{2}[M,M]_t^c\right)\prod_{s\le t}(1+\Delta M_s)\exp(-\Delta M_s)$$

If M is a quadratic pure jump martingale, then we have :

$$\mathcal{E}(M)_t = \exp\left(M_t - \sum_{s\leq t} \Delta M_s\right) \prod_{s\leq t} (1 + \Delta M_s)$$

Let X_t be a particle system of law $\mathbf{P}$, then we can define the following pure jump processes :

- The birth process $B_t(x) = \sum_{0<s\leq t} \Delta X_s(x)^+$;
- The death process $K_t(x) = \sum_{0<s\leq t} \Delta X_s(x)^-$.

Of course we have $X_t(x) - X_0(x) = B_t(x) - K_t(x)$. Moreover, the Markov property shows that the processes :

$$\widehat{B}_t(x) = B_t(x) - \int_0^t b_s(x) P(X_s)(x) ds \quad \text{and}$$

$$\widehat{K}_t(x) = K_t(x) - \int_0^t k_s(x) X_s(x) ds$$

are $\mathbf{P}$-martingales.

Then we have :

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(M)_t,$$

where we denote :

$$M_t = \sum_x (\alpha(x).\widehat{B}(x))_t + (\beta(x).\widehat{K}(x))_t.$$

2.2 Radon-Nikodým derivative for discrete time particle processes

We denote $\mathbf{P}$ the law of a discrete time particle process with offspring distribution Π, and $\mathbf{Q}$ a modification of this particle process with birthing rate Π'.

These Radon-Nikodym derivative are easier to compute, using the conditional independence of the number of offspring at time $n+1$ on each site. The Radon-Nikodým derivative can be expressed as a product over space and time :

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_N} (X) = \prod_{n=0}^{N} \prod_x L_{n,x},$$

where we denote :

$$L_{n,x} = \frac{\mathbf{Q}(\xi_{n+1}(x) = X_{n+1}(x))}{\mathbf{P}(\xi_{n+1}(x) = X_{n+1}(x))}.$$

Using the definition we gave of a discrete time particle system, we can finally write :

$$L_{n,x}(X) = \frac{\Pi'_{n,x}(X_{n+1})}{\Pi_{n,x}(X_{n+1})}.$$

3 Some computations for the branching random walk

As we said in the introduction, we will use a branching random walk as a reference law for all our other particle systems. In other words, to study the convergence of the Radon-Nikodým derivatives, we will study functions of the branching random walk (except for the Lotka-Volterra model, where we will use the voter model as a reference). So a few results about the computation of its moments will be useful later on.

3.1 Representation of the continuous time branching random walk

Let $\Lambda^n(x,y)$ and $\Lambda^n(x)$ be independent Poisson processes of intensity $\gamma p(x,y)$ and γ respectively. A rate γ branching random walk ξ_t is the unique strong solution of the following problem :

$$\xi_t(x) = \xi_0(x) + \sum_{y,n} \int_0^t \mathbf{1}_{\{\xi_{s-}(y)>n\}} d\Lambda_s^n(x,y) - \sum_n \int_0^t \mathbf{1}_{\{\xi_{s-}(x)>n\}} d\Lambda_s^n(x).$$

If Λ is a Poisson process with intensity λ, we will denote its compensated process by $\widehat{\Lambda}_t = \Lambda_t - \lambda t$, and for all functions ϕ,

$$\xi_t(\phi) = \sum_x \phi(x)\xi_t(x),$$

$$\mathcal{L}_\gamma\phi(x) = \gamma(\mathbf{P}\phi(x) - \phi(x)).$$

We can rewrite ξ in the following way :

$$\xi_t(\phi) = \xi_0(\phi) + M_t(\phi) + A_t(\phi) \tag{2}$$

where we denote :

$$M_t(\phi) = \left[\sum_{x,y,n} \int_0^t \phi(x)\mathbf{1}_{\{\xi_{s-}(y)>n\}} d\widehat{\Lambda}_s^n(x,y) - \sum_{x,n} \int_0^t \phi(x)\mathbf{1}_{\{\xi_{s-}(y)>n\}} d\widehat{\Lambda}_s^n(x,y)\right],$$

$$A_t(\phi) = \int_0^t \xi_s(\mathcal{L}_\gamma\phi)ds.$$

$M_t(\phi)$ is a martingale with quadratic variation :

$$\langle M(\phi)\rangle_t = \gamma \int_0^t \xi_s(\phi^2 + P(\phi^2))ds.$$

3.2 Computation of the moments

We want to extend (2) to functions $\phi(t,x) = \phi_t(x) \in \mathcal{C}^{1,3}([0,+\infty[\times\mathbb{R}^d)$. Then we have, by Riemann-Stieltjes equality :

$$\xi_t(x)\phi_t(x) = \xi_0(x)\phi_0(x) + \int_0^t \xi_s\dot{\phi}_s(x)ds + \int_0^t \phi_s(x)d\xi_s(x).$$

We can sum over x, and we obtain :

$$\xi_t(\phi_t) = \xi_0(\phi_0) + M_t(\phi) + A_t(\phi),$$

where we denote :

$$M_t(\phi) = \left[\sum_{x,y,n} \int_0^t \phi_s(x) \mathbf{1}_{\{\xi_{s-}(y)>n\}} d\widehat{\Lambda}_s^n(x,y) - \sum_{x,n} \int_0^t \phi_s(x) \mathbf{1}_{\{\xi_{s-}(y)>n\}} d\widehat{\Lambda}_s^n(x,y)\right]$$

and

$$A_t(\phi) = \int_0^t \sum_x \xi_s(\mathcal{L}_\gamma \phi_s + \dot{\phi}_s) ds.$$

$M_t(\phi)$ is a martingale with quadratic variation :

$$\langle M(\phi) \rangle_t = \gamma \int_0^t \xi_s(\phi_s^2 + P(\phi_s^2)) ds.$$

We call this martingale the orthogonal martingale of the branching random walk.

Definition 7. Let ξ_t be a rate γ branching random walk. The orthogonal martingale measure of ξ is the $\mathcal{M}_f(\mathbb{R}^d)$-valued process M_t such as for all $\phi \in \mathcal{C}^{1,3}([0,+\infty[\times\mathbb{R}^d)$, we have :

$$M(\phi)_t = \xi_t(\phi) - \xi_0(\phi) - \int_0^t \sum_x \xi_s(\mathcal{L}_\gamma \phi_s + \dot{\phi}_s) ds.$$

We let P_t denote the semi-group of the random walk B^γ on $\mathbf{Z}^d$ which jumps at rate γ in its neighbours with distribution p. In other words,

$$P_t(\phi)(x) = \mathbf{E}(\phi(x + B_t^\gamma)).$$

If we use the previous equation with the function $\phi_s(x) = P_{t-s}\phi(x)$, we have $A_t(\phi) = 0$, so :

$$\mathbf{E}(\xi_t(\phi)) = \xi_0(P_t(\phi)),$$

And moreover we can compute the other moments by recursion using the Ito formula. For the second moment :

$$\mathbf{E}(\xi_t(\phi)\xi_t(\psi)) = \xi_0(P_t(\phi))\xi_0(P_t(\psi)) + \int_0^t \mathbf{E}(\xi_s(P_{t-s}\phi P_{t-s}\psi)) ds.$$

3.3 Computation for the discrete time branching random walk

The same kind of computations for the moments for the discrete time branching random walk holds. Indeed we can easily prove by recurrence that :

$$\mathbf{E}(\xi_n(\phi)) = \xi_0(P_n(\phi)),$$

$$\mathbf{E}(\xi_n(\phi)\xi_n(\psi)) = \xi_0(P_n(\phi))\xi_0(P_n(\psi)) + \sum_{k=0}^{n-1} \mathbf{E}(\xi_k(P_{n-k}(\phi)P_{n-k}(\psi))),$$

where we denote $P_n(\phi)(x) = \mathbf{E}(\phi(x + B_n))$, with B_n a random walk on $\mathbf{Z}^d$ with kernel p.

4 Convergence for the branching radom walk with drift

In this section we will highlight one of the difficulties we have to deal with in the continuous time branching random walk, which doesn't exist in discrete time. To do this, we will study one of the simplest modifications of the branching random walk, where we just add a drift, a difference between the expectation of birth and death for each particle.

4.1 Continuous time branching random walk with symmetric drift θ

First we will study a model where the drift is symmetrically distributed on the birthing and the killing rate. For this model, the convergence of the Radon-Nikodým derivative can be proved, without too many problems.

Definition 8. A branching random walk with symmetric drift $\theta \in \mathcal{C}_b([0,+\infty[\times\mathbb{R}^d)$ is a particle system where each particle has a birth rate sped up and a death rate sped down by the same factor. The particle system evolves as the following :

$$\begin{cases} \xi_t(x) \to \xi_t(x)+1 & \text{at rate } \; V_t(x)(1+\theta(x,t)) \\ \xi_t(x) \to \xi_t(x)-1 & \text{at rate } \; \xi_t(x)(1-\theta(x,t)) \end{cases}$$

A sequence of these processes, suitably rescaled will converge to a Dawson-Watanabe super-process in the following way. We denote by θ^N a sequence of continuous bounded functions on $\mathbb{R}^+ \times \dfrac{\mathbf{Z}^d}{\sqrt{N}}$, that we extend by interpolation to $\mathcal{C}_b\left(\mathbb{R}^+\times\mathbb{R}^d\right)$ functions. We suppose that θ^N uniformly converges to θ.

Theorem 3. *Let ξ_t^N be a sequence of rate N branching random walks with symmetric drift $\dfrac{\theta^N}{N}$ on the rescaled lattice, we define :*

$$X_t^N = \frac{1}{N}\sum_{x\in\mathbf{Z}^d} \xi_t^N(x)\delta_{\frac{x}{\sqrt{N}}}.$$

If $X_0^N \underset{N\to+\infty}{\Longrightarrow} \mu$ where $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, then we have :

$$X^N \underset{N\to+\infty}{\Longrightarrow} X,$$

where X is a Dawson-Watanabe super-process with branching rate 2, dispersion σ^2 and drift θ.

Proof. Let P_θ^N be the law of the process X^N, and P^N the law of the branching random walk with rate N, rescaled in the same way, which we will also call ξ^N. The Radon-Nikodým derivative of these processes is equal to the derivative of particle systems they are extracted from :

$$\left.\frac{dP_\theta^N}{dP^N}\right|_{\mathcal{F}_t} = \mathcal{E}(\frac{1}{N}M^N(\theta^N))_t,$$

where M^N is the orthogonal martingale of the branching random walk ξ^N.

We use $\mathcal{C}_b^{1,3}\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$ approximations ϕ_ϵ^N of θ^N such as $\|\phi_\epsilon^N - \theta^N\|_\infty \leq \epsilon$. In the same way we denote ϕ_ϵ an approximation of θ

Moreover we know that the sequence of branching random walks with rate N, rescaled in space by $\sqrt{N}$ and in mass by N converge weakly to the super-Brownian motion. So we just have to prove the weak joint convergence for the Radon-Nikodým derivative. We have, for any function $\phi \in \mathcal{C}_b^{1,3}\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$:

$$M_t^N(\phi) = \xi_t^N(\phi) - \xi_0^N(\phi) - \int_0^t \xi_s^N(\mathcal{L}_N \phi) ds - \int_0^t \xi_s^N(\dot{\phi}_s) ds.$$

In the same way, we have :

$$[M^N(\phi)]_t = \sum_{x,y,n} \int_0^t \phi_{s-}(x)^2 \mathbf{1}_{\{\xi_{s-}^N(y) > n\}} d\widehat{\Lambda}_s^n(x,y) + \sum_{x,n} \phi_{s-}(x)^2 \mathbf{1}_{\{\xi_{s-}^N(x) > n\}} d\widehat{\Lambda}_s^n(x),$$

so we see that $[\frac{1}{N} M^N(\phi)]_t - \int_0^t X_s^N(\phi_s^2 + P^N(\phi_s^2)) ds$ is a martingale with quadratic variation converging to 0, so

$$[\frac{1}{N} M^N(\phi)]_t \xrightarrow[N \to +\infty]{L^2} 2 \int_0^t X_s(\phi_s^2) ds.$$

Furthermore the application :

$$X \in D([0,+\infty[, \mathcal{M}_f(\mathbb{R}^d)) \mapsto \int_0^t X_s(\phi) ds$$

is continuous, so let ϕ^n be a countable set of $\mathcal{C}_b^{1,3}\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$:

$$\left(X^N, \left(\frac{1}{N} M_t^N(\phi^n)\right), \left(\frac{1}{2N^2}\left[M^N(\phi^n)\right]_t\right)\right) \underset{N \to +\infty}{\Longrightarrow} \left(X, \left(M_t(\phi^n)\right), \int_0^t X_s({\phi^n}_s^2) ds\right).$$

Then we use this result with the countable set $\left(\phi_\epsilon^N\right)_{N \in \mathbf{N}, \epsilon \in \mathbf{Q}}$, which prove the joint convergence :

$$\left(X^N, \frac{1}{N} M_t^N(\theta^N), \frac{1}{2N^2}\left[M^N(\theta^N)\right]_t\right) \underset{N \to +\infty}{\Longrightarrow} \left(X, M_t(\theta), \int_0^t X_s(\theta_s^2) ds\right).$$

This convergence gives us in particular the convergence for the Radon-Nikodým derivative. □

We now consider the same kind of processes in the discrete time model.

4.2 Branching random walk with drift, discrete time model

Definition 9. A discrete time branching random walk with drift $\theta \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R}^d)$ is a particle system where each particle has a number of children increased by a factor θ. The particle system evolves as the following :

$$\xi_{n+1}(x) = k \text{ with probability } e^{-\lambda} \frac{\lambda^k}{k!}.$$

where we denote $\lambda = \lambda_n(x) = (1 + \theta_n(x)) V_n(x)$

As in the previous parts, we considere a sequence θ^N of continuous bounded functions on $\frac{\mathbf{N}}{N} \times \frac{\mathbf{Z}^d}{\sqrt{N}}$, that we extend by an interpolation to $\mathcal{C}_b\left(R^+ \times \mathbb{R}^d\right)$ functions. We suppose that θ^N converges uniformly to θ. The same theorem holds :

Theorem 4. *Let ξ^N be a sequence of discrete time branching random walks on the rescaled lattice with drift $\frac{\theta^N_{\frac{n}{N}}}{N}$. We define :*

$$X_t^N = \frac{1}{N} \sum_{x \in \mathbf{Z}^d} \xi^N_{\lfloor Nt \rfloor}(x) \delta_{\frac{x}{\sqrt{N}}}.$$

If $X_0^N \underset{N \to +\infty}{\Longrightarrow} \mu$ where $\mu \in \mathcal{M}_f(\mathbb{R}^d)$, then we have :

$$X^N \underset{N \to +\infty}{\Longrightarrow} X,$$

where X is a Dawson-Watanabe super-process with branching rate 1, dispersion σ^2 and drift θ.

Proof. We first suppose that the drift θ^N and θ are in $\mathcal{C}_b^{1,3}\left(\mathbb{R}^+ \times \mathbb{R}\right)$, and that the convergence is uniform in this space.

As in the previous part, we have a look on the Radon-Nikodým derivative of our process. We denote by $\mathbf{P}_\theta^N$ the law of $(X_t^N)_{t \geq 0}$ and $\mathbf{P}^N$ the law of the branching random walk without drift, rescaled in the same way. For the discrete time model, we have

$$\left.\frac{dP_\theta^N}{dP^N}\right|_{\mathcal{F}_t} = \prod_{k=0}^{\lfloor Nt \rfloor} \prod_x L_{n,x},$$

where we denote :

$$L_{n,x} = e^{-\frac{1}{N} V_n^N(x) \theta^N_{\frac{n}{N}}(x)} \left(1 + \frac{\theta^N_{\frac{n}{N}}(x)}{N}\right)^{\xi^N_{n+1}(x)}.$$

We can now rewrite

$$\left.\frac{dP_\theta^N}{dP^N}\right|_{\mathcal{F}_t} = (1 + o_P(1)) \exp\left[\frac{1}{N} \sum_{k=0}^{\lfloor Nt \rfloor} \xi^N_{n+1}\left(\theta^N_{\frac{n}{N}}\right) - \xi^N_n(P\theta^N_{\frac{n}{N}}) + \frac{1}{N^2} \sum_{k=0}^{\lfloor Nt \rfloor} \xi^N_{n+1}({\theta^N_{\frac{n}{N}}}^2))\right].$$

We study the convergence of each part of the Radon-Nikodým derivative. We begin with

$$\frac{1}{N^2} \sum_{k=0}^{\lfloor Nt \rfloor} \xi^N_{n+1}\left({\theta^N_{\frac{n}{N}}}^2\right) = \int_0^t X_s^N\left({\theta^N_s}^2\right) ds + o(1),$$

and in the same way as before we obtain the joint convergence for this part.

Now we have a look on :

$$\begin{aligned}
&\frac{1}{N}\sum_{k=0}^{\lfloor Nt\rfloor}\xi_{n+1}^N(\theta_{\frac{n}{N}}^N)-\xi_n^N(P\theta_{\frac{n}{N}}^N)\\
=\;&\frac{1}{N}\sum_{k=0}^{\lfloor Nt\rfloor}\left[\xi_{n+1}^N(\theta_{\frac{n+1}{N}}^N)-\xi_n^N(\theta_{\frac{n}{N}}^N)\right]-\left[\xi_{n+1}^N(\theta_{\frac{n+1}{N}}^N)-\xi_{n+1}^N(\theta_{\frac{n}{N}}^N)\right]-\left[\xi_n^N(P^N\theta_{\frac{n}{N}}^N)-\xi_n^N(\theta_{\frac{n}{N}}^N)\right]\\
=\;&X_{\lfloor Nt\rfloor}^N(\theta_{\frac{\lfloor Nt\rfloor}{N}}^N)-X_0^N(\theta_0^N)-\frac{1}{N^2}\sum_{k=0}^{\lfloor Nt\rfloor}\xi_{n+1}^N(\dot\theta_{\frac{n}{N}}^N)+\xi_n^N(\mathcal{L}_N(\theta_{\frac{n}{N}}^N))+o(1)\\
=\;&X_{\lfloor Nt\rfloor}^N(\theta_{\frac{\lfloor Nt\rfloor}{N}}^N)-X_0^N(\theta_0^N)-\int_0^t X_s^N(\dot\theta_s^N+\mathcal{L}_N\theta_s^N)ds+o(1).
\end{aligned}$$

So the joint convergence of the process and the derivative is easy to find.

We just have to use approximate θ by regular functions and to finish the proof. □

Remark 5. This discrete time process is very similar to the branching random walk with symmetrical drift. Here the expected number of children of each particle is equal to $\frac{1+\theta_t(x)}{2}$, this leads us to a drift $\frac{\theta}{2}$ in the discrete time process. Moreover, the speeding rate in the branching random walk with symmetrical drift is equal to 2, instead of 1 in the discrete time process.

But writing $\tilde X_t^N=\frac{1}{N}\sum_{x\in\mathbf{Z}^d}\xi_{\lfloor 2Nt\rfloor}^N(x)\delta_{\frac{x}{\sqrt N}}$ would have led us to a super-Brownian motion with rate 2 and drift θ, in the way that we expected.

We will now look at another branching random walk, where the drift isn't equally distributed on the birthing and the dying rate. Proving the convergence is more difficult.

4.3 Branching random walk with asymmetric drifts θ,θ'

Definition 10. A branching random walk with asymmetric drifts $(\theta,\theta')\in\mathcal{C}_b^{1,3}(\mathbb{R}^+\times\mathbb{R}^d)^2$ is a particle system where each particle has a birthing rate speeded up by some factor, and the killing rate by another one. The particle system evolves as the following :

$$\begin{cases}\xi_t(x)\to\xi_t(x)+1 & \text{at rate } V_t(x)(1+\theta_t(x))\\ \xi_t(x)\to\xi_t(x)-1 & \text{at rate } \xi_t(x)(1+\theta'_t(x))\end{cases}$$

The theorem we would like to prove is the following. Let (θ^N,θ'^N) be sequences of continuous bounded functions on $\mathbb{R}^+\times\frac{\mathbf{Z}^d}{\sqrt N}$, which we extend by an interpolation to $\mathcal{C}_b\left(R^+\times\mathbb{R}^d\right)$ functions. We suppose that θ^N and θ'^N uniformly converges to θ and θ'.

Theorem 5. *Let ξ_t^N a sequence of rate N branching random walks with asymmetric drifts $\left(\frac{\theta^N}{N},\frac{\theta'^N}{N}\right)$, we define :*

$$X_t^N=\frac{1}{N}\sum_{x\in\mathbf{Z}^d}\xi_t^N(x)\delta_{\frac{x}{\sqrt N}}.$$

If $X_0^N \underset{N\to+\infty}{\Longrightarrow} \mu$ *where* $\mu \in \mathcal{M}_f(\mathbb{R}^d)$*, then we have :*

$$X^N \underset{N\to+\infty}{\Longrightarrow} X,$$

where X *is a Dawson-Watanabe super-process with branching rate 2, dispersion* σ^2 *and drift* $\dfrac{\theta-\theta'}{2}$.

We would like to prove the theorem in the same way as before.

As in the previous part, we let $P'^N_{\theta,\theta'}$ denote the law of the process ξ^N, and recall that $\mathbf{P}^N$ denotes the law of the branching random walk with rate N. The Radon-Nikodým derivative of these processes is

$$\left.\frac{dP'^N_\theta}{d\mathbf{P}^N}\right|_{\mathcal{F}_t} = \mathcal{E}(B^N(\theta^N) - K^N(\theta'^N))_t,$$

where $B^N(\theta^N)_t = \sum_{0<s\leq t}\sum_x \theta_s^N(x)\Delta\xi_s^N(x)^+$, is defined as the birthing jumping process, which jump of $\theta_s(x)$ if there is a birth at site x at time s, and, in the same notations, $K^N(\theta'^N)_t = \sum_{0<s\leq t}\sum_x \theta_s^N(x)\Delta\xi_s^N(x)^-$.

We want to prove, in the same way as before the weak convergence of this derivative.

We begin by reminding that

$$B_t^N(\theta^N) \quad = \quad \sum_{x,y,n}\int_0^t \theta_{s-}^N(x)\mathbf{1}_{\{\xi_{s-}^N(y)>n\}}d\widehat{\Lambda}_s^n(x,y).$$

Moreover we have :

$$\begin{aligned}
[B^N(\theta^N)]_t &= \sum_{x,y,n}\int_0^t \theta_{s-}^N(x)^2\mathbf{1}_{\{\xi_{s-}^N(y)>n\}}d\Lambda_s^n(x,y) \\
&= \frac{1}{2}[M^N(\theta^N)]_t + \frac{1}{2}M_t^N(\theta^{N^2}).
\end{aligned}$$

So we see that the convergence for the quadratic variation is quite easy, using the previous arguments. What we now have to do is prove the weak convergence of $\dfrac{1}{N}B_t^N(\theta)$ to $\dfrac{1}{2}M_t(\theta)$. But this part seems problematic because of the difference between B^N and K^N.

Remark 6. To give the equivalent of the asymmetrical drift case, we have to consider the modified sequence of processes $Y_t^N = X^N_{t+\frac{\theta_t(x)}{N}}$, since we have a rate $2+\dfrac{\theta_t(x)}{N}$ in this process, and a drift which is similar to the previous one. We see that the difficulties of computation comes from a problem of time. Instead of keeping the same scale of time, an asymmetrical drift speeds it up a little, which leads to hard to control effects.

5 Convergence of the contact process

A contact process is a birth and death particle system, which represent the evolution of an epidemics in a population. The "particles" represent the infected individuals. At each site, there is only a finite number of persons which are susceptible to become infected. Each infected person can infect another one with a certain rate, and recover with another one. These dynamics have already been much studied, and the convergence has been proved in [3], but here we want to see where the use of the Radon-Nikodým derivative leads us.

5.1 Continuous time contact process

The continuous time contact process can be defined as follows. We suppose that in each site, there is a village of size M. Each infected individual tries to infect each of its neighbours at rate $\frac{1}{M}p(x,y)$. The target become infected if this was not already the case. If we compute this to the branching random walk, we see that the number of births is lower, so we also need to modify the death rate. Each particle will recover at rate $1-\theta$. More formally we can write the following :

Definition 11. A contact process with drift θ and village size M is a continuous time particle system with birth rate $(1 - V_{t-}(x))$ and death rate $(1-\theta)$, i.e. a Markov process ξ_t evolving as follows :

$$\begin{cases} \xi_t(x) \to \xi_t(x) + 1 \ \text{ at rate } \ V_t(x)(1 - \dfrac{X_t(x)}{M}) \\ \xi_t(x) \to \xi_t(x) - 1 \ \text{ at rate } \ \xi_t(x)(1-\theta). \end{cases}$$

We now want to study the convergence of a sequence of these processes. The following theorem holds :

Theorem 6. *Let ξ_t^N a sequence of contact processes with rate N, drift $\frac{\theta}{N}$ and village size N, we define :*

$$X_t^N = \frac{1}{N} \sum_{x \in \mathbf{Z}^d} \xi_t^N(x) \delta_{\frac{x}{\sqrt{N}}}.$$

If $X_0^N \underset{N \to +\infty}{\Longrightarrow} \mu$, where μ is an atomless finite measure on $\mathbb{R}^d$, then we have :

$$X^N \underset{N \to +\infty}{\Longrightarrow} X,$$

where X is a Dawson-Watanabe super-process with branching rate 2, dispersion σ^2 and drift $\theta - b$, where we denote :

$$b = \sum_{n=0}^{+\infty} \mathbf{E}(p(B_n)) = \sum_{n=1}^{+\infty} \mathbf{P}(B_n = 0),$$

with B_n a random walk on $\mathbf{Z}^d$ with kernel p.

This theorem has already be proved in [3], but in a different way. The existence of accumulation values is proved by using the tightness of the sequence, and then it is shown that all the limits satisfy the martingale problem (1). We would like to give another proof of this fact, using the Radon-Nikodým derivative.

If we denote the law of the process X^N by Q_θ^N, we have again, through Proposition 2 :

$$\left.\frac{dQ_\theta^N}{dP^N}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\frac{1}{N}\tilde{M}^N\right)_t,$$

where :

$$\tilde{M}_t^N = \sum_{x,y,n}\int_0^t \xi_s^N(x)\mathbf{1}_{\{\xi_s^N(y)>n\}}d\widehat{\Lambda}_s^n(x,y) - \sum_{x,n}\int_0^t \theta\mathbf{1}_{\{\xi_s^N(x)>n\}}d\widehat{\Lambda}_s^n(x).$$

We begin by treating the special case $\theta = b$, then we want to prove that the Radon-Nikodým derivative converge in law to 1.

But if we have this convergence, can compute the expectation of the logarithm of this derivative :

$$\begin{aligned}\frac{1}{N^2}\mathbf{E}([\tilde{M}^N]_t) &= \mathbf{E}\Big(\frac{1}{N^2}\sum_{x,y,n}\int_0^t \xi_s^N(x)^2\mathbf{1}_{\{\xi_s^N(y)>n\}}d\Lambda_s^n(x,y) - \frac{1}{N^2}\sum_{x,n}\int_0^t b^2\mathbf{1}_{\{\xi_s^N(x)>n\}}d\Lambda_s^n(x)\Big)\\ &= \int_0^t \frac{1}{N}\sum_{x,y}\mathbf{E}((\xi_s^N(x)^2 - b^2)\xi_s^N(y))p(x,y)ds \underset{N\to+\infty}{\longrightarrow} 0.\end{aligned}$$

But we can prove (it will be done in the annex) :

$$\mathbf{E}\Big(\int_0^t \frac{1}{N}\sum_{x,y}(\xi_s^N(x) - b)\xi_s^N(y)p(x,y)ds\Big) \underset{N\to+\infty}{\longrightarrow} 0. \tag{3}$$

This calculation leads us to the following statement. If the Radon-Nikodým derivative converge to 1, then we have :

$$\mathbf{E}\Big(\int_0^t \frac{1}{N}\sum_{x,y}(\xi_t^N(x) - b)^2)\xi_t^N(y)p(x,y)ds\Big) \underset{N\to+\infty}{\longrightarrow} 0.$$

But in the same time, since b is not an integer, if we take $\epsilon > 0$ such that $]b-\epsilon, b+\epsilon[\cap\mathbf{N} = \varnothing$, then we also have :

$$\mathbf{E}\Big(\int_0^t \frac{1}{N}\sum_{x,y}(\xi_t^N(x) - b)^2)\xi_t^N(y)p(x,y)ds\Big) \geq \mathbf{E}\Big(\int_0^t \frac{1}{N}\sum_{x,y}\epsilon^2\xi_t^N(y)p(x,y)ds\Big) = tX_0^N(\epsilon^2).$$

So we see that the proof of the convergence using the Radon-Nikodým derivative is impossible for this particle system.

Remark 7. Even if we had "symmetrized" our contact process by speeding up the recovery if the particle has a lot of infected neighbours, the proof of the convergence seems difficult because the "function" $X_t^N(x)$ we integrate against the orthogonal martingale measure does not have a limit in dimensions $d > 2$. In particular this limit certainly cannot be b.

We can't take the limit in the two different times (the martingale measure for one part and the function integrated for other part like in [5]). We cannot use the mean-field simplification of our process because we look at the microscopic square variation in the drift.

5.2 Discrete time contact processes

We now want to see what the discrete time difficulties with the contact process are. We want to give the distribution of particles offspring. First we notice that there are two possible ways to count a birth in the branching random walk which does not exist in the contact process. The first one is to infect a particle already infected, which arrives with probability $\frac{\xi_n(x)}{M}$, the second one is to infect several times the same particle. For now, we will forget this second term.

A simple calculation shows that the offspring distribution is :

$$\Pi_{n,x}(k) = e^{-(1-\frac{\xi_n(x)}{M})V_n(x)} \frac{\left((1-\frac{\xi_n(x))}{M})V_n(x)\right)^k}{k!}.$$

We can now give the following definition :

Definition 12. A discrete time modified contact process with drift θ and village size M is a discrete time particle system with offspring distribution :

$$\Pi_{n,x}(k) = e^{-\lambda}\frac{\lambda^k}{k!},$$

where $\lambda = \lambda_n(x) = (1 - \frac{\xi_n(x)}{M})V_n(x)(1+\theta)$.

Remark 8. This modified contact process is a good approximation. The number of particles at one site for the contact process is handled by the number of particle for a branching random walk with drift θ. Moreover, the conditional expectation of the number of particle infected by two or more other ones at time $n+1$ at site x is obviously bounded by a constant times the square number of neighbours of this particle. And it is an easy calculation to show that this number of errors until time t is a $o_P(N)$ when N grows to the infinity.

We study the possible convergence of a sequence ξ^N of modified contact processes with drift $\frac{\theta}{N}$ and village size N. We denote again, in the same way than for the branching random walk with drift :

$$X_t^N = \frac{1}{N}\sum_{x\in\mathbf{Z}^d} \xi^N_{\lfloor Nt\rfloor}(x)\delta_{\frac{x}{\sqrt{N}}}.$$

We can compute the Radon-Nikodým derivative of these processes against branching random walks. We begin by computing

$$L_{n,x} = \exp\left[-V_n(x)((1-\frac{\xi_n^N(x)}{N})(1+\frac{\theta}{N})-1)\right]((1-\frac{\xi_n^N(x)}{N})(1+\frac{\theta}{N}))^{\xi_{n+1}(x)}.$$

We now easily compute the logarithm of the Radon-Nikodým derivative until time t of this modified contact process against the branching random walk :

$$\begin{aligned}\ln(L_t^N) &= \frac{1}{N}\sum_{n=0}^{Nt}\sum_x(\xi^N_{n+1}(x) - V_n(x))(\theta - \xi_n^N(x)) \\ &- \frac{1}{2N^2}\sum_{n=0}^{Nt}\sum_x \xi^N_{n+1}(x)(\xi_n^N(x)^2+\theta^2) - 2V_n(x)\xi_n^N(x)\theta + o_P(1).\end{aligned}$$

The convergence of the part of this derivative that involves θ is already well known. So we just have to take interest to the part of the drift related to $\xi_n^N(x)$. What we can easily show, in the same way than is the continuous time branching random walk, is that :

$$\mathbf{E}\left(\frac{1}{N}\sum_{n=0}^{\lfloor Nt\rfloor}\frac{(\xi_n^N(x)-\frac{b}{2})V_n(x)}{N}\right)\underset{N\to+\infty}{\longrightarrow}0.$$

But the convergence of the Radon-Nikodým derivative leads to

$$\mathbf{E}(\frac{1}{N^2}\sum_{n=0}^{Nt}\sum_x\xi_{n+1}^N(x)(\xi_n^N(x)^2-\frac{b^2}{4}))=\mathbf{E}(\frac{1}{2N}\sum_{n=0}^{Nt}\sum_x V_n(x)(\xi_n^N(x)^2-\frac{b^2}{4}))\underset{N\to+\infty}{\longrightarrow}0,$$

and in the same way as before, we see that the two simultaneous convergences are impossible.

Remark 9. Once again we see that the main problem is to find a limit for the integral against the orthogonal martingale measure of a drift $\xi_n(x)$ for which we can't even give a càdlàg limit.

In dimension 1, in [4], as the super-Brownian motion has a continuous density, this limit exists and the limit of the Radon-Nikodým derivative of the particle process can be found, jointly with the process itself. In [5], these methods can be used because of the existence of a limit of the occupation measure of the branching random walk in dimension 2 and 3.

So we see that this new difficulty in the proof of the convergence of the Radon-Nikodým derivative is of a different kind that is for the branching random walks with drift. It is not only the local differences of speed of the particle processes which make the proof difficult, there is also the fact that the drift $\xi_n(x)$ does not have a càdlàg limit.

We will now give a few more exemples of the fact that the drift needs to have a limit and that the mean-term simplification isn't seen in the Radon-Nikodým derivative.

6 Convergence for the voter model

In this section we will introduce two kinds of voter models, with short and long range interactions. These processes have already been studied many times and the convergence to super-Brownian motions is proved in [1].

A voter model is a particle system where at each site, there is a certain number of individuals. Each of those individuals can have the opinion 0 or 1. At each time, they chose one of their neighbours at random and adopt its opinion. We have the following definitions :

Definition 13. A continuous time voter model with village size M is a particle systems ξ_t : $\mathbf{Z}^d\to\{0,\cdots M\}$ with birth rate $(1-\frac{\xi_t(x)}{M})$ and death rate $(1-\frac{V_t(x)}{M})$. As a consequence, it evolves as the following :

$$\begin{cases}\xi_t(x)\to\xi_t(x)+1 \text{ at rate } V_t(x)\left(1-\dfrac{X_t(x)}{M}\right)\\ \xi_t(x)\to\xi_t(x)-1 \text{ at rate } \xi_t(x)\left(1-\dfrac{V_t(x)}{M}\right).\end{cases}$$

Definition 14. A discrete time voter model with village size M is a particle system $\xi_n : \mathbf{Z}^d \to \{0, \cdots M\}$ with binomial $\left(M, \frac{V_n(x)}{M}\right)$ offspring distribution. As a consequence, it evolves as the following :

$$\xi_{n+1}(x) = k \text{ with probability } \binom{M}{k}\left(\frac{V_n(x)}{M}\right)^k\left(1 - \frac{V_n(x)}{M}\right)^{M-k}.$$

6.1 Long range voter model

The study for the long range contact process consider a sequence of voter models with village size M_N growing to infinity. In other words, each particle has a large number of neighbours. Let ξ_t^N a sequence of rate N, village size M_N continuous time voter model. Let :

$$X_t^N = \frac{1}{N} \sum_{x \in \mathbf{Z}^d} \xi_t^N(x) \delta_{\frac{x}{\sqrt{N}}}.$$

The following theorem is proved in [1] :

Theorem 7. *If* $X_0^N \underset{N \to +\infty}{\Longrightarrow} \mu \in \mathcal{M}_f(\mathbb{R}^d)$, *then we have :*

$$X^N \underset{N \to +\infty}{\Longrightarrow} X$$

where X is a super-Brownian motion with branching rate 2 and diffusion σ^2.

Once again we compute the Radon-Nikodým derivative of these processes, of law R^N :

$$\left.\frac{dR^N}{dP^N}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\frac{1}{M_N} V_t^N\right),$$

where :

$$V_t^N = \sum_{x,y,n} \int_0^t \xi_s^N(x) \mathbf{1}_{\{\xi_{s-}^N(y) > n\}} d\widehat{\Lambda}_s^n(x,y) + \sum_{x,n} \int_0^t P^N X_s^N(x) \mathbf{1}_{\{\xi_{s-}^N(X) > n\}} d\widehat{\Lambda}_s^n(x,y).$$

The difficulties we already found in the continuous time process arise very quickly in the same way than in the contact process.

6.2 Around the short range voter model

The short range voter model is defined as a voter model with village size 1. In this case, we have also convergence to super-Brownian motion, but this time with a different rate multiplied by $\gamma = \mathbf{P}(\forall n > 0, B_n \neq 0)$ (see [1]). We can compute again the Radon-Nikodým derivatives, in the same way as in the long range voter model, but against branching random walks with rate γN.

In the continuous time model we have :

$$\left.\frac{dR^N}{dP^N}\right|_{\mathcal{F}_t} = \mathcal{E}(\tilde{V}_t^N),$$

where we denote :

$$\begin{aligned}\tilde{V}_t^N &= \gamma\Bigg(\sum_{x,y,n}\int_0^t(\xi_s^N(x)-b)\mathbf{1}_{\{\xi_{s-}^N(y)>n\}}d\widehat{\Lambda}_s^n(x,y)\\ &\quad+\sum_{x,n}\int_0^t(P^N X_s^N(x)-b)\mathbf{1}_{\{\xi_{s-}^N(X)>n\}}d\widehat{\Lambda}_s^n(x,y)\Bigg),\end{aligned}$$

using the relation $b=\dfrac{\gamma-1}{\gamma}$. This is similar to the contact process results, and the same difficulties arise. We notice that solving this problem would also finish the proof for the long range voter model.

7 The Lotka-Volterra model, a modification of the voter model

Here we study the Lotka-Volterra model, which is a modification of the voter model. As for our previous system, the Radon-Nikodým derivative will be computed with respect to the voter model, not the branching random walk. And for the computations, we will use the duality with the coalescing random walk (B_t^x). A Lotka-Volterra model is a model of competition between two species 0 and 1. When one of those individuals dies, it is immediately replaced at a rate depending on the both concentration of 0 and 1 near this individual. In this part it will be convenient to use this notation for the densities :

$$f_n^i(x)=\sum_e p(e)\mathbf{1}_{\{\xi_n(x+e)=i\}},\ \ i\in\{0,1\}$$

The convergence of this process to a Dawson-Watanabe superprocess has already been proved in [2], using the characterisation by the martingale problem (1) and the tightness of the sequence of processes. Let's now have a look on the Radon-Nikodým derivative in the continuous time model.

7.1 Continuous time Lotka-Volterra model

Definition 15. A continuous time Lotka-Volterra model with interaction parameters α_0 and α_1 is a particle system $\xi_t:\mathbf{Z}^d\to\{0,1\}$ with birth rate $(1-\xi_t(x))(f_t^0(x)+\alpha_0 f_t^1(x))$ and death rate $f_t^0(x)(f_t^1(x)+\alpha_1 f_t^0(x))$. It evolves as follows :

$$\begin{cases}\xi_t(x)\to\xi_t(x)+1 \text{ at rate } (1-\xi_t(x))(f_t^1(x)+(\alpha_0-1)f_t^1(x)^2)\\ \xi_t(x)\to\xi_t(x)-1 \text{ at rate } \xi_t(x)(f_t^0(x)+(\alpha_1-1)f_t^0(x)^2)\end{cases}$$

Let ξ_t^N be a sequence Lotka-Volterra models with rate N and interaction parameters $1+\dfrac{\theta_0}{N},1+\dfrac{\theta_1}{N}$. We denote as usual the rescaled process with law $\tilde{R}^N$:

$$X_t^N=\frac{1}{N}\sum_{x\in\mathbf{Z}^d}\xi_t^N(x)\delta_{\frac{x}{\sqrt{N}}}.$$

We begin with a few notations. For all $x \in \mathbf{Z}^d$, B^x is a coalescent random walk starting at x. If B^x and B^y collide, then they coalesce, i.e. if $B_t^x = B_t^y$, then $\forall s > t B_s^x = B_s^y$. We define also :

$$\tau(x,y) = \inf\{t > 0 | B_t^x = B_t^y\}$$

$$\beta = \sum_{e,e'} p(e)p(e')\mathbf{P}(\tau(0,e) = \tau(0,e') = +\infty, \tau(e,e') < +\infty)$$

$$\delta = \sum_{e,e'} p(e)p(e')\mathbf{P}(\tau(0,e) = \tau(0,e') = +\infty)$$

Now we would like to prove this theorem :

Theorem 8. *If* $X_0^N \underset{N\to+\infty}{\Longrightarrow} \mu \in \mathcal{M}_f(\mathbb{R}^d)$, *then :*

$$X^N \underset{N\to+\infty}{\Longrightarrow} X,$$

where X *is a Dawson-Watanabe super-process with branching rate* 2γ *and drift* $\beta\theta_0 - \delta\theta_1$.

To compute its Radon-Nikodým derivative against the rescaled voter model, we have to give a representation of the voter model : in the same way that we did for the branching random walk, we see that a short range voter model ξ_t is the unique strong solution of the following problem :

$$\xi_t(x) = \xi_0(x) + \sum_{x,y} \int_0^t (\xi_{s-}(y) - \xi_{s-}(x)) d\Lambda_s(x,y).$$

This notation enables us to write the Radon-Nikodým derivative :

$$\left.\frac{d\tilde{R}^N}{dR^N}\right|_{\mathcal{F}_t} = \mathcal{E}(M_t^{1,N} - M_t^{0,N}),$$

where we denote :

$$M_t^{1,N} = \sum_{x,y} \int_0^t \xi_{s-}^N(y)(1 - \xi_{s-}^N(x)) f_s^1(x) d\widehat{\Lambda}_s(x,y),$$

$$M_t^{0,N} = \sum_{x,y} \int_0^t \xi_{s-}^N(x)(1 - \xi_{s-}^N(y)) f_s^0(x) d\widehat{\Lambda}_s(x,y).$$

The convergence of the compensator divided by N is established in the same way as in the contact process case, but we can show that it is impossible to have the convergence for the Radon-Nikodým derivative using the same kind of arguments.

7.2 Discrete time Lotka-Volterra model

Definition 16. A discrete time Lotka-Volterra model with interaction parameters α_0 and α_1 is a particle system $\xi_n : \mathbf{Z}^d \to \{0, 1\}$ with offspring distribution evolving as follows :

$$\xi_n(x) \to \begin{cases} \xi_{n+1}(x) = 0 & \text{with probability } f_n^0(x) - \epsilon_n(x) \\ \xi_{n+1}(x) = 1 & \text{with probability } f_n^1(x) + \epsilon_n(x) \end{cases}$$

where we denote $\epsilon_n(x) = (\alpha_0 - 1) f_n^0(x)^2 (1 - \xi_n(x)) - (\alpha_1 - 1) f_n^1(x)^2 \xi_n(x)$.

Let ξ_t^N be a sequence Lotka-Volterra models with interaction parameters $1 + \dfrac{\theta_0}{N}, 1 + \dfrac{\theta_1}{N}$. We denote as usual the rescaled process with law $\tilde{R}^N$:

$$X_t^N = \frac{1}{N} \sum_{x \in \mathbf{Z}^d} \xi_{\lfloor Nt \rfloor}^N(x) \delta_{\frac{x}{\sqrt{N}}}.$$

Theorem 9. *If* $X_0^N \underset{N \to +\infty}{\Longrightarrow} \mu \in \mathcal{M}_f(\mathbb{R}^d)$, *then :*

$$X^N \underset{N \to +\infty}{\Longrightarrow} X,$$

where X *is a Dawson-Watanabe super-process with branching rate* γ *and drift* $\beta\theta_0 - \delta\theta_1$*, where :*

$$\beta = \sum_{e, e'} p(e) p(e') \mathbf{P}(\tau(0, e) = \tau(0, e') = +\infty, \tau(e, e') < +\infty)$$

$$\delta = \sum_{e, e'} p(e) p(e') \mathbf{P}(\tau(0, e) = \tau(0, e') = +\infty)$$

We now compute the Radon-Nikodým derivative of this process against the rescaled voter model, and we see that :

$$L_t^N = \prod_{n=0}^{\lfloor Nt \rfloor} \prod_x \left(1 + \xi_{n+1}(x) \frac{\epsilon_n(x)}{f_n^1(x)} + (1 - \xi_{n+1}(x)) \frac{\epsilon_n(x)}{f_n^0(x)} \right),$$

which can immediately be rewritten, using the fact that the orthogonal martingale measure of the discrete time voter model is a pure atomic measure M, i.e. $\forall k \in \mathbf{N}, \forall x \in \mathbf{Z}^d, \quad M(k, x) = \xi_k(x) - f_{k-1}^1(x) = M_{k,x}$ as the product over space and time of :

$$L_{n,x} = \left(1 + M_{n,x} \frac{\epsilon_n(x)}{f_n^0(x) f_n^1(x)} \right).$$

Once again, we would like the convergence of the martingale and its quadratic variation, which seems difficult to get always because of the same problem : ϵ_n does not converge to this constant.

8 Conclusion

We saw in all those examples that two mains problems arise while trying to prove the convergence of the Radon-Nikodým derivative of our processes to the limits we expect. The first one arises only in the continuous time processes, where we see that the derivative takes into account the time shifting if the two rates are not symmetric. This does not converge as N grows to the infinity, so the derivative doesn't either.

The second and more general problem is that in our processes, when there is no càdlàg limit of the drift, proving the convergence becomes harder, and in some cases we saw, impossible.

As a consequence, we see that to prove the convergence of most of the processes we studied, it seems easier to prove the tightness of the sequence, then characterize the limit using the martingale problem (1).

A Appendix

In this section we will prove the previous equation (3). Here, we will denote ξ_t^N a sequence of branching random walks with rate N and symmetric kernel $p_N = p(\frac{\cdot}{\sqrt{N}})$ on $\frac{\mathbf{Z}^d}{\sqrt{N}}$, such as :

$$\frac{1}{N}\xi_0^N \underset{N\to+\infty}{\Longrightarrow} \mu.$$

We denote $X_t^N = \frac{1}{N}\xi_t^N$, which is understood as a measure on $\mathbb{R}^d$.

For further calculations, we will write η a branching random walk with rate N and kernel p on $\mathbf{Z}^d$ starting with a unique particle at $t=0$ in position 0. Let $\left(\eta^{x,i}\right)_{x,i}$ denote a sequence of independent rate N branching random walks starting with one particle at x at time $t=0$.

We denote too :

- $(B_n)_{n\in\mathbf{N}}$ a simple random walk on $\mathbf{Z}^d$ with kernel p,
- $\Pi(t)$ a Poisson process with intensity 1,
- $V_t = B_{\Pi(t)}$ a simple (continuous time) random walk on $\mathbf{Z}^d$ with kernel p,
- $V_t', V_t'', \cdots$ independent copies of V_t

A.1 About the random walk

Here we will take interest of V_t the trajectory of each particle of a branching random walk. The upper bounds we give here will be very useful later.

Lemma 2. *There exists $C>0$ such as for any $s>0$ and $x\in\mathbf{Z}^d$:*

$$\mathbf{E}(p(x+V_s)) < C(1+s)^{-d/2}$$

Proof. We have $E(p(x+V_s)) = \sum_{k=0}^{+\infty} P(\Pi(s)=k)\mathbf{E}(p(x+V_k))$, which can be rewritten using the fact that :

$$\mathbf{E}(p(V_k+x)) = \mathbf{P}(V_{k+1}=-x) < C(1+k)^{-d/2}.$$

Then let's apply a large deviations result for the Poisson process, we have :

$$\mathbf{P}\left(\Pi(s) < \frac{s}{2}\right) \leq e^{-cs},$$

for some $c>0$. Then we have :

$$\mathbf{E}(p(x+V_s)) \leq \mathbf{P}(\Pi_s < \frac{s}{2}) + \left(1+\frac{s}{2}\right)^{-d/2}.$$

□

A.2 Average number of neighbours of a particle

We now have a look on the comportment of the average number of neighbours of a particle. As at the limit, only the local densities has a mean, we will study the following quantity :

$$Z_t^N(\phi) = \frac{1}{N}\sum_{x,y}\phi(x)\xi_t^N(x)\xi_t^N(y)p_N(x,y).$$

To do this, we will first take interest of this value when the branching random walk starts with a single particle at $t=0$, we denote :

$$Z_t'^N(\phi) = \sum_{x,y}\phi(x)\eta_t(x)\eta_t(y)p(x,y).$$

Lemma 3. *There exists $b>0$ such as for all sequence $\tau_N \to 0$ such as $N\tau_N \to +\infty$, we have :*

$$\mathbf{E}(Z_{\tau_N}'^N(1)) \underset{N\to+\infty}{\longrightarrow} b.$$

Moreover we have :

$$\begin{aligned} b &= \int_0^{+\infty}\mathbf{E}(p(V_s))ds = \int_0^{+\infty}\mathbf{P}(V_s+W=0)ds \\ &= \sum_{n=0}^{+\infty}\mathbf{E}(p(B_n)) = \sum_{n=1}^{+\infty}\mathbf{E}(\mathbf{1}_{\{B_n=0\}}). \end{aligned}$$

Proof. We know that for all $t>0$,

$$\begin{aligned} \mathbf{E}(\sum_{x,y}\eta_t(x)\eta_t(y)p(x,y)) &= \sum_{x,y}\eta_0(P_t(x))\eta_0(P_t(y))p(x,y) \\ &\quad +N\sum_{x,y}\int_0^t \mathbf{E}(\eta_u(P_{t-u}(x)P_{t-u}(y)) \\ &\quad +\eta_u(P^N(P_{t-u}(x)P_{t-u}(y))))p(x,y)du, \end{aligned}$$

which can be rewritten, exchanging the sum and the expectation, and using the initial condition :

$$\mathbf{E}(Z_t'^N(1)) = \mathbf{E}(p(V_{2Nt})) + 2N\int_0^t \mathbf{E}(p(V_{2Ns}))ds.$$

We use Lemma 2. to see that :

$$\mathbf{E}(p(V_{2N\tau_N})) \leq C(1+2N\tau_N)^{-d/2} \underset{N\to+\infty}{\longrightarrow} 0.$$

Moreover we can compute the other term :

$$\begin{aligned} 2N\int_0^{\tau_N} \mathbf{E}(p(V_{2Ns}))ds &= \int_0^{2Nt} \mathbf{E}(p(V_s))ds \\ &\underset{N\to+\infty}{\longrightarrow} \int_0^{+\infty} \mathbf{E}(p(V_s))ds, \end{aligned}$$

which is finite using again the bound given by Lemma 2.

Then we have :

$$\begin{aligned} b &= \int_0^{+\infty} \mathbf{E}(p(V_s))ds \\ &= \sum_{n=0}^{+\infty} \mathbf{E}(p(B_n)) \int_0^{+\infty} P(\Pi(s)=n)ds \\ &= \sum_{n=0}^{+\infty} \mathbf{E}(p(B_n)) \\ &= \sum_{n=0}^{+\infty} \sum_e p(e)\mathbf{P}(B_n = e) \\ &= \sum_{n=1}^{+\infty} \mathbf{P}(B_n = 0) \\ &= \sum_{n=0}^{+\infty} \mathbf{P}(B_{n+1} = 0) \int_0^{+\infty} P(\Pi(s)=n)ds \\ &= \int_0^{+\infty} \mathbf{P}(V_s + W = 0)ds \end{aligned}$$

□

We see that starting with a single particle, the total number of neighbours become fast equal to b. We will now have interest to the variance of this quantity, and try to find an upper bound.

Lemma 4. *There exists a positive constant $C > 0$ such as for all $t > 0$ and $N \in \mathbf{N}$:*

$$\mathbf{E}(Z_t'^N(1)^2) \leq CNt.$$

Proof. In this proof, we will denote C a finite number, which can be different during the calculations, but is independent of N and t. We have to compute the second moment of $Z_t'^N(1)$, :

$$\mathbf{E}(Z_t'^N(1)^2) = \sum_{a,b,c,d} \mathbf{E}(\eta_t(a)\eta_t(b)\eta_t(c)\eta_t(d))p(a,b)p(c,d).$$

We have to compute and upper bound for some forth moments of the branching random walk. We will use symmetries to reduce as far as possible the further calculations. We will often use the bound of Lemma 2. First let's reduce this computation to a computation of third order moment :

$$\begin{aligned}
& \mathbf{E}(Z_t'^N(1)^2) \\
= \quad & \mathbf{E}(p(V_{2Nt}))^2 \\
+ \quad & 4N \sum_{a,b} \int_0^t \mathbf{E}(\eta_s(a)\eta_s(b)\eta_s(1)) \\
& \mathbf{E}(p(b-a+V_{2N(t-s)}))\mathbf{E}(p(V_{2N(t-s)}))ds \\
+ \quad & 8N \sum_{a,b,c} \int_0^t \mathbf{E}(\eta_s(a)\eta_s(b)\eta_s(c)) \\
& \mathbf{E}(p(b-a+V_{2N(t-s)}))\mathbf{E}(p(c-a+V_{2N(t-s)}))ds.
\end{aligned}$$

So we can give the following inequality :

$$\begin{aligned}
\mathbf{E}(Z_t'^N(1)^2) \quad \leq \quad & 1 + CN \int_0^t \frac{1}{(1+N(t-s))^{d/2}} \\
& \sum_{a,b} \mathbf{E}(\eta_s(a)\eta_s(b)\eta_s(1))\mathbf{E}(p(b-a+V_{2N(t-s)}))ds.
\end{aligned}$$

Then we just need to give a good upper bound for mean under the integral :

$$\begin{aligned}
& \sum_{a,b} \mathbf{E}(\eta_s(a)\eta_s(b)\eta_s(1))\mathbf{E}(p(b-a+V_{2N(t-s)}) \\
= \quad & \mathbf{E}(p(V_{2Nt})) \\
& +2N \int_0^s \mathbf{E}(\eta_u(1)^2)\mathbf{E}(p(V_{2N(t-u)}))du \\
& +4N \sum_{a,b} \int_0^s \mathbf{E}(\eta_u(a)\eta_u(b))\mathbf{E}(p(b-a+V_{2N(t-u)}))du.
\end{aligned}$$

Now we can use the fact that $\mathbf{E}(\eta_u(1)^2) = 1 + 2Nu$, to bound this term by :

$$C(1 + \int_{N(t-s)}^{Nt} \frac{1+Nt}{(1+u)^{d/2}} du) \leq C(1+Nt).$$

Using this, we have finally :

$$E(Z_t'^N(1)^2) \leq C(1+Nt)(1 + \int_0^t \frac{1}{(1+N(t-s))^{d/2}}) \leq C(1+Nt).$$

This end the proof of our lemma $\square$

Remark 10. Have the same kind of bounds for the case $d = 2$ is not so complicate, we just have to take care of some logarithmic modifications.

We have now enough tools to prove the main result of this section : that the quantities X_t^N and Z_t^N are closely related, which gives us a convergence, at least for $t \geq \epsilon$ of the average number of neighbours of a particle to b. Let put this in words :

Theorem 10. *For all ϕ continuous Lipschitz, for all $t > 0$ we have :*

$$\mathbf{E}(\sup_{s\in[0,t]} | \int_0^s Z_u^N(\phi) - bX_u^N(\phi) du|) \underset{N\to+\infty}{\longrightarrow} 0.$$

Proof. The proof of the following will be done in a few step. First we will replace the quantities $X_t^N(\phi)$ and $Z_(^N\phi)$ by approximations : instead of counting the particles where they are, we will count them where were their ancestor a few time ago, and multiply by the number of descendants or the average neighbours of a descendant in the descendants. We have already seen that those quantities are not far for each other.

We will also replace b by an approximation, and prove the theorem for the modified quantities.

Finally we will prove that the approximation we took are good enough for this problem.

Let begin by fixing a sequence $\tau_N \underset{N\to+\infty}{\longrightarrow} 0$ such as $N\tau_N \underset{N\to+\infty}{\longrightarrow} +\infty$. τ_N is the time we will go back to find the ancestors of $\xi_t^N(x)$, it is fixed such as there is a lot of jumps in the time τ_N, but this interval becomes short.

By having a look at the definition, we see that a branching random walk starting with k particle is nothing else than k independent branching random walks starting with one particle.

For $t > 0$, we can use the Markov property to find a family $(\eta^{z,i})_{z\in\frac{\mathbf{Z}^d}{\sqrt{N}}, i\in\mathbf{N}}$ of independent branching random walks starting with one particle in $\sqrt{N}z$, such as :

$$\xi_t^N(x) = \sum_z \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \eta_{\tau_N\wedge t}^{z,i}(\sqrt{N}x),$$

where we take the convention that for $s < 0$, $\xi_s^N(x) = \xi_0^N(x)$.

Then the approximation we described before are clearly the following :

$$\tilde{X}_t^N(\phi) = \frac{1}{N}\sum_z \phi(z) \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \eta_{\tau_N\wedge t}^{z,i}(1)$$

for X^N, and :

$$\tilde{Z}_t^N(\phi) = \frac{1}{N}\sum_z \phi(z) \sum_{i=0}^{\xi_{t-\tau_N}^N(z)} \sum_{x,y} \eta_{\tau_N\wedge t}^{z,i}(x)\eta_{\tau_N\wedge t}^{z,i}(y)p(x,y)$$

We denote last $\tilde{b}_N = \mathbf{E}(Z'^N_{\tau_N\wedge t}(1))$, let's now prove the theorem for those quantities.

Lemma 5. *For all ϕ continuous Lipschitz, for all $t > 0$ we have :*

$$\mathbf{E}(\sup_{s\in[0,t]} |\int_0^s \tilde{Z}_u^N(\phi) - \tilde{b}_N \tilde{X}_u^N(\phi)du|) \underset{N\to+\infty}{\longrightarrow} 0.$$

Proof. We begin with a notation :

$$X_s^{z,i} = \eta_{s\wedge t}^{z,i}(1) \text{ and}$$
$$Z_s^{z,i} = \sum_{x,y} \eta_{s\wedge t}^{z,i}(x)\eta_{s\wedge t}^{z,i}(y)p(x,y).$$

Then we can rewrite the formula for the difference we are computing in this way :

$$\begin{aligned} & \tilde{Z}_t^N(\phi) - \tilde{b}_N \tilde{X}_t^N(\phi) \\ = \ & \frac{1}{N}\sum_z \phi(z) \sum_{i=0}^{\xi_{t-\tau}^N(z)} Z_{\tau_N}^{z,i} - \tilde{b}_N X_{\tau_N}^{z,i}. \end{aligned}$$

So $\tilde{Z}_t^N(\phi) - \tilde{b}_N\tilde{X}_t^N(\phi)$ is the sum of independent variable of mean 0. So it's mean is 0, and its variance is easily computed as :

$$\begin{aligned} & \mathbf{E}((\tilde{Z}_t^N(\phi) - \tilde{b}_N\tilde{X}_t^N(\phi))^2) \\ = \ & \frac{1}{N^2}\sum_z \phi(z)^2 \mathbf{E}(\xi_{t-\tau_N}^N(z))\mathbf{E}((Z_{\tau_N}^{z,i} - \tilde{b}_N X_{\tau_N}^{z,i})^2) \\ = \ & \frac{X_0^N(P_{t-\tau_N}(\phi^2))}{N}\mathbf{E}((Z'^N_{\tau_N\wedge t}(1) - \tilde{b}_N\eta_{\tau_N\wedge t}(1))^2) \\ \leq \ & C\|\phi\|_\infty^2 X_0^N(1)\tau_N. \end{aligned}$$

In the last inequality we used Lemma 3., which says that $\tilde{b}_N \underset{N\to+\infty}{\longrightarrow} b$, Lemma 4., the fact that $N\tau_N \underset{N\to+\infty}{\longrightarrow} +\infty$ and the Cauchy-Schwarz inequality to give the following upper bound :

$$\begin{aligned} & \mathbf{E}((Z'^N_{\tau_N\wedge t}(1) - \tilde{b}_N\eta_{\tau_N\wedge t}(1))^2) \\ = \ & \mathbf{E}((Z'^N_{\tau_N\wedge t}(1))^2) - 2\tilde{b}_N\mathbf{E}(Z'^N_{\tau_N\wedge t}(1)\eta_{\tau_N\wedge t}(1)) + \mathbf{E}((\eta_{\tau_N\wedge t}(1))^2) \\ \leq \ & C(1 + (\tau_N\wedge t)N) \leq CN\tau_N \end{aligned}$$

We can now handle the integral for $t \geq \tau_N$ with this bound :

$$\begin{aligned} & \int_{\tau_N}^t \mathbf{E}(|\tilde{Z}_s^N(\phi) - \tilde{b}_N\tilde{X}_s^N(\phi)|)ds \\ \leq \ & \int_{\tau_N}^t \mathbf{E}((\tilde{Z}_s^N(\phi) - \tilde{b}_N\tilde{X}_s^N(\phi))^2)^{1/2}ds \\ \leq \ & C\tau_N^{1/2}\|\phi\|_\infty X_0^N(1)^{1/2}t \underset{N\to+\infty}{\longrightarrow} 0. \end{aligned}$$

To handle the integral from 0 to τ_N, we just have to use trivial inequalities :

$$\begin{aligned}
&\int_0^{\tau_N} \mathbf{E}(|\tilde{Z}_s^N(\phi) - \tilde{b}_N \tilde{X}_s^N(\phi)|)ds \\
\leq\ & \|\phi\|_\infty \int_0^{\tau_N} \tilde{b}_N \mathbf{E}(\tilde{X}_s^N(1)) + \mathbf{E}(\tilde{Z}_s^N(1))ds \\
\leq\ & \|\phi\|_\infty (X_0^N(1)\tau_N b_{\tau_N} + X_0^N(1)\mathbf{E}(Z'^N_{\tau_N}(1)) \\
\leq\ & C\|\phi\|_\infty X_0^N(1)\tau_N \underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

These two bounds end the proof since we have :

$$\begin{aligned}
\mathbf{E}(\sup_{s\in[0,t]} |\tilde{Z}_s^N(\phi) - \tilde{b}_N \tilde{X}_s^N(\phi)du| &\leq \mathbf{E}(\int_0^t |\tilde{Z}_s^N(\phi) - \tilde{b}_N \tilde{X}_s^N(\phi)|du) \\
&\leq \int_0^t \mathbf{E}(|\tilde{Z}_s^N(\phi) - \tilde{b}_N \tilde{X}_s^N(\phi)|)du \\
&\leq C\|\phi\|_\infty X_0^N(1)(\tau_N)^{1/2} t \underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

□

We now have to prove that the approximations of the quantities we took are really close to what they approximate.

Lemma 6. *For all ϕ continuous Lipschitz, for all $t > 0$, we have :*

$$\mathbf{E}(\int_0^t |X_s^N(\phi) - X_s^{N,\tau_N}(\phi)|ds) \underset{N\to+\infty}{\longrightarrow} 0.$$

Proof. Let begin by giving an estimate of the difference between the two terms :

$$\begin{aligned}
|X_t^N(\phi) - \tilde{X}_t^N(\phi)| &= |\frac{1}{N}\sum_z \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \sum_x (\phi(x) - \phi(z))\eta_{\tau_N\wedge t}^{z,i}(x\sqrt{N})| \\
&\leq \frac{1}{N}\sum_z \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \sum_x |\phi(x) - \phi(z)|\eta_{\tau_N\wedge t}^{z,i}(x\sqrt{N}) \\
&\leq \frac{C}{N}\sum_z \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \sum_x \|z - x\|\eta_{\tau_N\wedge t}^{z,i}(x\sqrt{N}).
\end{aligned}$$

But we see that this last bound is a sum of iid variables, so its mean is at most :

$$\frac{C}{N}\mathbf{E}(\xi_{t-\tau_N}^N(1))\mathbf{E}(\sum_x \frac{\|x\|}{\sqrt{N}}\eta_{\tau_N\wedge t}(x)),$$

and moreover we have :

$$\mathbf{E}(\sum_x \|x\|\eta_{\tau_N\wedge t}(x)) = \mathbf{E}(|V_{N\tau_N}|) \leq C\sqrt{N\tau_N}.$$

These bound allow us to end the proof since :

$$\mathbf{E}(\int_0^t |X_s^N(\phi) - \tilde{X}_s^N(\phi)|ds) \leq CtX_0^N(1)\sqrt{\tau_N} \underset{N\to+\infty}{\longrightarrow} 0.$$

□

Last but not least, we have to prove that $\tilde{Z}^N$ is a good approximation of Z^N. But if we write Z^N in terms of $\eta^{z,i}$, we can break this proof in two parts :

$$\begin{aligned}
Z_t^N(\phi) &= \sum_{x,y} \phi(x)\xi_t^N(x)\xi_t^N(y)p(x,y) \\
&= \sum_{z,z'} \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \sum_{j=1}^{\xi_{t-\tau_N}^N(z')} \sum_{x,y} \phi(x)\eta_{\tau_N\wedge t}^{z,i}(x)\eta_{\tau_N\wedge t}^{z',j}(y)p(x,y) \\
&= \sum_{z} \sum_{i=1}^{\xi_{t-\tau}^N(z)} \sum_{x,y} \phi(x)\eta_{\tau_N\wedge t}^{z,i}(x)\eta_{\tau_N\wedge t}^{z,i}(y)p(x,y) \\
&+ \sum_{z,z'} \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \sum_{j=1,(z,i)\neq(z',j)}^{\xi_{t-\tau_N}^N(z')} \sum_{x,y} \phi(x)\eta_{\tau_N\wedge t}^{z,i}(x)\eta_{\tau_N\wedge t}^{z',j}(y)p(x,y).
\end{aligned}$$

We will begin by evaluate the first of these two terms which is close to $\tilde{Z}_t^N(\phi)$ on its own.

Lemma 7. *We denote in the following :*

$$Z_t^{N,1}(\phi) = \sum_{z} \sum_{i=1}^{\xi_{t-\tau}^N(z)} \sum_{x,y} \phi(x)\eta_{\tau_N\wedge t}^{z,i}(x)\eta_{\tau_N\wedge t}^{z,i}(y)p(x,y)$$

For all ϕ continuous Lipschitz, for all $t > 0$, we have :

$$\mathbf{E}(\int_0^t |Z_s^{N,1}(\phi) - \tilde{Z}_s^N(\phi)|ds) \underset{N\to+\infty}{\longrightarrow} 0.$$

Proof. In the same way than the previous one, we begin to handle the difference between the two evaluated quantities :

$$\begin{aligned}
|Z_t^{N,1}(\phi) &- \tilde{Z}_t^N(\phi)| \\
&\leq \frac{1}{N}\sum_{z} \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \sum_{x,y} |\phi(\frac{x}{\sqrt{N}}) - \phi(z)|\eta_{\tau_N\wedge t}^{z,i}(x)\eta_{\tau_N\wedge t}^{z,i}(y)p(x,y) \\
&\leq \frac{C}{N}\sum_{z} \sum_{i=1}^{\xi_{t-\tau_N}^N(z)} \sum_{x,y} \|z - \frac{x}{\sqrt{N}}\|\eta_{\tau_N\wedge t}^{z,i}(x)\eta_{\tau_N\wedge t}^{z,i}(y)p(x,y)
\end{aligned}$$

Once again, this last bound is a sum of iid variables, so its mean is at most :

$$\frac{C}{N}\mathbf{E}(\xi^N_{t-\tau_N}(1))\mathbf{E}(\sum_{x,y}\frac{\|x\|}{\sqrt{N}}\eta_{\tau_N\wedge t}(x)\eta_{\tau_N\wedge t}(y)p(x,y)),$$

and moreover we have :

$$\begin{aligned}
& \mathbf{E}(\sum_{x,y}\|x\|\eta_s(x)\eta_s(y)p(x,y)) \\
= \ & \mathbf{E}(\|V_{Ns}\|p(V_{2Ns})) \\
+ \ & N\int_0^s \mathbf{E}((\|V_{N(s-u)+V'_{Nu}}\| + \|V_{N(s-u)} + V'_{Nu} + W\|)p(V_{2N(s-u)}))du \\
\leq \ & C\sqrt{Ns}(1 + \int_0^s \frac{Ndu}{(1+Nu)^{d/2}} \\
\leq \ & C\sqrt{Ns},
\end{aligned}$$

where in the first upper bound, we used the following inequality :

$$\begin{aligned}
\mathbf{E}(\|V_{t-s} + V'_s\|p(V_{2(t-s)})) &= \mathbf{E}(\|V_{t-s} + V'_s\|\mathbf{E}(p(V_{2(t-s)})|V_{t-s})) \\
&\leq \frac{C}{(1+t-s)^{d/2}}\mathbf{E}(\|V_t\|).
\end{aligned}$$

So we now see that :

$$\mathbf{E}(\int_0^t |Z^{N,1}_s(\phi) - \tilde{Z}^N_s(\phi)|ds) \leq CtX^N_0(1)\sqrt{\tau_N}\underset{N\to+\infty}{\longrightarrow} 0.$$

Which end this proof, very similar to the previous one. □

We now need one last theorem to handle the term due to the interferences between particle which are not close relative (i.e. if they have a common ancestor, this one died more than τ_N ago).

Lemma 8. *We denote in the following :*

$$Z^{N,2}_t(\phi) = \frac{1}{N}\sum_{z,z'}\sum_{i=1}^{\xi^N_{t-\tau_N}(z)}\sum_{j=1,(z,i)\neq(z',j)}^{\xi^N_{t-\tau_N}(z')}\sum_{x,y}\phi(x)\eta^{z,i}_{\tau_N\wedge t}(x)\eta^{z',j}_{\tau_N\wedge t}(y)p(x,y).$$

For all ϕ continuous Lipschitz, for all $t>0$, we have :

$$\mathbf{E}(\sup_{s\in[0,t]}|\int_0^s Z^{N,2}_u(\phi)du|)\underset{N\to+\infty}{\longrightarrow} 0.$$

Proof. We begin to handle $\mathbf{E}(|Z_t^{N,2}(\phi)|)$ for $t > \tau_N$:

$$\begin{aligned}
\mathbf{E}(|Z_t^{N,2}(\phi)|) &\leq \frac{\|\phi\|_\infty}{N}\sum_{z,z'}\mathbf{E}(\xi^N_{t-\tau_N}(z)\xi^N_{t-\tau_N}(z')) \\
&\quad \sum_{x,y}\mathbf{E}(\eta_{\tau_N}(x))\mathbf{E}(\eta_{\tau_N}(y))p(x+z,y+z') \\
&\leq \frac{\|\phi\|_\infty}{N}\sum_{z,z'}\mathbf{E}(\xi^N_{t-\tau_N}(z)\xi^N_{t-\tau_N}(z'))\mathbf{E}(p(z'-z+V_{2N\tau_N})) \\
&\leq \frac{C}{N}\sum_{z,z'}\xi_0^N(z)\xi_0^N(z')\mathbf{E}(p(z'-z+V_{2Nt})) \\
&+ C\int_{\tau_N}^t \xi_0^N(1)\mathbf{E}(p(V_{2Ns}))ds \\
&\leq CNX_0^N(1)^2\frac{1}{(1+Nt)^{d/2}} + CX_0^N(1)\int_{2N\tau_N}^{2Nt}\frac{ds}{(1+s)^{d/2}}.
\end{aligned}$$

So we have :

$$\begin{aligned}
\int_{\tau_N}^t \mathbf{E}(|Z_s^{N,2}(\phi)|)ds &\leq CX_0^N(1)^2\int_{N\tau_N}^{Nt}\frac{ds}{(1+s)^{d/2}} \\
&\quad +\frac{C}{N}X_0^N(1)\int_{2N\tau_N}^{2Nt}\int_{2N\tau_N}^{s}\frac{du}{(1+u^{d/2})} \\
&\leq C(X_0^N(1)^2 + X_0^N(1)t\int_{N\tau_N}^{+\infty}\frac{ds}{(1+s)^{d/2}} \\
&\quad \underset{N\to+\infty}{\longrightarrow} 0
\end{aligned}$$

We now take care to the other integral, using the fact that μ is atomless to give the conclusion :

$$\begin{aligned}
&|\int_0^{\tau_N}\mathbf{E}(Z_s^{N,2}(\phi))ds| \\
\leq\ & \frac{C}{N^2}\sum_{z,z'}\xi_0^N(z)\xi_0^N(z')\int_0^{2N\tau_N}\mathbf{E}(p(z'-z+V_s))ds \\
\leq\ & \frac{C}{N^2}\sum_{z,z'}\xi_0^N(z)\xi_0^N(z')\sum_{n=0}^{+\infty}\mathbf{E}(p(z'-z+B_n))\int_0^{2N\tau_N}\mathbf{P}(\Pi(s)=n)ds \\
\leq\ & CX_0^N(1)^2\frac{1}{(1+N)^{d/2}}\int_0^{2N\tau_N}\mathbf{P}(\Pi(s)>N)ds \\
& +CX_0^N \times X_0^N(\{\|y-x\|<\epsilon\}) + CX_0^N(1)^2\sum_{n=\epsilon\sqrt{N}}^{N}(1+n)^{-d/2}ds \\
& \underset{N\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

□

This last lemma end the proof of the Theorem 10, and in particular the proof of (3) □

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