A short proof of the asymptotic of the minimum of the branching random walk after time n

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Abstract

We write R_n for the minimal position attained after time n by a branching random walk in the boundary case. In this article, we prove that $R_n - \frac{1}{2}\log n$ converges in law toward a shifted Gumbel distribution.

1 Introduction

A branching random walk on $\mathbb{R}$ is particle system defined as follows. It starts with a unique individual located at position 0 at time 0. At each time $n \in \mathbb{N}$, every individual currently alive in the process dies, giving birth to children that are positioned around their parent according to i.i.d. versions of a point process. We write $\mathbf{T}$ for the genealogical tree of the process. For any $u \in \mathbf{T}$, we denote by $V(u)$ the position of individual u and $|u|$ the generation to which u belongs. The branching random walk is the random marked tree $(\mathbf{T}, V)$. In this article, we take interest in the asymptotic behaviour of the smallest position reached after time n, defined by

$$R_n = \inf_{u \in \mathbf{T}, |u| \geq n} V(u). \tag{1.1}$$

We assume that the branching random walk is in the boundary case:

$$\mathbb{E}\left(\sum_{|u|=1} 1\right) > 1, \ \ \mathbb{E}\left(\sum_{|u|=1} e^{-V(u)}\right) = 1 \text{ and } \mathbb{E}\left(\sum_{|u|=1} V(u)e^{-V(u)}\right) = 0, \tag{1.2}$$

and that the displacement law is non-lattice. It is well-known (see e.g. the discussions in [BG11, Jaf12]) that under mild integrability assumptions, a branching random walk can be mapped with a branching random walk in the boundary case by an affine transformation. We also assume the following, classical, integrability assumptions on the reproduction law of the process

$$\sigma^2 := \mathbb{E}\left(\sum_{|u|=1} V(u)^2 e^{-V(u)}\right) \in (0, +\infty) \tag{1.3}$$

$$\mathbb{E}\left(\sum_{|u|=1} e^{-V(u)} \log_+\left(\sum_{|u|=1} (1 + V(u)_+)e^{-V(u)}\right)^2\right) < +\infty. \tag{1.4}$$

These conditions replace, in some sense, the $L \log L$ integrability condition for the Galton-Watson process.

Under assumption (1.2), the Galton-Watson tree $\mathbf{T}$ is supercritical, and the survival event $\{\#\mathbf{T} = +\infty\}$ occurs with positive probability. For any $n \in \mathbb{N}$, we introduce

$$W_n = \sum_{|u|=n} e^{-V(u)} \quad \text{and} \quad Z_n = \sum_{|u|=n} V(u) e^{-V(u)}. \tag{1.5}$$

By (1.2) and the branching property of the branching random walk, (W_n) and (Z_n) are martingales. They are respectively called the additive martingale and the derivative martingale. Under assumptions (1.3) and (1.4), there exists a random variable Z_∞ which is a.s. positive on S such that

$$\lim_{n \to +\infty} W_n = 0 \quad \text{and} \quad \lim_{n \to +\infty} Z_n = Z_\infty \quad \text{a.s.} \tag{1.6}$$

This result was proved in [BK04] under stronger integrability assumptions, then extended in [Aïd13]. It is proved in [Che15] that assumption (1.4) is necessary and sufficient for (1.6) to hold. Aïdékon and Shi [AS10] obtained the precise asymptotic behaviour of W_n. This result is recalled in Fact 2.2.

The asymptotic behaviour of the extremal individuals in the branching random walk has been an object of interest in the recent years. We denote by $M_n = \min_{|u|=n} V(u)$ the minimal displacement at time n. Hammersley [Ham74], Kingman [Kin75] and Biggins [Big76] proved that $\frac{M_n}{n}$ converges almost surely to 0. The second order has been obtained independently by Addario-Berry and Reed [ABR09], who proved that $M_n - \frac{3}{2} \log n$ is tight, and by Hu and Shi [HS09], who obtained that $\frac{M_n}{\log n}$ converges in probability toward $\frac{3}{2}$, while experiencing almost sure fluctuations. Finally, Aïdékon [Aïd13] completed the picture by proving the convergence in law of $M_n - \frac{3}{2} \log n$. A more precise statement of this result is given in Fact 2.1.

In this article, we study the asymptotic behaviour of $R_n = \min_{k \geq n} M_k$, the lowest position reached after time n. Using previous sharp estimates on the branching random walks from [Aïd13, AS10, Mad16], we compute the joint convergence in law of Z_n, M_n and R_n.

Theorem 1.1. *Under assumptions* (1.2)*,* (1.3) *and* (1.4)*, we have*

$$\lim_{n \to +\infty} \left(Z_n, M_n - \tfrac{3}{2} \log n, R_n - \tfrac{1}{2} \log n \right) = (Z_\infty, W, L) \quad \text{in law,}$$

where $\mathbb{P}(W \geq x, L \geq y | Z_\infty) = \exp\left(-c_* Z_\infty e^x - c' Z_\infty e^y\right)$*,* c_* *is the constant defined in Fact 2.1,* c_M *the constant defined in Fact 2.3 and* $c' = \sqrt{\frac{2}{\pi\sigma^2}} c_M$*.*

As a side result, we observe that

$$\lim_{n \to +\infty} \frac{R_n}{\log n} = \frac{1}{2} \quad \text{a.s. on the event} \quad \{\#\mathbf{T} = +\infty\}. \tag{1.7}$$

In particular, we observe that $\frac{R_n}{\log n}$ does not fluctuates almost surely at scale n, contrarily to M_n the smallest displacement at time n (see [HS09]).

2 Proof of Theorem 1.1

In a first time, we recall some branching random walk estimates that we use to prove Theorem 1.1. We first give a precise statement for the convergence in law of $M_n - \frac{3}{2}\log n$.

Fact 2.1 (Aïdékon [Aïd13]). *Under assumptions* (1.2), (1.3) *and* (1.4) *we have*

$$\lim_{n\to+\infty} \left(Z_n, M_n - \tfrac{3}{2}\log n\right) = (Z_\infty, W) \quad \textit{in law,}$$

where $\mathbb{P}(W \geq x | Z_\infty) = \exp\left(-c_* Z_\infty e^x\right)$ *and* c_* *is a fixed constant, that depends only on the reproduction law of the branching random walk.*

More precisely, in [Aïd13, Theorem 1.1] we have only the convergence in law of $M_n - \frac{3}{2}\log n$, but the joint convergence of (Z_n, M_n) follows immediately from the proof. This result can also be obtained as a straightforward consequence of [Mad15, Theorem 1.1].

By (1.6), we have $\lim_{n\to+\infty} W_n = 0$ a.s. Aïdékon and Shi [AS10] computed the rate of decay for this quantity.

Fact 2.2 (Aïdékon and Shi [AS10]). *Under assumptions* (1.2), (1.3) *and* (1.4), *we have*

$$\lim_{n\to+\infty} n^{1/2} W_n = \sqrt{\frac{2}{\pi\sigma^2}} Z_\infty \quad \textit{in probability.}$$

Finally, Madaule [Mad16] obtained a precise estimate for the left tail asymptotic of R_0 the minimal position attained by the branching random walk.

Fact 2.3 (Madaule [Mad16]). *Under assumptions* (1.2), (1.3) *and* (1.4), *there exists* $c_M > 0$ *such that* $\lim_{x\to+\infty} e^x \mathbb{P}(R_0 \leq -x) = c_M$.

Using these two additional results, we are able to compute the joint asymptotic behaviour of Z_n, M_n and R_n.

Proof of Theorem 1.1. For any $n \in \mathbb{N}$, we denote by $\mathcal{F}_n = \sigma(u, V(u), |u| \leq n)$. For any $x, y \in \mathbb{R}$, we write

$$r_n(x) = \frac{1}{2}\log n + x \quad \text{and} \quad m_n(y) = \frac{3}{2}\log n + y.$$

For any $u, v \in \mathbf{T}$, we write $u < v$ if u is an ancestor of v. We observe that

$$R_n = \min_{|u|\geq n} V(u) = \min_{|u|=n} \left(V(u) + \min_{v>u} V(v) - V(u) \right),$$

where, by the branching property $(\min_{v>u} V(v) - V(u), |u| = n)$ are i.i.d. copies of R_0 that are independent with $\mathcal{F}_n$. Consequently, we have

$$\mathbb{P}\left(R_n \geq r_n(x), M_n \geq m_n(y) | \mathcal{F}_n\right) = \mathbf{1}_{\{M_n \geq m_n(y)\}} \prod_{|u|=n} \phi(r_n(x) - V(u)) \quad \text{a.s.,}$$

where we set $\phi(z) = \mathbb{P}(R_0 \geq z)$. By Fact 2.3, for any $\varepsilon > 0$, there exists $A > 0$ such that for any $z \leq -A$, we have

$$1 - (c_M + \varepsilon)e^z \leq \phi(z) \leq 1 - (c_M - \varepsilon)e^z.$$

Consequently, for any $n \geq e^{y-x+A}$, we have a.s. on the event $\{M_n \geq m_n(y)\}$,

$$\mathbb{P}\left(R_n \geq r_n(x), M_n \geq m_n(y) | \mathcal{F}_n\right) \geq \prod_{|u|=n} \left(1 - (c_M + \varepsilon) e^{r_n(x) - V(u)}\right)$$

$$\geq \exp\left(-\sum_{|u|=n} (c_M + \varepsilon) e^{r_n(x) - V(u)}\right) = \exp\left(-(c_M + \varepsilon) e^x n^{1/2} W_n\right).$$

Moreover, there exists $\delta > 0$ such that $(1 - h) \leq e^{-(1-\varepsilon)h}$ for any $0 \leq h < \delta$. Therefore, a.s. on the event $\{M_n \geq m_n(y)\}$ we have for any n large enough,

$$\mathbb{P}\left(R_n \geq r_n(x), M_n \geq m_n(y) | \mathcal{F}_n\right) \leq \prod_{|u|=n} \left(1 - (c_M - \varepsilon) e^{r_n(x) - V(u)}\right)$$

$$\leq \exp\left(-(1-\varepsilon)(c_M + \varepsilon) e^x n^{1/2} W_n\right).$$

By Fact 2.2 and Slutsky's lemma, we can extend Fact 2.1 into

$$\lim_{n \to +\infty} (n^{1/2} W_n, Z_n, M_n - m_n(0)) = \left(\sqrt{\frac{2}{\pi\sigma^2}} Z_\infty, Z_\infty, W\right)$$

As a consequence, for any continuous bounded function ϕ, letting $n \to +\infty$ then $\varepsilon \to 0$, we have

$$\lim_{n \to +\infty} \mathbb{E}\left(\phi(Z_n) \mathbf{1}_{\{M_n \geq m_n(y), R_n \geq r_n(x)\}}\right) = \mathbb{E}\left(\phi(Z_\infty) \mathbf{1}_{\{W \geq y\}} e^{-c' e^x Z_\infty}\right),$$

which concludes the proof. □

In a second time, we prove (1.7). For any $n \in \mathbb{N}$, we write

$$\tau_n = \inf\{k \geq n : M_k = R_n\}.$$

We observe easily that $\frac{R_n}{\log n} \geq \frac{M_{\tau_n}}{\log n} \geq \frac{M_{\tau_n}}{\log \tau_n}$. Thus by [HS09, Theorem 1.2], we have $\liminf_{n \to +\infty} \frac{R_n}{\log n} \geq \liminf_{n \to +\infty} \frac{M_n}{n} = \frac{1}{2}$ a.s.

To prove the upper bound, we use [Hu15, Lemma 3.5]. There exists $c > 0$ and $K > 0$ such that for any $n \in \mathbb{N}$ and $0 \leq \lambda \leq \frac{\log n}{3}$, we have

$$\mathbb{P}\left(\exists u \in \mathbf{T} : |u| \in [n, 2n], V(u) - \tfrac{1}{2}\log n + \lambda \in [0, K]\right) \geq c e^{-\lambda}.$$

In particular $\mathbb{P}(R_n \leq \frac{1}{2}\log n + K) \geq c$. We conclude with a cutting argument.

By [Mal16, Lemma 2.4], there exists $a > 0$ and $\varrho > 1$ such that almost surely on $\{\#\mathbf{T} = +\infty\}$, for any k large enough we have

$$\#\{u \in \mathbf{T} : |u| = k, V(u) \leq ka\} \geq \varrho^k.$$

As each individual u alive at time k starts an independent branching random walk from position $V(u)$, for any $\varepsilon > 0$, we have almost surely on $\{\#\mathbf{T} = +\infty\}$, for all n large enough,

$$\mathbb{P}\left(R_{n+\varepsilon \log n} \geq (\tfrac{1}{2} + a\varepsilon)\log n + K \middle| \mathcal{F}_{\varepsilon \log n}\right) \leq (1-c)^{\varrho^{\varepsilon \log n}}.$$

We conclude by Borel-Cantelli lemma that $\limsup_{n \to +\infty} \frac{R_n}{\log n} \leq \frac{1}{2}$ a.s, which completes the proof of (1.7).

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