THE DERRIDA-RETAUX MODEL ON A GEOMETRIC GALTON–WATSON TREE

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ABSTRACT. We consider a generalized Derrida-Retaux model on a Galton-Watson tree with a geometric offspring distribution. For a class of recursive systems, including the Derrida-Retaux model with either a geometric or exponential initial distribution, we characterize the critical curve using an involution-type equation and prove that the free energy satisfies the Derrida-Retaux conjecture.

1. INTRODUCTION

The Derrida-Retaux model, henceforth referred to as the DR model, is a max-type recursive equation in distribution introduced by the physicists Derrida and Retaux [16] as a toy model for studying the depinning transition in the limit of strong disorder. After a simple change of variables, the model can be described as a family of recursively defined probability distributions $(\nu_n, n \geq 1)$ on $\mathbb{R}_+$. Specifically, for all $n \in \mathbb{Z}_+$, we consider two independent random variables $Y_n^{(1)}$ and $Y_n^{(2)}$ with law ν_n. Then the random variable Y_{n+1} is defined as

$$Y_{n+1} \;=\; \big(Y_n^{(1)} + Y_n^{(2)} - 1\big)_+\,, \tag{1.1}$$

where $x_+ := \max\{x,\, 0\}$ denotes the positive part of $x \in \mathbb{R}$, and it follows the law ν_{n+1}. This model was previously studied by Collet, Eckmann, Glaser, and Martin [12] for probability distributions on $\mathbb{Z}_+$, where it served as a toy model for spin glass studies.

One of the most notable features of the DR model is its connection to an infinite-order Berezinskii-Kosterlitz-Thouless (BKT) phase transition. This phenomenon, observed in disordered systems, is characterized by the absence of a stable distributional fixed point under renormalization. This connection has attracted significant interest, as it provides insight into the critical thresholds of hierarchical pinning models. The free energy of the model $(\nu_n)_{n\geq 0}$, starting from an initial distribution $\nu_0 = \nu$, is given by

$$F_Y(\nu) \;:=\; \lim_{n\to\infty} 2^{-n} \int x\nu_n(\mathrm{d}x) \;=\; \lim_{n\to\infty} 2^{-n}\,\mathbb{E}(Y_n) \;\in\; [0,\infty), \tag{1.2}$$

where Y_n is a random variable with distribution ν_n. Note that the existence of the above limit follows directly from the fact that $\mathbb{E}(Y_{n+1}) \leq 2\,\mathbb{E}(Y_n)$, as implied by equation (1.1). The critical point of the system is defined as

$$p_c \;:=\; \inf\{p > 0 : F_Y((1-p)\delta_0 + p\nu) > 0\}.$$

The results of Collet et al. [12] show that, for $\nu = \delta_2$, the critical point is

$$p_c \;=\; \frac{1}{5}.$$

For further details and references on the recursive equation (1.1), we refer to [16] and [18]. While the definition of the model $\{Y_n\}$ is relatively simple, it exhibits intricate

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behavior at criticality, making its rigorous analysis challenging and leaving many fundamental questions unresolved. It is widely believed that for a broad class of recursive models, including those described by (1.1), the hierarchical renormalization model, and the pinning model (see [15], [19], and [5]), the transition at the critical point is of infinite order. Specifically, for the model in equation (1.1), Derrida and Retaux [16] conjectured the existence of a constant $C > 0$ such that

$$F_Y((1-p)\delta_0 + p\nu) \;=\; \exp\left(-(C+o(1))(p-p_c)^{-1/2}\right), \quad \text{as } p \downarrow p_c. \tag{1.3}$$

Naturally, instead of the sum $Y_n^{(1)} + Y_n^{(2)}$ in (1.1), we may generalize the model by considering the sum $Y_n^{(1)} + ... + Y_n^{(m)}$ for any integer $m \geq 2$, where $Y_n^{(1)}, ..., Y_n^{(m)}$ are independent copies of Y_n. In this case, for all $n \geq 0$, the recursion becomes

$$Y_{n+1} \overset{\text{(law)}}{=} (Y_n^{(1)} + ... + Y_n^{(m)} - 1)_+, \tag{1.4}$$

where the law of Y_0 is given by $(1-p)\delta_0 + p\nu$, with ν a probability distribution on $\mathbb{N}$. The corresponding critical value p_c will naturally depend on the initial distribution ν. A weaker form of the conjecture in equation (1.3) was proved in [7] for the model in equation (1.4). Specifically, assuming that $\int x^3 m^x \nu(\mathrm{d}x) < \infty$ (an integrability condition that is also necessary), it was shown that

$$F_Y((1-p)\delta_0 + p\nu) \;=\; \exp\left(-\,(p-p_c)^{-(1/2)+o(1)}\right), \quad \text{as } p \downarrow p_c. \tag{1.5}$$

Moreover, Chen [6] recently established the infinite differentiability of the function $p \mapsto F_Y((1-p)\delta_0 + p\nu)$, thereby proving that the phase transition is of infinite order.

The upper bound for the free energy obtained in [7] is more precise than the expression in (1.5), as it shows that the free energy is bounded by $Ae^{-\delta(p-p_c)^{1/2}}$ for some constants $A, \delta > 0$. However, eliminating the $o(1)$ term in the lower bound of [7], or determining the exact constant in (1.3), remains a much more challenging task. Nevertheless, an exactly solvable version of a continuous-time generalization of the DR model has been described in [21]. The integrability of the continuous model allows for a more detailed study of the phase transition near criticality, confirming the corresponding version of (1.3). We also refer to [8] for the associated partial differential equations.

This work aims to investigate and analyze a class of exactly solvable discrete-time Derrida-Retaux models and to prove the Derrida-Retaux conjecture (1.3) for them. Specifically, we consider a class where the parameter m in (1.4) is replaced by a random variable with a geometric distribution, independent of the sequence $(Y_n^{(j)})_{j\geq 1}$. From a recursive tree-based perspective, this modification corresponds to replacing a regular m-ary tree with a Galton-Watson tree that has a geometric offspring distribution.

More precisely, we consider the *generalized DR model* $(\nu_n)_{n\geq 0}$, which is recursively defined as follows. Let $\mathtt{R}$ be a geometric random variable with parameter p, written shorthand as $\mathtt{R} \overset{\text{(d)}}{=} \mathcal{G}(p)$, and let $\mathtt{Z}$ be an independent nonnegative random variable. For $n \in \mathbb{N}$, let $(X_n^{(k)}, k \geq 1)$ be i.i.d. random variables with common law ν_n, independent of $(\mathtt{R}, \mathtt{Z})$. Then ν_{n+1} is defined as the law of X_{n+1}, where

$$X_{n+1} \;=\; \left(\sum_{j=1}^{\mathtt{R}} X_n^{(k)} \;-\; \mathtt{Z}\right)_+ . \tag{1.6}$$

Observe that, in particular, if the laws of ν_0 and $\mathtt{Z}$ are supported on $\mathbb{Z}_+$, the process can be interpreted as a parking process on a Galton-Watson tree as follows. Start with a Galton-Watson tree of height n with a geometric offspring law of parameter p. Assign to each leaf a random number of cars according to the law ν_0, and to each internal node a random number of parking spots according to the law of $\mathtt{Z}$. Cars then drive toward the root, parking as early as possible at any available spot. The number of cars reaching the root without finding a suitable parking spot follows the law ν_n. The process is supercritical if this number grows exponentially as n becomes large, and is critical or subcritical if the number converges in probability to 0. For detailed studies of parking models on trees, see Goldschmidt and Przykucki [20], Aldous et al. [2] and Contat and Curien [13].

We are interested in the free energy of the generalized DR model (X_n), which is defined as

$$F_X(\nu_0) \;=\; \lim_{n\to\infty} p^n \int x\,\nu_n(\mathrm{d}x) \;=\; \lim_{n\to\infty} p^n\,\mathbb{E}(X_n) \;\in\; [0,\infty]. \tag{1.7}$$

Similarly to the case (1.1), this limit exists because, from (1.6), we have the inequality $\mathbb{E}(X_{n+1}) \le \mathbb{E}(\mathtt{R})\,\mathbb{E}(X_n)$, with $\mathbb{E}(\mathtt{R}) = p^{-1}$, thus $(p^n\,\mathbb{E}(X_n))$ is nonincreasing and nonnegative.

We will consider the generalized DR model (1.6) for the following two families of initial distributions ν_0:

- *Linear fractional distributions,* which are mixtures of the Dirac measure at 0 and a geometric distribution;
- *Continuous linear fractional distributions,* which are mixtures of the Dirac measure at 0 and an exponential distribution on $\mathbb{R}_+$.

The special case where $\mathtt{Z} = 1$ was recently studied by Li and Zhang in [23] and [24]. Our main results for these two families of generalized DR models are as follows: We first provide a characterization of the critical curve that separates the regions where $F_X(\nu_0) > 0$ and $F_X(\nu_0) = 0$. We then establish the Derrida–Retaux conjecture by proving (1.3) for F_X as ν_0 approaches the critical value.

The precise statements are provided in Theorems 2.6 and 2.12. To the best of our knowledge, this is the first time the precise asymptotics of the free energy at criticality for a discrete-time Derrida-Retaux model have been computed. Notably, this result is not limited to integer-valued systems or to iterations where $\mathtt{Z} = 1$ a.s.

The common feature of the two families of discrete-time DR models given above is that their evolution can be explicitly represented in a two-dimensional parameter space via the following iteration (up to a change of variables)

$$(u_0, v_0) \in \mathbb{R}_+ \times \mathbb{R} \quad \text{and} \quad \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} \;=\; \begin{pmatrix} u_n \Psi(v_{n+1}) \\ u_n + v_n \end{pmatrix}, \tag{1.8}$$

where Ψ is a nonnegative nondecreasing function satisfying $\Psi(0) = \Psi'(0) = 1$. More specifically, we work under the following assumptions for Ψ:

(A) $\quad \Psi : \mathbb{R} \to (0,\infty)$ is a bounded, nondecreasing $\mathcal{C}^2$ function with $\Psi(0) = \Psi'(0) = 1$.

We will show in the next section that when ν_0 is a mixture of the Dirac measure at 0 and a geometric or exponential distribution, the parameters describing the model at each step evolve according to (1.8). As a consequence, the phase transition observed in the DR model can be understood by examining the asymptotic properties of the sequence $(u_n, v_n)_{n\ge 0}$. Note that $(v_n)_{n\ge0}$ is increasing, whereas $(u_n)_{n\ge0}$ decreases for $n < N_0$ and increases for $n \ge N_0$, where $N_0 = \sup\{n \in \mathbb{N} : v_n \le 0\}$. Thus, the point $(0,0)$ plays a particular role in this dynamic, as it is the farthest point on the horizontal axis that can serve as a limiting point for the evolution. More precisely, we have

$$\lim_{n\to\infty} (u_n, v_n) \;\in\; \{(0,v) : v \in (-\infty,0)\} \cup \{(0,0)\} \cup \{(\infty,\infty)\}, \tag{1.9}$$

as will be shown in Lemma 3.1. This decomposition of the possible asymptotic behaviors of the equation enables us to divide the parameter space into three distinct domains:

$$\begin{aligned} \mathcal{P} &:= \Big\{(u_0,v_0) : \lim_{n\to\infty} v_n = \infty\Big\}, \quad \mathcal{C} := \Big\{(u_0,v_0) : \lim_{n\to\infty} v_n = 0\Big\} \\ \text{and} \quad \mathcal{U} &:= \Big\{(u_0,v_0) : \lim_{n\to\infty} v_n < 0\Big\}. \end{aligned} \tag{1.10}$$

These domains are inspired by the depinning transition of polymers, with $\mathcal{P}$ corresponding to the pinned state (associated with a positive free energy), and $\mathcal{U}$ corresponding to the unpinned state (associated with a null free energy). By analogy with the associated DR models, we refer to:

- $\mathcal{P}$ as the supercritical domain, where ν_n converges in distribution to ∞,
- $\mathcal{U}$ as the subcritical domain, where ν_n converges in distribution to δ_0, the Dirac measure at 0,
- $\mathcal{C}$ as the critical domain, which forms the boundary between $\mathcal{P}$ and $\mathcal{U}$.

Our first main result concerning the recursion (1.8) is about the crucial properties of the critical curve h, which describes the boundary of the set $\mathcal{P}$. This function is defined as follows:

$$h(v) \;:=\; \inf\{u \in \mathbb{R}_+ : (u,v) \in \mathcal{P}\} \quad \text{for all } v \in \mathbb{R}. \tag{1.11}$$

We derive a functional equation that h satisfies, along with several of its regularity properties and its asymptotic behavior near the critical point $(0,0)$. We finally show that the set $\mathcal{C}$ is the graph of h.

Theorem 1.1. *Under the assumptions* (A)*, the function h is nonincreasing, Lipschitz continuous and satisfies $0 \le h(x) \le (-x)_+$ for all $x \in \mathbb{R}$. Moreover, h is the unique nonzero solution to the functional equation*

$$h(x + h(x)) \;=\; \Psi(x + h(x))h(x) \quad \textit{for all } x \in \mathbb{R}_-, \tag{1.12}$$

and satisfies

$$h(x) \;\sim\; \frac{x^2}{2}, \quad \textit{as } x \uparrow 0. \tag{1.13}$$

Finally, the domains $\mathcal{P}, \mathcal{C}$ and $\mathcal{U}$ can be characterized as follows:

$$\begin{aligned} \mathcal{P} \;&=\; \{(u,v) : u > h(v)\}, \quad \mathcal{C} \;=\; \{(u,v) : u = h(v)\}, \\ \textit{and} \quad \mathcal{U} \;&=\; \{(u,v) : u < h(v)\}. \end{aligned} \tag{1.14}$$

We do not have an explicit solution for the involution-type equation (1.12), even for some natural choices of the function Ψ associated with the stochastic equations described in Section 2. However, it is relatively straightforward to first select h and then choose a function Ψ such that h satisfies (1.12), as demonstrated in Figure 1.

We use the function h to quantify the distance from a given starting point to the critical curve and introduce an analogue of the free energy for the recursion (1.8) that allows us to state and prove the analog of the Derrida-Retaux conjecture. Specifically, we define

$$\Psi(\infty) \;=\; \lim_{x\to\infty} \Psi(x),$$

which always exists as Ψ is assumed to be monotone. The free energy is then given by

$$F(u_0,v_0) \;:=\; \liminf_{n\to\infty} \Psi(\infty)^{-n} u_n \;\in\; [0,\infty). \tag{1.15}$$

Some observations regarding this quantity are presented in the next result.

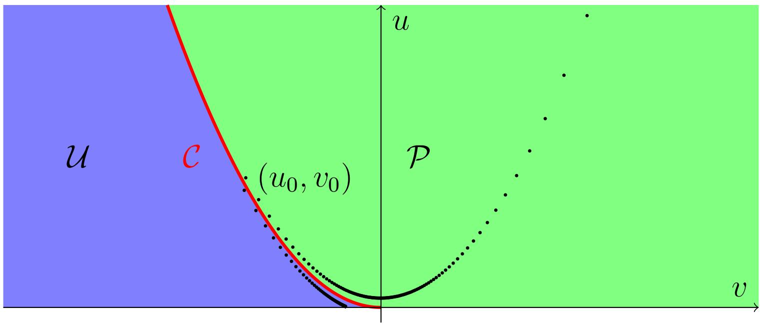

FIGURE 1. Decomposition of the phase space for a function Ψ defined by $\Psi(x) = x^2/2(1 + x - \sqrt{1+2x})$ for $x \in [-0.5, 0.5]$, extended to $\mathbb{R}$ in such a way that (A) holds. For this function, the critical curve h is given by $x \mapsto x^2/2$ on $[-0.5, 0.5]$ and drawn in red. Additionally, slightly supercritical and subcritical trajectories for (u, v) are depicted.

Proposition 1.2. *Under assumptions* (A)*, we have*

$$F(u_0, v_0) \;=\; \lim_{n\to\infty} \Psi(\infty)^{-n} u_n \;=\; \inf_{n\in\mathbb{N}} \Psi(\infty)^{-n} u_n.$$

Furthermore, if

$$\sum_{j=1}^{\infty} (\Psi(\infty) - \Psi(\kappa^j)) < \infty \quad \textit{for all } \kappa > 1, \tag{B}$$

then we have the equivalence

$$(u_0, v_0) \in \mathcal{P} \iff F(u_0, v_0) > 0.$$

Observe that the (primarily technical) assumption (B) expresses that $\Psi(x)$ converges sufficiently fast to $\Psi(\infty)$ as $x \to \infty$. This is implied, for example, by the condition

$$\lim_{x\to\infty} (\log x)^{1+\delta} (\Psi(x) - \Psi(\infty)) = 0 \quad \text{for some } \delta > 0.$$

Under assumption (B), we can identify the supercritical domain with the set of parameters such that the free energy is positive.

We can now state the Derrida-Retaux conjecture for the recursive equation (1.8) as follows.

Theorem 1.3. *Under the assumptions* (A) *and* (B)*, for all $v \le 0$, there exists $C_v > 0$ such that*

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2} \log F(h(v) + \varepsilon, v) \;=\; -C_v.$$

Moreover, we have

$$C_0 \;=\; \frac{1}{2} \lim_{v\to 0} C_v \;=\; \frac{\pi}{\sqrt{2}} \log \Psi(\infty). \tag{1.16}$$

The last statement particularly highlights the universal constant $\lim_{v\to 0} \frac{C_v}{\log \Psi(\infty)} = \pi\sqrt{2}$. It is also worth noting the partial symmetry breaking around the critical point: starting just to the left of the critical point results in a much smaller free energy than starting above it. Additionally, when $v > 0$, we observe that $F(\varepsilon, v) \to 0$ at a polynomial rate as $\varepsilon \to 0$. These results are in agreement with the findings in [21] for the solvable continuous-time DR model.

Finally, we give a result that describes the behavior of the recursive equation along the critical curve, obtaining results that are consistent with previous literature on DR models.

Theorem 1.4. *Assuming* (A), *let* $v < 0$ *and* $(u_0, v_0) = (h(v), v)$. *Then*

$$u_n \sim \frac{2}{n^2} \quad \textit{and} \quad v_n \sim -\frac{2}{n} \quad \textit{as } n \to \infty.$$

We conclude this introduction with some comments. The generalized DR model (1.6) in the case $\mathtt{Z} = 1$, with a general initial distribution ν, was studied in [22]. In that work, the asymptotic behavior of the free energy was examined in a domain that, in our normalized coordinates, corresponds to $u_0 \to 0$ and $v_0 > 0$. With specific assumptions on the tail of ν, different asymptotic behaviors from (1.3) can emerge at $v_0 = 0$.

The model (1.6) with $\mathtt{Z} = 1$, where the initial distribution is a mixture of a Dirac measure at 0 and either a geometric or exponential distribution, has been studied in [23] and [24]. These articles classify different regimes and provide precise estimates for the distribution of X_n in the critical regime [23]. In addition, [23] explores the scaling limit, which leads to a continuous-time model.

Previous studies of the stochastic recursions (1.1) or (1.4) have heavily relied on the explicit computation of the critical point p_c as derived in Collet et al. [12]. This computation requires two key assumptions: that v_0 is supported on $\mathbb{Z}_+$, and that the subtraction term $\mathtt{Z}$ equals 1 almost surely. For instance, if the constant 1 in (1.1) or (1.4) is replaced by 2, even the computation of the critical value becomes an open problem.

Our method, however, proves to be robust. By reducing the study of the law of X_n to that of the recursive equation (1.8), we can establish universality without needing to know the explicit value of the critical point. The critical point is characterized by the function h, which is the solution of the involution-type equation (1.12). Specifically, as long as the critical curve h satisfies $h(x) \sim x^2/2$ as $x \to 0$, the free energy defined for the recursive equation (1.8) will satisfy the corresponding Derrida-Retaux conjecture.

Finally, we mention studies of the Derrida-Retaux system in other contexts: as a spin glass model in in Collet et al. [11], as an iteration function of random variables in Li and Rogers [26] and Jordan [25], and as part of the max-type recursion families in the seminal paper by Aldous and Bandyopadhyay [1].

The rest of the article is organized as follows. In the next section, we introduce two families of two-parameter DR models, whose evolution is governed by equation (1.8). Specifically, we discuss the implications of Theorems 1.3 and 1.4 for these processes. Section 3 explores some basic properties of the recursion (1.8), including a proof of Proposition 1.2 and an analysis of the backward evolution of this equation. In Section 4, we investigate the critical domain, proving Theorems 1.1 and 1.4. Finally, we establish Theorem 1.3 in Section 5, thereby confirming the Derrida-Retaux conjecture for (1.8).

Notation. Throughout, we use the following standard notation: $\mathbb{N}$ for the set of positive integers, $\mathbb{Z}_+$ for the set of nonnegative integers, $\mathbb{R}_+$ for the set of nonnegative reals and $(0, \infty)$ the set of positive reals. We also recall that $x_+ = \max(x, 0)$ is the positive part of x, and we write $x_- = (-x)_+$ for the negative part of x.

2. Solvable discrete-time DR models with two parameters

We introduce in this section two families of two-parameters DR models. As mentioned in the introduction, these models can be thought of as stochastic recursions on a Galton-Watson tree with a geometric offspring distribution. Using several equalities in distribution involving sums of a geometric number of i.i.d. random variables, we are able to describe the families of laws by tracking two parameters that evolve according to (1.8).

Recall (1.6): For $n \in \mathbb{N}$, writing $(X_n^{(k)}, k \geq 1)$ for i.i.d. random variables of law ν_n, independent of $(\mathtt{R}, \mathtt{Z})$, then

$$X_{n+1} = \left(\sum_{j=1}^{\mathtt{R}} X_n^{(k)} - \mathtt{Z} \right)_+ \quad \text{is a random variable of law } \nu_{n+1}.$$

We introduce in Section 2.1 a generalized DR model such that ν_n is a probability distribution on $\mathbb{Z}_+$ for all $n \in \mathbb{N}$, then in Section 2.2 a generalized DR model such that $\nu_{n|(0,\infty)}$ has density with respect to the Lebesgue measure on $(0, \infty)$. We relate these two models to the recursion equation (1.8), which allows us to apply Theorems 1.3 and 1.4.

2.1. **Generalized DR model on $\mathbb{Z}_+$ with linear fractional input.** We assume in this section that $\mathtt{Z}$ takes values in $\mathbb{N}$. In this case, if ν_0 is supported in $\mathbb{Z}_+$, then ν_n will be supported in $\mathbb{Z}_+$ for all $n \geq 1$. A solvable DR model will be obtained by choosing for the input distribution ν_0 a *linear fractional distribution* that we now introduce, see also [3] for further details about this class of distributions that is of particular interest in the theory of branching processes.

Definition 2.1 (*Linear fractional distribution*)**.** Let $\alpha, \beta > 0$ be such that $\alpha + \beta \geq 1$ and Y be a random variable. We say that Y has a linear fractional distribution with parameters α, β and write $Y \stackrel{\text{(d)}}{=} \mathsf{LF}(\alpha, \beta)$ if Y takes values in $\mathbb{Z}_+$ and has probability generating function $f_Y(s) := \mathbb{E}(s^Y)$ which satisfies

$$\frac{1}{1 - f_Y(s)} = \frac{\alpha}{1 - s} + \beta, \quad \text{i.e.} \quad f_Y(s) = 1 - \frac{1 - s}{\alpha + \beta(1 - s)}$$

for $0 \leq s \leq 1$. In particular, $\mathbb{E}(Y) = \alpha^{-1}$.

In this section, we consider the generalized DR model (1.6) with geometrically distributed $\mathtt{R}$ and initial distribution $\nu_0 = \mathsf{LF}(\alpha, \beta)$, where α and β are as specified. We demonstrate that, for each $n \in \mathbb{N}$, the distribution ν_n remains linear fractional, i.e. $\nu_n = \mathsf{LF}(\alpha_n, \beta_n)$ for appropriate values of (α_n, β_n) that will be further specified below. Additionally, we show that, up to a reparametrization, the evolution of this sequence can be mapped to the recursion (1.8).

If $Y \stackrel{\text{(d)}}{=} \mathsf{LF}(\alpha, \beta)$, then we infer from Definition 2.1 that

$$\mathbb{P}(Y = k) = \frac{\alpha}{(\alpha + \beta)^2} \left(\frac{\beta}{\alpha + \beta} \right)^{k-1} \text{ for } k \geq 1 \text{ and } \quad \mathbb{P}(Y = 0) = 1 - \frac{1}{\alpha + \beta}.$$

The geometric distribution appears as a special case of linear fractional distribution, namely $\mathcal{G}(p) = \mathsf{LF}(p, 1 - p)$ for all $p \in (0, 1)$. Two well-known distributional identities for linear fractional distributions are next.

Fact 2.2. *Let $\alpha, \beta > 0$ be such that $\alpha + \beta \geq 1$ and $\mathtt{R} \stackrel{\text{(d)}}{=} \mathcal{G}(p)$ for some $p \in (0, 1)$. If $Y_1, Y_2, \ldots$ are i.i.d. random variables with common law $\mathsf{LF}(\alpha, \beta)$ and independent of $\mathtt{R}$, then*

$$\sum_{j=1}^{\mathtt{R}} Y_j \stackrel{\text{(d)}}{=} \mathsf{LF}(p\alpha, 1 - p + p\beta).$$

Proof. Denote by $f_{\mathtt{R}}, f_Y$ and f_Σ the probability generating functions of $\mathtt{R}, Y_1$ and $\sum_{j=1}^{\mathtt{R}} Y_j$ respectively. Then $f_\Sigma(s) = f_{\mathtt{R}}(f_Y(s))$ for $0 \le s \le 1$ and therefore

$$\begin{aligned}\frac{1}{1-f_\Sigma(s)} &= \frac{1}{1-f_{\mathtt{R}}(f_Y(s))} = 1-p+\frac{p}{1-f_Y(s)}\\ &= 1-p+p\left(\beta+\frac{\alpha}{1-s}\right) = (1-p+p\beta)+\frac{p\alpha}{1-s},\end{aligned}$$

which completes the proof. □

Fact 2.3. *If Y, Z are independent integer-valued random variables such that $Y \stackrel{(d)}{=} \mathsf{LF}(\alpha,\beta)$, then*

$$(Y-\mathtt{Z})_+ \stackrel{(d)}{=} \mathsf{LF}\left(\frac{\alpha}{\varphi(\beta/(\alpha+\beta))}, \frac{\beta}{\varphi(\beta/(\alpha+\beta))}\right),$$

where $\varphi(s) = \mathbb{E}(s^{\mathtt{Z}})$ for $0 \le s \le 1$.

Proof. Since $\mathbb{P}(Y \ge k) = p_0\lambda^{k-1}$ for all $k \ge 1$, where $\lambda = \frac{\beta}{\alpha+\beta}$ and $p_0 = \frac{1}{\alpha+\beta}$, it follows that

$$\mathbb{P}((Y-\mathtt{Z})_+ \ge k) = \mathbb{E}\left(p_0\lambda^{k-1+\mathtt{Z}}\right) = p_0\lambda^{k-1}\varphi(\lambda).$$

and thus $(Y-\mathtt{Z})_+ \stackrel{(d)}{=} \mathsf{LF}(\gamma,\delta)$ with

$$\frac{\delta}{\gamma+\delta} = \lambda = \frac{\beta}{\alpha+\beta} \quad\text{and}\quad \frac{1}{\gamma+\delta} = p_0\varphi(\lambda) = \frac{1}{\alpha+\beta}\varphi\left(\frac{\beta}{\alpha+\beta}\right).$$

The asserted values of γ and δ are now easily obtained by simple algebra. □

A combination of the two facts now enables us to show that the generalized DR model $(\nu_n)_{n\ge0}$ with linear fractional input can be explicitly described by the evolution of a two-dimensional recursive sequence.

Lemma 2.4. *Let $\nu_0 = \mathsf{LF}(\alpha,\beta)$ for some $\alpha,\beta > 0$ with $\alpha+\beta \ge 1$ and $\mathtt{R} \stackrel{(d)}{=} \mathcal{G}(p)$ for some $p \in (0,1)$. Then $\nu_n = \mathsf{LF}(\alpha_n,\beta_n)$ for any $n \ge 0$, where the (α_n,β_n) are given by the recursion $(\alpha_0,\beta_0) = (\alpha,\beta)$ and*

$$(\alpha_{n+1},\beta_{n+1}) = \left(\frac{p\alpha_n}{d_n}, \frac{1-p+p\beta_n}{d_n}\right) \quad\text{with}\quad d_n = \varphi\left(\frac{1-p+p\beta_n}{1-p+p(\alpha_n+\beta_n)}\right). \tag{2.1}$$

Proof. This is a direct consequence of the Facts 2.2 and 2.3. □

To transform the recursive equation (2.1) into (1.8), which is the next step, we introduce the function

$$\psi(x) := \frac{1}{p}\varphi\left(\frac{x}{x+1}\right), \quad x \ge 0. \tag{2.2}$$

which is nonnegative, strictly increasing, $\mathcal{C}^\infty$ and bounded, with $\psi(0) = 0$ and $\lim_{x\to\infty}\psi(x) = 1/p$. In particular, there exists a unique $\xi > 0$ that satisfies the equation

$$\psi(\xi) = 1.$$

We use this parameter to renormalize the recursion satisfied by (α_n,β_n), transforming it into the form given in (1.8).

Proposition 2.5. *Let $(\alpha_n,\beta_n)_{n\ge0}$ be a sequence satisfying (2.1). For all $n \in \mathbb{N}$, we define the following reparametrization:*

$$u_n := \psi'(\xi)\frac{1-p}{p\alpha_n} \quad\text{and}\quad v_n := \psi'(\xi)\left(\frac{\beta_n}{\alpha_n}-\xi\right). \tag{2.3}$$

Then $(u_n, v_n)_{n\geq 0}$ *satisfies the recursion in* (1.8)*, with* Ψ *defined by*

$$\Psi(x) := \psi\left(\frac{x}{\psi'(\xi)} + \xi\right) \quad \text{for } x \in I := (-\psi'(\xi)\xi, \infty). \tag{2.4}$$

Straightforward calculations provide that $\Psi(0) = \Psi'(0) = 1$ (by construction) and also $\Psi(\infty) = p^{-1}$. Moreover, for all $\delta > 0$, the restriction of Ψ to the interval $[-\psi'(\xi)\xi + \delta, \infty)$ can be extended to a function that satisfies (A). Since $(v_n)_{n\geq 0}$ is nondecreasing, we can therefore apply general results on solutions of (1.8) to this generalized DR model.

Proof. Introducing the parameterization

$$(x_n, y_n) := \left(\frac{\beta_n}{\alpha_n}, \frac{1-p}{p\alpha_n}\right) \quad \text{for } n \geq 0,$$

it follows that $x_{n+1} = x_n + y_n$ and

$$\frac{1 - p + p(\alpha_n + \beta_n)}{1 - p + p\beta_n} = 1 + \frac{p\alpha_n}{1 - p + p\beta_n} = 1 + \frac{1}{y_n + x_n} = 1 + \frac{1}{x_{n+1}}$$

for each $n \geq 0$. From this, we obtain

$$\frac{y_{n+1}}{y_n} = \frac{\alpha_n}{a_{n+1}} = \frac{1}{p}\varphi\left(\frac{x_{n+1}}{1 + x_{n+1}}\right)$$

and then the recursive equation

$$(x_{n+1}, y_{n+1}) = (x_n + y_n, y_n\psi(x_{n+1})),$$

valid for all $n \geq 0$. Finally, the proof is concluded by noting that $v_n = \psi'(\xi)(x_n - \xi)$ and $u_n = \psi'(\xi)y_n$. □

Observe that the free energy of the generalized DR model with linear fractional input, starting from $\nu_0 = \mathsf{LF}(\alpha, \beta)$, is defined by

$$F_{\mathsf{LF}}(\alpha, \beta) = \lim_{n\to\infty} \frac{p^n}{\alpha_n} = \frac{p}{(1-p)\psi'(\xi)} \lim_{n\to\infty} p^n u_n.$$

As a consequence, the Theorems 1.1 and 1.3 can be translated as follows.

Theorem 2.6. *Assuming* $\mathbb{E}(\log \mathrm{Z}) < \infty$*, the following assertions hold:*
(a) There exists a unique nonincreasing, Lipschitz continuous function h *which satisfies*

$$h(x + h(x)) = \Psi(x + h(x))h(x) \quad \text{for all } x \in (-\psi'(\xi)\xi, 0],$$

and $h(x) \sim x^2/2$ *as* $x \uparrow 0$*. Furthermore, for any* $\alpha, \beta > 0$ *with* $\alpha + \beta \geq 1$,

$$h\left(\psi'(\xi)\left(\frac{\beta}{\alpha} - \xi\right)\right) < \psi'(\xi)\frac{1-p}{p\alpha} \quad \Longleftrightarrow \quad F_{\mathsf{LF}}(\alpha, \beta) > 0.$$

(b) For $\alpha, \beta > 0$ *such that* $\alpha + \beta \geq 1$ *and* $\beta/\alpha \leq \xi$*, let*

$$\gamma_* := \inf\{\gamma \in (0, \alpha + \beta] : F_{\mathsf{LF}}(\alpha/\gamma, \beta/\gamma) > 0\}.$$

Then

$$h\left(\psi'(\xi)\left(\frac{\beta}{\alpha} - \xi\right)\right) < \psi'(\xi)\frac{1-p}{p\alpha}(\alpha + \beta), \tag{2.5}$$

implies

$$\gamma_* = \frac{p\,\alpha\, h\left(\psi'(\xi)\left(\frac{\beta}{\alpha} - \xi\right)\right)}{\psi'(\xi)(1-p)} \in [0, \alpha + \beta),$$

and there exists $C_{\alpha,\beta} > 0$ such that

$$\lim_{\gamma\downarrow\gamma_*}(\gamma-\gamma_*)^{1/2}\log F_{\mathsf{LF}}(\alpha/\gamma,\beta/\gamma) \;=\; -C_{\alpha,\beta}.$$

Note that if (2.5) fails, then $F_{\mathsf{LF}}(\alpha/\gamma,\beta/\gamma)=0$ for any $\gamma\in(0,\alpha+\beta]$.

Proof. By a standard Tauberian argument, $\mathbb{E}(\log \mathsf{Z})<\infty$ can be used to obtain asymptotic estimates of the probability generating function φ of Z about 1. In particular, it implies that condition (B) holds (see (2.2) and (2.4) for the connection between φ and the function Ψ appearing in this condition). We can therefore apply Theorem 1.1 and Proposition 1.2 to infer (a). For part (b), we apply Theorem 1.3, observing that with our notation, we have

$$u_0 \;=\; \psi'(\xi)\frac{1-p}{p\alpha}\gamma \quad\text{and}\quad v_0 \;=\; \psi'(\xi)\left(\frac{\beta}{\alpha}-\xi\right),$$

i.e., u_0 varies with γ. The assertions now follow by immediate translation. $\square$

The following result about the critical regime follows in a similar manner by a translation of Theorem 1.4.

Corollary 2.7. *Let $\alpha,\beta>0$ be such that $\alpha+\beta\ge 1$ and $\beta/\alpha\le\xi$, and assume* (2.5)*. Let $(X_n)_{n\ge 0}$ be a sequence of random variables such that X_n has law ν_n for each n, where $\nu_0=\mathsf{LF}(\alpha/\gamma_*,\beta/\gamma_*)$. Then*

$$\mathbb{P}(X_n\ge 1) \;\sim\; \frac{2p}{(1-p)\psi'(\xi)(1+\xi)}\frac{1}{n^2}, \quad \text{as } n\to\infty$$

and X_n, conditioned on $\{X_n\ge 1\}$, converges in law to $\mathcal{G}(\frac{1}{1+\xi})=\mathsf{LF}(\frac{1}{1+\xi},\frac{\xi}{1+\xi})$.

2.2. Generalized DR model on $\mathbb{R}_+$ with continuous linear fractional input.

The *solvable* generalized DR model on $\mathbb{R}_+$, which will be studied in this section, can be viewed as the continuous counterpart to the model with linear fractional input. It is based on the assumption that Z is a generic $\mathbb{R}_+$-valued random variable, with ν_0 taken as a mixture of an exponential distribution and a Dirac mass at 0. This mixture is referred to as a continuous linear fractional distribution in [4], as it corresponds to a continuous measure with a linear fractional Laplace transform (as opposed to the probability generating function used in the discrete case). For the sake of symmetry with the previous section, let us define these random variables formally.

Definition 2.8 (*Continuous linear fractional distributions*)**.** Let $\lambda>0$, $\varrho\in[0,1]$ and X be a random variable. We say that X has a continuous linear fractional distribution with parameters λ,ϱ and write $X\stackrel{(d)}{=}\mathsf{CLF}(\lambda,\varrho)$ if X takes values in $\mathbb{R}_+$ and

$$\mathbb{P}(X>x) \;=\; \varrho e^{-\lambda x} \quad \text{for all } x>0. \tag{2.6}$$

This implies $\mathbb{P}(X=0)=1-\varrho$, $\mathbb{E}(X)=\varrho/\lambda$, and

$$\mathbb{E}\left(e^{-\mu X}\right) \;=\; 1-\varrho+\frac{\varrho\lambda}{\lambda+\mu} \quad \text{for all } \mu>-\lambda. \tag{2.7}$$

Note that continuous linear fractional distributions actually form the unique two-parameter family of probability measures on $\mathbb{R}_+$ whose Laplace transforms are linear fractional. Therefore, it is not surprising that, similar to the integer-valued linear fractional distributions, these continuous distributions define a solvable family within the context of the generalized DR model.

We also remark that this family corresponds to the family of probability distributions introduced in [21] for the continuous-time DR model. There does not seem to be a direct link between the continuous- and the discrete-time models, however [23] proved that one can obtain the continuous-time DR model as a scaling limit of the discrete-time model.

Similar to the previous section, we begin by assembling a couple of facts about the laws $\mathsf{CLF}(\lambda, \varrho)$.

Fact 2.9. *Let $\varrho \in [0,1]$, $\lambda > 0$, $\mathtt{R} \overset{(d)}{=} \mathcal{G}(p)$ for some $p \in (0,1)$ and $(X_n, n \geq 1)$ be a sequence of i.i.d. random variables with common law $\mathsf{CLF}(\lambda, \varrho)$ and independent of $\mathtt{R}$. Then*

$$\sum_{j=1}^{\mathtt{R}} X_j \overset{(d)}{=} \mathsf{CLF}\left(\frac{\lambda p}{p+(1-p)\varrho}, \frac{\varrho}{p+(1-p)\varrho}\right).$$

The proof of this fact follows by computing the Laplace transform of the randomized sum and its identification using (2.7). The second fact to notice is the following counterpart of Fact 2.3.

Fact 2.10. *Let $\varrho \in [0,1]$ and $\lambda > 0$. If $X \overset{(d)}{=} \mathsf{CLF}(\lambda, \varrho)$ and $\mathtt{Z}$ is an independent random variable taking values in $\mathbb{R}_+$ and with Laplace transform φ, then*

$$(X - \mathtt{Z})_+ \overset{(d)}{=} \mathsf{CLF}(\lambda, \varrho\,\varphi(\lambda)).$$

In view of (2.6), it suffices to note that $\mathbb{P}((X-\mathtt{Z})_+ > x) = \mathbb{P}(X > x+\mathtt{Z}) = \varrho\,\mathbb{E}(e^{-\lambda(x+\mathtt{Z})}) = \varrho\,\varphi(\lambda)e^{-\lambda x}$ for all $x > 0$.

As a consequence of these two facts, we directly infer that, if $\nu_0 = \mathsf{CLF}(\lambda_0, \varrho_0)$, then the ν_n in the generalized DR model defined by (1.6) are all continuous linear fractional distributions, namely

$$\nu_n = \mathsf{CLF}(\lambda_n, \varrho_n). \tag{2.8}$$

for suitable recursively defined $(\lambda_n, \varrho_n) \in (0,\infty) \times [0,1]$. However, we need to introduce a new parametrization of $\mathsf{CLF}(\lambda, \varrho)$ such that the sequence $(\nu_n)_{n\geq 0}$ under this new parametrization satisfies (1.8).

Let us define the function

$$\gamma(\theta) := \frac{1}{p}\,\mathbb{E}(e^{-\mathtt{Z}/\theta}) \quad \text{for } \theta > 0, \tag{2.9}$$

and $\tau > 0$ as the unique number satisfying

$$\gamma(\tau) = 1. \tag{2.10}$$

Moreover, we let Ψ be given by

$$\Psi(x) := \gamma\left(\frac{x}{\gamma'(\tau)} + \tau\right) \quad \text{for } x \in I := (-\tau\gamma'(\tau), \infty), \tag{2.11}$$

and note that this function satisfies $\Psi(0) = \Psi'(0) = 1$.

Proposition 2.11. *Let $(\varrho_n, \lambda_n)_{n\geq 0}$ be the sequence determined by (2.8) and define*

$$u_n := \gamma'(\tau)\,\frac{1-p}{p}\,\frac{\varrho_n}{\lambda_n} \quad \text{and} \quad v_n := \gamma'(\tau)\left(\frac{1}{\lambda_n} - \tau\right) \quad \text{for } n \geq 0.$$

Then $(u_n, v_n)_{n\geq 0}$ satisfies (1.8) with the function Ψ in (2.11) provided that $v_0 > -\tau\gamma'(\tau)$.

We note that for all $\delta > 0$, the restriction of Ψ to $[-\psi'(\xi)\xi + \delta, \infty)$ can be extended to a bounded $\mathcal{C}^2$ function on $\mathbb{R}$. Using that $(v_n)_{n\geq 0}$ is nondecreasing, we can once again apply general results on solutions to (1.8) to this generalized DR model. The proof of Proposition 2.11 involves straightforward computations, similar to those for Proposition 2.5, and we therefore omit the details.

As in the previous subsection, we can compute the free energy of the generalized DR model starting from $\nu_0 = \mathsf{CLF}(\lambda, \varrho)$, defined by

$$F_{\mathsf{CLF}}(\lambda, \varrho) \;=\; \lim_{n\to\infty} \frac{p^n \varrho_n}{\lambda_n} \;=\; \frac{p}{(1-p)\gamma'(\tau)} \lim_{n\to\infty} p^n u_n.$$

Therefore, Theorems 1.1 and 1.3 can once again be applied to this model, yielding the following result.

Theorem 2.12. *Assuming $\mathbb{E}(\log \mathrm{Z}) < \infty$, the following assertions hold:*
(a) There exists a unique nontrivial function h satisfying

$$h(x + h(x)) = \Psi(x + h(x))h(x) \quad \text{for all } x \in (-\psi'(\xi)\xi, 0].$$

Furthermore, for any $\varrho \in [0,1]$ and $\lambda > 0$,

$$h\left(\gamma'(\tau)\left(\frac{1}{\lambda} - \tau\right)\right) \;<\; \gamma'(\tau)\,\frac{1-p}{p}\,\frac{\varrho}{\lambda} \qquad \Longleftrightarrow \qquad F_{\mathsf{CLF}}(\lambda, \varrho) \;>\; 0.$$

(b) For $\lambda \geq 1/\tau$, let

$$\varrho^*_\lambda \;:=\; \inf\{\varrho \in (0,1] : F_{\mathsf{CLF}}(\lambda, \varrho) > 0\}.$$

Then

$$\lambda\, p\, h\left(\gamma'(\tau)\left(\frac{1}{\lambda} - \tau\right)\right) \;<\; \gamma'(\tau)(1-p), \tag{2.12}$$

implies

$$\varrho^*_\lambda \;=\; \frac{\lambda p}{\gamma'(\tau)(1-p)} h\left(\gamma'(\tau)\left(\frac{1}{\lambda} - \tau\right)\right),$$

and there exists $C_\lambda > 0$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2} \log F(\varrho^*_\lambda + \varepsilon, \lambda) \;=\; -C_\lambda.$$

Note that if (2.12) fails, then $F_{\mathsf{CLF}}(\lambda, \varrho) = 0$.

Once again, we can apply Theorem 1.4 to obtain the following counterpart of Corollary 2.7 in the present situation when ν_0 lies on the critical curve.

Corollary 2.13. *Let $\lambda \geq 1/\tau$ and assume (2.12). Let $(X_n)_{n\geq 0}$ be a sequence of random variables such that X_n has law ν_n for each n, where $\nu_0 = \mathsf{CLF}(\varrho^*_\lambda, \lambda)$. Then X_n, conditioned on $\{X_n > 0\}$, converges in law to the exponential distribution with parameter $1/\tau$, i.e., $\mathsf{CLF}(1/\tau, 1)$.*

3. Simple properties of the Derrida-Retaux recursion

In this section, we present some straightforward properties of the solutions to the recursive equation (1.8). Let Ψ be a bounded, nonnegative, nondecreasing $\mathcal{C}^2$ function such that $\Psi(0) = \Psi'(0) = 1$, i.e. that Ψ satisfies (A). We begin by proving (1.9) through a characterization of the limits of a sequence $(u_n, v_n)_{n\geq 0}$ verifying (1.8).

Lemma 3.1. *Given a solution $(u_n, v_n)_{n\geq 0}$ to (1.8), the following dichotomy holds: Either*

$$\lim_{n\to\infty} v_n \;=\; \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log u_n \;=\; \log \Psi(\infty) \;>\; 0$$

or

$$\lim_{n\to\infty} v_n \;\leq\; 0 \quad \text{and} \quad \lim_{n\to\infty} u_n \;=\; 0.$$

Proof. We remark that $v_{n+1} - v_n = u_n \geq 0$ for all $n \in \mathbb{N}$, thus $(v_n)_{n\geq 0}$ is nondecreasing. As a result, $(v_n)_{n\geq 0}$ either converges to a nonpositive limit, or all v_n are nonnegative for sufficiently large n.

In the first situation, since v_n converges to a finite limit, we have $u_n = v_{n+1} - v_n \to 0$ as $n \to \infty$, which is the desired conclusion.

Let us now assume that there exists $N \in \mathbb{N}$ such that $v_n > 0$ for all $n \geq N$. In this case, we infer $u_{n+1} = u_n \Psi(v_{n+1}) > u_n$ for all $n \geq N$, using that $\Psi'(0) = 1$. Therefore, both u_n and v_n are strictly increasing for $n \geq N$ and, as a consequence, we have $v_n - v_N \geq (n - N)\, u_N$, which shows that $\lim_{n\to\infty} v_n = \infty$. With this, we conclude

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \Psi(v_{n+1}) = \Psi(\infty)$$

and then

$$\lim_{n\to\infty} \frac{1}{n} \log u_n = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{u_{k+1}}{u_k} = \log \Psi(\infty).$$

by an appeal to the Stolz-Cesàro lemma. □

Next, we prove that the free energy is well-defined and characterizes the supercritical domain when Ψ satisfies (B), thereby proving Proposition 1.2.

Proof of Proposition 1.2. Recall that the free energy is defined as

$$F(u_0, v_0) = \liminf_{n\to\infty} \Psi(\infty)^{-n} u_n$$

and note that, if $\lim_{n\to\infty} v_n \leq 0$, then $F(u_0, v_0) = 0$ by Lemma 3.1. Therefore, we need only consider the case where $v_n \to \infty$.

The monotonicity of Ψ implies

$$\frac{u_{n+1}}{\Psi(\infty)^{n+1}} = \frac{u_n}{\Psi(\infty)^n} \frac{\Psi(v_{n+1})}{\Psi(\infty)} \leq \frac{u_n}{\Psi(\infty)^n},$$

and since $(\Psi(\infty)^{-n} u_n)_{n\geq 0}$ is nonincreasing, we infer that $F(u_0, v_0)$ is well-defined as the limit of this sequence.

We now assume that, in addition, (B) holds and demonstrate that $F(u_0, v_0) > 0$ whenever $v_n \to \infty$. By Lemma 3.1, for any $1 < \varrho < \Psi(\infty)$, we have $u_n > \varrho^n$ for all n large enough. Thus, as $v_n = v_0 + \sum_{i=0}^{n-1} u_i$, we have

$$\liminf_{n\to\infty} \varrho^{-n} v_n \geq \frac{1}{\varrho - 1} \liminf_{n\to\infty} \varrho^{-n} u_n \geq \frac{1}{\varrho - 1}.$$

Next, we use (1.8) to write

$$\frac{u_n}{\Psi(\infty)^n} = u_0 \prod_{j=1}^{n} \frac{\Psi(v_n)}{\Psi(\infty)} = u_0 \exp\left(\sum_{j=1}^{n} \log \frac{\Psi(v_j)}{\Psi(\infty)} \right) \tag{3.1}$$

and then conclude from (B) that $\sum_{n\in\mathbb{N}} \log \frac{\Psi(v_n)}{\Psi(\infty)}$ converges and thus that $u_n/\Psi(\infty)^n$ converges to a positive limit as $n \to \infty$. □

The following monotonicity lemma, which will frequently be used in our analysis, can be succinctly summarized by stating that (1.8) preserves the order when replacing Ψ by a smaller or larger function.

Lemma 3.2. *Let $(u_n, v_n)_{n\geq 0}$ be a solution to (1.8) and $\underline{\Psi}, \overline{\Psi}$ be two nonnegative nondecreasing functions such that*

$$\underline{\Psi}(x) \leq \Psi(x) \leq \overline{\Psi}(x) \quad \textit{for all } x \in \mathbb{R}.$$

If $(\underline{u}_n, \underline{v}_n)_{n\geq 0}$ *and* $(\overline{u}_n, \overline{v}_n)_{n\geq 0}$ *are solutions to* (1.8) *with* Ψ *replaced by* $\underline{\Psi}$ *and* $\overline{\Psi}$*, respectively, and* $0 \leq \underline{u}_0 \leq u_0 \leq \overline{u}_0$ *and* $\underline{v}_0 \leq v_0 \leq \overline{v}_0$*, then*

$$0 \leq \underline{u}_n \leq u_n \leq \overline{u}_n \quad \textit{and} \quad \underline{v}_n \leq v_n \leq \overline{v}_n \tag{3.2}$$

for all $n \in \mathbb{N}$.

Proof. The proof follows by a simple induction. Assuming that (3.2) holds for some $n \in \mathbb{N}$, we obtain by summation

$$\underline{v}_{n+1} \leq v_{n+1} \leq \overline{v}_{n+1}.$$

Then, by using that $\underline{u}_n$ and $\underline{\Psi}(v_{n+1})$ are nonnegative, we find by immediate comparison that

$$\underline{u}_{n+1} \leq u_{n+1} \leq \overline{u}_{n+1}.$$

□

We now examine a duality relationship for the recursion equation (1.8), which is based on time reversal. Specifically, we show that the type of the recursion remains unchanged when time is reversed.

Proposition 3.3. *Let* $(u_n, v_n)_{n\geq 0}$ *be a sequence defined recursively by* (1.8)*. We fix* $N \in \mathbb{N}$ *and define*

$$\breve{u}_n := u_{N-n} \quad \textit{and} \quad \breve{v}_n := -v_{N-n+1} \quad \textit{for } 0 \leq n \leq N.$$

Then, for all $0 \leq n < N$*, we have*

$$\begin{pmatrix} \breve{u}_{n+1} \\ \breve{v}_{n+1} \end{pmatrix} = \begin{pmatrix} \breve{u}_n/\Psi(-\breve{v}_{n+1}) \\ \breve{u}_n + \breve{v}_n \end{pmatrix}. \tag{3.3}$$

We see from Proposition 3.3 that the backward evolution of $(u_n, v_n)_{n\geq 0}$ is a solution to (1.8) with Ψ replaced by $\breve{\Psi}(x) = 1/\Psi(-x)$. We note that if Ψ satisfies (A) and if $\lim_{x\to-\infty} \Psi(x) > 0$, then $\breve{\Psi}$ satisfies (A) as well and satisfies $\lim_{x\to-\infty} \breve{\Psi}(x) > 0$.

Proof. The proof follows by simple computations and can be omitted. □

4. Evolution along the critical line

The main goal of this section is to describe the set $\mathcal{C}$, defined in (1.10), as the set of initial conditions (u_0, v_0) such that $\lim_{n\to\infty} v_n = 0$. As noted in Lemma 3.1, the sequence $(v_n)_{n\geq 0}$ either converges to a nonpositive limit or diverges to ∞. Moreover, by Lemma 3.2, we observe that the function ϕ_v, which assigns to each $u \in \mathbb{R}_+$ the quantity $\lim_{n\to\infty} v_n$ with the initial conditions $(u_0, v_0) = (u, v)$, is nondecreasing. The range of ϕ_v is $(-\infty, 0] \cup \{\infty\}$.

We show that $\mathcal{C}$ is the graph of a continuous function, which can be described as follows:

$$h : v \mapsto \inf\{u \in (0, \infty) : \phi_v(u) = \infty\}. \tag{4.1}$$

Note that this definition of h coincides with the one given in (1.11). Since ϕ_v is nondecreasing, we immediately deduce that $\lim_{n\to\infty} v_n = \infty$ if $h(u_0) > v_0$ and $\lim_{n\to\infty} v_n \leq 0$ if $h(u_0) < v_0$. The behavior of the limit when $u_0 = h(v_0)$ remains unclear at this stage. In other words, we have:

$$\{(u, v) : u < h(v)\} \subset \mathcal{C} \cup \mathcal{U} \quad \text{and} \quad \{(u, v) : u > h(v)\} \subset \mathcal{P}, \tag{4.2}$$

which is a first step toward proving (1.14). Finally, using Lemma 3.2 again, we observe that h is nonincreasing. Furthermore, since $(u_n)_{n\geq 0}$ is nondecreasing when $v_0 \geq 0$, we see that $h(x) = 0$ for all $x \geq 0$.

The rest of the section is organized as follows. We prove in Subsection 4.1 that the function h forms the only nontrivial solution to the functional equation stated in Theorem 1.1. This equation then allows us to identify the subcritical, critical and supercritical domains of the dynamic via (1.14). We then study the regularity of this function h and finally, in Subsection 4.2, the asymptotic behaviour of (u_n, v_n) provided that $u_0 = h(v_0)$.

4.1. **Functional equation and analysis of the critical curve.** The main result of the section is the following proposition, which establishes the existence of a unique function h satisfying (1.12). Additionally, we show that the function h corresponds to (4.1).

Proposition 4.1. *Let $A > 0$ and $\Psi : [-A, \infty) \mapsto \mathbb{R}_+$ be a nondecreasing $\mathcal{C}^2$ function, such that $\Psi(0) = \Psi'(0) = 1$. There exists a unique function $g : [-A, \infty) \mapsto \mathbb{R}$ such that $g(x) = x$ for all $x \geq 0$, $g(x) > x$ for all $x \in [-A, 0)$ and*

$$g(g(x)) \;=\; g(x) + \Psi(g(x))(g(x) - x) \quad \text{for } x \in [-A, 0]. \tag{4.3}$$

Furthermore,

(i) the function g is nondecreasing, 1-Lipschitz, that is, Lipschitz continuous with Lipschitz constant 1, and satisfies

$$g(x) - x \;\sim\; \frac{x^2}{2} \quad \text{as } x \uparrow 0, \tag{4.4}$$

(ii) given a solution $(u_n, v_n)_{n\geq 0}$ to (1.8),

$$\lim_{n\to\infty} v_n \begin{cases} = \infty & \text{if } g(u_0) - u_0 \;>\; v_0, \\ = 0 & \text{if } g(u_0) - u_0 \;=\; v_0, \\ < 0 & \text{if } g(u_0) - u_0 \;<\; v_0. \end{cases} \tag{4.5}$$

Proof. We observe that the fact that $x \mapsto g(x) - x$ satisfies (4.5) implies $g(x) = x + h(x)$ for all $x \in \mathbb{R}$, with h the function defined in (4.1). Consequently, demonstrating that a solution to (4.3) satisfies (4.5) establishes the uniqueness of the function.

In the first part of the proof, we show the existence of a function g satisfying (4.3) via an approximation argument, using a fixed-point approach. Next, we establish the regularity of g stated in part (i) by analyzing the properties of its approximating sequence. Finally, we prove that g satisfies (4.5).

Proof of (4.3). We fix an arbitrary constant $K > 0$ such that

$$K \;\geq\; \sup_{x\in[-A,0]} \left(\Psi(x) + (x + A)\Psi'(x)\right)$$

and introduce the auxiliary function

$$\sigma : [-A, 0] \ni x \;\mapsto\; Kx - (x + A)\Psi(x).$$

It is straightforward to verify that $\sigma'(x) \geq 0$ for all $x \in [-A, 0]$, implying that σ is nondecreasing. We also note that (4.3) can be rewritten as

$$(K + 1)g(x) \;=\; g(g(x)) + \sigma(g(x)) + (x + A)\Psi(g(x)) \quad \text{for } x \in [-A, 0]. \tag{4.6}$$

We now define recursively a sequence of functions $(g_n)_{n\geq 1}$ on $[-A, 0]$ as follows. We begin by fixing g_1 as the unique solution to the differential equation $y' = \Psi(y)$ on $[-A, 0]$ with the initial condition $y(0) = 0$. Then for any $n \geq 1$ and $x \in [-A, 0]$, we define g_{n+1} by the relation

$$(K + 1)g_{n+1}(x) := g_n(g_n(x)) \;+\; \sigma(g_n(x)) \;+\; (x + A)\Psi(g_n(x)) \tag{4.7}$$

$$= g_n(g_n(x)) \;+\; Kg_n(x) \;-\; (g_n(x) - x)\Psi(g_n(x)). \tag{4.8}$$

In the second line, we used the definition of σ. We prove by induction on n that the sequence $(g_n)_{n\geq 1}$ is nondecreasing on $[-A, 0]$ and consists of nondecreasing functions that are $\mathcal{C}^2$, 1-Lipschitz, and satisfy $g_n(0) = 0$ and $g_n'(0) = 1$ for all n.

We immediately observe from the definition of g_1 that it is a $\mathcal{C}^2$ function with $g_1(0) = 0$, using the fact that Ψ is $\mathcal{C}^1$, and that $g_1'(0) = \Psi(0) = 1$. Moreover, we have

$$0 \;\le\; \Psi(g_1(x)) \;=\; g_1'(x) \;\le\; 1 \quad \text{for all } x \in [-A, 0].$$

Therefore g_1 is nondecreasing and 1-Lipschitz, and since $g_1' = \Psi \circ g_1$ is also nondecreasing, we have that g_1 is convex. As a result, by the definition in (4.8),

$$(K+1)(g_2(x) - g_1(x)) \;=\; g_1(g_1(x)) - (g_1(x) + (g_1(x) - x)\Psi(g_1(x))) \;\ge\; 0$$

for all $x \in [-A, 0]$, where we used the fact that $y \mapsto (y - x)g_1'(x) + g_1(x)$ is the tangent of g_1 at point x, and hence smaller than g_1, in particular at $y = g_1(x)$.

Turning to the inductive step, we fix $n \in \mathbb{N}$ and assume that g_n is a nondecreasing function that is $\mathcal{C}^2$ and 1-Lipschitz with $g_n(0) = 0$ and $g_n'(0) = 1$. We also assume that $g_{n+1}(x) \ge g_n(x)$ for all $x \in [-A, 0]$. By (4.8), g_{n+1} is then clearly $\mathcal{C}^2$ as well, and we have

$$(K+1)g_{n+1}(0) \;=\; g_n(g_n(0)) + Kg_n(0) - g_n(0)\Psi(g_n(0)) \;=\; 0$$

since $g_n(0) = 0$.

Regarding the first derivative of g_{n+1}, we compute using (4.8)

$$\begin{aligned}
&(K+1)g_{n+1}'(x)\\
&\quad=\; g_n'(g_n(x))g_n'(x) + Kg_n'(x) - (g_n'(x) - 1)\Psi(g_n(x)) - (g_n(x) - x)\Psi'(g_n(x))g_n'(x)\\
&\quad= g_n'(g_n(x))g_n'(x) + \Psi(g_n(x)) + g_n'(x)\big(K - \Psi(g_n(x)) - (g_n(x) - x)\Psi'(g_n(x))\big).
\end{aligned}$$

We first observe that $g_{n+1}'(0) = 1$. Moreover, since g_n is nondecreasing and 1-Lipschitz, we know that $g_n(x) - x \ge 0$ for all $x \in [-A, 0]$ and can therefore estimate

$$(K+1)g_{n+1}'(x) \;\le\; 1 + \Psi(g_n(x)) + (K - \Psi(g_n(x))) \;\le\; K + 1.$$

Additionally, since $g_n(x) \in [-A, 0]$, we can use the definition of K to obtain the following lower bound:

$$K - \Psi(g_n(x)) - (g_n(x) - x)\Psi'(g_n(x)) \;\ge\; K - \sup_{y \in [-A,0]} \Psi(y) - (y + A)\Psi'(y) \;\ge\; 0. \tag{4.9}$$

Thus, we conclude that $g_n'(x) \in [0, 1]$ for all $x \in [-A, 0]$, which shows that g_n is nondecreasing and 1-Lipschitz.

Finally, we show that $g_{n+2} \ge g_{n+1}$, using (4.7). We have

$$\begin{aligned}
(K+1)(g_{n+2}(x) - g_{n+1}(x)) \;&=\; g_{n+1}(g_{n+1}(x)) - g_n(g_n(x))\\
&\quad +\; \sigma(g_{n+1}(x)) - \sigma(g_n(x)) + (x + A)(\Psi(g_{n+1}(x)) - \Psi(g_n(x))).
\end{aligned}$$

Since g_n and g_{n+1} are monotone with $g_{n+1} \ge g_n$, we have $g_{n+1}(g_{n+1}(x)) - g_n(g_n(x)) \ge 0$. Similarly, using the monotonicity of σ and Ψ, and noting that $x + A \ge 0$, we infer that $g_{n+2}(x) \ge g_{n+1}(x)$ for all $x \in [-A, 0]$. Next, we define g as the increasing limit of g_n, i.e.,

$$g(x) \;:=\; \lim_{n\to\infty} g_n(x) \quad \text{for all } x \in [-A, 0].$$

Using the properties of the sequence $(g_n)_{n\ge1}$, we observe that g is nondecreasing, 1-Lipschitz, and satisfies $g(0) = 0$. Moreover, for all $x \in [-A, 0)$, we have $g(x) \ge g_1(x) > x$. By continuity of Ψ, we also have

$$(K+1)g(x) = g(g(x)) + Kg(x) - (g(x) - x)\Psi(g(x)) \quad \text{for all } x \in [-A, 0],$$

which shows that g satisfies equation (4.3).

Proof of (4.4). It remains to show that $g(x) - x \sim x^2/2$ as $x \to 0$. First, observe that by construction, $g \geq g_1$, where g_1 is a $\mathcal{C}^2$ function with $g_1(0) = 0$, and $g_1'(0) = g_1''(0) = 1$. Therefore, by a Taylor expansion of g_1 around $x = 0$, we have

$$g(x) - x \ \geq \ g_1(x) - x \ = \ \frac{x^2}{2}(1 + o(1)).$$

To complete the proof, we therefore only need to show that for any $w > 1/2$, there exists $\delta > 0$ such that

$$g(x) \ \leq \ x + wx^2 \quad \text{for all } x \in [-\delta, 0].$$

To this end, we fix $\delta \in (0, A/3)$ such that $g_1(x) \leq x + wx^2$ and $\Psi(x) \leq (1 + wx)^2$ for all $x \in [-\delta, 0]$, using the fact that $\Psi(1) = \Psi'(1) = 1$. We will prove by induction that for all $n \geq 1$ and $x \in [-\delta, 0]$, the inequality $g_n(x) \leq Q(x) := x + wx^2$ holds, from which (4.4) will follow by passage to the limit.

Assuming $g_n(x) \leq Q(x)$ for all $x \in [-\delta, 0]$ and using formula (4.7), it follows that

$$\begin{aligned}(K+1)g_{n+1}(x) \ &\leq \ Q(Q(x)) + \sigma(Q(x)) + (x + A)\Psi(Q(x)) \\ &\leq \ Q(Q(x)) + KQ(x) - (Q(x) - x)\Psi(Q(x)),\end{aligned}$$

since g_n is 1-Lipschitz, thus $g_n(x) \geq -\delta$ for all $x \in [-\delta, 0]$. Now consider the expression $Q(Q(x)) - (Q(x) - x)\Psi(Q(x))$:

$$Q(Q(x)) - (Q(x) - x)\Psi(Q(x)) \ = \ Q(x) + wQ(x)^2 - wx^2\Psi(Q(x)).$$

This simplifies to

$$Q(x) + wx^2 \left((1 + wx^2) - \Psi(Q(x))\right).$$

Since $Q(x) \geq x$, we have $(1 + wx^2) - \Psi(Q(x)) \leq 0$, which shows that

$$Q(Q(x)) - (Q(x) - x)\Psi(Q(x)) \ \leq \ Q(x).$$

Hence, we conclude that

$$(K+1)g_{n+1}(x) \ \leq \ (K+1)Q(x),$$

which completes the induction step and the proof of (4.4).

Proof of (4.5). To complete the proof of Proposition 4.1, it remains to show that the function $h^* : x \mapsto g(x) - x$ and the function h in (4.1) are identical on $[-A, \infty)$. Using the properties of g, we observe that h^* is a continuous solution of the functional equation

$$h^*(x + h^*(x)) \ = \ h^*(x)\Psi(x + h^*(x)) \quad \text{for all } x \in [-A, 0]. \tag{4.10}$$

such that $0 \leq h^*(x) \leq (-x)_+$ for all $x \in \mathbb{R}$ and $h(x) > 0$ for $x < 0$. We underscore that to prove h^* coincides with h on $[-A, \infty)$, no additional regularity conditions on h^*, such as monotonicity or 1-Lipschitz continuity, are required. Let $v_0^* < 0$ and define $(u_n^*, v_n^*)_{n\geq 0}$ as the solution of the recursive equation (1.8) with initial conditions $(u_0^*, v_0^*) = (h^*(v_0^*), v_0^*)$. By induction, we immediately observe that $u_n^* = h^*(v_n^*)$ for all $n \geq 1$. This follows from the recurrence relation

$$u_{n+1}^* \ = \ u_n^*\Psi(v_{n+1}^*) \ = \ h^*(v_n^*)\Psi(h^*(v_n^*) + v_n^*) \ = \ h^*(v_n^* + h^*(v_n^*)) \ = \ h^*(v_{n+1}^*),$$

where we used (4.10) for the last equality. Note that if $v_n^* \leq 0$, then $v_{n+1}^* = v_n^* + h^*(v_n^*) \leq 0$ as well since $h^*(x) \leq (-x)_+$. Therefore, $\sup_{n\geq 0} v_n^* \leq 0$. Since $(v_n^*)_{n\geq 0}$ is a nondecreasing sequence, we have $v_\infty^* := \lim_{n\to\infty} v_n^* \leq 0$. Furthermore, $u_\infty^* := \lim_{n\to\infty} u_n^* = h^*(v_\infty^*)$, and thus $u_\infty^* > 0$ if $v_\infty^* < 0$. But this contradicts (1.9) and we conclude that $\lim_{n\to\infty} v_n^* = 0$.

Next, let $(u_n, v_n)_{n\geq 0}$ be a solution to (1.8) with $u_0 > h^*(v_0)$. By Lemma 3.2, we know that $u_n \geq u_n^*$ and $v_n \geq v_n^*$ for all $n \in \mathbb{N}$. Moreover, we have the inequality

$$v_{n+1} - v_{n+1}^* \;=\; v_n - v_n^* + u_n - u_n^* \;\geq\; v_n - v_n^* \;\geq\; v_1 - v_1^*.$$

Since $v_1 - v_1^* = u_0 - h(v_0^*) > 0$, it follows that $\lim_{n\to\infty} v_n > 0$. By Lemma 3.1, we conclude that $\lim_{n\to\infty} v_n = \infty$.

Similarly, if $u_0 < h(v_0)$, then for all $n \in \mathbb{N}$, we have $u_n \leq u_n^*$ and $v_n \leq v_n^*$, with

$$v_{n+1} - v_{n+1}^* \;\leq\; v_n - v_n^* \;\leq\; v_1 - v_1^* \;<\; 0.$$

Thus, we deduce that $\lim_{n\to\infty} v_n < 0$. □

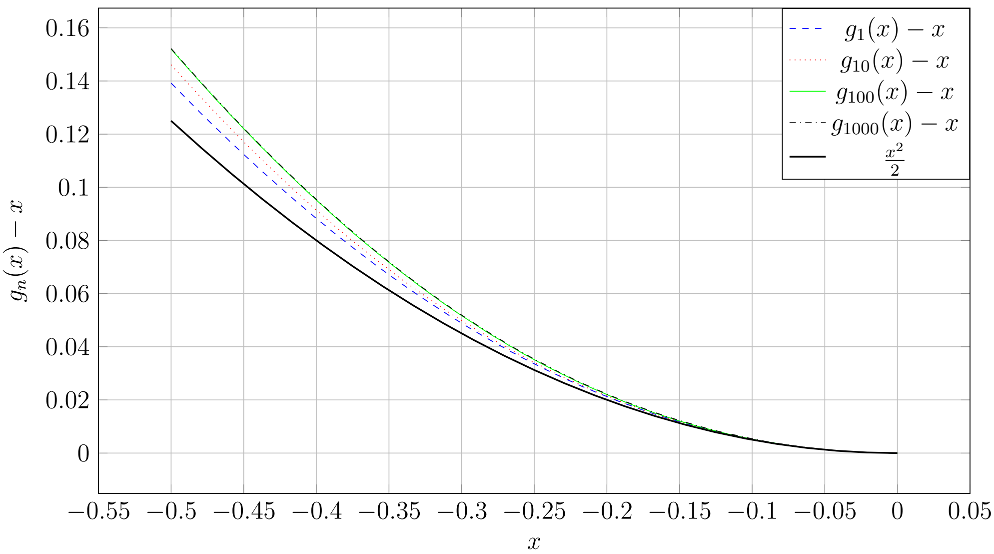

FIGURE 2. Numerical computations of $g_n(x) - x$ and $x^2/2$, where $\eta = -0.5, K = 10$, and $\Psi(x) = \frac{1+2x}{1+x}$ corresponding to the generalized DR model described in Section 2.1 with $\mathtt{Z} = 1$ and $p = 0.5$.

We conclude this section with the proof of Theorem 1.1.

Proof of Theorem 1.1. For each $A > 0$, we apply Proposition 4.1 to construct the unique function g^A on $[-A, \infty)$ that satisfies (4.3). By the compatibility property, for any $B > A$, the restriction of g^B to $[-A, \infty)$ equals g^A. Thus, we can construct g on $\mathbb{R}$ by taking the projective limit.

Next, define the function $h^* : x \mapsto g(x) - x$. We deduce from Proposition 4.1 that this is the unique nonincreasing 1-Lipschitz, non-trivial solution to the functional equation (1.12), and by the same result, we know that $h^*(x) \sim x^2/2$ as $x \uparrow 0$. Finally, for any $(u_0, v_0) \in \mathbb{R}_+ \times \mathbb{R}$, we have the following characterizations:

$$\begin{aligned} &(u_0, v_0) \in \mathcal{C} \iff u_0 = h^*(v_0), \quad (u_0, v_0) \in \mathcal{P} \iff u_0 > h^*(v_0) \\ \text{and} \quad &(u_0, v_0) \in \mathcal{U} \iff u_0 < h^*(v_0). \end{aligned}$$

This proves that $h = h^*$ and establishes the decomposition in (1.14). □

The duality relationship outlined in Proposition 3.3 enables the definition of a curve $\check{h}$, which plays the same role as h in the backward evolution of the dynamics. Heuristically, it can be described as the trajectory of (u_n, v_n) such that (u_0, v_0) lies within a small neighborhood of $(0, 0)$.

Corollary 4.2. *Assume* (A). *There exists a unique nondecreasing and continuous function* $\breve{h}$*, with* $0 \le \breve{h}(x) \le \Psi(x)\, x_+$ *for all* $x \in \mathbb{R}$*, that satisfies the functional equation*

$$\breve{h}(x + \breve{h}(x)) \;=\; \Psi(x + \breve{h}(x))\breve{h}(x) \quad \textit{for all } x \in \mathbb{R}_+, \tag{4.11}$$

with the asymptotics $\breve{h}(x) \sim x^2/2$ *as* $x \downarrow 0$*, and* $\breve{h}(x) \to \infty$ *as* $x \to \infty$*.*

Proof. Using the notation of Proposition 3.3, we first apply Proposition 4.1 to the function

$$\breve{\Psi} : x \mapsto \frac{1}{\Psi(-x)}.$$

This defines a function $\widetilde{h}$ such that if $(\breve{u}_k, \breve{v}_k)_{k \ge 0}$ is the solution to (1.8) with $\breve{\Psi}$ in place of Ψ, and with initial condition $\breve{u}_0 = \widetilde{h}(\breve{v}_0)$, then $\breve{u}_k = \widetilde{h}(\breve{v}_k)$ for all $k \ge 1$. Let $N \ge 2$. Applying the duality relationship from Proposition 3.3 to $u_n := \breve{u}_{N-n}$ and $v_n := -\breve{v}_{N-n+1}$ for $0 \le n \le N$, we obtain $u_n = \widetilde{h}(\breve{v}_{N-n}) = \widetilde{h}(-v_{n+1})$. Since $u_{n+1} = u_n \Psi(v_{n+1})$, we get

$$u_{n+1} \;=\; \breve{h}(v_{n+1}), \quad \text{where } \breve{h}(x) \;:=\; \widetilde{h}(-x)\Psi(x) \text{ for } x \in \mathbb{R}.$$

Thus, $\breve{h}$ satisfies (4.11) and exhibits regularity similar to that of $\widetilde{h}$ near 0. Moreover, since $\breve{h}$ is nondecreasing and $\Psi(\infty) > 1$, it follows that $\breve{h}(x) \to \infty$ as $x \to \infty$.

Finally, to prove the uniqueness of the solution to (4.11), we proceed as follows. Let $\widehat{h}$ be another solution to (4.11), satisfying the same regularity conditions as $\breve{h}$. We aim to show that the function $x \to \widehat{h}(-x)/\Psi(-x)$ defines the critical curve associated with $\breve{\Psi}$. By the uniqueness of critical curve, this implies $\widehat{h}(-x)/\Psi(-x) = \widetilde{h}(x)$, and therefore $\widehat{h} = \breve{h}$. To this end, we verify that the function $f(y) := \widehat{h}(-y)/\Psi(-y)$ satisfies the equation

$$f(y + f(y)) \;=\; \breve{\Psi}(y + f(y)) f(y), \quad y \le 0.$$

We proceed by making the change of variables $y = -(x + \widehat{h}(x))$ with $x \ge 0$. Such an x exists and is unique because $x + \widehat{h}(x)$ is increasing and tends to ∞ as $x \to \infty$. From (4.11), we have $\widehat{h}(-y) = \Psi(-y)\widehat{h}(x)$, which implies $f(y) = \widehat{h}(-y)/\Psi(-y) = \widehat{h}(x)$. Next, observe that $y + f(y) = y + \widehat{h}(x) = -x$, by the definition of y. Therefore

$$f(y + f(y)) \;=\; f(-x) \;=\; \frac{\widehat{h}(x)}{\Psi(x)}.$$

Using the facts that $\widehat{h}(x) = f(y)$ and $1/\Psi(x) = \breve{\Psi}(-x) = \breve{\Psi}(y + f(y))$, we conclude that

$$f(y + f(y)) \;=\; \breve{\Psi}(y + f(y))\, f(y),$$

showing that f is the critical curve associated with $\breve{\Psi}$. This completes the proof. □

Remark 4.3. *The above proof shows that if* $(u_n, v_n)_{n \ge 0}$ *is a solution of* (1.8) *satisfying* $v_0 \ge 0$ *and* $u_n = \breve{h}(v_n)$ *for all* $n \ge 0$*, then for any* $N \ge 2$*, the dual system defined by*

$$(\breve{u}_n, \breve{v}_n) \;:=\; (u_{N-n}, -v_{N-n+1}) \quad \textit{for } 0 \le n \le N,$$

is also a solution to (1.8)*, but with* $\breve{\Psi}$ *in place of* Ψ*. Moreover, we have* $\breve{u}_n = \widetilde{h}(\breve{v}_n)$ *for* $0 \le n \le N$*, where* $\widetilde{h}(x) := \breve{h}(-x)\breve{\Psi}(x)$*. In other words, the dual system* $(\breve{u}_n, \breve{v}_n)_{0 \le n \le N}$ *moves along the critical curve associated with* $\breve{\Psi}$*. This observation will be helpful in the proof of Lemma 4.5.*

The regularity of the critical curve h will play a crucial role in the proof of the Derrida-Retaux conjecture in the nearly supercritical regime. In this section, we prove that the function h is $\mathcal{C}^1$ and convex in a neighborhood of 0, with a Lipschitz first derivative.

Similarly to the previous section, we establish this result for the function g satisfying (4.3), by analyzing its approximation sequence.

Lemma 4.4. *Assume* (A)*, and let g be the unique nontrivial solution to* (4.3)*. For all $b > 1$, there exists $\eta > 0$ such that g is convex and $\mathcal{C}^1$ on $[-\eta, 0]$, with g' being b-Lipschitz on $[-\eta, 0]$.*

Proof. Let $(g_n)_{n\geq 1}$ be the sequence of functions defined inductively by (4.8) in the proof of Proposition 4.1. Recall that g_1 is the solution of the equation $y' = \Psi(y)$ with initial condition $y(0) = 0$. Therefore, g_1 is convex and $\mathcal{C}^2$, with $g_1(0) = 0$ and $g_1'(0) = g_1''(0) = 1$.

Let $b \in (1, 4/3)$ and set $a = 2 - b \in (2/3, 1)$. We will prove by induction that there exists $\eta > 0$ such that

$$g_n''(x) \in [a, b] \quad \text{for all } x \in [-\eta, 0]. \tag{4.12}$$

We first choose $\delta > 0$ such that $g_1'(x) \in [a, b]$ for all $x \in [-\delta, 0]$. The value of $\eta \leq \delta$ will be determined later. Additionally, define $C = \sup_{x\in[-\delta,0]} |\Psi''(x)|$. If necessary, we reduce δ so that $C\delta < 1$.

Now assume that (4.12) holds for some fixed $n \in \mathbb{N}$. By direct integration, we observe that $g_n'(x) \in [1 + bx, 1 + ax]$ holds for all $x \in [-\eta, 0]$. Moreover, since $|\Psi''|$ is bounded by C, we have

$$\Psi'(x) \in [1 + Cx, 1 - Cx] \quad \text{and} \quad \Psi(x) \in [1 + x - Cx^2/2, 1 + x + Cx^2/2].$$

Next, differentiating (4.8) twice yields the equation

$$(K + 1)g_{n+1}''(x) = g_n''(g_n(x))g_n'(x)^2 + g_n'(g_n(x))g_n''(x) + g_n''(x)U_n(x) + g_n'(x)V_n(x), \tag{4.13}$$

where we have defined

$$U_n(x) := K - \Psi(g_n(x)) - (g_n(x) - x)\Psi'(g_n(x)),$$
$$V_n(x) := 2(1 - g_n'(x))\Psi'(g_n(x)) - (g_n(x) - x)g_n'(x)\Psi''(g_n(x)).$$

Bounding U_n. We first bound U_n using the known bounds for Ψ. We have:

$$\begin{aligned} U_n(x) &\leq K - 1 - g_n(x) + \frac{Cg_n(x)^2}{2} - (g_n(x) - x)(1 + Cg_n(x)) \\ &\leq K - 1 + x - 2g_n(x) + Cxg_n(x), \end{aligned}$$

and similarly

$$U_n(x) \geq K - 1 + x - 2g_n(x) - Cxg_n(x).$$

Let us introduce $D > 0$ such that for all $x \in [-\delta, 0]$, we have

$$Cxg_n(x) \leq Cxg_1(x) \leq Dx^2.$$

Thus, we obtain

$$K - 1 + x - 2g_n(x) - Dx^2 \leq U_n(x) \leq K - 1 + x - 2g_n(x) + Dx^2.$$

Bounding V_n. Next, we bound $V_n(x)$ in a similar manner, using the bounds on Ψ', Ψ''. We write:

$$2(1 - g_n'(x))(1 - Cg_n(x)) - \frac{Cbx^2}{2} \leq V_n(x) \leq 2(1 - g_n'(x))(1 + Cg_n(x)) + \frac{Cbx^2}{2}.$$

Since $1 - g_n'(x) \in [-ax, -bx]$ and $g_n(x) \in [x - bx^2/2, x - ax^2/2]$, we conclude that there exists $E > 0$, depending only on C, a, b, such that

$$2(1 - g_n'(x)) - Ex^2 \leq V_n(x) \leq 2(1 - g_n'(x)) + Ex^2.$$

Bounding g''_{n+1} from above. Since $g''_n(x) \le b$, equation(4.13) yields

$$(K+1)g''_{n+1}(x) \;\le\; bg'_n(x)^2 + bg'_n(g_n(x)) \\ + b(K-1+x-2g_n(x)) + 2g'_n(x)(1-g'_n(x)) + Rx^2 \;=:\; P_n(x)$$

where $R>0$ is a sufficiently large constant, depending only on a,b,C,D and E. We observe that $P_n(0) = b(K+1)$, and that P'_n is increasing on $[-\eta_2, 0]$. To see this, we compute the derivative

$$P'_n(x) \;=\; 2bg'_n(x)g''_n(x) + bg''_n(g_n(x))g'_n(x) + b(1-2g'_n(x)) + 2g''_n(x)(1-2g'_n(x)) + 2Rx.$$

We then bound this from below as follows:

$$P'_n(x) \;\ge\; 2bg''_n(x) + bg''_n(g_n(x)) - b - 2g''_n(x) + Sx$$

where S is a constant depending on R,a,b, and we used the fact that $g'_n(x) \in [1+bx, 1+ax]$ for all $x \in [-\delta, 0]$. Therefore, by the inductive hypothesis that (4.12) holds for g''_n, we have

$$P'_n(x) \ge 2(b-1)g''_n(x) + ab - b + 2Sx \ge 2(b-1)a + ab - b + Sx \;\ge\; 3ab - 2a - b + Sx.$$

Since $a = 2-b$ and $b \in (1, 4/3)$, we have $3ab - 2a - b = (b-1)(4-3b) > 0$. Thus, we can choose $\eta_2 > 0$ small enough such that $3ab - 2a + Sx \ge 0$ for all $x \in [-\eta_2, 0]$. We conclude that P_n is nondecreasing on $[-\eta_2, 0]$, with $P_n(0) = (K+1)b$. Therefore, for all $x \in [-\eta_2, 0]$, we have

$$(K+1)g''_{n+1}(x) \;\le\; P_n(x) \;\le\; P_n(0) \;=\; (K+1)b.$$

Bounding g''_{n+1} from below. The lower bound is treated in the same manner. Using $g''(x) \ge a$, we obtain from (4.13) the following lower bound for g''_{n+1}

$$(K+1)g''_{n+1}(x) \;\ge\; ag'_n(x)^2 + ag'_n(g_n(x)) \\ + a(K-1+x-2g_n(x)) + 2g'_n(x)(1-g'_n(x)) - Rx^2 \;=:\; Q_n(x),$$

where $R>0$ is a sufficiently large constant, depending only on a,b,D and E. We note that $Q_n(0) = a(K+1)$. Next, we compute the derivative of $Q_n(x)$:

$$Q'_n(x) \;=\; 2ag'_n(x)g''_n(x) + ag''_n(g_n(x))g'_n(x) + a(1-2g'_n(x)) + 2g''_n(x)(1-2g'_n(x)) - 2Rx.$$

Using again that $g'_n(x) \in [1+bx, 1+ax]$, we can bound this expression, namely

$$Q'_n(x) \;\le\; 2ag''_n(x) + ag''_n(g_n(x)) - a - 2g''_n(x) - Sx,$$

with a constant S depending on R,a,b. Since $g''_n(g_n(x)) \le b$ and $g''_n(x) \ge a$ we get

$$Q'_n(x) \;\le\; 2(a-1)a + ab - a - Sx \le a(2a+b-3) - Sx$$

Using $a = 2-b$, we find that $a(2a+b-3) = (2-b)(b-1) < 0$ for all $b \in (1,2)$. Therefore, we can choose $\eta_3 > 0$ small enough such that $Q'_n(x) \le 0$ for all $x \in [-\eta_3, 0]$. We conclude that Q_n is nonincreasing on $[-\eta_3, 0]$, and thus

$$(K+1)g''_{n+1}(x) \ge Q_n(x) \ge Q_n(0) = (K+1)a.$$

Finally, by setting $\eta = \min(\eta_1, \eta_2, \eta_3)$, we observe that we have proved (4.12) for g''_{n+1}.

Convexity of g. Now, since (4.12) holds for all $n \in \mathbb{N}$, the sequence $(g'_n)_{n\ge1}$ forms a family of continuous, increasing, and b-Lipschitz functions on $[-\eta, 0]$. By the Arzelà-Ascoli theorem, we can therefore extract a subsequence $(g'_{n_k})_{k\ge1}$ that converges pointwise to a continuous, increasing, and b-Lipschitz limit function f. Using the dominated convergence theorem, we then obtain

$$g(x) = -\lim_{k\to\infty} \int_x^0 g'_{n_k}(y)\,\mathrm{d}y \;=\; -\int_x^0 f(y)\,\mathrm{d}y.$$

This shows that g is $\mathcal{C}^1$ on $[-\eta, 0]$ with $g' = f$. As g' is increasing, we conclude that g is convex. This completes the proof. □

4.2. **The critical regime.** In this subsection, we prove Theorem 1.4. For a fixed initial condition $(u_0, v_0) \in \mathcal{C}$, we examine the precise asymptotic behavior of (u_n, v_n) as $n \to \infty$.

Proof of Theorem 1.4. Let $v_0 < 0$, and fix $u_0 = h(v_0)$. Let $(u_n, v_n)_{n\geq 0}$ the solution to (1.8). From Proposition 4.1, we know that $u_n = h(v_n)$ for all $n \geq 1$, and that

$$\lim_{n\to\infty} u_n \;=\; \lim_{n\to\infty} v_n \;=\; 0.$$

Therefore, using the asymptotic relation $h(x) \sim x^2/2$ as $x \uparrow 0$, we immediately obtain $u_n \sim 2/n^2$ once we establish that $v_n \sim -2/n$.

To this end, let $0 < w_1 < \frac{1}{2} < w_2$ and $\delta > 0$ such that $w_1 x^2 \leq h(x) \leq w_2 x^2$ for all $x \in [-\delta, 0]$. Thus, for sufficiently large n, we have

$$v_{n+1} \;=\; v_n + u_n \;=\; v_n + h(v_n) \;\in\; [v_n + w_1 v_n^2, v_n + w_2 v_n^2],$$

which implies the inequality

$$\frac{1}{v_{n+1}} \;\leq\; \frac{1}{v_n(1 + w_1 v_n)} \;\leq\; \frac{1}{v_n}(1 - w_1 v_n) \;=\; \frac{1}{v_n} - w_1,$$

using $\frac{1}{1+x} \geq 1 - x$ for all $x > -1$. Hence, we deduce that

$$\limsup_{n\to\infty} \frac{1}{n v_n} \;\leq\; -w_1.$$

Similarly, for $w_2' > w_2$, there exists an n large enough such that

$$\frac{1}{v_{n+1}} \;\geq\; \frac{1}{v_n(1 + w_2 v_n)} \;\geq\; \frac{1}{v_n} - w_2',$$

which gives $\liminf_{n\to\infty} 1/n v_n \geq -w_2'$. We conclude that $\lim_{n\to\infty} n v_n = -2$, completing the proof. □

We conclude this section by applying the duality relationship to analyze the time it takes for the sequence (u_n, v_n) to evolve from the first time when $v_n \geq 0$ to the first time when $u_n \geq 1$. For all initial points $(u_0, v_0) \in \mathcal{P}$, we define

$$n_* = n_*(u_0, v_0) := \inf\{n \geq 0 : v_n \geq 0 \text{ and } u_n \geq 1\}. \tag{4.14}$$

Lemma 4.5. *Let $v_0 < 0$ and $u_0 > h(v_0)$. Defining $N_0' := \inf\{n \geq 0 : v_n \geq 0\}$, there exists a positive constant $c > 0$ such that*

$$(n_* - k)_+ \;\leq\; \frac{c}{v_k} \quad \text{for any } k > N_0'.$$

Proof. Recall the function $\check{h}$ from Corollary 4.2. We begin by verifying that $u_{N_0'} > \check{h}(v_{N_0'})$. If $v_{N_0'} = 0$, this is trivial since $\check{h}(0) = 0$. If $v_{N_0'} > 0$, there exists a unique $x > 0$ such that $v_{N_0'} = x + \check{h}(x)$. By definition, we have

$$v_{N_0'} \;=\; v_{N_0'-1} + u_{N_0'-1} \;<\; u_{N_0'-1},$$

which implies $u_{N_0'-1} > \check{h}(x)$ and

$$u_{N_0'} = u_{N_0'-1}\Psi(v_{N_0'}) > \check{h}(x)\Psi(x+\check{h}(x)) = \check{h}(x+\check{h}(x)) = \check{h}(v_{N_0'}).$$

By induction, we get that $u_n > \check{h}(v_n)$ for any $n \geq N_0'$.

Now consider $k > N_0'$. If $u_k \geq 1$, $n_* \leq k$ and there is nothing to prove. Otherwise, assume that $u_k < 1$, thus $\check{h}(v_k) < 1$. Then v_k is bounded from above, because $\check{h}(x) \to \infty$ as $x \to \infty$.

We now examine the system $(\widetilde{u}_j^*, \widetilde{v}_j^*)_{j\geq 0}$, where $(\widetilde{u}_0^*, \widetilde{v}_0^*) = (\check{h}(v_k), v_k)$ and

$$(\widetilde{u}_j^*, \widetilde{v}_j^*) = (\check{h}(\widetilde{v}_j^*), \widetilde{v}_{j-1}^* + \widetilde{u}_{j-1}^*) \quad \text{for } j \geq 1.$$

This system satisfies the recursive equation (1.8) with the same function Ψ. An induction shows that $u_{j+k} \geq \widetilde{u}_j^*$ and $v_{j+k} \geq \widetilde{v}_j^*$ for all $j \geq 0$. Therefore, $n_* - k \leq \underline{n}_k$, where

$$\underline{n}_k := \inf\{j \geq 0 : \widetilde{u}_j^* \geq 1\}.$$

To complete the proof, it therefore suffices that $\underline{n}_k \leq c/v_k$ for some positive constant c. The idea is to apply the duality in Proposition 3.3 to

$$(\check{u}_j, \check{v}_j)_{0\leq j\leq \underline{n}_k} := (\widetilde{u}_{\underline{n}_k-j}^*, -\widetilde{v}_{\underline{n}_k-j+1}^*)_{0\leq j\leq \underline{n}_k}.$$

By Remark 4.3, the system $(\check{u}_j, \check{v}_j)_{j\geq 0}$ is a solution to (1.8) with $\check{\Psi}(x) := 1/\Psi(-x), x \in \mathbb{R}$, replacing Ψ. Moreover, $\underline{n}_k$ is the time it takes for this system to evolve along its critical curve from the initial position $(\widetilde{u}_{\underline{n}_k}^*, -\widetilde{v}_{\underline{n}_k+1}^*)$ to $(\widetilde{u}_0^*, -\widetilde{v}_1^*)$.

Next, observe that $\widetilde{v}_1^* > \widetilde{v}_0^* = v_k$. If we can show that $\widetilde{v}_{\underline{n}_k+1}^* \leq c'$ for some positive constant c', then by the monotonicity in the starting point (see Lemma 3.2), we have

$$\underline{n}_k \leq \inf\{j \geq 0 : \check{v}_j \geq -v_k\},$$

where $(\check{u}_j, \check{v}_j)_{j\geq 0}$ now lies on the critical curve with $\check{v}_0 = -c'$. We obtain $\underline{n}_k \leq c/v_k$ for some positive constant c by applying Theorem 1.4 to the critical system $(\check{u}_j, \check{v}_j)_{j\geq 0}$.

We are thus left with proving that $\widetilde{v}_{\underline{n}_k+1}^* \leq c'$. Since $\widetilde{v}_{\underline{n}_k+1}^* = \widetilde{v}_{\underline{n}_k}^* + \check{h}(\widetilde{v}_{\underline{n}_k}^*)$, it suffices to show that $\widetilde{v}_{\underline{n}_k}^*$ is bounded from above. By the definition of $\underline{n}_k$, we have $\check{h}(\widetilde{v}_{\underline{n}_k-1}^*) = \widetilde{u}_{\underline{n}_k-1}^* < 1$. Since $\check{h}(x) \to \infty$ as $x \to \infty$, it follows that $\widetilde{v}_{\underline{n}_k-1}^*$ must be bounded from above. Furthermore, note that

$$\widetilde{v}_{\underline{n}_k}^* = \widetilde{v}_{\underline{n}_k-1}^* + \widetilde{u}_{\underline{n}_k-1}^* < \widetilde{v}_{\underline{n}_k-1}^* + 1.$$

Consequently, $\widetilde{v}_{\underline{n}_k}^*$ is bounded from above by a constant. This completes the proof. □

5. The Derrida-Retaux conjecture

The primary goal of this section is to prove Theorem 1.3, which establishes the Derrida-Retaux conjecture for the recursive equation (1.8). This result, in turn, implies that the free energy of both solvable Derrida-Retaux models described in Section 2 undergoes an infinite-order BKT-type phase transition.

As a first step, we relate the free energy of (u_0, v_0) to the number of steps n_* required by the recursion (1.8) to bring (u_n, v_n) into the domain $[1, \infty) \times \mathbb{R}_+$.

Lemma 5.1. *Assume* (A) *and* (B). *Then there exists a constant* $c > 0$ *such that*

$$F(1,0)\Psi(\infty)^{-n_*} \leq F(u_0, v_0) \leq \max(u_0, 1)\Psi(\infty)^{-n_*+1},$$

where n_* *is defined in* (4.14).

Observe that if n_* is large, then $\log F(u_0, v_0)$ is comparable to $n_* \log \Psi(\infty)$, up to a correction of order $O(1)$. Consequently, the estimate for the free energy $\log F(h(v_0)+\varepsilon, v_0)$ as $\varepsilon \to 0$ reduces to a proper estimate for the asymptotic behavior of $n_* = n_*(h(v_0)+\varepsilon, v_0)$ as $\varepsilon \to 0$.

Proof. Recall that $u_n/\Psi(\infty)^n$ is nonincreasing. Therefore, we have two possible cases:

(1) $n_* = 0$ and $F(u_0, v_0) \le u_0$.

(2) $n_* > 0$ and

$$F(u_0, v_0) \;\le\; \frac{u_{n_*-1}}{\Psi(\infty)^{n_*-1}} \;\le\; \max(1, u_0)\Psi(\infty)^{-n_*+1}.$$

Here, we used that either $u_{n_*-1} < 1$ or $v_{n_*-1} < 0$. In the second case, u is decreasing for $k < n_*$ and thus $u_{n_*-1} < u_0$.

Next, we establish a lower bound for $F(u_0, v_0)$. We observe that

$$F(u_0, v_0) \;=\; \lim_{n\to\infty} \frac{u_n}{\Psi(\infty)^n} \;=\; \lim_{n\to\infty} \frac{u_{n+k}}{\Psi(\infty)^{n+k}} \;=\; F(u_k, v_k)\Psi(\infty)^{-k}.$$

Hence,

$$F(u_0, v_0) \;\ge\; \Psi(\infty)^{-n_*} F(u_{n_*}, v_{n_*}) \;\ge\; \Psi(\infty)^{-n_*} F(1, 0),$$

where the final inequality follows from the fact that F is nondecreasing with respect to both u and v, as stated in Lemma 3.2. ☐

Let $(u_n^{(\varepsilon)}, v_n^{(\varepsilon)})$ be a solution of (1.8), where $v_0^{(\varepsilon)} = v_0 < 0$ is fixed and $u_0^{(\varepsilon)} = h(v_0) + \varepsilon$. We define

$$N_0^{(\varepsilon)} \;=\; N_0(u_0^{(\varepsilon)}, v_0^{(\varepsilon)}) \;:=\; \max\{n \in \mathbb{N} : v_n^{(\varepsilon)} \le 0\}, \tag{5.1}$$

and observe that $u^{(\varepsilon)}_{N_0^{(\varepsilon)}} = \inf_{n\in\mathbb{N}} u_n^{(\varepsilon)}$, because $u_{n+1}^{(\varepsilon)}/u_n^{(\varepsilon)} = \Psi(v_{n+1}^{(\varepsilon)}) \le 1$ if $n+1 \le N_0^{(\varepsilon)}$ and $u_{n+1}^{(\varepsilon)}/u_n^{(\varepsilon)} = \Psi(v_{n+1}^{(\varepsilon)}) > 1$ if $n+1 > N_0^{(\varepsilon)}$.

The proof of Theorem 1.3 will proceed in three main steps. First, we establish the existence of a constant $c_\star > 0$ such that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} u^{(\varepsilon)}_{N_0^{(\varepsilon)}} \;=\; c_\star. \tag{5.2}$$

This critical step allows us to connect the distance to the critical curve at the initial point of the evolution to its distance at the minimum, where the analysis is simpler (since the critical point is 0 at that stage).

For $A > 0$, we observe that n_*, defined in (4.14), can be decomposed as

$$n_* \;=\; n_A^{(1)} + (n_A^{(2)} - n_A^{(1)}) + (n_* - n_A^{(2)}),$$

where $n_A^{(1)}$ and $n_A^{(1)}$ are defined as

$$n_A^{(1)} \;:=\; \inf\{n \in \mathbb{N} : v_n^{(\varepsilon)} > -A\varepsilon^{1/2}\} \quad \text{and} \quad n_A^{(2)} \;:=\; \inf\{n \in \mathbb{N} : v_n^{(\varepsilon)} > A\varepsilon^{1/2}\}. \tag{5.3}$$

The second step is to show that

$$\lim_{A\to\infty} \lim_{\varepsilon\to 0} (c_\star \varepsilon)^{1/2} \left(n_A^{(2)} - n_A^{(1)}\right) \;=\; \pi\sqrt{2}, \tag{5.4}$$

which will rely on the fact that, on this time-interval, $(u_n^{(\varepsilon)}, v_n^{(\varepsilon)})$ is well-approximated by an Eulerian scheme for the function tan, with an initial condition given by (5.2). Details will follow later, see (5.22)–(5.24)

Finally, we will prove that

$$\lim_{A\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^{1/2} \left(n_A^{(1)} + (n_* - n_A^{(2)})\right) = 0, \tag{5.5}$$

and thus show that n_* is well-approached by $n_A^{(2)} - n_A^{(1)}$, for sufficiently large A. The proof of Theorem 1.3 will then be completed by applying Lemma 5.1.

The proofs of (5.2), (5.4), and (5.5) are provided in the following three steps. For simplicity, we will omit the superscript (ε) in u_n, v_n, and N_0 where there is no risk of confusion.

Step 1: Proof of (5.2). We begin by proving a lemma that enables us to restrict our attention to the case where v_0 lies in any neighbourhood of 0. For $\delta > 0$, we define

$$n_\delta^{(3)} = n_\delta^{(3)}(u_0, v_0) := \inf\{n \in \mathbb{N} : v_n > -\delta\}. \tag{5.6}$$

Lemma 5.2. *Assume* (A), *and let* $v_0 < 0$ *and* $u_0 = h(v_0) + \varepsilon$. *Then*

$$\limsup_{\varepsilon \to 0} n_\delta^{(3)} < \infty \quad \textit{for any } \delta > 0.$$

Moreover, for any sufficiently small $\delta > 0$*, there exists a constant* $C = C(\delta) > 0$ *such that*

$$u_{n_\delta^{(3)}} - h(v_{n_\delta^{(3)}}) \sim C\varepsilon \quad \textit{as } \varepsilon \to 0.$$

The second part of this lemma allows us to describe the relationship between the parameter $\varepsilon = u_0 - h(v_0)$ and the corresponding parameter for the sequence $(u_{n_\delta^{(3)}+n}, v_{n_\delta^{(3)}+n})_{n \ge 0}$.

Proof. Recall from Lemma 4.4 that there exists some $\eta > 0$ such that $h(x) = g(x) - x$ is C^1 on $[-\eta, 0]$. Let $0 < \delta < \eta$. For each $k \in \mathbb{N}$, we consider (u_k, v_k) as a function of (u_0, v_0) and write $u_k = u_k(u_0, v_0)$ and $v_k = v_k(u_0, v_0)$. We observe that $(u_k(\cdot, \cdot), v_k(\cdot, \cdot))$, as an iteration of $\mathcal{C}^2$ functions, is also $\mathcal{C}^2$. Let $K = K(\delta)$ be the smallest positive integer such that

$$v_K(h(v_0), v_0) \ge -\delta.$$

Note that this constant K does not depend on $\varepsilon > 0$. We will now prove that $\lim_{\varepsilon \to 0} n_3^{(\delta)} = K$. By the monotonicity of the v_k, we have $v_K(h(v_0)+\varepsilon, v_0) > -\delta$ for any $\varepsilon > 0$. Moreover, by continuity,

$$v_{K-1}(h(v_0) + \varepsilon, v_0) < -\delta \quad \text{for all sufficiently small } \varepsilon > 0.$$

Thus, $n_\delta^{(3)} = K$ for all sufficiently small ε. This completes the first part of Lemma 5.2.

Next, for all sufficiently small $\varepsilon > 0$, we have

$$u_{n_\delta^{(3)}} - h(v_{n_\delta^{(3)}}) = u_K(\varepsilon) - h(v_K(\varepsilon)),$$

where, for notational convenience, we define

$$u_K(\varepsilon) := u_K(h(v_0) + \varepsilon, v_0) \quad \text{and} \quad v_K(\varepsilon) := v_K(h(v_0) + \varepsilon, v_0)).$$

It is easy to check that $u_K'(0) > 0$ and $v_K'(0) > 0$. The function $\varepsilon \mapsto u_K(\varepsilon) - h(v_K(\varepsilon))$ is C^1 in a neighborhood of 0, and vanishes there. Therefore, we have

$$u_K(\varepsilon) - h(v_K(\varepsilon)) \sim C\,\varepsilon, \quad \text{as } \varepsilon \to 0,$$

with $C := u_K'(0) - h'(v_K(0))v_K'(0)$. Since $h'(v_K(0)) \le 0$, it follows that $C \ge u_K'(0) > 0$. This completes the proof of Lemma 5.2. □

Let us introduce some preliminary notation and concepts. Recall the definition of N_0 from (5.1). Specifically, we have

$$N_0 = N_0(u_0, v_0) = n_\delta^{(3)} + N_0(u_{n_\delta^{(3)}}, v_{n_\delta^{(3)}}).$$

By considering the recursive system $(u_{n_\delta^{(3)}+n}, v_{n_\delta^{(3)}+n})_{n \ge 0}$ and applying Lemma 5.2, we can extend (5.2) to all $v_0 < 0$, with the constant $c_\star$ potentially being multiplied by $C(\delta)$, provided we can establish (5.2) for $v_0 \in (-\delta, 0)$. In particular, using Lemma 4.4, we choose $\delta > 0$ small enough such that h is $\mathcal{C}^1$, convex with a Lipschitz continuous derivative h' on $(-\delta, 0)$. The precise value of δ will be determined later (see (5.12) and (5.13)).

From this point on, we assume that $v_0 \in (-\delta, 0)$ and $u_0 - h(v_0) = \varepsilon$, and we define

$$\Delta_n := u_n - h(v_n) \quad \text{for } 0 \le n \le N_0.$$

Note that $\Delta_n > 0$ for any $n \ge 0$ with the initial condition $\Delta_0 = \varepsilon$.

Let (u_n^*, v_n^*) denote the solution of (1.8) with $u_0^* := h(v_0^*)$ and $v_0^* = v_0$, which we refer to as the critical system. By comparison, we have $0 > v_n \ge v_n^* \sim -2/n$. This implies that for some positive constant c, uniformly in $\varepsilon \in (0,1)$, we have the bound

$$|v_n| \le \frac{c}{n+1} \quad \text{for all } 0 \le n \le N_0. \tag{5.7}$$

We further note that by monotonicity, N_0 is an increasing function of ε, and since $\lim_{\varepsilon \to 0} v_n = v_n^* < 0$ for each fixed n, we conclude $\lim_{\varepsilon \to 0} N_0 = \infty$.

In particular, the bound in (5.7) implies that $\lim_{\varepsilon \to 0} v_{N_0} = 0$, and using $v_{N_0+1} = u_{N_0} + v_{N_0} > 0$, we see that $0 \ge v_{N_0} > -u_{N_0}$. Since $h(x) = o(x)$ as $x \uparrow 0$, we obtain

$$\Delta_{N_0} = u_{N_0} - h(v_{N_0}) \sim u_{N_0} \quad \text{as } \varepsilon \to 0. \tag{5.8}$$

Therefore, to complete the proof of (5.2), it suffices to prove the existence of a positive constant $c_\star$ such that

$$\lim_{\varepsilon \to 0} \frac{\Delta_{N_0}}{\varepsilon} = c_\star. \tag{5.9}$$

The proof of (5.9) involves on studying the variation of the Δ_n. For $n < N_0$, by definition, we have

$$\Delta_{n+1} = u_n \Psi(v_{n+1}) - h(v_{n+1}) = (h(v_n) + \Delta_n)\Psi(v_{n+1}) - h(v_{n+1}).$$

Recall from Section 4 that we have set $g(x) = x + h(x)$ (for $x \le 0$). Therefore, we can express $v_{n+1} = u_n + v_n = g(v_n) + \Delta_n$. Substituting into the expression for $\Psi(v_{n+1})$, a Taylor expansion provides

$$\Psi(v_{n+1}) = \Psi(g(v_n)) + \Psi'(g(v_n))\Delta_n + B_n \Delta_n^2,$$

with

$$B_n := \int_0^1 (1-s)\Psi''(g(v_n) + s\Delta_n)\, \mathrm{d}s$$

Now fix $b \in (1, 4/3)$. By Lemma 4.4, we can choose δ small enough such that h is $\mathcal{C}^1$, convex and h' is b-Lipschitz. Then for any $-\delta \le x \le y \le 0$ with $\delta \in (0, \eta)$, we have the estimate

$$0 \le h(y) - [h(x) + h'(x)(y-x)] \le \frac{b}{2}(y-x)^2. \tag{5.10}$$

Using this, we obtain the approximation

$$h(v_{n+1}) = h(g(v_n)) + h'(g(v_n))\Delta_n + C_n \Delta_n^2$$

for some $C_n \in [0, b/2]$. Recall that for $x \le 0$, we have $h(g(x)) = \Psi(g(x))h(x)$. Thus, for any $n \le N_0$, we arrive at the following expression for Δ_{n+1},

$$\Delta_{n+1} = A_n \Delta_n + D_n \Delta_n^2, \tag{5.11}$$

where

$$A_n := h(v_n)\Psi'(g(v_n)) + \Psi(g(v_n)) - h'(g(v_n)),$$
$$D_n := h(v_n)B_n - C_n + \Psi'(g(v_n)) + B_n\Delta_n.$$

Observe that $|B_n| \le \sup_{x\in[-\delta,0]} |\Psi''(x)|$ and $h(v_n) \le \sup_{x\in[-\delta,0]} h(x)$. Since $\Psi'(0) = 1$ and $0 \le C_n \le \frac{b}{2} < \frac{2}{3}$, we can choose (and fix) a sufficiently small $\delta > 0$ such that for all $n \le N_0$,

$$D_n \in [1/3, 2]. \tag{5.12}$$

We further assume that δ is small enough to satisfy the following conditions:

$$h(x) \ge \frac{x^2}{3} \quad \text{and} \quad b^2 h(x) \le g'(x)x^2, \quad \text{for } x \in [-\delta, 0], \tag{5.13}$$

which is possible since $h(x) \sim x^2/2$ and $g'(x) \to 1$ as $x \uparrow 0$ and $b^2 < 2$.

We now compare A_n with 1 by using the convexity of h. For $x < 0$, differentiating the expression $h(g(x)) = \Psi(g(x))h(x)$ gives

$$h'(g(x))g'(x) = \Psi'(g(x))g'(x)h(x) + \Psi(g(x))h'(x).$$

Since $g'(x) = 1 + h'(x)$, we get

$$A_n = \frac{\Psi(g(v_n))}{g'(v_n)} = \frac{h(g(v_n))}{h(v_n)g'(v_n)}.$$

Next, we examine the expression $\frac{h(g(x))}{h(x)} - g'(x)$. Using the definition of $g(x) = x + h(x)$, we can expand as follows:

$$\frac{h(g(x))}{h(x)} - g'(x) = \frac{h(x + h(x)) - h(x)}{h(x)} - h'(x) = h'(y_x) - h(x)$$

for some $y_x \in [x, x + h(x)]$. Since h is convex, we have $h'(y_x) - h(x) \ge 0$ and therefore conclude that

$$A_n \ge 1.$$

This together with (5.12) shows that Δ_n is increasing on $[0, N_0]$, in particular $\Delta_n \ge \Delta_0 = \varepsilon$ for all $n \le N_0$.

We now turn to deriving an upper bound of A_n. Again, using (5.10),

$$h(g(x)) = h(x + h(x)) \le h(x) + h'(x)h(x) + \frac{b}{2}h^2(x).$$

Thus, we obtain

$$A_n - 1 = \frac{h(g(v_n)) - h(v_n)g'(v_n)}{h(v_n)g'(v_n)} \le \frac{b^2 h(v_n)}{2g'(v_n)} < v_n^2,$$

where the last inequality follows from (5.13). Using (5.11), we get that for any $0 \le k < N_0$,

$$\Delta_{N_0} = \Delta_k \prod_{n=k}^{N_0-1} (1 + (A_n - 1) + D_n\Delta_n). \tag{5.14}$$

This equation allows us to complete the proof of (5.2). Specifically, using the same reasoning as in Lemma 5.2, we can deduce that for all (fixed) $k \in \mathbb{N}$,

$$\Delta_k \sim C_k\varepsilon \quad \text{as } \varepsilon \to 0,$$

for some constant $C_k > 0$. Since $\Delta_{N_0} \ge \Delta_k$, we obtain

$$\liminf_{\varepsilon\to 0} \frac{\Delta_{N_0}}{\varepsilon} \ge C_k \tag{5.15}$$

for all $\varepsilon > 0$ sufficiently small such that $N_0 \geq k$ and all $k \in \mathbb{N}$. On the other hand, using (5.14) and noting that $A_n - 1 < v_n^2$ and $D_n \leq 2$, we get the bound

$$\Delta_{N_0} \leq \Delta_k \exp\left(\sum_{n=k}^{N_0-1} v_n^2 + 2\Delta_n\right).$$

We estimate the sum as follows: since $\sum_{n=k}^{N_0} \Delta_n \leq \sum_{n=k}^{N_0} u_n \leq v_{N_0} - v_k$, and $v_n^2 \leq \frac{c^2}{(n+1)^2}$ (from (5.7)), there exists decreasing null sequence $(R_n)_{n\geq 1}$ such that for all $k \in \mathbb{N}$,

$$\Delta_{N_0} \ \leq \ \Delta_k e^{R_k + v_{N_0} - v_k}.$$

Taking the limit as $\varepsilon \to 0$, we obtain

$$\limsup_{\varepsilon\to 0} \frac{\Delta_{N_0}}{\varepsilon} \leq C_k e^{R_k - v_k^*} < \infty, \tag{5.16}$$

using that $v_{N_0} \to 0$ and $v_k \to v_k^*$ as $\varepsilon \to 0$.

Observe that $(C_k)_{k\geq 1}$ is a nondecreasing sequence because $\Delta_{k+1} \geq \Delta_k$ for all $k \in \mathbb{N}$. From (5.15) and (5.16), we also have $C_k \leq C_1 e^{R_1 - v_1}$ for all k. Consequently, the C_k converge to a finite constant $c_\star$. Moreover, since $\lim_{k\to\infty}(R_k - v_k^*) = 0$, we can apply (5.15) and (5.16) to conclude that (5.9) holds, thereby completing the proof. ☐

Step 2: Proof of (5.4). The main idea here to approximate the function Ψ, in a neighborhood of 0 of width $\varepsilon^{1/2}$, by the simpler function $x \mapsto 1 + x$. Specifically, we substitute this approximation into the recursion (1.8), resulting in the simplified system:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} a_n(1 + b_{n+1}) \\ b_n + a_n \end{pmatrix}. \tag{5.17}$$

obtained by specifying (1.8) to the function $\Psi : x \mapsto 1 + x$. This simplified system has been considered in [17]. Let for $\delta > 0$,

$$n_\delta^{(4)} = n_\delta^{(4)}(u_0, v_0) := \inf\{n \in \mathbb{N} : v_n > \delta\}. \tag{5.18}$$

Lemma 5.3. *Assume* (A)*, let $\eta \in (0, 1)$ be small and fix $\delta > 0$ such that*

$$\frac{\Psi(x) - 1}{x} \in [(1 - \eta), (1 + \eta)] \quad \textit{for all } x \in [0, \delta]. \tag{5.19}$$

Let $(u_n, v_n)_{n\geq 0}$ satisfy (1.8) *with initial conditions $u_0 > 0$ and $v_0 \in (-u_0, 0]$. Then for all $1 \leq k < n_\delta^{(4)}$, we have*

$$a_k^{(\eta,-)} < u_k < a_k^{(\eta,+)} \quad \textit{and} \quad b_k^{(\eta,-)} < v_k < b_k^{(\eta,+)}, \tag{5.20}$$

where $(a_k^{(\eta,\pm)}, b_k^{(\eta,\pm)})_{k\geq 0}$ are the solutions to the simplified recursion (5.17) *with the respective initial conditions $(a_0^{(\eta,\pm)}, b_0^{(\eta,\pm)}) := ((1 \pm \eta)u_0, (1 \pm \eta)v_0)$.*

Lemma 5.3 enables us to compare the system $(u_n, v_n)_{n\geq 0}$ with $(a_n, b_n)_{n\geq 0}$ up to a multiplicative factor $1 \pm \eta$ on the initial conditions. In particular, the evolution of the system $(u_{N_0+k}, v_{N_0+k})_{k\geq 0}$ can be controlled by the two systems $(a_k^{(\eta,\pm)}, b_k^{(\eta,\pm)})_{k\geq 0}$ with initial conditions $(a_0, b_0) = ((1 \pm \eta)u_{N_0}, (1 \pm \eta)v_{N_0})$.

Proof. The proof follows from a direct application of Lemma 3.2 with the functions

$$\underline{\Psi}(x) = 1 + x - \eta|x| \quad \text{and} \quad \overline{\Psi}(x) = 1 + x + \eta|x|.$$

In particular, if $u_0 > 0$ and $v_0 \in (-u_0, 0]$, we have (using the notation of that lemma)

$$\begin{pmatrix} \underline{u}_{n+1} \\ \underline{v}_{n+1} \end{pmatrix} = \begin{pmatrix} \underline{u}_n(1 + (1 - \eta)\underline{v}_{n+1}) \\ \underline{v}_n + \underline{u}_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \overline{u}_{n+1} \\ \overline{v}_{n+1} \end{pmatrix} = \begin{pmatrix} \overline{u}_n(1 + (1 + \eta)\overline{v}_{n+1}) \\ \overline{v}_n + \overline{u}_n \end{pmatrix}.$$

We then observe that the sequence $((1-\eta)a_n, (1-\eta)b_n)_{n\geq 0}$ satisfies the same recursion as $(\underline{u}_n, \underline{v}_n)_{n\geq 0}$, and similarly $((1+\eta)a_n, (1+\eta)b_n)_{n\geq 0}$ follows the same recursion as $(\underline{u}_n, \underline{v}_n)_{n\geq 0}$. This completes the argument for the direct application of Lemma 3.2. □

By (5.2), we have $u_{N_0} \sim c_\star \varepsilon$, and therefore the study of $n_A^{(2)} - N_0$ reduces to that of the corresponding quantity for the system $(a_n, b_n)_{n\geq 0}$. For $N_0 - n_A^{(1)}$, we consider the dual system $(u_{N_0-n}, v_{N_0+1-n})_{0\leq n\leq N_0}$ as in Proposition 3.3, and note that the function $\check{\Psi}(x) := 1/\Psi(-x)$ also satisfies (5.19). Thus, we can again apply Lemma 5.3 to the dual system, reducing the study of $N_0 - n_A^{(1)}$ to that of $(a_n, b_n)_{n\geq 0}$. Both $n_A^{(2)} - N_0$ and $N_0 - n_A^{(1)}$ contribute equally to the order in (5.4), which explains why the constant on the right-hand-side of (5.21) below is half that of (5.4).

From the previous discussion, it suffices to show that for $(a_n, b_n)_{n\geq 0}$ defined by (5.17) with initial condition $a_0 = \varepsilon$ and $b_0 \in (-\varepsilon, 0]$, we have

$$\lim_{A\to\infty}\lim_{\varepsilon\to 0} \varepsilon^{1/2} m_A^{(\varepsilon)} \;=\; \frac{\pi}{\sqrt{2}}, \tag{5.21}$$

where

$$m_A^{(\varepsilon)} \;=\; m_A^{(\varepsilon)}(a_0, b_0) \;:=\; \inf\{n \geq 0 : b_n > A\sqrt{\varepsilon}\}.$$

Note that, if $a_1 > \varepsilon$ and $b_1 > 0$, then

$$m_A^{(\varepsilon)}(\varepsilon, 0) \;\leq\; m_A^{(\varepsilon)}(a_0, b_0) \;\leq\; 1 + m_A^{(\varepsilon)}(\varepsilon, 0)$$

by monotonicity. Therefore, we assume without loss of generality that $a_0 = \varepsilon$ and $b_0 = 0$ from now on.

Next, define $x_n^{(\varepsilon)} = a_n/\varepsilon$ and $y_n^{(\varepsilon)} = b_n/\varepsilon^{1/2}$. We observe that

$$\begin{pmatrix} x_{n+1}^{(\varepsilon)} - x_n^{(\varepsilon)} \\ y_{n+1}^{(\varepsilon)} - y_n^{(\varepsilon)} \end{pmatrix} \;=\; \begin{pmatrix} \varepsilon^{1/2} x_n^{(\varepsilon)} y_{n+1}^{(\varepsilon)} \\ \varepsilon^{1/2} x_n^{(\varepsilon)} \end{pmatrix}. \tag{5.22}$$

with initial conditions $x_0^{(\varepsilon)} = 1$ and $y_0^{(\varepsilon)} = 0$. This is an Euler scheme for the differential system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \;=\; \begin{pmatrix} y \\ xy \end{pmatrix}. \tag{5.23}$$

The solution to this system is given by

$$x(t) \;=\; 1 + \tan(t\sqrt{2})^2, \quad y(t) \;=\; \sqrt{2}^{-1}\tan(t\sqrt{2}) \quad \text{for all } t < T \;:=\; \frac{\pi}{\sqrt{2}}. \tag{5.24}$$

We then use the following classical result related to Euler schemes for solving differential systems (see [14, Chapter V.2.3]).

Fact 5.4. *For all $\delta > 0$, we have*

$$\limsup_{\varepsilon\to 0} \sup_{k\leq (T-\delta)\varepsilon^{-1/2}} \left| x_k^{(\varepsilon)} - x_{k\varepsilon^{1/2}} \right| + \left| y_k^{(\varepsilon)} - y_{k\varepsilon^{1/2}} \right| \;<\; \infty. \tag{5.25}$$

Since $m_A(\varepsilon) = \inf\{n \geq 0 : y_n^{(\varepsilon)} > A\}$, it follows from (5.25) that

$$\liminf_{A\to\infty} \liminf_{\varepsilon\to 0} \varepsilon^{1/2} m_A(\varepsilon) \geq \frac{\pi}{\sqrt{2}} - \delta \quad \text{for any } \delta > 0.$$

Letting $\delta \to 0$ gives the lower bound. For the upper bound, note that for all $t < \pi/\sqrt{2}$,

$$\lim_{\varepsilon\to 0} \frac{b_{\lfloor t\varepsilon^{-1/2}\rfloor}}{\varepsilon^{1/2}} \;=\; \lim_{\varepsilon\to 0} y_{\lfloor t\varepsilon^{-1/2}\rfloor}^{(\varepsilon)} \;=\; \frac{\tan(t\sqrt{2})}{\sqrt{2}}. \tag{5.26}$$

For future reference, we also have the following limits for any $A > 0$:

$$\lim_{\varepsilon\to 0} b_{m_A(\varepsilon)}/\varepsilon^{1/2} = A, \qquad \text{and} \quad \lim_{\varepsilon\to 0} a_{m_A(\varepsilon)}/\varepsilon = 1 + A^2/2. \tag{5.27}$$

By monotonicity, we deduce from (5.26) that

$$\lim_{\varepsilon\to 0} b_{\lfloor T\varepsilon^{-1/2}\rfloor}/\varepsilon^{1/2} = \infty.$$

This implies that

$$\limsup_{A\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^{1/2} m_A(\varepsilon) \le \frac{\pi}{\sqrt{2}}.$$

We have then proved (5.21) and completed the proof of (5.4). □

Step 3: Proof of (5.5). We begin by considering $n_A^{(1)}$, comparing the supercritical system $(u_n, v_n)_{n\ge 0}$ with the critical system. Let $(u_n^*, v_n^*)_{n\ge 0}$ satisfy the recursive equation (1.8) with $v_0^* := v_0 < 0$ and $u_0^* := h(v_0)$. Since $u_0 = h(v_0) + \varepsilon > u_0^*$, Lemma 3.2 implies that $v_n \ge v_n^*$ for all n. Therefore, we have

$$n_A^{(1)} \;\le\; \inf\{n \ge 1 : v_n^* > -A\varepsilon^{1/2}\}.$$

By Theorem 1.4, we know that $v_n^* \sim -2/n$ as $n \to \infty$. Hence

$$\limsup_{\varepsilon\to 0} \varepsilon^{1/2} n_A^{(1)} \;\le\; \frac{2}{A}.$$

Next, we consider $n_A^{(2)}$, using (5.27) to obtain

$$v_{n_A^{(2)}} \;\sim\; A\varepsilon^{1/2} \quad \text{as } \varepsilon \to 0.$$

An application of Lemma 4.5 provides the bound

$$\limsup_{\varepsilon\to 0} \varepsilon^{1/2}(n_* - n_A^{(2)}) \;\le\; \frac{c}{A}.$$

Therefore, (5.5) follows. □

Proof of Theorem 1.3. By combining (5.4) and (5.5), we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2} n_*(h(v_0) + \varepsilon, v_0) = c_\star^{-1/2}\pi\sqrt{2}.$$

The result then follows from Lemma 5.1. □

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