Submitted to the Annals of Probability

ON THE DERIVATIVE MARTINGALE IN A BRANCHING RANDOM WALK

BY DARIUSZ BURACZEWSKI[1], ALEXANDER IKSANOV[2] AND BASTIEN MALLEIN[3]

[1]*Mathematical Institute, University of Wrocław, Wrocław, Poland, dbura@math.uni.wroc.pl*

[2]*Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine, iksan@univ.kiev.ua*

[3]*Laboratoire Algèbre, Géométrie et Applications UMR 7539, Université Sorbonne Paris Nord, Villetaneuse, France, mallein@math.univ-paris13.fr*

We work under the Aïdékon-Chen conditions which ensure that the derivative martingale in a supercritical branching random walk on the line converges almost surely to a nondegenerate nonnegative random variable that we denote by Z. It is shown that $\mathbb{E}Z\,\mathbb{1}_{\{Z\leq x\}}=\log x+o(\log x)$ as $x\to\infty$. Also, we provide necessary and sufficient conditions under which $\mathbb{E}Z\,\mathbb{1}_{\{Z\leq x\}}=\log x+\text{const}+o(1)$ as $x\to\infty$. This more precise asymptotics is a key tool for proving distributional limit theorems which quantify the rate of convergence of the derivative martingale to its limit Z. The methodological novelty of the present paper is a three terms representation of a subharmonic function of at most linear growth for a killed centered random walk of finite variance. This yields the aforementioned asymptotics and should also be applicable to other models.

1. Introduction: a branching random walk and the derivative martingale. We consider a discrete-time supercritical *branching random walk* (BRW) on the real line $\mathbb{R}$. The distribution of the branching random walk is governed by a point process $\mathcal{Z}:=\sum_{j=1}^{N}\delta_{X_j}$ on $\mathbb{R}$. The number of offspring, $N=\mathcal{Z}(\mathbb{R})$, is a random variable taking values in $\mathbb{N}_0\cup\{+\infty\}:=\{0,1,2,\dots\}\cup\{+\infty\}$.

It is convenient to associate the evolution of BRW with that of some population of individuals. At time 0, the population starts with one individual, the ancestor, which resides at the origin. At time 1, the ancestor dies and simultaneously places offspring on the real line with positions given by the points of the point process $\mathcal{Z}$. The offspring of the ancestor form the first generation of the underlying population. At time 2, each particle of the first generation dies and has offspring with positions relative to their parent's position given by an independent copy of $\mathcal{Z}$. The individuals produced by the first generation particles form the second generation of the population, and so on.

More formally, let $\mathcal{I}=\bigcup_{n\geq 0}\mathbb{N}^n$ be the set of all possible individuals. The ancestor label is the empty word $\varnothing$, its position is $S(\varnothing)=0$. On some probability space let $(\mathcal{Z}(u))_{u\in\mathcal{I}}$ be a family of independent copies of the point process $\mathcal{Z}$. An individual of the nth generation with label $u=u_1\dots u_n$ and position $S(u)$ produces a random number $N(u)$ of offspring at time $n+1$. The offspring of the individual u are placed at random locations on $\mathbb{R}$ given by the positions of the point process

$$\delta_{S(u)}*\mathcal{Z}(u)=\sum_{j=1}^{N(u)}\delta_{S(u)+X_j(u)},$$

Primary 60G50, 60J80; secondary 60F05, 60G42

Keywords and phrases: Branching random walk, Derivative martingale, Killed random walk, Rate of convergence, Subharmonic function, Tail behavior

where $\mathcal{Z}(u)=\sum_{j=1}^{N(u)}\delta_{X_j(u)}$ and $N(u)$ is the number of points in $\mathcal{Z}(u)$. The offspring of the individual u are enumerated by $uj=u_1\ldots u_n j$, where $j\in\{1,\ldots,N(u)\}$ (assuming that $N(u)<\infty$) or $j\in\mathbb{N}$ (if $N(u)=\infty$), and the positions of the offspring are denoted by $S(uj)$. No assumptions are imposed on the dependence structure of the random variables $N(u),X_1(u),X_2(u),\ldots$ for fixed $u\in\mathcal{I}$. The point process of the positions of the nth generation individuals will be denoted by $\mathcal{Z}_n$ so that $\mathcal{Z}_0=\delta_0$ and

$$\mathcal{Z}_{n+1}=\sum_{|u|=n}\sum_{j=1}^{N(u)}\delta_{S(u)+X_j(u)}=\sum_{|u|=n}\sum_{j=1}^{N(u)}\delta_{S(uj)},\quad n\in\mathbb{N}_0.$$

Here and hereafter, $|u|=n$ means that the sum is taken over all individuals of the nth generation rather than over all $u\in\mathbb{N}^n$. The sequence of point processes $(\mathcal{Z}_n)_{n\in\mathbb{N}_0}$ is then called a branching random walk. Throughout the article, we assume that $\mathbb{E}N\in(1,\infty]$ (supercriticality) which implies that the population survives with positive probability. Notice that the sequence of generation sizes in the BRW forms a Galton-Watson process provided that $N<\infty$ almost surely (a.s.).

In what follows we always assume that

$$\mathbb{E}\sum_{i=1}^{N}e^{-X_i}\;=\;1. \tag{1.1}$$

On the other hand, the situation is not excluded that $\mathbb{E}\sum_{i=1}^{N}e^{-\gamma X_i}=\infty$ for all $\gamma\neq 1$. Put

$$W_n\;:=\;\sum_{|u|=n}e^{-S(u)},\;\;n\in\mathbb{N}_0$$

and let $\mathcal{F}_n$ be the σ-algebra generated by the first n generations, i.e. $\mathcal{F}_n=\sigma(\mathcal{Z}(u):|u|<n)$ where $|u|<n$ means that $u\in\mathbb{N}^k$ for some $k<n$. It is a straightforward consequence of (1.1) and the branching property that the sequence $(W_n,\mathcal{F}_n)_{n\in\mathbb{N}_0}$ is a nonnegative martingale and thus converges a.s. to a random variable that we denote by W. This martingale is called *additive* or *Biggins' martingale*.

In addition to (1.1) we shall assume that

$$\mathbb{E}\sum_{i=1}^{N}e^{-X_i}X_i=0 \tag{1.2}$$

which means that we are focussed on the so called *boundary case*. Observe that, under (1.2), we have $W=0$ a.s. (see, for instance Theorem on p. 218 in [28]). Putting

$$Z_n\;:=\;\sum_{|u|=n}e^{-S(u)}S(u),\;\;n\in\mathbb{N}_0,$$

we obtain another martingale $(Z_n,\mathcal{F}_n)_{n\in\mathbb{N}_0}$ which is known in the literature as *derivative martingale*. Let $\mathrm{i}:=\sqrt{-1}$ and $\gamma\in\mathbb{R}$. Differentiating formally

$$\sum_{|u|=n}e^{-(1-\mathrm{i}\gamma)S(u)}/\mathbb{E}\sum_{|u|=n}e^{-(1-\mathrm{i}\gamma)S(u)}$$

in γ and putting $\gamma=0$ yields $\mathrm{i}Z_n$ which justifies the term 'derivative martingale'.

Put

$$\widetilde{W}_1:=\sum_{i=1}^{N}e^{-X_i}(X_i)_+.$$

Here and hereafter, we use the standard notation: for $x \in \mathbb{R}$, $x_+ := x \vee 0$, $x_- := (-x) \vee 0$ and $\log_+ x := \log(x \vee 1)$. It is well-known (see e.g. Proposition A.3 (iii) in [1]) that the a.s. limit $Z := \lim_{n\to\infty} Z_n$ exists and is nonnegative and nondegenerate, that is, $\mathbb{P}\{Z > 0\} > 0$ provided that conditions (1.1), (1.2),

$$\sigma^2 := \mathbb{E} \sum_{i=1}^{N} e^{-X_i} X_i^2 < \infty \tag{1.3}$$

and

$$\mathbb{E} W_1 (\log_+ W_1)^2 + \mathbb{E}\widetilde{W}_1 \log_+ \widetilde{W}_1 < \infty \tag{1.4}$$

hold. Further, according to Theorem 1.1 in [12], under (1.1), (1.2) and (1.3), condition (1.4) is also necessary for the existence of $Z \geq 0$ which is positive with positive probability.

In some of our main results we shall assume that the distribution of the displacements of the BRW is nonarithmetic, that is, for all $\delta > 0$,

$$\mathbb{P}\{\mathcal{Z}(\mathbb{R} \backslash \delta\mathbb{Z}) > 0\} > 0, \tag{1.5}$$

where $\mathbb{Z}$ is the set of integers.

Conditions (1.3) and (1.4) are standard assumptions which are imposed in articles dealing with the derivative martingale, see, for instance, [1, 2, 12]. The additional assumption (1.5) is often needed for proving distributional convergence or convergence of moments, see [1] for an analysis of the maximal displacement in a BRW. Conditions (1.1), (1.2), (1.3), (1.4) are our standing assumptions throughout the paper, sometimes referred to thereafter as Condition $\mathcal{S}$. Condition $\mathcal{S}$ in conjunction with the nonarithmeticity assumption (1.5) will be called Condition $\mathcal{S}_{\rm na}$.

2. Main results.

2.1. *Tail behavior of the derivative martingale limit.* Our purpose is to provide a two terms asymptotic expansion for $\mathbb{E} Z\, \mathbb{1}_{\{Z \leq x\}}$ as $x \to \infty$. While investigating the relevant literature we have realized that even the first order asymptotics of that expectation is not given under optimal assumptions. Thus, we start by filling up this gap.

THEOREM 2.1. *Assume that Condition $\mathcal{S}$ holds. Then*

$$\mathbb{E} Z\, \mathbb{1}_{\{Z \leq x\}} \sim \log x, \quad x \to \infty. \tag{2.1}$$

To formulate our main result, put

$$W_1^+ := \sum_{i=1}^{N} e^{-X_i}\, \mathbb{1}_{\{X_i \geq 0\}}, \quad W_1^- := \sum_{i=1}^{N} e^{-X_i}\, \mathbb{1}_{\{X_i < 0\}} \tag{2.2}$$

and $X_{\min} := \min_{1 \leq i \leq N} X_i$, so that, $X_{\min}$ is the position of the leftmost individual in the first generation. Further, we introduce the following conditions

$$\mathbb{E} W_1^+ (\log_+ W_1^+)^3 + \mathbb{E}\widetilde{W}_1 (\log_+ \widetilde{W}_1)^2 < \infty; \tag{2.3}$$

$$\mathbb{E} W_1^- (\log W_1^-)^3\, \mathbb{1}_{\left\{\sum_{i=1}^{N} (1 + X_i - X_{\min}) e^{X_{\min} - X_i}\, \mathbb{1}_{\{X_i < 0\}} > C_0\right\}} < \infty \quad \text{for some } C_0 > 0 \tag{2.4}$$

and

$$\mathbb{E} \sum_{i=1}^{N} e^{-X_i} (X_i)_-^3 < \infty. \tag{2.5}$$

In what follows, we refer to the union of (2.3), (2.4) and (2.5) as Condition $\mathcal{S}^*$.

THEOREM 2.2. *Under Condition $\mathcal{S}_{\rm na}$, we have*

$$\mathbb{E} Z\,\mathbb{1}_{\{Z\leq x\}} = \log x + c + o(1), \quad x\to\infty \tag{2.6}$$

for a finite constant c if, and only if, Condition $\mathcal{S}^*$ holds. Formula (2.6) *particularly entails*

$$\lim_{x\to\infty} x\mathbb{P}\{Z>x\} = 1. \tag{2.7}$$

We proceed with a number of remarks.

REMARK 2.3. 1) We start by giving one particular example in which condition (2.4) holds true. Assume that the number of the first generation individuals positioned on the negative halfline is bounded a.s., that is, $\sum_{i=1}^N \mathbb{1}_{\{X_i<0\}} \leq C_0$ a.s. for some $C_0>0$. Then $\sum_{i=1}^N (1+X_i-X_{\min})e^{X_{\min}-X_i}\,\mathbb{1}_{\{X_i<0\}} \leq C_0$ a.s. which entails (2.4). Of course, if $\sum_{i=1}^N \mathbb{1}_{\{X_i<0\}}=0$ a.s., then (2.4) holds trivially.
2) A sufficient condition for (2.6) is

$$\mathbb{E}W_1(\log_+ W_1)^3 + \mathbb{E}\widetilde{W}_1(\log_+\widetilde{W}_1)^2<\infty.$$

Observe that it has a form similar to (1.4).
3) In a frequently encountered and mathematically tractable setting, the random variables $X_1, X_2,\ldots$ (displacements) are independent and identically distributed and also independent of N (the number of offspring). Direct calculation reveals that Conditions $\mathcal{S}_{\rm na}$ and $\mathcal{S}^*$ are ensured by

$$\mathbb{E}N\in(1,\infty), \quad \mathbb{E}N(\log_+ N)^2<\infty;$$

$$\mathbb{E}e^{-X_1}=(\mathbb{E}N)^{-1}, \quad \mathbb{E}e^{-X_1}X_1=0, \quad \mathbb{E}e^{-X_1}X_1^2<\infty;$$

$$\text{the distribution of } X_1 \text{ is nonarithmetic}$$

and

$$\mathbb{E}N(\log_+ N)^3<\infty, \quad \mathbb{E}e^{-X_1}(X_1)_-^3<\infty, \tag{2.8}$$

respectively. Alternatively, but a bit informally, this can be seen by identifying the nth generation of the BRW described above with the $(n+1)$st generation of a BRW driven by a point process $\mathcal{Z}^*:=N\delta_{X_1}$ (the correspondence is set by replacing the position of each parent in the latter BRW with the position of its children). Thus, neglecting the numbering of generations one may replace, for instance, the condition $\mathbb{E}W_1(\log_+ W_1)^2<\infty$ which is a part of (1.4) with $\mathbb{E}Ne^{-X_1}(\log_+ Ne^{-X_1})^2<\infty$. The latter is equivalent to $\mathbb{E}N(\log_+ N)^2<\infty$ and $\mathbb{E}e^{-X_1}(X_1)_-^2<\infty$.

2.2. *The rate of convergence of the derivative martingale to its limit.* Recall that the characteristic function of a general nondegenerate 1-stable distribution ν takes the form

$$t\mapsto \exp({\rm i}at - b|t|(1+{\rm i}\beta\,{\rm sgn}\,t(2/\pi)\log|t|)), \quad t\in\mathbb{R},$$

where $a\in\mathbb{R}$, $b>0$ and $\beta\in\mathbb{R}$, $|\beta|\leq 1$, and that ν is uniquely determined by the generating triple (a,b,β). The Lévy spectral function M^* of ν is given by $M^*(x)=b_1|x|^{-1}$ for $x<0$ and $M^*(x)=-b_2x^{-1}$ for $x>0$, where $b_1,b_2\geq 0$ are defined by $b=(b_1+b_2)\pi/2$ and $\beta=(b_2-b_1)/(b_2+b_1)$. When $b_1=0$, $b_2>0$, so that $\beta=1$ the distribution ν is called *spectrally positive*.

As an application of Theorem 2.2 which is a result on the tail behavior of Z we state a one-dimensional limit theorem. Set $\mathcal{F}_\infty:=\sigma(\mathcal{F}_n : n\in\mathbb{N}_0)$ and note that Z, the a.s. limit of Z_n, is an $\mathcal{F}_\infty$-measurable random variable. As usual, $\stackrel{\mathbb{P}}{\to}$ and $\stackrel{\rm d}{\to}$ will denote convergence in probability and in distribution, respectively.

THEOREM 2.4. *Assume that Conditions $\mathcal{S}_{\rm na}$ and $\mathcal{S}^*$ hold. Then, for every bounded continuous function $f:\mathbb{R}\to\mathbb{R}$,*

$$\mathbb{E}\big(f(n^{1/2}(Z-Z_n+(2^{-1}\log n)W_n))\big|\mathcal{F}_n\big)\ \overset{\mathbb{P}}{\to}\ \mathbb{E}(f(ZL)|\mathcal{F}_\infty),\quad n\to\infty, \tag{2.9}$$

which particularly entails

$$n^{1/2}(Z-Z_n+(2^{-1}\log n)W_n)\ \overset{\rm d}{\to}\ ZL,\quad n\to\infty. \tag{2.10}$$

Here, a random variable L is assumed independent of $\mathcal{F}_\infty$ and has a 1-stable distribution with the generating triple $((c+1-\gamma)(2/(\pi\sigma^2))^{1/2},(\pi/(2\sigma^2))^{1/2},1)$, γ is the Euler-Mascheroni constant, and c is the same constant as in (2.6). Thus, the distribution of L is spectrally positive with characteristic function

$$\mathbb{E}e^{{\rm i}tL}=\exp\big({\rm i}(c+1-\gamma)(2/(\pi\sigma^2))^{1/2}t-(\pi/(2\sigma^2))^{1/2}|t|(1+{\rm i}\,{\rm sgn}\,(t)(2/\pi)\log|t|)\big),\ t\in\mathbb{R}.$$

Plainly, Theorem 2.4 is a result on the rate of convergence of the derivative martingale to its a.s. limit.

REMARK 2.5. Mimicking the proof of Theorem 2.4 one can also show that, for every bounded continuous function $f:\mathbb{R}\to\mathbb{R}$, on the set of survival $\{\mathcal{Z}_n(\mathbb{R})>0\quad\text{for all } n\in\mathbb{N}\}$,

$$\mathbb{E}\Big(f\Big(\frac{n^{1/2}}{Z_n}(Z-Z_n+(2^{-1}\log n)W_n)\Big)\Big|\mathcal{F}_n\Big)\ \overset{\mathbb{P}}{\to}\ \mathbb{E}f(L),\quad n\to\infty. \tag{2.11}$$

As a consequence, a counterpart of (2.10) holds, namely, conditionally on the survival,

$$\frac{n^{1/2}}{Z_n}(Z-Z_n+(2^{-1}\log n)W_n)\ \overset{\rm d}{\to}\ L,\quad n\to\infty. \tag{2.12}$$

We omit further details.

The rest of the article is structured as follows. In Section 3.1 we explain our approach which is based on a novel look at a Poisson equation on the halfline. Also in the section is a brief survey of some earlier papers dealing with a general Poisson equation. In Section 3.2 we compare our results to similar ones available in the literature. In Section 4 we introduce a standard random walk associated with the BRW and lay down the frequently used notation. In Section 5, which is the core of our work, we prove a representation of subharmonic functions of at most linear growth for killed centered standard random walks with finite variance. As a corollary, we show that actually such functions grow linearly. While Theorems 2.1 and 2.2 are proved in Section 6, Theorem 2.4 is proved in Section 7.2. The appendix collects several Abelian and Tauberian theorems related to the de Haan class of slowly varying functions and some auxiliary facts about standard random walks, Lebesgue integrable and directly Riemann integrable functions.

3. Discussion.

3.1. *Our approach.* To determine the tail behavior of Z we work with its Laplace transform. Formula (6.9) written in terms of this Laplace transform is an instance of a Poisson equation. In view of this, our principal purpose is to develop an approach towards understanding the asymptotics of solutions to a *general* Poisson equation

$$K(x)=\mathbb{E}K(x+\eta)-L(x),\quad x\in\mathbb{R}, \tag{3.1}$$

where η is a random variable and $L:\mathbb{R}\to\mathbb{R}$ is a given function. Especially, we are interested in situations in which K exhibits a linear growth.

When $\mathbb{E}\eta \neq 0$ and $\mathbb{E}|\eta| < \infty$, (3.1) is called *renewal equation*. In this case,

$$K(x) = -\int_{\mathbb{R}} L(x+y)U^*(\mathrm{d}y), \quad x \in \mathbb{R},$$

where, with $\eta_1, \eta_2, \ldots$ being independent copies of η, U^* is the (locally finite) renewal measure defined by $U^*(\mathrm{d}y) = \sum_{k\geq 0} \mathbb{P}\{\eta_1 + \ldots + \eta_k \in \mathrm{d}y\}$. Furthermore, the asymptotics of K is well-understood and driven by the key renewal theorem in which case

$$\lim_{x\to\pm\infty} K(x) = -(\mathbb{E}\eta)^{-1} \int_{\mathbb{R}} L(y)\mathrm{d}y$$

(depending on the sign of $\mathbb{E}\eta$ the limit is as $x \to -\infty$ or $x \to +\infty$) or its relatives, see, for instance, Section 6.2 in [23].

In this article our focus is on the centered case $\mathbb{E}\eta = 0$ in which the renewal measure (potential) U^* is not locally finite. This makes things more complicated, and one has to find a proper replacement for U^*. This task was accomplished by Spitzer in Section 28 of [42] for centered random walks on integers and then by Port and Stone in [37] in a general setting. Assuming that the distribution of η is spread-out (that is, some convolution power of it has a nontrivial absolutely continuous component) and that L is a bounded function of compact support these authors proposed a limiting procedure yielding the potential kernel A defined by

$$AL(x) := \int_{\mathbb{R}} L(x-y)a(y)\mathrm{d}y - \int_{\mathbb{R}} L(x-y)\varrho(\mathrm{d}y) + b\int_{\mathbb{R}} L(y)\mathrm{d}y - L(x), \quad x \in \mathbb{R}.$$

Here, $a : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $\lim_{x\to\pm\infty}(a(x-y) - a(x)) = \mp\mathtt{s}^{-2}y$, where $\mathtt{s}^2 = \mathbb{E}\eta^2$; ϱ is a finite measure and b is a constant. As a consequence, it was shown in Theorem 10.3 of [37] that any positive (or more generally bounded from below) solution to (3.1) is of the form

$$K(x) = AL(x) + \Big(c\mathtt{s}^{-2}\int_{\mathbb{R}} L(y)\mathrm{d}y\Big)x + d, \quad x \in \mathbb{R}, \tag{3.2}$$

where d is any constant and $|c| \leq 1$. It is known that either $K(x)$ converges to a positive constant or behaves linearly as $x \to \infty$ depending on whether $\int_{\mathbb{R}} L(y)\mathrm{d}y$ is zero or not. There is an extension of the results discussed above to the case where L is not necessarily compactly supported and rather satisfies an integral condition, see Theorem 3.1 in [9] or Theorem 3.2 in [10].

While investigating a particular Poisson equation related to a smoothing transform (see the beginning of Section 6.1 for the definition and some more details) Durrett and Liggett in [19] were concerned with the asymptotic behavior of a given solution to (3.1) rather than in description of the set of all solutions. These authors invented a novel approach based on Feller's duality principle (Lemma 1 on p. 609 in [20]). This enabled them to employ the key renewal theorem for describing the asymptotic behavior of the given solution. In a more general setting similar ideas were exploited by Liu in [27].

The main methodological achievement of the present work is an explicit formula, other than (3.2), for solutions of at most linear growth to a Poisson equation on the halfline. Among other things this provides a way to easily obtain the precise asymptotic behavior of those solutions. Roughly speaking, the idea is as follows. We are interested in the asymptotics of a solution f at ∞, so that the values $f(x)$ for $x \leq 0$ should play no role. Thus, we regard f as a solution to a Dirichlet problem: given the values of f on $(-\infty, 0]$ (which can be thought of as *boundary values*) we intend to recover f on $(0, \infty)$ which is nothing else but a subharmonic function of at most linear growth for a recurrent standard random walk killed upon entering $(-\infty, 0]$.

3.2. *Comparison to earlier literature.* COMMENTS ON SECTION 2.1. Theorem 2.1 provides an improvement over Theorem 2.18 in [19] and Theorem 4.2 in [27] obtained for Z being a fixed point of the smoothing transform. In the former, relation (2.1) is proved in the situation that $N \geq 2$ is a deterministic integer, that conditions (1.1), (1.2) and (1.5) hold, and that $\mathbb{E}W_1^\gamma < \infty$ for some $\gamma > 1$. In the latter, while N is random with $\mathbb{E}N > 1$, the other conditions ensuring (2.1) are comparable to those in [19].

Theorem 2.2 strengthens several results on the tail behavior of Z available in the literature. The best previously known sufficient conditions for (2.7) that we are aware of are in Theorem 1.4 of [29]. In addition to Condition $\mathcal{S}_{\rm na}$ the author requires

$$\mathbb{E}\Big(\sum_{i=1}^N e^{-X_i} + \sum_{i=1}^N e^{-X_i}(X_i)_+\Big)\log_+\Big(\sum_{i=1}^N e^{-X_i} + \log \sum_{i=1}^N e^{-X_i}(X_i)_+\Big)^5 < \infty.$$

To be more precise, in the last cited theorem it is claimed that

$$\lim_{x\to\infty} x\mathbb{P}\{Z > x\} = b,$$

where b is the product of two positive constants expressed in terms of the minimal position of BRW's individuals over the whole population and the random variable Z. Our Theorem 2.2 reveals that b is actually equal to one, thereby giving an explicit relationship between these two constants. Under stronger moment assumptions a relation like (2.7) was also proved in Theorem 1.2 of [11] for Z being a fixed point of the smoothing transform. Last but not least, a counterpart of (2.6) in the context of branching Brownian motion was proved in Proposition 4.1 of [31]. Our condition (2.8) is reminiscent of Maillard's condition.

COMMENTS ON SECTION 2.2. Limit theorems providing a rate of convergence have been and still are quite popular in the area of branching processes. Surveys of the relevant literature can be found in [24] and [32]. The latter article discusses, among others, limit theorems for some models of statistical mechanics. A large selection of rate of convergence results for more complicated branching processes, including branching diffusions and superprocesses, can be traced via the references given in [38].

Theorem 2.4 is a counterpart of Proposition 2.1 in [32] obtained for the derivative martingale which corresponds to a branching Brownian motion. Observing the martingale at nonnegative integer times only yields a particular version of $(Z_n, \mathcal{F}_n)_{n\in\mathbb{N}_0}$ investigated here, with $\sigma^2 = 1$. According to Theorem 2.4, the random variable L appearing in (2.10) has a 1-stable distribution with the generating triple $((c+1-\gamma)(2/\pi)^{1/2}, (\pi/2)^{1/2}, 1)$, whereas according to Proposition 2.1 in [32] the generating triple is $((c-\gamma)(2/\pi)^{1/2}, (\pi/2)^{1/2}, 1)$, that is, 1 is lost. The error in [32] is caused by missing the term $x\mathbb{P}\{Z > x\}$ which converges to 1 as $x \to \infty$ in the equality

$$\int_0^x \mathbb{P}\{Z > y\}{\rm d}y = \mathbb{E}Z\, \mathbb{1}_{\{Z\leq x\}} + x\mathbb{P}\{Z > x\}, \quad x > 0$$

(see formula (1.9) and Lemma C.1 in [32]).

4. A standard random walk associated with BRW.

Under (1.1), denote by ξ a random variable with distribution given by

$$\mathbb{E}t(\xi) = \mathbb{E}\sum_{i=1}^N e^{-X_i} t(X_i) \tag{4.1}$$

for any measurable bounded function $t : \mathbb{R} \to \mathbb{R}^+$, where $\mathbb{R}^+ := [0,\infty)$. Note that (4.1) also holds for real-valued t whenever the left- or right-hand side of (4.1) is well-defined, possibly infinite.

Observe that Condition $\mathcal{S}$ implies that $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = \sigma^2 < \infty$. Further, we stress that supercriticality in combination with (1.1) guarantees that $\mathbb{P}\{\xi = 0\} < 1$ (taken together with $\mathbb{E}\xi = 0$ the latter means that the distribution of ξ is nondegenerate, whence $\sigma^2 > 0$). Indeed, assuming the contrary

$$1 = \mathbb{P}\{\xi = 0\} = \mathbb{E} \sum_{i=1}^{N} e^{-X_i} \mathbb{1}_{\{X_i = 0\}}$$

we conclude that $N = 1$ and $X_1 = 0$ a.s., a contradiction to supercriticality. Additionally, note that Condition $\mathcal{S}_{\rm na}$ implies that the distribution of ξ is nonarithmetic, that is, concentrated on $d\mathbb{Z}$ for no $d > 0$.

We denote by $S := (S_n)_{n\in\mathbb{N}_0}$ a standard random walk defined by $S_n - S_0 := \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$, where $\xi_1, \xi_2, \ldots$ are independent copies of ξ which are also independent of S_0. For $x \in \mathbb{R}$, we denote by $\mathbb{P}_x$ the distribution of the random walk $(S_n)_{n\in\mathbb{N}_0}$ when $S_0 = x$ a.s. As usual, we write $\mathbb{P}$ for $\mathbb{P}_0$.

It is a well-known fact that the behavior of BRW is driven, among others, by the random walk S. A classical example of this connection is the so-called *many-to-one lemma* which can be traced back at least to Kahane and Peyrière [26, 36]. We quote it from Theorem 1.1 in [41].

LEMMA 4.1 (Many-to-one). *For each $n \in \mathbb{N}$ and a measurable bounded function $t : \mathbb{R}^n \to \mathbb{R}^+$,*

$$\mathbb{E} \sum_{|u|=n} e^{-S(u)} t(S(u_1), \ldots, S(u_1 \ldots u_n)) = \mathbb{E} t(S_1, \ldots, S_n),$$

where $u = u_1 \ldots u_n$.

Let $(\tau_k)_{k\in\mathbb{N}_0}$ be the sequence of weak descending ladder epochs, defined by $\tau_0 := 0$ and, for $k \in \mathbb{N}$, $\tau_k := \inf\{j > \tau_{k-1} : S_j \le S_{\tau_{k-1}}\}$. Also, let $(\sigma_n)_{n\in\mathbb{N}_0}$ be the sequence of strict ascending ladder epochs, defined by $\sigma_0 := 0$ and, for $n \in \mathbb{N}$, $\sigma_n := \inf\{i > \tau_{n-1} : S_i > S_{\sigma_{n-1}}\}$. In view of $\mathbb{E}\xi = 0$, all these random variables are a.s. finite. Under $\mathbb{P}$, $(S_{\tau_k})_{k\in\mathbb{N}_0}$ and $(S_{\sigma_n})_{n\in\mathbb{N}_0}$, being the sequences of weak descending and strict ascending ladder heights, form standard random walks with independent nonpositive and nonnegative jumps having the same distribution as S_{τ_1} and S_{σ_1}, respectively. Under $\mathbb{P}$, denote by U and V the renewal functions of $(-S_{\tau_k})_{k\in\mathbb{N}_0}$ and $(S_{\sigma_k})_{n\in\mathbb{N}_0}$, respectively, that is,

$$U(x) := \sum_{k\ge0} \mathbb{P}\{-S_{\tau_k} < x\} \quad \text{and} \quad V(x) := \sum_{k\ge0} \mathbb{P}\{S_{\sigma_k} \le x\}, \quad x \in \mathbb{R}. \tag{4.2}$$

Plainly, $U(x) = V(x) = 0$ for $x < 0$. Observe that U is a left-continuous renewal function which is a slight digression, for typically renewal functions are defined to be right-continuous. Nevertheless, the so defined U shares all the standard asymptotic properties of right-continuous renewal functions.

5. Subharmonic functions of at most linear growth for the killed random walk. Throughout this section we retain the notation $S := (S_n)_{n\in\mathbb{N}_0}$ for a standard random walk, not necessarily related to the BRW. All the other notation introduced in Section 4 is also retained but associated to the S as above. We shall assume, without further notice, until the end of this section that the distribution of ξ is nondegenerate, that $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 < \infty$ (the only exception is Lemma 5.3 in which finiteness of the second moment is not assumed). The following formulae which are ensured by Lemma A.4 (a,b) will be often used

$$\mu := (-\mathbb{E}S_{\tau_1}) \in (0, \infty) \quad \text{and} \quad \nu := \mathbb{E}S_{\sigma_1} \in (0, \infty) \tag{5.1}$$

and

$$\lim_{x\to\infty}(U(x)/x)=\mu^{-1}\quad\text{and}\quad\lim_{x\to\infty}(V(x)/x)=\nu^{-1}. \tag{5.2}$$

5.1. *Auxiliary results.* Set $\tau:=\inf\{n\in\mathbb{N}_0:S_n\le 0\}$ and note that $\tau=\tau_1$ under $\mathbb{P}_x$ for $x>0$ whereas $\tau=0$ under $\mathbb{P}_x$ for $x\le 0$. For all $x\in\mathbb{R}$, denote by

$$\sigma(x):=\inf\{n\in\mathbb{N}_0:S_n>x\}$$

the first passage of S into (x,∞). We now present an alternative formula for the renewal function U.

LEMMA 5.1. *For all $x\ge 0$,*

$$\mu U(x)=\lim_{y\to\infty}y\mathbb{P}_x\{\sigma(y)<\tau\}=\lim_{y\to\infty}\mathbb{E}_xS_{\sigma(y)}\,\mathbb{1}_{\{\sigma(y)<\tau\}},$$

where $\mu=-\mathbb{E}S_{\tau_1}<\infty$.

PROOF. When $x=0$, the second equality appears in the proof of Theorem (part (i)) in [17]. Under the assumption that S is an integer-valued random walk, formula (1.6) in [15] states that, for all $x\ge 0$,

$$\lim_{y\to\infty}y\mathbb{P}_x\{S_k=y\text{ for some }k<\tau\}=\mu U(x).$$

This is a result similar to our first equality.

In full generality, these equalities can be found in [3]. Namely, equation (32) there gives, for $x\ge 0$,

$$\mu U(x)=x-\mathbb{E}_xS_\tau.$$

Then, the first equality follows from Corollary 4.4 and equation (35) in Lemma 4.3 (both in the cited article) can be written as

$$\lim_{y\to\infty}\mathbb{E}_x(S_{\sigma(y)}-y)\,\mathbb{1}_{\{\sigma(y)<\tau\}}=0$$

which completes the proof of the second equality. □

Lemma 5.2 is a restatement of Proposition 4.1 in [3] which characterizes right-continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$\begin{cases} f(x)=\mathbb{E}f(x+\xi)\,\mathbb{1}_{\{\xi>-x\}}=\mathbb{E}_xf(S_1)\,\mathbb{1}_{\{S_1>0\}}, & \text{if } x>0,\\ f(x)=0, & \text{if } x\le 0,\\ \limsup_{x\to\infty}(|f(x)|/x)<\infty. \end{cases} \tag{5.3}$$

In words, the so defined f are harmonic functions of at most linear growth for the killed centered random walk with finite variance.

For $d>0$, we say that the distribution of ξ is d-arithmetic if $\mathbb{P}\{\xi\in d\mathbb{Z}\}=1$, and d is the largest number with this property. With a slight abuse of notation, we say that the distribution of ξ is 0-arithmetic if it is nonarithmetic, and that a function $\kappa(\cdot)$ is 0-periodic if it is a constant.

LEMMA 5.2. *Assume that the distribution of ξ is d-arithmetic for $d\ge 0$. Then if f satisfies* (5.3)*, there exists a right-continuous d-periodic function $\kappa(\cdot)$ such that $f(x)=\kappa(x)U(x)$ for $x>0$.*

In particular, note that any solution to (5.3) is a scalar multiple of the renewal function U provided that the distribution of ξ is nonarithmetic. When the distribution of ξ is arithmetic, Lemma 5.2 is a particular case of Theorem 1 in [18].

Given next is a formula which represents the expectation of an additive functional of the killed random walk in terms of renewal functions.

LEMMA 5.3. *Not assuming that $\mathbb{E}\xi^2 < \infty$, for all measurable functions $p : \mathbb{R}^+ \to \mathbb{R}^+$ and $x > 0$,*

$$\mathbb{E}_x \sum_{k=0}^{\tau-1} p(S_k) = \int_{[0,\,x]} \mathrm{d}U(y) \int_{[0,\,\infty)} \mathrm{d}V(z) p(x-y+z),$$

where V is the renewal function defined in (4.2). Here, both sides of the equality may be infinite.

PROOF. Set $r(x) := \int_{[0,\,\infty)} p(x+z)\mathrm{d}V(z)$ for $x \ge 0$. We use a standard decomposition of S into cycles: for $x > 0$,

$$\mathbb{E}_x \sum_{j=0}^{\tau-1} p(S_j) = \mathbb{E} \sum_{j\ge 0} p(x+S_j)\, \mathbb{1}_{\{x+S_1>0,\ldots,x+S_j>0\}}$$

$$= \mathbb{E} \sum_{k\ge 0} \sum_{j=\tau_k}^{\tau_{k+1}-1} p(x+S_j)\, \mathbb{1}_{\{x+S_1>0,\ldots,x+S_j>0\}} = \mathbb{E} \sum_{k\ge 0} \mathbb{1}_{\{x+S_{\tau_k}>0\}} \sum_{j=\tau_k}^{\tau_{k+1}-1} p(x+S_j)$$

$$= \mathbb{E} \sum_{k\ge 0} \mathbb{1}_{\{x+S_{\tau_k}>0\}} \sum_{j=0}^{\tau_{k+1}-\tau_k-1} p(x+S_{\tau_k}+(S_j-S_{\tau_k}))$$

$$= \mathbb{E} \sum_{k\ge 0} \mathbb{1}_{\{-S_{\tau_k}<x\}}\, r(x-(-S_{\tau_k})) = \int_{[0,\,x]} r(x-y)\mathrm{d}U(y).$$

Here, the third equality follows from the fact that $0 \le -S_{\tau_1} \le -S_{\tau_2} \le \ldots$ are the weak record values of the sequence $(-S_j)_{j\in\mathbb{N}_0}$, whence, for integer $j \in [\tau_k, \tau_{k+1}-1]$,

$$\mathbb{1}_{\{x+S_1>0,\ldots,x+S_j>0\}} = \mathbb{1}_{\{x+S_{\tau_1}>0,\ldots,x+S_{\tau_k}>0\}} = \mathbb{1}_{\{x+S_{\tau_k}>0\}}.$$

To explain the penultimate equality, note that given S_{τ_k}, for any $y \in \mathbb{R}$, by the strong Markov property, $\sum_{j=0}^{\tau_{k+1}-\tau_k-1} p(y+(S_j-S_{\tau_k}))$ has the same $\mathbb{P}$-distribution as $\sum_{j=0}^{\tau_1-1} p(y+S_j)$ which, in its turn, has the same $\mathbb{P}$-distribution as $\sum_{k\ge 0} p(y+S_{\sigma_k})$ by the duality principle (see Lemma 1 on p. 609 in [20]). In particular,

$$\mathbb{E}\Big(\sum_{j=0}^{\tau_{k+1}-\tau_k-1} p(x+S_{\tau_k}+(S_j-S_{\tau_k})) \Big| S_{\tau_k} \Big) = r(x+S_{\tau_k}).$$

The proof of Lemma 5.3 is complete. □

5.2. *New results.* In this section, we extend Lemma 5.2 by characterizing right-continuous subharmonic functions of at most linear growth for the killed random walk. More precisely, given $g : \mathbb{R}^+ \to \mathbb{R}^+$ a càdlàg function and $h : (-\infty, 0] \to \mathbb{R}$ a right-continuous bounded function, we aim at finding all right-continuous functions f that satisfy

$$\begin{cases} f(x) = \mathbb{E}f(x+\xi) - g(x), & \text{if } x > 0 \\ f(x) = h(x), & \text{if } x \le 0 \end{cases} \tag{5.4}$$

and

$$\limsup_{x\to\infty}(|f(x)|/x)<\infty. \tag{5.5}$$

The definition of directly Riemann integrable (dRi) functions which are mentioned below can be found in Section A.3.

THEOREM 5.4. *Assume that the distribution of ξ is d-arithmetic for $d\geq 0$. If solutions f to* (5.4) *exist, then, for each $x>0$,*

$$\mathbb{E}_x\sum_{k=0}^{\tau-1}g(S_k)<\infty. \tag{5.6}$$

Conversely, if (5.6) *holds for some $x>0$ and the function g is dRi on $\mathbb{R}^+$, then there exist solutions f to* (5.4) *with $\lim_{x\to\infty}(f(x)/x)=0$. Furthermore, to any solution f satisfying* (5.5) *there corresponds a d-periodic right-continuous function $\kappa(\cdot)$ such that, for all $x>0$,*

$$f(x)=\kappa(x)U(x)+\mathbb{E}_x h(S_\tau)-\mathbb{E}_x\sum_{k=0}^{\tau-1}g(S_k). \tag{5.7}$$

REMARK 5.5. Assume that the distribution of ξ is nonarithmetic (the arithmetic case is discussed in Remark 5.10). Then so is the distribution of S_{σ_1}, see Lemma A.4(c). According to Lemmas 5.3 and A.7(c), condition (5.6) holding for some $x>0$ does not even guarantee that the function g is Lebesgue integrable on $\mathbb{R}^+$. However, by Lemma A.7 (d), it does under an additional uniformity condition. Conversely, while by Lemma A.7(a), (5.6) may fail to hold for each $x>0$ if g is Lebesgue integrable, by Lemma A.7(b), direct Riemann integrability of g is a sufficient condition ensuring that (5.6) holds for each $x>0$. Summarizing, we think that condition (5.6) alone is not sufficient for proving formula (5.12) given below, which states that the remainder term $\mathbb{E}_x\sum_{k=0}^{\tau-1}g(S_k)$ exhibits a sublinear growth. This is the reason behind introducing in Theorem 5.4 the additional assumption that g is dRi which in conjunction with (5.6) guarantees that (5.12) holds, see Lemma 5.8.

REMARK 5.6. Our equality (5.7) is reminiscent of formula (5) in [14]. The authors of the cited article construct a particular harmonic function for a multidimensional random walk killed upon exiting a cone. In view of this similarity it is likely that a counterpart of Theorem 5.4 holds for solutions to a Poisson equation in a cone.

The proof of Theorem 5.4 consists of the three steps. First, in Lemma 5.7, we prove that condition (5.6) is necessary for the existence of a solution. Second, in Lemma 5.9, we exhibit a particular solution to (5.4) which is a subharmonic function of sublinear growth. Third, in the proof of Theorem 5.4, using the linearity of (5.4) we show that any solution to (5.4) is the sum of a harmonic function of linear growth and the subharmonic function obtained at the second step.

LEMMA 5.7. *Assume that condition* (5.6) *does not hold for $x=x_0>0$. Then no solution to* (5.4) *exists.*

PROOF. Assume on the contrary that there exists a solution to (5.4) and denote it by f. We define g for negative arguments by $g(x):=\mathbb{E}f(x+\xi)-h(x)$, $x\leq 0$. For $n\in\mathbb{N}_0$, introduce $M_n:=f(S_n)-\sum_{k=0}^{n-1}g(S_k)$ and, for $n\in\mathbb{N}$, let $\mathcal{G}_n$ denote the σ-algebra generated by $\xi_1,\ldots,\xi_n$, with $\mathcal{G}_0$ being the trivial σ-algebra. The sequence $(M_n,\mathcal{G}_n)_{n\in\mathbb{N}_0}$ is a $\mathbb{P}_{x_0}$-martingale. Since, for each $n\in\mathbb{N}_0$, $\tau\wedge\sigma(y)\wedge n$ is a stopping time with respect to the filtration $(\mathcal{G}_k)_{k\in\mathbb{N}}$, then, for $y\geq x_0$, the sequence $(M_{\tau\wedge\sigma(y)\wedge n},\mathcal{G}_n)_{n\in\mathbb{N}_0}$ is also a $\mathbb{P}_{x_0}$-martingale. In particular,

$$f(x_0)=\mathbb{E}_{x_0}M_0=\mathbb{E}_{x_0}M_{\tau\wedge\sigma(y)\wedge n},\quad n\in\mathbb{N}_0. \tag{5.8}$$

We intend to show that

$$\lim_{n\to\infty}\mathbb{E}_{x_0}M_{\tau\wedge\sigma(y)\wedge n}=\mathbb{E}_{x_0}M_{\tau\wedge\sigma(y)}. \tag{5.9}$$

Note that $\lim_{n\to\infty}M_{\tau\wedge\sigma(y)\wedge n}=M_{\tau\wedge\sigma(y)}$ $\mathbb{P}_{x_0}$-a.s. Hence, according to the Lebesgue dominated convergence theorem, it is enough to check that

$$\mathbb{E}_{x_0}\sup_{n\geq 0}|M_{\tau\wedge\sigma(y)\wedge n}|<\infty. \tag{5.10}$$

To this end, write, for $n\in\mathbb{N}_0$,

$$\begin{aligned}|M_{\tau\wedge\sigma(y)\wedge n}|&\leq|f(S_n)|\,\mathbb{1}_{\{\tau\wedge\sigma(y)>n\}}+|f(S_{\tau\wedge\sigma(y)})|\,\mathbb{1}_{\{\tau\wedge\sigma(y)\leq n\}}+\sum_{k=0}^{\tau\wedge\sigma(y)-1}g(S_k)\\&\leq\sup_{z\in[0,\,y]}|f(z)|+|h(S_\tau)|+|f(S_{\sigma(y)})|+(\tau\wedge\sigma(y))\sup_{z\in[0,\,y]}g(z)\quad\mathbb{P}_{x_0}-\text{a.s.}\end{aligned}$$

having utilized the fact that $S_k\in(0,y]$ on the event $\{\tau\wedge\sigma(y)>k\}$ for the first and the last summands, and $f(x)=h(x)$ for $x\leq 0$ in combination with $S_\tau\leq 0$ $\mathbb{P}_{x_0}$-a.s. for the second summand. To prove inequality (5.10) we have to show that the right-hand side of the last centered formula (which does not depend on n) is $\mathbb{P}_{x_0}$-integrable.

Since h is bounded on $(-\infty,0]$ by assumption and $S_\tau\leq 0$ $\mathbb{P}_{x_0}$-a.s., we trivially infer $\mathbb{E}_{x_0}|h(S_\tau)|<\infty$. Further, since, by assumption, f is a right-continuous function of at most linear growth, there exists $C>0$ such that $|f(z)|\leq C(z+1)$ for $z\geq 0$. Hence, $\sup_{z\in[0,\,y]}|f(z)|<\infty$, and also

$$\mathbb{E}_{x_0}|f(S_{\sigma(y)})|\leq C(\mathbb{E}_{x_0}S_{\sigma(y)}+1)=C(\nu V(y-x_0)+1)<\infty,$$

where V is the renewal function defined in (4.2). The last inequality is justified by (5.1). The inequality $\sup_{z\in[0,\,y]}g(z)<\infty$ is secured by our assumption that g is a càdlàg function. So, it remains to prove that

$$\mathbb{E}_{x_0}(\tau\wedge\sigma(y))<\infty. \tag{5.11}$$

Since the distribution of ξ is nondegenerate, there exists $\delta>0$ such that $\mathbb{P}\{\xi>\delta\}\in(0,1)$. Set $N_y:=\lceil y/\delta\rceil$, where $z\mapsto\lceil z\rceil$ for $z\in\mathbb{R}$ is the ceiling function. Then

$$\sup_{z\in[0,\,y]}\mathbb{P}_z\{\tau\wedge\sigma(y)\leq N_y\}\geq\sup_{z\in[0,\,y]}\mathbb{P}_z\{\inf_{1\leq j\leq N_y}\xi_j>\delta\}=(\mathbb{P}\{\xi>\delta\})^{N_y}=:\varrho_y\in(0,1).$$

Now an application of the Markov property yields, for $k\in\mathbb{N}$,

$$\mathbb{P}_{x_0}\{\tau\wedge\sigma(y)>kN_y\}\leq\Big(1-\sup_{z\in[0,\,y]}\mathbb{P}_z\{\tau\wedge\sigma(y)\leq N_y\}\Big)^k\leq(1-\varrho_y)^k.$$

This shows that the $\mathbb{P}_{x_0}$-distribution of $\tau\wedge\sigma(y)$ has an exponential tail which particularly implies (5.11). Thus, formula (5.9) has been proved.

A combination of (5.8) and (5.9) gives

$$f(x_0)=\mathbb{E}_{x_0}f(S_{\sigma(y)})\,\mathbb{1}_{\{\sigma(y)<\tau\}}+\mathbb{E}_{x_0}h(S_\tau)\,\mathbb{1}_{\{\sigma(y)\geq\tau\}}-\mathbb{E}_{x_0}\sum_{k=0}^{\tau\wedge\sigma(y)-1}g(S_k).$$

Using Lemma 5.1 and the estimate for $|f|$ we arrive at

$$\limsup_{y\to\infty}\mathbb{E}_{x_0} f(S_{\sigma(y)})\,\mathbb{1}_{\{\sigma(y)<\tau\}} \le C(\limsup_{y\to\infty}\mathbb{E}_{x_0} S_{\sigma(y)}\,\mathbb{1}_{\{\sigma(y)<\tau\}}+1) = C(U(x_0)+1).$$

Since h is a bounded function on $(-\infty,0]$ we infer

$$\lim_{y\to\infty}\mathbb{E}_{x_0} h(S_\tau)\,\mathbb{1}_{\{\sigma(y)\ge\tau\}} = \mathbb{E}_{x_0} h(S_\tau) =: C_1 \in (-\infty,\infty)$$

by the Lebesgue dominated convergence theorem. Invoking the Lévy monotone convergence theorem yields

$$\lim_{y\to\infty}\mathbb{E}_{x_0}\sum_{k=0}^{\tau\wedge\sigma(y)-1} g(S_k) = \mathbb{E}_{x_0}\sum_{k=0}^{\tau-1} g(S_k)$$

By assumption, the right-hand side is infinite. We conclude that necessarily

$$f(x_0) \le C(U(x_0)+1) + C_1 - \mathbb{E}_{x_0}\sum_{k=0}^{\tau-1} g(S_k) = -\infty,$$

a contradiction which completes the proof of Lemma 5.7. □

LEMMA 5.8. *Assume that condition (5.6) holds for some $x>0$ and that the function g is dRi on $\mathbb{R}^+$. Then (5.6) holds for each $x>0$ and*

$$\lim_{x\to\infty}\Big(\mathbb{E}_x\sum_{k=0}^{\tau-1} g(S_k)\Big)/x = 0. \tag{5.12}$$

PROOF. We start by recalling that $\mu,\nu\in(0,\infty)$ according to (5.1). By Lemma 5.3,

$$\mathbb{E}_x\sum_{k=0}^{\tau-1} g(S_k) = \int_{[0,\,x]} r(x-y)\mathrm{d}U(y), \quad x>0, \tag{5.13}$$

where $r(x)=\int_{[0,\,\infty)} g(x+z)\mathrm{d}V(z)$ for $x\ge 0$. Thus, if

$$\lim_{x\to\infty} r(x) = 0,$$

then using the first part of (5.2), relation (5.12) follows with the help of a simple (Stolz-Cesàro like) argument.

By Lemma A.7 (b), we infer $r(x)<\infty$ for each $x\ge 0$ which implies that (5.6) holds for each $x>0$. The function V is subadditive on $\mathbb{R}$ (see, for instance, formula (6.3) in [23]). Armed with this we obtain, for each $x\ge 0$,

$$\begin{aligned} r(x) &\le \int_{[\lfloor x\rfloor,\,\infty)} g(y)\mathrm{d}V(y-x) \\ &\le \sum_{n\ge\lfloor x\rfloor+1}\sup_{n-1\le y<n} g(y)(V(n-x)-V(n-1-x)) \le V(1)\sum_{n\ge\lfloor x\rfloor+1}\sup_{n-1\le y<n} g(y), \end{aligned} \tag{5.14}$$

where $z\mapsto\lfloor z\rfloor$ is the floor function. Since g is dRi on $\mathbb{R}^+$ and thereupon

$$\overline{\sigma}(1) = \sum_{n\ge 1}\sup_{n-1\le y<n} g(y) < \infty,$$

the right-hand side converges to 0 as $x\to\infty$. □

LEMMA 5.9. *Assume that condition* (5.6) *holds for each* $x>0$ *and that* g *is dRi on* $\mathbb{R}^+$. *Then the function* f *defined by*

$$f(x) := \mathbb{E}_x h(S_\tau) - \mathbb{E}_x \sum_{k=0}^{\tau-1} g(S_k), \quad x \in \mathbb{R}$$

is a solution to (5.4), *and* $\lim_{x\to\infty}(f(x)/x)=0$.

PROOF. Let us check that f is a solution to (5.4) which exhibits at most linear growth. Using the fact that, under $\mathbb{P}_x$, $x\in\mathbb{R}$, $(S_k-S_1)_{k\in\mathbb{N}}$ has the same distribution as $(S_n-x)_{n\in\mathbb{N}_0}$ and is independent of S_1 and that, by definition,

$$\tau = 0 \quad \mathbb{P}_x - \text{a.s.} \quad \text{for } x \le 0, \tag{5.15}$$

we obtain

$$f(x) = \mathbb{E} f(x+\xi) - g(x), \quad x>0.$$

Also,

$$f(x) = h(x), \quad x \le 0$$

by another appeal to (5.15) (in particular, $\mathbb{E}_x \sum_{k=0}^{\tau-1} g(S_k) = 0$ for $x<0$).

Next, we note that $\lim_{x\to\infty}(f(x)/x)=0$ is a consequence of Lemma 5.8 and boundedness of h.

Finally, we show that the function f is right-continuous. By assumption, h is a right-continuous bounded function. Hence, the function $x\mapsto \mathbb{E}_x h(S_\tau)$ is right-continuous by the Lebesgue dominated convergence theorem. To prove right-continuity of $x\mapsto \mathbb{E}_x\sum_{k=0}^{\tau-1} g(S_k)$ on $(0,\infty)$ we are going to use representation (5.13). For $x,y\ge 0$ and $z\in[0,1]$,

$$\begin{aligned} g(x+z+y) &\le \sum_{n\ge 1} \sup_{n-1+z\le y<n+z} g(x+y)\,\mathbb{1}_{[n-1,n)}(y) \\ &\le \sum_{n\ge 1} \sup_{n-1\le y<n+1} g(x+y)\,\mathbb{1}_{[n-1,n)}(y) =: L_x(y). \end{aligned}$$

Also,

$$\begin{aligned} \int_{[0,\infty)} L_x(y)\mathrm{d}V(y) &= \sum_{n\ge1} \sup_{n-1\le y<n+1} g(x+y)(V(n-)-V((n-1)-)) \\ &\le V(1)\sum_{n\ge 1}\sup_{n-1\le y<n+1} g(x+y) < \infty, \end{aligned}$$

where the finiteness is secured by the fact that g is dRi and the penultimate inequality is justified by subadditivity of V on $\mathbb{R}$. Hence,

$$\lim_{z\to 0+} r(x+z) = \int_{[0,\infty)} \lim_{z\to0+} g(x+z+y)\mathrm{d}V(y) = \int_{[0,\infty)} g(x+y)\mathrm{d}V(y) = r(x)$$

by right-continuity of g and the Lebesgue dominated convergence theorem. According to the proof of Lemma 5.8, $\lim_{x\to\infty} r(x)=0$, whence

$$r(x+z-y)\,\mathbb{1}_{[0,x+z]}(y) \le c\,\mathbb{1}_{[0,x+1]}(y)$$

for $x \geq 0$, $z \in [0,1]$, $y \in [0, x+z]$ and a constant $c>0$. Thus, we infer

$$\lim_{z\to 0+} \int_{[0,\infty)} r(x+z-y)\, \mathbb{1}_{[0,x+z]}(y) \mathrm{d}U(y) = \int_{[0,\infty)} \lim_{z\to 0+} r(x+z-y)\, \mathbb{1}_{[0,x+z]}(y) \mathrm{d}U(y) = \int_{[0,x]} r(x-y) \mathrm{d}U(y)$$

by another appeal to the Lebesgue dominated convergence theorem. Thus, right-continuity on $(0,\infty)$ has been proved. One can also check that $\lim_{x\to 0+} \mathbb{E}_x \sum_{k=0}^{\tau-1} g(S_k) = 0$ by a similar reasoning. □

REMARK 5.10. Assume that the distribution of ξ is d-arithmetic for $d>0$ and the function g is not dRi. Then it can be checked (details are simple, hence omitted) that if (5.6) holds for some $x>0$, then $\sum_{n\geq 0} g(x+nd) < \infty$ and thereupon

$$\lim_{n\to\infty} (H(x+nd)/(nd)) = 0, \text{ where } H(y) := \mathbb{E}_y \sum_{k=0}^{\tau-1} g(S_k) \text{ for } y>0.$$

However, this does not seem to imply $\limsup_{x\to\infty}(H(x)/x) < \infty$ which is needed for proving that f defined in Lemma 5.9 satisfies $\limsup_{x\to\infty}(|f(x)|/x) < \infty$, let alone $\lim_{x\to\infty}(H(x)/x) = 0$. On the other hand, the assumption that g is dRi comfortably ensures the latter.

We now turn to the proof of the main result of the section.

PROOF OF THEOREM 5.4. In view of Lemma 5.7 it remains to consider the case when condition (5.6) holds for $x>0$ and g is dRi on $\mathbb{R}^+$. Then, by Lemma 5.8, (5.6) holds for each $x>0$. Hence, Lemma 5.9 applies and ensures that $x \mapsto \mathbb{E}_x h(S_\tau) - \mathbb{E}_x \sum_{k=0}^{\tau-1} g(S_k)$, $x \in \mathbb{R}$ is a solution to (5.4) of sublinear growth.

Let f be any solution to (5.4) for which (5.5) holds. Lemma 5.9 in combination with the linearity of (5.4) enables us to conclude that the function $\hat{f}$ defined by

$$\hat{f}(x) := f(x) - \mathbb{E}_x h(S_\tau) + \mathbb{E}_x \sum_{k=0}^{\tau-1} g(S_k), \quad x \in \mathbb{R}$$

satisfies

$$\begin{cases} \hat{f}(x) = \mathbb{E}\hat{f}(x+\xi)\, \mathbb{1}_{\{\xi > -x\}}, & \text{if } x>0, \\ \hat{f}(x) = 0, & \text{if } x \leq 0 \\ \limsup_{x\to\infty}(|\hat{f}(x)|/x) < \infty. \end{cases}$$

In other words, $\hat{f}$ is a harmonic function of at most linear growth for the random walk S killed upon entering $(-\infty, 0]$. Therefore, the proof is completed by an application of Lemma 5.2. □

As a consequence of Theorem 5.4, we conclude that subharmonic functions of at most linear growth for the killed random walk exhibit exactly a linear growth rate (at least along the closure of the group generated by the support of the distribution of ξ).

COROLLARY 5.11. *Assume that the distribution of ξ is d-arithmetic for $d \geq 0$ and that the function g is dRi on $\mathbb{R}^+$. Let f be a solution to (5.4) satisfying (5.5) and $\kappa(\cdot)$ the corresponding d-periodic function from (5.7). Then*

$$\begin{cases} \lim_{n\to\infty}(f(x+nd)/nd) = \kappa(x)/\mu & \text{for all } x \in [0,d), \quad \text{if } d > 0 \\ \lim_{x\to\infty}(f(x)/x) = \kappa/\mu, & \text{if } d = 0. \end{cases}$$

Furthermore, if $\kappa(x) = \kappa$ for all $x \in \mathbb{R}$ in the case $d > 0$, then $\lim_{x\to\infty}(f(x)/x) = \kappa/\mu$.

This result follows from Theorem 5.4, Lemma 5.9 and (5.2).

6. Proofs related to tail behavior. Recall that the random variable Z is the a.s. limit of the derivative martingale $(Z_n, \mathcal{F}_n)_{n\in\mathbb{N}_0}$. In this section we prove Theorems 2.1 and 2.2 by investigating the asymptotic behavior of the Laplace transform of Z near zero and using Tauberian theorems given in the Appendix.

6.1. *Decomposition of Z.* Let $\eta_1, \eta_2, \ldots$ be independent copies of a random variable η which are independent of $\mathcal{Z} = \sum_{j=1}^N \delta_{X_j}$. The mapping which maps the distribution of η to the distribution of $\sum_{i=1}^N e^{-X_i}\eta_i$ is an instance of *smoothing transform*. The distribution of η is a *fixed point* of this smoothing transform if

$$\eta \overset{\mathrm{d}}{=} \sum_{i=1}^N e^{-X_i}\eta_i,$$

where $\overset{\mathrm{d}}{=}$ denotes equality of distributions. Recent advances concerning fixed points of general smoothing transforms can be found in [3, 4, 25, 35], the list is far from being complete.

Denote by ϕ the Laplace transform of Z, that is,

$$\phi(s) = \mathbb{E}e^{-sZ}, \quad s \geq 0.$$

Below we provide an a.s. decomposition of Z over the individuals of any fixed generation. The distributional version of formula (6.3) in the case $k = 1$ shows that the distribution of Z is a fixed point of the particular smoothing transform. This fact reformulated in terms of ϕ reads

$$\phi(s) = \mathbb{E}\prod_{i=1}^N \phi(se^{-X_i}), \quad s \geq 0. \tag{6.1}$$

As a preparation, we recall from Lemma 3.1 in [41] that, under (1.1) and (1.2), we have

$$\lim_{n\to\infty} \inf_{|u|=n} S(u) = \infty \quad \text{a.s.}, \tag{6.2}$$

that is, the minimal position of the nth generation individuals diverges to ∞ as $n \to \infty$. Here, the infimum is defined to be $+\infty$ if the population dies out by the nth generation. Further, for $u, v \in \mathcal{I}$ we write $v > u$ if u is an ancestor of v, that is, $u = u_1 \ldots u_k$ and $v = u_1 \ldots u_k \ldots u_n$ for some $k \in \mathbb{N}_0$ and integer $n > k$. Given $u \in \mathcal{I}$, set

$$Z_n(u) := \sum_{|v|=n+|u|,\, v>u} e^{-(S(v)-S(u))}(S(v) - S(u)), \quad n \in \mathbb{N},$$

so that $(Z_n(u))_{n\in\mathbb{N}}$ is a version of $(Z_n)_{n\in\mathbb{N}}$. Then

$$Z(u) := \lim_{n\to\infty} Z_n(u)$$

is the a.s. limit of the derivative martingale defined on the subtree of $\mathcal{I}$ rooted at u. For fixed $k \in \mathbb{N}$, the random variables $(Z(u))_{|u|=k}$ are independent copies of Z which are also independent of $(S(u))_{|u|=k}$.

LEMMA 6.1. *Assume that Condition $\mathcal{S}$ holds. Then, for each $k \in \mathbb{N}$,*

$$Z = \sum_{|u|=k} e^{-S(u)} Z(u) \quad \text{a.s.} \tag{6.3}$$

REMARK 6.2. In the situation where $N < \infty$ a.s. this fact was proved in Theorem 5.1 of [7]. However, we work under weaker assumptions, in particular, the case $\mathbb{P}\{N = \infty\} > 0$ is not excluded in the present work. Since we did not find an appropriate reference in the literature, we give a complete proof.

PROOF. Let $(\tau_k^*)_{k \in \mathbb{N}_0}$ be the sequence of strict descending ladder epochs, that is, $\tau_0^* := 0$, $\tau_1^* := \inf\{j \in \mathbb{N} : S_j < 0\}$ and $\tau_k^* := \inf\{j > \tau_{k-1}^* : S_j < S_{\tau_{k-1}^*}\}$ for $k \geq 2$. Put

$$R(x) := \sum_{n \geq 0} \mathbb{P}\{-S_{\tau_n^*} \leq x\}, \quad x \in \mathbb{R},$$

that is, R is the renewal function for the standard random walk formed by strict descending ladder heights. Note that $R(x) = 0$ for $x < 0$. For fixed $\alpha \geq 0$, put

$$D_n^{(\alpha)} = \sum_{|u|=n} e^{-S(u)} R(S(u) + \alpha)\, \mathbb{1}_{\{S(u_1) \geq -\alpha, S(u_1 u_2) \geq -\alpha, \ldots, S(u_1 \ldots u_n) \geq -\alpha\}}, \quad n \in \mathbb{N}_0$$

and let $A_\alpha := \{S(u) \geq -\alpha \quad \text{for all ever born individuals } u\}$ denote the event of nonextinction of the branching random walk killed below $-\alpha$. According to Lemma A.1 in [1], the sequence $(D_n^{(\alpha)}, \mathcal{F}_n)_{n \in \mathbb{N}_0}$ forms a nonnegative martingale called *truncated martingale*. Furthermore, by Proposition A.3 in [1], $D_n^{(\alpha)}$ converges a.s. and in L^1 as $n \to \infty$ to a random variable that we denote by $D^{(\alpha)}$, and $D^{(\alpha)} > 0$ a.s. on A_α.

By Lemma A.4 (a,b),

$$\lim_{x \to \infty} x^{-1} R(x) = (-\mathbb{E} S_{\tau_1^*})^{-1} =: \mathtt{m}^{-1} > 0. \tag{6.4}$$

This together with (6.2) enables us to conclude that, a.s. on A_α,

$$D^{(\alpha)} = \lim_{n \to \infty} \sum_{|u|=n} e^{-S(u)} R(S(u) + \alpha) = \lim_{n \to \infty} \sum_{|u|=n} \mathtt{m}^{-1} e^{-S(u)} S(u) = \mathtt{m}^{-1} Z \tag{6.5}$$

(we note in passing that these random variables are not equal a.s. because $\mathbb{E} D^{(\alpha)} < \infty$, whereas $\mathbb{E} Z = \infty$). We extend the definition of the truncated martingale to the subtrees rooted at $u \in \mathcal{I}$ as follows: for all $n \in \mathbb{N}$,

$$D_n^{(\alpha)}(u) := \sum_{|v|=n+|u|,\, v > u} e^{-S(v)} R(S(v) + \alpha)\, \mathbb{1}_{\{S(uv_1) \geq -\alpha, S(uv_1 v_2) \geq -\alpha, \ldots, S(uv_1 \ldots v_n) \geq -\alpha\}}.$$

Fix $u \in \mathcal{I}$. The sequence $(e^{S(u)} D_n^{(\alpha)}(u))_{n \in \mathbb{N}}$ has the same distribution as $(D_{n,*}^{(S(u)+\alpha)})_{n \in \mathbb{N}}$, where for $\beta \geq 0$, $(D_{n,*}^{(\beta)})_{n \in \mathbb{N}}$ is a distributional copy of $(D_n^{(\beta)})_{n \in \mathbb{N}}$ which is independent of $S(u)$; and for $\beta < 0$, $D_{n,*}^{(\beta)} = 0$ for each $n \in \mathbb{N}$. From this we conclude that $D_n^{(\alpha)}(u)$ converges a.s. and in L^1, as $n \to \infty$, to a random variable $D^{(\alpha)}(u)$, say which satisfies

$$D^{(\alpha)}(u) = \mathtt{m}^{-1} e^{-S(u)} Z(u) \qquad \text{a.s. on } A_\alpha \tag{6.6}$$

and

$$\mathbb{E}\big(D^{(\alpha)}(u) | \mathcal{F}_{|u|}\big) = \mathbb{E}\big(D_n^{(\alpha)}(u) | \mathcal{F}_{|u|}\big) = e^{-S(u)} R(S(u) + \alpha) \qquad \text{a.s.} \tag{6.7}$$

Decomposing $D_n^{(\alpha)}$ over the kth generation yields

$$D_n^{(\alpha)}=\sum_{|u|=k}D_{n-k}^{(\alpha)}(u),\quad n>k\quad \text{a.s.}$$

By Fatou's lemma, for $k\in\mathbb{N}$,

$$D^{(\alpha)}\geq\sum_{|u|=k}D^{(\alpha)}(u)\geq 0\quad \text{a.s.}$$

Also, for $k\in\mathbb{N}$,

$$\mathbb{E}D^{(\alpha)}=R(\alpha)=\mathbb{E}R(S_k+\alpha)=\mathbb{E}\sum_{|u|=k}e^{-S(u)}R(S(u)+\alpha)=\mathbb{E}\sum_{|u|=k}D^{(\alpha)}(u),$$

where the first and the last equalities follow from (6.7), the second equality expresses the known fact that R is a harmonic function of the random walk S killed upon entering $(-\infty,0)$ (see Lemma 1 in [43]), and the third equality is a consequence of Lemma 4.1. The last two centered formulae together ensure that, for $k\in\mathbb{N}$,

$$D^{(\alpha)}=\sum_{|u|=k}D^{(\alpha)}(u)\quad \text{a.s.}$$

Using (6.5) and (6.6) yields, for each $\alpha\geq 0$ and $k\in\mathbb{N}$,

$$Z=\sum_{|u|=k}e^{-S(u)}Z(u)\qquad \text{a.s. on } A_\alpha,$$

hence just a.s. as $(A_\alpha)_{\alpha\geq 0}$ is a nondecreasing family of events with $\lim_{\alpha\to\infty}\mathbb{P}(A_\alpha)=1$. □

6.2. *Asymptotic behavior of the Laplace transform.* Recall that ϕ denotes the Laplace transform of Z and put

$$D(x)=e^x(1-\phi(e^{-x})),\quad x\in\mathbb{R}.$$

REMARK 6.3. Assume that Condition $\mathcal{S}$ holds. Then, according to Lemma 5.1 in [3],

$$\sup_{x>0}\frac{D(x)}{1+x}<\infty. \tag{6.8}$$

Theorems 6.5 and 6.6 given below in this section can be thought of as a strengthening of (6.8).

Following Durrett and Liggett [19] and many of their successors, we put, for $x\in\mathbb{R}$,

$$\begin{aligned}G(x)&=\mathbb{E}\sum_{i=1}^N e^{-X_i}D(x+X_i)-D(x)=\mathbb{E}D(x+\xi)-D(x)\\&=e^x\mathbb{E}\Big(\prod_{i=1}^N\phi(e^{-x-X_i})-1+\sum_{i=1}^N\big(1-\phi(e^{-x-X_i})\big)\Big),\end{aligned}$$

where ξ is a random variable with distribution defined in (4.1). To obtain the second equality we have used (6.1). For later needs, we note the following.

LEMMA 6.4. *(a) $G(x)\geq 0$ for $x\in\mathbb{R}$;*
(b) the function $x\mapsto e^{-x}G(x)$ is nonincreasing on $\mathbb{R}$.

These two properties were given in Lemma 2.4 of [19] under the assumption that N is deterministic. However, the proof of the cited result extends verbatim to the more general situation treated here.

From the definition of G and formula (6.8) it follows that D satisfies

$$\begin{cases} D(x) = \mathbb{E}D(x+\xi) - G(x), & x \in \mathbb{R}; \\ \sup_{x\in\mathbb{R}} \frac{D(x)}{1+|x|} < \infty. \end{cases} \tag{6.9}$$

In particular, D is a nonnegative subharmonic function of at most linear growth for the random walk S. Therefore, invoking Theorem 5.4 we can give an alternative formula for D. Below we use the notation introduced in Section 4.

THEOREM 6.5. *Assume that Condition $\mathcal{S}$ holds. Then, for each $x > 0$,*

$$D(x) = \mu U(x) + \mathbb{E}_x D(S_\tau) - \mathbb{E}_x \sum_{k=0}^{\tau-1} G(S_k) \tag{6.10}$$

and

$$D(x) \sim x, \quad x \to \infty. \tag{6.11}$$

Also, if the distribution of ξ is nonarithmetic, then the limit $\lim_{x\to\infty} \mathbb{E}_x D(S_\tau)$ exists and is finite. If the distribution of ξ is d-arithmetic for $d > 0$, then the limit does not exist but

$$\lim_{x\to\infty} (\mathbb{E}_x D(S_\tau) - c_{11}(x)) = 0 \tag{6.12}$$

for a bounded d-periodic function $c_{11}(\cdot)$ which is not a constant.

Theorem 2.1 is an immediate consequence of (6.11) and Corollary 8.1.7 in [8] which states that relations (6.11) and (2.1) are equivalent.

THEOREM 6.6. *Assume that Condition $\mathcal{S}_{\rm na}$ holds. Then Condition $\mathcal{S}^*$ ensures*

$$D(x) = \mu U(x) + c_1 + o(1) = x + c_2 + o(1), \quad x \to \infty, \tag{6.13}$$

where $c_2 = c_1 + (2\mu)^{-1}\mathbb{E}S_\tau^2 = c + 1 - \gamma$, γ is the Euler-Mascheroni constant, and c is the same as in (2.6). Conversely, the second equality in (6.13) entails Condition $\mathcal{S}^*$.

At this point it is convenient to prove Theorem 2.2.

PROOF OF THEOREM 2.2. Assume first that Condition $\mathcal{S}^*$ holds. While formula (2.6) of Theorem 2.2 follows from the second equality in (6.13) and the implication (I) $\Rightarrow$ (III) of Lemma A.3, formula (2.7) is a consequence of the fact that (6.13) entails that for each $y \in \mathbb{R}$, $\lim_{x\to\infty}(D(x+y) - D(x)) = y$ and the implication (ii)$\Rightarrow$ (i) of Lemma A.1. Assume now that representation (2.6) holds true. By the implication (III) $\Rightarrow$ (I) of Lemma A.3, the second equality in (6.13) holds. With this at hand, the necessity of Condition $\mathcal{S}^*$ follows from Theorem 6.6. □

REMARK 6.7. Assume that Conditions $\mathcal{S}$ and $\mathcal{S}^*$ hold and that the distribution of ξ is d-arithmetic for $d > 0$. Although the limit relation (6.13) cannot hold, there exists a bounded d-periodic function $c_1(\cdot)$ which is not a constant such that

$$D(x) = \mu U(x) + c_1(x) + o(1), \quad x \to \infty. \tag{6.14}$$

Details can be found in Remark 6.9.

6.3. *Proof of Theorem 6.5*. By Lemma A.4 (a), the assumption

$$\mathbb{E}\xi_-^2 = \mathbb{E}\sum_{i=1}^{N} e^{-X_i}(X_i)_-^2 < \infty$$

which is one half of (1.3) ensures that $\mu = -\mathbb{E}S_{\tau_1}$ is finite.

In view of (6.9), the function D satisfies (5.5) and is a continuous solution to (5.4) with $h(x) = D(x)$ for $x \le 0$ and $g = G$. Note that D is bounded on $(-\infty, 0]$ in view of

$$D(x) = e^x(1-\phi(e^{-x})) \le e^x \le 1, \quad x \le 0,$$

and that G is continuous. Let us show that G is dRi on $\mathbb{R}^+$. Let $h_0 = d$ if the distribution of ξ is d-arithmetic for $d > 0$ and $h_0 > 0$ be arbitrary if the distribution of ξ is nonarithmetic. By Theorem 5.4, $\mathbb{E}_x \sum_{k=0}^{\tau-1} G(S_k) < \infty$ for each $x > 0$, hence for $x = h_0$. Then using Lemma 5.3 with $p = G$ to justify the first inequality we infer

$$\infty > r(h_0) = \int_{[h_0,\infty)} G(y)\mathrm{d}V(y-h_0)$$
$$\ge \sum_{n\ge 1} \inf_{(n-1)h_0 \le y < nh_0} G(y)(V((n-1)h_0) - V((n-2)h_0)).$$

By the Blackwell theorem (see, for instance, formulae (6.8) and (6.9) in [23]),

$$\lim_{n\to\infty}(V((n-1)h_0) - V((n-2)h_0)) = h_0/\nu \in (0,\infty).$$

Therefore,

$$\underline{\sigma}(h_0) = h_0 \sum_{n\ge 1} \inf_{(n-1)h_0 \le y < nh_0} G(y) < \infty.$$

By Lemma A.5, this together with the fact that the function $x \mapsto e^{-x}G(x)$ is nonincreasing (see Lemma 6.4) enables us to conclude that g is dRi on $\mathbb{R}^+$.

By Theorem 5.4, there exists a d-periodic function $\kappa(\cdot)$ such that, for all $x > 0$,

$$D(x) = \kappa(x)U(x) + \mathbb{E}_x D(S_\tau) - \mathbb{E}_x \sum_{k=0}^{\tau-1} G(S_k) =: \kappa(x)U(x) + r(x).$$

To complete the proof of (6.10) we have to show that $\kappa(x) = \mu$ for all $x > 0$. Relation (6.11) will then follow by Lemma 5.9 and Corollary 5.11.

The subsequent argument is close to the discussion in [6], particularly to Theorem 3.1 and Lemma 5.1 therein. Using Lemma 6.1 yields, for $\lambda > 0$,

$$\mathbb{E}(e^{-\lambda Z}|\mathcal{F}_n) = \mathbb{E}\Big(\exp\Big(-\sum_{|u|=n} \lambda e^{-S(u)} Z(u)\Big)\Big|\mathcal{F}_n\Big)$$
$$= \prod_{|u|=n} \phi(\lambda e^{-S(u)}) = \prod_{|u|=n} (1 - \lambda D(S(u) - \log\lambda)e^{-S(u)}).$$

We have $\lim_{n\to\infty} \mathbb{E}(e^{-\lambda Z}|\mathcal{F}_n) = e^{-\lambda Z}$ a.s. since $\big(\mathbb{E}(e^{-\lambda Z}|\mathcal{F}_n), \mathcal{F}_n\big)_{n\in\mathbb{N}_0}$ is a right closable martingale, and thereupon

$$Z = (1/\lambda)\lim_{n\to\infty} \sum_{|u|=n} -\log\big(1 - \lambda D(S(u) - \log\lambda)e^{-S(u)}\big) \quad \text{a.s.}$$

On the other hand, for all $\lambda > 0$ and u with $|u| = n$,

$$D(S(u) - \log\lambda) = \kappa(-\log\lambda)U(S(u) - \log\lambda) + r(S(u) - \log\lambda).$$

We have used the equality $\kappa(S(u) - \log\lambda) = \kappa(-\log\lambda)$ which is trivial if the distribution of ξ is nonarithmetic, for $\kappa(\cdot)$ is then a constant. If the distribution of ξ is d-arithmetic for $d > 0$, the equality is secured by $S(u) \in d\mathbb{Z}$ a.s. Further, we conclude that, for all $\lambda > 0$,

$$\lim_{n\to\infty} \sum_{|u|=n} -\log\Big(1 - \lambda D(S(u) - \log\lambda)e^{-S(u)}\Big) = \lim_{n\to\infty} \sum_{|u|=n} (\lambda/\mu)\kappa(-\log\lambda)S(u)e^{-S(u)}$$
$$= (\lambda/\mu)\kappa(-\log\lambda)Z \quad \text{a.s.}$$

having utilized $e^{-x}D(x) = 1 - \phi(e^{-x}) \to 0$ as $x \to \infty$, $\lim_{n\to\infty} \inf_{|u|=n} S(u) = \infty$ a.s., the last centered formula, Lemma A.4(b) and $\lim_{x\to\infty}(r(x)/x) = 0$ (see Lemma 5.9) for the first equality. Since $\mathbb{P}\{Z > 0\} > 0$, we infer $\kappa(-\log\lambda) = \mu$ for all $\lambda > 0$.

It remains to investigate the existence of the limit $\lim_{x\to\infty} \mathbb{E}_x D(S_\tau)$. For $x > 0$, put $\tau(-x) := \inf\{k \in \mathbb{N} : S_k \le -x\}$ and $\nu(x) := \inf\{k \in \mathbb{N} : -S_{\tau_k} \ge x\}$, so that $\nu(x)$ is the first passage time into $[x, \infty)$ of $(-S_{\tau_k})_{k\in\mathbb{N}_0}$. Then

$$\mathbb{E}_x D(S_\tau) = \mathbb{E} D(x + S_{\tau(-x)}), \quad x > 0. \tag{6.15}$$

Since the first passage into $(-\infty, -x]$ of $(S_k)_{k\in\mathbb{N}_0}$ can only occur at a weakly descending ladder epoch we infer

$$S_{\tau(-x)} = S_{\tau_{\nu(x)}} \quad \text{a.s.} \tag{6.16}$$

Hence,

$$\mathbb{E}_x D(S_\tau) = \mathbb{E} D(-(-S_{\tau_{\nu(x)}} - x)) = \int_{[0,\,x)} m(x-y)\mathrm{d}U(y),$$

where $m(x) := \mathbb{E} D(-(-S_{\tau_1} - x))\,\mathbb{1}_{\{-S_{\tau_1} \ge x\}}$ for $x \ge 0$. Boundedness and continuity of D on $(-\infty, 0]$ implies that m is locally Riemann integrable on $(-\infty, 0]$. Since we have $m(x) \le \mathbb{P}\{-S_{\tau_1} \ge x\}$ for $x \ge 0$ and $x \mapsto \mathbb{P}\{-S_{\tau_1} \ge x\}$ is dRi on $\mathbb{R}^+$ as a nonincreasing and Lebesgue integrable function (note that $\int_0^\infty \mathbb{P}\{-S_{\tau_1} \ge x\}\mathrm{d}x = \mu < \infty$) we conclude that m is dRi on $\mathbb{R}^+$. It is known (see Lemma A.4 (c)) that the distribution of S_{τ_1} is d-arithmetic because so is the distribution of ξ. Thus, invoking the key renewal theorem yields

$$\lim_{x\to\infty} \mathbb{E}_x D(S_\tau) = \mu^{-1} \int_0^\infty m(y)\mathrm{d}y = \mu^{-1} \int_0^\infty D(-y)\mathbb{P}\{-S_{\tau_1} \ge y\}\mathrm{d}y < \infty$$

in the nonarithmetic case $d = 0$ (see, for instance, Proposition 6.2.3 in [23]), whereas, for each $x \in [0, d)$,

$$\lim_{n\to\infty} \mathbb{E}_{x+nd} D(S_\tau) = d\mu^{-1} \sum_{k\ge 0} m(x + kd) =: \tilde{m}(x) < \infty$$

in the arithmetic case $d > 0$ (see, for instance, Proposition 6.2.6 in [23]). Writing $\{y\}$ for the fractional part of y, we set $c_{11}(x) := \tilde{m}(d\{x/d\})$ for $x \ge 0$. We observe that the last limit relation is equivalent to (6.12). It is clear that $c_{11}(\cdot)$ is a bounded d-periodic function. To see that it is not a constant (which implies that the limit $\lim_{x\to\infty} \mathbb{E}_x D(S_\tau)$ does not exist) one may use, for instance, the fact $x \mapsto e^{-x}m(x)$ is a nonincreasing function which follows from Lemma 6.4 (b). The proof of Theorem 6.5 is complete.

6.4. *Proof of Theorem 6.6.* For the proof of Theorem 6.6 we need some more preparations.

LEMMA 6.8. *Assume that Condition $\mathcal{S}_{\rm na}$ holds. Then the limit $\lim_{x\to\infty}\mathbb{E}_x\sum_{k=0}^{\tau-1}G(S_k)$ exists and is finite if, and only if, $\int_0^\infty yG(y){\rm d}y<\infty$.*

PROOF. The definitions of the strict ascending ladder epochs $(\sigma_n)_{n\in\mathbb{N}_0}$ and the renewal functions U and V are given in Section 4. We first recall that by Lemma 5.3

$$\mathbb{E}_x\sum_{j=0}^{\tau-1}G(S_j)=\int_{[0,\,x]}r(x-y){\rm d}U(y),\quad x>0,\tag{6.17}$$

where $r(z)=\int_{[0,\,\infty)}G(z+y){\rm d}V(y)=\mathbb{E}\sum_{k\geq 0}G(z+S_{\sigma_k})$ for $z\geq 0$.

Next, we are going to prove that r is Lebesgue integrable on $\mathbb{R}^+$ if, and only if, $\int_0^\infty yG(y){\rm d}y<\infty$. By a multiple use of Fubini's theorem,

$$\begin{aligned}\int_0^\infty r(y){\rm d}y&=\int_0^\infty\mathbb{E}\sum_{k\geq 0}G(y+S_{\sigma_k}){\rm d}y=\mathbb{E}\sum_{k\geq 0}\int_{S_{\sigma_k}}^\infty G(y){\rm d}y\\&=\sum_{k\geq 0}\mathbb{E}\int_0^\infty G(y)\,\mathbb{1}_{\{S_{\sigma_k}\leq y\}}\,{\rm d}y=\int_0^\infty G(y)V(y){\rm d}y,\end{aligned}$$

where all the integrals are either convergent or divergent simultaneously. By Lemma A.4(a), the condition $\mathbb{E}\xi_+^2<\infty$ guarantees $\mathbb{E}S_{\sigma_1}<\infty$, and we obtain with the help of Lemma A.4(b) that $\lim_{y\to\infty}y^{-1}V(y)=(\mathbb{E}S_{\sigma_1})^{-1}\in(0,\infty)$. Hence, the last integral converges if, and only if, so does the integral $\int_0^\infty yG(y){\rm d}y$.

Assume now that $\int_0^\infty yG(y){\rm d}y<\infty$, hence $\int_0^\infty r(y){\rm d}y<\infty$. According to Lemma 6.4(b), $x\mapsto e^{-x}G(x)$ is a nonincreasing function on $\mathbb{R}$. Hence, so is $x\mapsto e^{-x}r(x)$ which implies that r is a dRi function on $\mathbb{R}^+$, see Lemma A.5. Recalling that the distribution of S_{τ_1} is nonarithmetic (because so is the distribution of ξ), that $\mu=-\mathbb{E}S_{\tau_1}<\infty$ and invoking the key renewal theorem we infer

$$\mathbb{E}_x\sum_{j=0}^{\tau-1}G(S_j)=\int_{[0,\,x]}r(x-y){\rm d}U(y)\ \to\ \mu^{-1}\int_0^\infty r(y){\rm d}y\in(0,\infty),\quad x\to\infty.$$

Finally, assume that $\int_0^\infty yG(y){\rm d}y=\infty$, so that $\int_0^\infty r(y){\rm d}y=\infty$. We already know that the function $x\mapsto e^{-x}r(x)$ is nonincreasing on $\mathbb{R}^+$. Hence, r is locally bounded and a.e. continuous on $\mathbb{R}^+$. This implies that, for each $b>0$, the function $x\mapsto r(x)\,\mathbb{1}_{[0,\,b]}(x)$ is dRi. Now an application of the key renewal theorem yields

$$\liminf_{x\to\infty}\int_{[0,\,x]}r(x-y){\rm d}U(y)\geq\lim_{x\to\infty}\int_{(x-b,x]}r(x-y){\rm d}U(y)=\mu^{-1}\int_0^b r(y){\rm d}y.$$

Letting $b\to\infty$ we conclude that $\lim_{x\to\infty}\int_{[0,\,x]}r(x-y){\rm d}U(y)=\infty$. □

REMARK 6.9. The constant in (6.13) is given by

$$c_1=\lim_{x\to\infty}(\mathbb{E}_xD(S_\tau)-\mathbb{E}_x\sum_{k=0}^{\tau-1}G(S_k)).$$

Assume that Condition $\mathcal{S}$ holds and that the distribution of ξ is d-arithmetic for $d>0$. Then, according to Theorem 6.5, the limit $\lim_{x\to\infty}\mathbb{E}_xD(S_\tau)$ does not exist which implies that

(6.13) cannot hold. Under the additional assumption $\int_0^\infty yG(y)\mathrm{d}y<\infty$ a minor modification of the proof of Lemma 6.8, along the lines of the argument leading to (6.12), yields

$$\lim_{x\to\infty}\Big(\mathbb{E}_x\sum_{k=0}^{\tau-1}G(S_k)-c_{12}(x)\Big)=0$$

for a bounded d-periodic function $c_{12}(\cdot)$ which is not a constant. This in combination with (6.12) justifies (6.14) with $c_1(\cdot):=c_{11}(\cdot)-c_{12}(\cdot)$. Here, $c_{11}(\cdot)$ is the same as in (6.12).

LEMMA 6.10. *Assume that Condition $\mathcal{S}$ holds. Then* (2.3) *and* (2.4) *are sufficient for* $\int_0^\infty yG(y)\mathrm{d}y<\infty$.

Before giving a proof of Lemma 6.10 we need an auxiliary result.

LEMMA 6.11. *Let a,b and ε be real numbers satisfying $a>0$, $b\geq 0$, $c:=\log a-b/a\geq 0$ and $\varepsilon\in(0,1/e)$. The equation*

$$ay-b=\varepsilon e^y \tag{6.18}$$

has two solutions

$$y_1=d+b/a\quad\text{and}\quad y_2=-\log\varepsilon+\log a+\log\big(-\log\varepsilon+\log a-b/a\big)+o(1) \tag{6.19}$$

where $d\in(0,1)$ and the term $o(1)$ converges to 0 as εe^{-c} does so.

PROOF. Set $f(z):=ze^z$ for $z\in\mathbb{R}$. Changing the variable $z=-(y-b/a)$ transforms (6.18) into an equivalent form

$$f(z)=-\varepsilon':=-(\varepsilon/a)e^{b/a}=-\varepsilon e^{-c}, \tag{6.20}$$

where $-\varepsilon'\in(-1/e,0)$ by assumption. According to Section 4 in [13]), equation (6.20) has two solutions $z_1\in(-1,0)$ and $z_2=\log(-\varepsilon')-\log(-\log(-\varepsilon'))+o(1)$. Equivalently, equation (6.18) has two solutions given in (6.19). □

PROOF OF LEMMA 6.10. According to (6.11) there exist constants $\delta,M>0$ such that

$$\delta z(-\log z)\leq 1-\phi(z)\leq Mz(-\log z),\quad z\in(0,1/2). \tag{6.21}$$

Without loss of generality we assume that the positions of individuals in the first generation are ordered $X_i\leq X_{i+1}$ for all $i\in\mathbb{N}$, so that $X_1=X_{\min}$. Recall from Section 6.2 that G is a nonnegative function given by $G(y)=e^y\mathbb{E}H(y)$ for $y\in\mathbb{R}$, where

$$H(y)=\prod_{j=1}^N\phi(e^{-y-X_j})-1+\sum_{j=1}^N(1-\phi(e^{-y-X_j}))\geq 0,\quad y\in\mathbb{R}. \tag{6.22}$$

Note that, for each $y\in\mathbb{R}$, $H(y)\,\mathbb{1}_{\{N\leq 1\}}=0$ a.s. In view of this, in what follows we work on the event $\{N\geq 2\}$ without further notice.

For $y\geq 0$ and $\varepsilon\in(0,\min(\delta/e,1/2,1-\phi(1/e)))$ with the same δ as in (6.21), we introduce the set $A_y(\varepsilon):=\big\{\sum_{j=1}^N(1-\phi(e^{-y-X_j}))<\varepsilon\big\}$ and write

$$\int_0^\infty yG(y)\mathrm{d}y=\mathbb{E}\int_0^\infty ye^yH(y)\,\mathbb{1}_{A_y(\varepsilon)}\,\mathrm{d}y+\mathbb{E}\int_0^\infty ye^yH(y)\,\mathbb{1}_{(A_y(\varepsilon))^c}\,\mathrm{d}y=:I_1+I_2.$$

PROOF OF $I_1<\infty$. We shall show that I_1 is finite under Condition $\mathcal{S}$. Fix any $y\geq 0$. Note that $1-\phi(e^{-y-X_1})>\varepsilon$ provided that $y+X_1<1$, whence

$$A_y(\varepsilon)\subseteq A_y:=\{y+X_i\geq 1\text{ for }1\leq i\leq N\}.$$

We have, a.s. on A_y,

$$\sum_{j=1}^{N}(1-\phi(e^{-y-X_j}))\le M\sum_{j=1}^{N}e^{-y-X_j}(y+X_j) \le Me^{-y}(y(W_1^+ + W_1^-)+\widetilde{W}_1+\widetilde{W}_1^-) \tag{6.23}$$

and similarly

$$\sum_{j=1}^{N}(1-\phi(e^{-y-X_j}))\ge \delta e^{-y}(y(W_1^+ + W_1^-)+\widetilde{W}_1+\widetilde{W}_1^-), \tag{6.24}$$

where $\widetilde{W}_1^- := -\sum_{j=1}^{N}e^{-X_j}(X_j)_-$. Recall that the random variables W_1^+ and W_1^- were defined in (2.2). Observe that $yW_1^- + \widetilde{W}_1^- \ge 0$ a.s. on A_y, although $\widetilde{W}_1^- \le 0$ a.s.

On the event $A_y(\varepsilon)$, we have $\sum_{j=1}^{N}(1-\phi(e^{-y-X_j}))<\varepsilon<1/2$ a.s. which entails that $1-\phi(e^{-y-X_j})<1/2$ for $j=1,2,\dots,N$ a.s. In particular, using that for $z\in[0,1/2]$, the inequality $-\log(1-z)\le 2z$ holds, we obtain, a.s. on $A_y(\varepsilon)$,

$$-\log\phi(e^{-y-X_j})\le 2(1-\phi(e^{-y-X_j})),\quad j=1,2,\dots,N \tag{6.25}$$

and thereupon

$$\sum_{j=1}^{N}(-\log\phi(e^{-y-X_j}))\le 2\sum_{j=1}^{N}(1-\phi(e^{-x-X_j}))<1.$$

An appeal to the inequality $e^{-z}\le 1-z+z^2$ for $z\in[0,1]$ enables us to conclude that, a.s. on $A_y(\varepsilon)$,

$$\begin{aligned}\prod_{j=1}^{N}\phi(e^{-y-X_i}) &= \exp\Big(\sum_{j=1}^{N}\log\phi(e^{-y-X_i})\Big)\\ &\le 1+\sum_{j=1}^{N}\log\phi(e^{-y-X_i})+\Big(\sum_{j=1}^{N}\log\phi(e^{-y-X_i})\Big)^2\\ &\le 1-\sum_{j=1}^{N}(1-\phi(e^{-y-X_i}))+\Big(\sum_{j=1}^{N}(1-\phi(e^{-y-X_i}))\Big)^2.\end{aligned}$$

Combining (6.22) and (6.23) yields

$$H(y)\,\mathbb{1}_{A_y(\varepsilon)}\le M^2e^{-2y}\big(\widetilde{W}_1+(yW_1^-+\widetilde{W}_1^-)+yW_1^+\big)^2 \quad \text{a.s.} \tag{6.26}$$

Further, we decompose I_1 into three parts depending on which of the terms $\widetilde{W}_1$, $yW_1^-+\widetilde{W}_1^-$ or yW_1^+ dominates. In view of (6.24), $A_\varepsilon(y)$ is a subset of each of the three following sets $\{\widetilde{W}_1<\varepsilon_1 e^y\}$, $\{yW_1^-+\widetilde{W}_1^-<\varepsilon_1 e^y\}$ and $\{yW_1^+<\varepsilon_1 e^y\}$ for $\varepsilon_1:=\varepsilon/\delta$. Note that $\varepsilon_1<1/e$ by our choice of ε. The inequalities $y>-X_1$ a.s. on A_y and (6.26) together entail

$$I_1\le 9M^2(I_{1,1}+I_{1,2}+I_{1,3}),$$

where

$$I_{1,1}:=\mathbb{E}\widetilde{W}_1^2\int_0^\infty ye^{-y}\,\mathbb{1}_{\{\widetilde{W}_1<\varepsilon_1 e^y\}}\,\mathrm{d}y,$$

$$I_{1,2} := \mathbb{E}\int_{-X_1}^{\infty} ye^{-y}(yW_1^- + \widetilde{W}_1^-)^2\, \mathbb{1}_{A_y}\, \mathbb{1}_{\{yW_1^- + \widetilde{W}_1^- < \varepsilon_1 e^y\}}\, \mathrm{d}y,$$

$$I_{1,3} := \mathbb{E}(W_1^+)^2 \int_0^\infty y^3 e^{-y}\, \mathbb{1}_{\{yW_1^+ < \varepsilon_1 e^y\}}\, \mathrm{d}y.$$

As for $I_{1,1}$, write

$$I_{1,1} \le \mathbb{E}\widetilde{W}_1^2 \int_{\log_+(\widetilde{W}_1/\varepsilon_1)}^{\infty} ye^{-y}\mathrm{d}y = \mathbb{E}\widetilde{W}_1^2(\log_+(\widetilde{W}_1/\varepsilon_1)+1)e^{-\log_+(\widetilde{W}_1/\varepsilon_1)}$$
$$\le \varepsilon_1 \mathbb{E}\widetilde{W}_1(\log(\widetilde{W}_1/\varepsilon_1)+1)\, \mathbb{1}_{\{\widetilde{W}_1 \ge \varepsilon_1\}} + \varepsilon_1^2 < \infty.$$

Here, the finiteness is secured by (1.4) which is a part of Condition $\mathcal{S}$.

To deal with $I_{1,2}$, we intend to use Lemma 6.11 with $a = W_1^-$, $b = -\widetilde{W}_1^-$ and $\varepsilon = \varepsilon_1$. Since

$$-\frac{\widetilde{W}_1^-}{W_1^-} \le -X_1 \le \log W_1^- \tag{6.27}$$

and $\varepsilon_1 < 1/e$, the lemma applies and justifies the inclusion $\{y > 0 :\, \widetilde{W}_1^- + yW_1^- < \varepsilon_1 e^y\} \subseteq (0, Y_1) \cup (Y_2, \infty)$. Here, (random variables) Y_1 and Y_2 are solutions to the equation

$$\widetilde{W}_1^- + yW_1^- = \varepsilon_1 e^y \tag{6.28}$$

given by

$$\begin{aligned} Y_1 &= V - \widetilde{W}_1^-/W_1^-, \\ Y_2 &= -\log\varepsilon_1 + \log W_1^- + \log\big(-\log\varepsilon_1 + \log W_1^- + \widetilde{W}_1^-/W_1^-\big) + o(1), \end{aligned} \tag{6.29}$$

where V is a nonnegative random variable bounded by 1 a.s. In view of these observations we are going to consider the two integrals $I'_{1,2}$ and $I''_{1,2}$ with the integration sets being $(0, Y_1) \cap (-X_1, \infty)$ and (Y_2, ∞), respectively. Inequality (6.27) tells us that $Y_1 \le -X_1 + V$ a.s., so that $(0, Y_1) \cap (-X_1, \infty) \subseteq (-X_1, -X_1 + V)$ and thereupon

$$I'_{1,2} \le \mathbb{E}\int_{-X_1}^{-X_1+V} ye^{-y}(yW_1^- + \widetilde{W}_1^-)^2\, \mathbb{1}_{A_y}\, \mathbb{1}_{\{yW_1^- + \widetilde{W}_1^- < \varepsilon_1 e^y\}}\, \mathrm{d}y$$
$$\le \varepsilon_1^2 \mathbb{E}\int_{-X_1}^{-X_1+V} ye^{y}\mathrm{d}y \le \varepsilon_1^2\mathbb{E}(-X_1+1)e^{-X_1+1} \le \varepsilon_1^2 e(\mathbb{E}W_1^- \log_+ W_1^- + \mathbb{E}W_1^-) < \infty.$$

Here, the finiteness is guaranteed by (1.4). Further, recalling that Y_2 solves equation (6.28) and changing the variable we obtain

$$\begin{aligned} I''_{1,2} &\le \mathbb{E}\int_{Y_2}^{\infty} ye^{-y}(yW_1^- + \widetilde{W}_1^-)^2\mathrm{d}y \\ &= \mathbb{E}\int_0^\infty (y+Y_2)e^{-y}e^{-Y_2}(yW_1^- + Y_2W_1^- + \widetilde{W}_1^-)^2\mathrm{d}y \\ &= \mathbb{E}\int_0^\infty (y+Y_2)e^{-y}e^{-Y_2}(yW_1^- + \varepsilon_1 e^{Y_2})^2\mathrm{d}y \\ &\le C\mathbb{E}(1+Y_2)\big((W_1^-)^2e^{-Y_2} + W_1^- + e^{Y_2}\big). \end{aligned}$$

Here and hereafter, C denotes a constant whose value is of no importance and may change from line to line. Using the inequalities

$$1 + Y_2 \le C(1 + \log W_1^-), \quad e^{Y_2} \le CW_1^-(C + \log W_1^-), \quad e^{-Y_2} \le C/W_1^-$$

which follow from (6.29) we infer $I''_{1,2} \le C\mathbb{E}(1+\log W_1^-)^2 W_1^-$. Condition $\mathcal{S}$ ensures that the right-hand side is finite.

Finally, to check that $I_{1,3} < \infty$ we proceed in the same way as above. One needs to determine precisely the integration domain, that is, to solve the equation $yW_1^+ = \varepsilon_1 e^y$. Lemma 6.11 (with $a = W_1^+$ and $b = 0$) ensures the existence of two solutions to this equation: an a.s. bounded nonnegative random variable Y_1 and

$$Y_2 = -\log\varepsilon_1 + \log W_1^+ + \log\big(-\log\varepsilon_1 + \log W_1^+\big) + o(1). \tag{6.30}$$

We skip further details. The proof of $I_1 < \infty$ is complete.

PROOF OF $I_2 < \infty$. The function G is bounded on $[0,1]$. Therefore, it is sufficient to consider the integral I_2 over the set $[1,\infty)$. For $y \ge 1$, put $N^-(y) := \max\{j \le N : y + X_j < 1\}$ with the standard convention that $N^-(y) := 0$ if $y + X_1 \ge 1$. Plainly, $N^- := N^-(1)$ denotes the number of the first generation individuals located on the negative halfline, and, provided that $N^- \ge 1$, X_{N^-} denotes the position of the rightmost first generation individual located on the negative halfline.

We shall use the inequality which follows directly from (6.21) (compare (6.23)):

$$H(y) \le \sum_{j=1}^{N}(1-\phi(e^{-y-X_j})) \le Me^{-y}\big(\widetilde{W}_1 + yW_1^+ + F(y)\big) + N^-(y), \tag{6.31}$$

where $F(y) := \widetilde{W}_1(y) + yW_1(y)$,

$$\widetilde{W}_1(y) := \sum_{j=N^-(y)+1}^{N^-} e^{-X_j}X_j \quad \text{and} \quad W_1(y) := \sum_{j=N^-(y)+1}^{N^-} e^{-X_j}. \tag{6.32}$$

We note in passing that $F(y) = N^-(y) = 0$ a.s. on $\{N^- = 0\}$ and that, in general, $F(y) \ge 0$ a.s., but $\widetilde{W}_1(y) \le 0$ a.s. As we did before for I_1, we shall investigate the contribution of each term on the right-hand side of (6.31) separately. To this end, we use the easily checked inequality

$$(a_1 + \ldots + a_m)\,\mathbb{1}_{\{a_1+\ldots+a_m > \varrho\}} \le m(a_1\,\mathbb{1}_{\{a_1 > \varrho/m\}} + \ldots + a_m\,\mathbb{1}_{\{a_m > \varrho/m\}})$$

for $m \in \mathbb{N}$, nonnegative $a_1, \ldots, a_m$ and $\varrho > 0$, to obtain

$$I_2 \le 4M(I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4}).$$

Here, with $\varepsilon' := \varepsilon/(4M)$,

$$I_{2,1} := \mathbb{E}\widetilde{W}_1 \int_1^\infty y\,\mathbb{1}_{\{\widetilde{W}_1 > \varepsilon' e^y\}}\,\mathrm{d}y, \quad I_{2,2} := \mathbb{E}W_1^+ \int_1^\infty y^2\,\mathbb{1}_{\{yW_1^+ > \varepsilon' e^y\}}\,\mathrm{d}y,$$

$$I_{2,3} := \mathbb{E}\int_1^\infty ye^yN^-(y)\mathrm{d}y, \quad I_{2,4} := \mathbb{E}\int_1^\infty yF(y)\,\mathbb{1}_{\{F(y) > \varepsilon' e^y\}}\,\mathrm{d}y.$$

The analysis of $I_{2,1}$ is simple:

$$I_{2,1} \le \mathbb{E}\widetilde{W}_1 \int_0^{\log_+(\widetilde{W}_1/\varepsilon')} y\mathrm{d}y = (1/2)\mathbb{E}\widetilde{W}_1(\log_+(\widetilde{W}_1/\varepsilon'))^2 < \infty,$$

where the finiteness is a consequence of the second part of (2.3).

To treat $I_{2,2}$ we use the same Y_2 as in (6.30) which gives

$$I_{2,2} \le C + C\mathbb{E}\,\mathbb{1}_{\{W_1^+ > \varepsilon'\}}\,W_1^+ \int_0^{Y_2} y^2\mathrm{d}y \le C(1 + \mathbb{E}W_1^+(\log_+ W_1^+)^3) < \infty.$$

Here, the finiteness is justified by the first part of (2.3).

Next, we work with $I_{2,3}$. For notational simplicity, let $X_0 := -\infty$ and $X_{N^-+1} := 0$. Put $g(y) = (y-1)e^y$ for $y \in \mathbb{R}$ and note that $g'(y) = ye^y$. Since $N^-(y) = j$ for $y \in (-X_{j+1}, -X_j)$, we have

$$I_{2,3} = \mathbb{E}\sum_{j=0}^{N^-} j\int_{-X_{j+1}+1}^{-X_j+1} g'(y)\mathrm{d}y = \mathbb{E}\sum_{j=1}^{N^-} j(g(-X_j+1) - g(-X_{j+1}+1))$$

$$= \mathbb{E}\sum_{j=1}^{N^-} g(-X_j+1) \le e\mathbb{E}W_1^-\log_+ W_1^- < \infty.$$

The finiteness follows from (1.4).

It remains to prove that $I_{2,4} < \infty$. Put $B := \left\{\sum_{j=1}^{N^-} e^{-\Delta_j}(1+\Delta_j) \le C_0\right\}$, where $\Delta_j := X_j - X_1$ and C_0 is the same as in (2.4). Write

$$I_{2,4} \le \mathbb{E}\,\mathbb{1}_B\int_0^{\max(-X_1,1)} yF(y)\mathrm{d}y + \mathbb{E}\,\mathbb{1}_{B^c}\int_0^{\max(-X_1,1)} yF(y)\mathrm{d}y$$

$$+\,\mathbb{E}\,\mathbb{1}_B\int_{\max(-X_1,1)}^{\infty} yF(y)\,\mathbb{1}_{\{F(y)>\varepsilon' e^y\}}\,\mathrm{d}y + \mathbb{E}\,\mathbb{1}_{B^c}\int_{\max(-X_1,1)}^{\infty} F(y)\,\mathbb{1}_{\{F(y)>\varepsilon' e^y\}}\,\mathrm{d}y$$

$$=: J_1 + J_2 + J_3 + J_4.$$

For $y \ge 1$, we have, a.s. on $B \cap \{y < -X_1\}$,

$$F(y) = \sum_{j=1}^{N^-} e^{-X_j}(y+X_j)_+ \le e^{-X_1}\sum_{j=1}^{N_-} e^{-\Delta_j}\Delta_j \le C_0 e^{-X_1}$$

and thereupon

$$J_1 \le (C_0/2)\mathbb{E}(-X_1)^2 e^{-X_1} \le (C_0/2)\mathbb{E}W_1^-(\log W_1^-)^2 < \infty.$$

Here, the finiteness is ensured by (1.4). Next, using $(-X_1) \le W_1^-$ a.s. we infer

$$J_2 \le \mathbb{E}W_1^-\,\mathbb{1}_{B^c}\int_0^{-X_1} y^2\mathrm{d}y \le (1/3)\mathbb{E}W_1^-\,\mathbb{1}_{B^c}(-X_1)^3 \le (1/3)\mathbb{E}W_1^-(\log W_1^-)^3\,\mathbb{1}_{B^c} < \infty,$$

where the finiteness is a consequence of (2.3).

It holds, a.s. on B, that

$$e^{-y+X_1}F(y-X_1) = \sum_{j=1}^{N^-} e^{-y-\Delta_j}(y+\Delta_j)$$

$$= e^{-y}y\sum_{j=1}^{N^-} e^{-\Delta_j} + e^{-y}\sum_{j=1}^{N^-} e^{-\Delta_j}\Delta_j \le C_0(y+1)e^{-y}.$$

With this at hand, we obtain

$$J_3 \le (1/\varepsilon')\mathbb{E}\,\mathbb{1}_B\int_{\max(-X_1,1)}^{\infty} ye^{-y}(F(y))^2\mathrm{d}y$$

$$= (1/\varepsilon')\mathbb{E}\,\mathbb{1}_B\int_{\max(0,1+X_1)}^{\infty} (y-X_1)e^{-y+X_1}(F(y-X_1))^2\mathrm{d}y$$

$$\leq C_0^2\mathbb{E}e^{-X_1}\int_0^\infty (y-X_1)(y+1)^2e^{-y}\mathrm{d}y$$
$$\leq C\mathbb{E}(1+(-X_1)e^{-X_1}) \leq C\mathbb{E}(1+W_1^-\log_+ W_1^-) < \infty,$$

where the finiteness is secured by (1.4).

Finally, to deal with J_4 we denote by Y_2 the larger solution to the equation $yW_1^- = \varepsilon' e^y$. According to Lemma 6.11, $Y_2 = -\log\varepsilon' + \log W_1^- + \log(-\log\varepsilon' + \log W_1^-) + o(1)$ which entails

$$J_4 \leq \mathbb{E}\,\mathbb{1}_{B^c}\int_{-X_1}^\infty y^2 W_1^-\,\mathbb{1}_{\{yW_1^->\varepsilon' e^y\}}\,\mathrm{d}y$$
$$\leq C + C\mathbb{E}\,\mathbb{1}_{B^c}\,\mathbb{1}_{\{W_1^->\varepsilon'\}}\,W_1^-\int_{-X_1}^{Y_2} y^2\mathrm{d}y \leq C\mathbb{E}\,\mathbb{1}_{B^c}\,W_1^-(\log W_1^-)^3 < \infty.$$

The finiteness is ensured by (2.4). The proof of $I_2 < \infty$ is complete. □

LEMMA 6.12. *Assume that Condition $\mathcal{S}$ holds. Then $\int_0^\infty yG(y)\mathrm{d}y < \infty$ implies (2.3).*

PROOF. In view of (6.11) there exists $\delta_1 > 0$ such that

$$1-\phi(z) \geq \delta_1 z(-\log z)_+, \quad z \geq 0.$$

Hence, for $y \geq 0$,

$$\sum_{j=1}^N (1-\phi(e^{-y-X_j})) \geq \delta_1\sum_{j=1}^N e^{-y-X_j}(y+X_j)_+ \geq \delta_1 e^{-y}\big(\widetilde{W}_1 + yW_1^+\big). \tag{6.33}$$

For each $y > 0$, define the event $D_y := \{\delta_1 e^{-y}\big(\widetilde{W}_1 + yW_1^+\big) > 2\}$. If $D_y = \oslash$ for all $y > 0$, then both $\widetilde{W}_1$ and W_1^+ are a.s. bounded random variables which entails that (2.3) holds. Thus, in what follows we assume that $D_y \neq \oslash$ for some $y > 0$. For such y, we conclude with the help of (6.33) that, a.s. on D_y,

$$\prod_{j=1}^N \phi(e^{-y-X_j}) - 1 + \sum_{j=1}^N (1-\phi(e^{-y-X_j})) \geq -1 + \sum_{j=1}^N (1-\phi(e^{-y-X_j})) \geq (\delta_1/2)e^{-y}\big(\widetilde{W}_1 + yW_1^+\big). \tag{6.34}$$

This in combination with the inclusions $\{\widetilde{W}_1 > 2e^y/\delta_1\} \subseteq D_y$ and $\{yW_1^+ > 2e^y/\delta_1\} \subseteq D_y$ yields

$$\infty > \int_0^\infty yG(y)\mathrm{d}y \geq (\delta_1/2)\mathbb{E}\int_0^\infty y\big(\widetilde{W}_1 + yW_1^+\big)\,\mathbb{1}_{D_y}\,\mathrm{d}y$$
$$\geq (\delta_1/2)\Big(\mathbb{E}\widetilde{W}_1\int_0^\infty y\,\mathbb{1}_{\{\widetilde{W}_1>2e^y/\delta_1\}}\,\mathrm{d}y + \mathbb{E}W_1^+\int_1^\infty y^2\,\mathbb{1}_{\{W_1^+>2e^y/\delta_1\}}\,\mathrm{d}y\Big)$$
$$= (\delta_1/4)\mathbb{E}\widetilde{W}_1\big(\log_+(\delta_1\widetilde{W}_1/2)\big)^2 + (\delta_1/6)\big(\mathbb{E}W_1^+\big(\log((\delta_1 W_1^+/2)\vee e)\big)^3 - \mathbb{E}W_1^+\big).$$

This proves the necessity of (2.3). □

LEMMA 6.13. *Assume that Condition $\mathcal{S}$ and (2.5) hold. Then (2.4) is necessary for* $\int_0^\infty yG(y)dy<\infty$.

PROOF. We retain, for the most part, the notation from the proof of Lemma 6.10. Additionally, we put $B:=\big\{\sum_{j=1}^{N^-}e^{-\Delta_j}(1+\Delta_j)\leq 2e/\delta_1\big\}$ (with δ_1 as in (6.33)) and, for $y>0$, $D_y:=\{\delta_1 e^{-y}F(y)>2\}$. Assume that $D_y=\oslash$ for all $y>0$. Then taking $y=-X_1+1$ we conclude that $\mathbb{P}(B)=1$ which implies that (2.4) holds with any $C_0>2e/\delta_1$. Therefore, from now on we assume that $D_y\neq\oslash$ for some $y>0$. By the argument leading to (6.34), we have, a.s. on D_y, $H(y)\geq(\delta_1/2)e^{-y}F(y)$, whence

$$\infty>\int_0^\infty yG(y)\mathrm{d}y\geq(\delta_1/2)\mathbb{E}\int_0^\infty yF(y)\,\mathbb{1}_{D_y}\,\mathrm{d}y.$$

This particularly yields

$$I_1:=\mathbb{E}\int_0^{-X_1}yF(y)\,\mathbb{1}_{D_y}\,\mathrm{d}y<\infty\quad\text{and}\quad I_2:=\mathbb{E}\int_{-X_1+1}^\infty yF(y)\,\mathbb{1}_{D_y}\,\mathbb{1}_{B^c}\,\mathrm{d}y<\infty. \tag{6.35}$$

We first prove that $I_1<\infty$ entails

$$\mathbb{E}(-X_1)^3W_1^-<\infty. \tag{6.36}$$

Indeed, observe that

$$\begin{aligned}I_1^*:=\mathbb{E}\int_0^{-X_1}yF(y)\,\mathbb{1}_{D_y^c}\,\mathrm{d}y&=\mathbb{E}\int_0^{-X_1}yF(y)\,\mathbb{1}_{\{F(y)\leq(2/\delta_1)e^y\}}\,\mathrm{d}y\\&\leq(2/\delta_1)\mathbb{E}\int_0^{-X_1}ye^y\mathrm{d}y=(2/\delta_1)(\mathbb{E}(-X_1)e^{-X_1}-\mathbb{E}e^{-X_1}+1)<\infty\end{aligned}$$

as a consequence of (1.4). Summing up I_1 and I_1^* we obtain

$$\begin{aligned}\infty>\mathbb{E}\int_0^{-X_1}yF(y)\mathrm{d}y&=\mathbb{E}\int_0^{-X_1}y\sum_{j=1}^{N^-}e^{-X_j}(y+X_j)_+\mathrm{d}y\\&=\mathbb{E}\sum_{j=1}^{N^-}e^{-X_j}\int_{-X_j}^{-X_1}y(y+X_j)\mathrm{d}y\\&=\mathbb{E}\sum_{j=1}^{N^-}e^{-X_j}\Big[(1/3)((-X_1)^3-(-X_j)^3)+(1/2)((-X_1)^2-(-X_j)^2)X_j\Big]\\&=(1/6)\mathbb{E}\sum_{j=1}^{N^-}e^{-X_j}\Big[2(-X_1)^3+(-X_j)^3-3(-X_1)^2(-X_j)\Big]\\&=(1/6)\mathbb{E}\sum_{j=1}^{N^-}e^{-X_j}\Delta_j^2(2(-X_1)+(-X_j)),\end{aligned}$$

where we have used the identity $2a^3+b^3-3a^2b=(a-b)^2(2a+b)$, $a,b\in\mathbb{R}$ for the last equality. Hence,

$$\mathbb{E}\sum_{j=1}^{N^-}e^{-X_j}(-X_j)\Delta_j^2<\infty. \tag{6.37}$$

The inequality $a^3 \le 8b^3 + 4a(a-b)^2$ holds for any $a > b > 0$. Using it with $a = -X_1$ and $b = -X_i$ we infer

$$\mathbb{E}(-X_1)^3 \sum_{j=1}^{N^-} e^{-X_j} \le 8 \sum_{j=1}^{N^-} e^{-X_j}(-X_j)^3 + 4 \sum_{j=1}^{N^-} e^{-X_j}(-X_j)\Delta_j^2.$$

This reveals that (6.36) is a consequence of (2.5) and (6.37).

After these preparations we are ready to show the necessity of condition (2.4). To this end, we first observe that, a.s. on B^c, $y_0 W_1^- + \widetilde{W}_1^- > (2/\delta_1)e^{y_0}$, where $y_0 := -X_1 + 1$. This implies that $D_y \cap (-X_1+1, \infty) = (-X_1+1, Y_2)$ with Y_2 being the larger solution to the equation $yW_1^- + \widetilde{W}_1^- = (2/\delta_1)e^y$. Recall that the Y_2 is given by (6.29) with $2/\delta_1$ replacing ε_1. As a consequence, we obtain

$$\begin{aligned}\infty > I_2 &= \mathbb{E}\, \mathbb{1}_{B^c} \int_{-X_1+1}^{Y_2} y(yW_1^- + \widetilde{W}_1^-)\mathrm{d}y \\ &= (1/6)\mathbb{E}\, \mathbb{1}_{B^c} \Big[2W_1^-(Y_2^3 - (-X_1+1)^3) + 3\widetilde{W}_1^-(Y_2^2 - (-X_1+1)^2)\Big],\end{aligned}$$

and inequality (6.36) ensures that

$$\infty > J := \mathbb{E}\, \mathbb{1}_{B^c} \Big[2W_1^- Y_2^3 + 3\widetilde{W}_1^- Y_2^2\Big].$$

Put $A := \{\log W_1^- < 2(-X_1)\}$. We have, a.s. on $B^c \cap A$,

(6.38)
$$\big|2W_1^- Y_2^3 + 3\widetilde{W}_1^- Y_2^2\big| \le C\big(W_1^-(\log W_1^-)^3 + (-X_1)W_1^-(\log W_1^-)^2\big) \le CW_1^-(-X_1)^3,$$

whereas, a.s. on $B^c \cap A^c$,

(6.39)
$$\begin{aligned}2W_1^- Y_2^3 + 3\widetilde{W}_1^- Y_2^2 &\ge 2W_1^- Y_2^3 + 3X_1 W_1^- Y_2^2 \ge W_1^- Y_2^2(2Y_2 + 3X_1) \\ &\ge W_1^- Y_2^2(2\log W_1^- + 3X_1) \ge (1/2)W_1^-(\log W_1^-)^3.\end{aligned}$$

Combining (6.38) and (6.39) yields

$$\begin{aligned}\infty > 2J &\ge \mathbb{E}\, \mathbb{1}_{B^c} \mathbb{1}_{A^c} W_1^-(\log W_1^-)^3 - C\mathbb{E}W_1^-(-X_1)^3 \\ &= \mathbb{E}\, \mathbb{1}_{B^c} W_1^-(\log W_1^-)^3 - \mathbb{E}\, \mathbb{1}_{B^c} \mathbb{1}_A W_1^-(\log W_1^-)^3 - C\mathbb{E}W_1^-(-X_1)^3 \\ &\ge \mathbb{E}\, \mathbb{1}_{B^c} W_1^-(\log W_1^-)^3 - (C+8)\mathbb{E}W_1^-(-X_1)^3.\end{aligned}$$

Invoking (6.36) we conclude that condition (2.4) holds. □

Now we are ready to prove Theorem 6.6.

PROOF OF THEOREM 6.6. We first note that, by Lemma A.4 (b), the distribution of S_{τ_1} is nonarithmetic.

In view of Theorem 6.5 and Lemma 6.8, the first equality in (6.13) holds if, and only if, $\int_0^\infty yG(y)\mathrm{d}y < \infty$. Thus, the second equality in (6.13) holds if, and only if

(6.40)
$$\lim_{x\to\infty} (\mu U(x) - x) = c_3$$

for some finite constant c_3 and $\int_0^\infty yG(y)\mathrm{d}y < \infty$. By Lemma A.4 (d), relation (6.40) holds if, and only if, $\mathbb{E}\xi_-^3 < \infty$ (which is nothing else but (2.5)).

Now we conclude with the help of Lemmas 6.12 and 6.13 that the second equality in (6.13) also entails (2.3) and (2.4), hence Condition $\mathcal{S}^*$. Sufficiency of (2.3) and (2.4) for the first equality in (6.13) is justified by Lemma 6.10. □

7. Proofs related to the rate of convergence.

7.1. *Auxiliary results.* We start with a few auxiliary facts which can be lifted from the existing literature.

LEMMA 7.1. *Assume that Condition $\mathcal{S}$ holds. Then,*
(a) $n^{1/2}\sum_{|u|=n} e^{-S(u)} \overset{\mathbb{P}}{\to} (2/(\pi\sigma^2))^{1/2} Z$ *as* $n\to\infty$*;*
(b) $M_n^* := \inf_{|u|=n} S(u) - 2^{-1}\log n \overset{\mathbb{P}}{\to} +\infty$ *as* $n\to\infty$*;*
(c) for $\beta>1$ *and* $m\in\mathbb{N}$*,*

$$n^{\beta/2}\sum_{|u|=n} e^{-\beta S(u)}(S(u)-2^{-1}\log n)^m \overset{\mathbb{P}}{\to} 0, \quad n\to\infty.$$

PROOF. (a) This is Theorem 1.1 in [2].
(b) This follows from Theorem 1.1 in [33] which states that the sequence of distributions of $(M_n^* - \log n)_{n\in\mathbb{N}}$ is tight. Noting that the cited result considers maxima rather than minima we refer to Lemma A.1 in [34] for a proof of the fact that the assumptions imposed in [33] are equivalent to Condition $\mathcal{S}$.
(c) Let $\alpha>1$. The sequence of distributions of $(\sum_{|u|=n} e^{-\alpha(S(u)-(3/2)\log n)})_{n\in\mathbb{N}}$ is tight by Proposition 2.1 in [30]. This implies that

$$\sum_{|u|=n} e^{-\alpha(S(u)-2^{-1}\log n)} \overset{\mathbb{P}}{\to} 0, \quad n\to\infty. \tag{7.1}$$

Pick any $\varepsilon\in(0,\beta-1)$. Then, for all $x>0$, $e^{-\beta x}x^m \le \varepsilon^{-m} m! e^{-(\beta-\varepsilon)x}$. Using this we obtain, for any $\delta>0$,

$$\begin{aligned}\mathbb{P}\Big\{\Big|\sum_{|u|=n} e^{-\beta(S(u)-2^{-1}\log n)}(S(u)-2^{-1}\log n)^m\Big| > \delta\Big\}\\ \le \mathbb{P}\Big\{\sum_{|u|=n} e^{-\beta(S(u)-2^{-1}\log n)}(S(u)-2^{-1}\log n)^m > \delta, M_n^*>0\Big\} + \mathbb{P}\{M_n^*\le 0\}\\ \le \mathbb{P}\Big\{\sum_{|u|=n} e^{-(\beta-\varepsilon)(S(u)-2^{-1}\log n)} > \delta\varepsilon^m/m!\Big\} + \mathbb{P}\{M_n^*\le 0\}.\end{aligned}$$

Sending $n\to\infty$ we conclude that each summand on the right-hand side converges to 0 by (7.1) and part (b) of the lemma, respectively. □

7.2. *Proof of Theorem 2.4.* Put $H(x) := \mathbb{E}Z\,\mathbb{1}_{\{Z\le x\}}$ for $x\ge 0$ and let L^* be a random variable which is independent of $\mathcal{F}_\infty$ and has a 1-stable distribution with the generating triple $((1-\gamma)(2/(\pi\sigma^2))^{1/2}, (\pi/(2\sigma^2))^{1/2}, 1)$. Note that L has the same distribution as $L^* + (2/(\pi\sigma^2))^{1/2}c$.

Only assuming that Condition $\mathcal{S}_{\rm na}$ and (2.7) hold we shall prove more general results

$$\mathbb{E}\Big(f\Big(n^{1/2}\Big(Z-\sum_{|u|=n} e^{-S(u)}H(e^{S(u)}n^{-1/2})\Big)\Big)\Big|\mathcal{F}_n\Big) \overset{\mathbb{P}}{\to} \mathbb{E}(f(ZL^*)|\mathcal{F}_\infty), \quad n\to\infty \tag{7.2}$$

and

$$n^{1/2}\Big(Z-\sum_{|u|=n} e^{-S(u)}H(e^{S(u)}n^{-1/2})\Big) \overset{\rm d}{\to} ZL^*, \quad n\to\infty, \tag{7.3}$$

and then obtain (2.9) and (2.10) as corollaries. Our argument is based on the following representation

$$\Theta_n := n^{1/2}\Big(Z - \sum_{|u|=n} e^{-S(u)} H(e^{S(u)} n^{-1/2})\Big)$$
$$= n^{1/2} \sum_{|u|=n} e^{-S(u)}\big(Z(u) - H(e^{S(u)} n^{-1/2})\big), \quad n \in \mathbb{N} \quad \text{a.s.}$$

which follows from (6.3).

In view of Lemma 7.1, from any deterministic increasing sequence which diverges to ∞ we can extract a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \big(\inf_{|u|=n_k} S(u) - 2^{-1}\log n_k\big) = +\infty \quad \text{a.s.}; \tag{7.4}$$

$$\lim_{k\to\infty} n_k^{1/2} \sum_{|u|=n_k} e^{-S(u)} = (2/(\pi\sigma^2))^{1/2} Z \quad \text{a.s.}; \tag{7.5}$$

for $m = 1, 2$,

$$\lim_{k\to\infty} n_k \sum_{|u|=n_k} e^{-2S(u)} (S(u) - 2^{-1}\log n_k)^m = 0 \quad \text{a.s.} \tag{7.6}$$

For $n \in \mathbb{N}_0$ and the σ-algebra $\mathcal{F}_n$ defined in Section 1, we shall use the following notation $\mathbb{P}_n\{\cdot\} := \mathbb{P}\{\cdot|\mathcal{F}_n\}$ and, for a random variable θ, $\mathbb{E}_n\theta := \mathbb{E}(\theta|\mathcal{F}_n)$ and $\mathrm{Var}_n\theta := \mathrm{Var}\,(\theta|\mathcal{F}_n) = \mathbb{E}(\theta^2|\mathcal{F}_n) - (\mathbb{E}(\theta|\mathcal{F}_n))^2$.

Suppose we can check that the triangular array

$$(T_{u,k})_{|u|=n_k,\, k\in\mathbb{N}} := \big(n_k^{1/2} e^{-S(u)}\big(Z(u) - H(e^{S(u)} n_k^{-1/2})\big)\big)_{|u|=n_k,\, k\in\mathbb{N}}$$

is a null array, that is, for every $\delta > 0$,

$$\lim_{k\to\infty} \sup_{|u|=n_k} \mathbb{P}_{n_k}\big\{|T_{u,k}| > \delta\big\} = 0 \quad \text{a.s.}, \tag{7.7}$$

that, for every $x > 0$,

$$M(x) := -\lim_{k\to\infty} \sum_{|u|=n_k} \mathbb{P}_{n_k}\big\{T_{u,k} > x\big\} = -(2/(\pi\sigma^2))^{1/2} Z x^{-1} \quad \text{a.s.} \tag{7.8}$$

$$M(-x) := \lim_{k\to\infty} \sum_{|u|=n_k} \mathbb{P}_{n_k}\big\{T_{u,k} \le -x\big\} = 0 \quad \text{a.s.}; \tag{7.9}$$

$$\sigma^2 := \lim_{\varepsilon\to 0+} \lim_{k\to\infty} \sum_{|u|=n_k} \mathrm{Var}_{n_k}\big[T_{u,k}\, \mathbb{1}_{\{|T_{u,k}|\le\varepsilon\}}\big] = 0 \quad \text{a.s.} \tag{7.10}$$

and, for every $\tau > 0$,

$$a_0(\tau) := \lim_{k\to\infty} \sum_{|u|=n_k} \mathbb{E}_{n_k}\big[T_{u,k}\, \mathbb{1}_{\{|T_{u,k}|\le\tau\}}\big] = (2/(\pi\sigma^2))^{1/2} Z \log\tau \quad \text{a.s.} \tag{7.11}$$

Then, according to Theorem 1 on p. 116 in [21],

$$\lim_{k\to\infty} \mathbb{E}_{n_k}\big[\mathrm{i}t\Theta_{n_k}\big] = \exp\left(\mathrm{i}at - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right) \mathrm{d}M(x)\right) \tag{7.12}$$
$$= \exp\left((2/(\pi\sigma^2))^{1/2} Z \int_0^\infty \left(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right) x^{-2}\,\mathrm{d}x\right)$$
$$= \exp\big(Z(\mathrm{i}(1-\gamma)(2/(\pi\sigma^2))^{1/2} t - (\pi/(2\sigma^2))^{1/2} |t|(1 + \mathrm{i}\,\mathrm{sgn}\,(t)(2/\pi)\log|t|)\big) \quad \text{a.s.}$$

for $t\in\mathbb{R}$, where γ is the Euler-Mascheroni constant. Here,

$$a := a_0(\tau) - \int_{[-\tau,\tau]} \frac{x^3}{1+x^2}\,\mathrm{d}L(x) + \int_{\mathbb{R}\setminus[-\tau,\tau]} \frac{x}{1+x^2}\,\mathrm{d}L(x)$$
$$= (2/(\pi\sigma^2))^{1/2} Z\Big(\log\tau - \int_0^\tau \frac{x}{1+x^2}\,\mathrm{d}x + \int_\tau^\infty \frac{1}{x(1+x^2)}\,\mathrm{d}x\Big) = 0,$$

and the last equality in (7.12) follows from calculations given on p. 170 in [21]. However, the constant $1-\gamma$ is not given explicitly in [21] and rather represented as the integral

$$\Gamma := \int_0^\infty \Big(\frac{\sin x}{x^2} - \frac{1}{x(1+x^2)}\Big)\mathrm{d}x.$$

To evaluate it, write

$$\Gamma = \int_0^\infty \Big(\frac{\sin x}{x^2} - \frac{1}{x(1+x)}\Big)\mathrm{d}x + \int_0^\infty \Big(\frac{1}{x(1+x)} - \frac{1}{x(1+x^2)}\Big)\mathrm{d}x.$$

While the first integral is equal to $1-\gamma$ by formula (3.781.1) in [22], the second is equal to 0 which can be seen by direct calculation. Equivalently, we have shown that, for every bounded continuous function $f:\mathbb{R}\to\mathbb{R}$,

$$\lim_{k\to\infty} \mathbb{E}\Big(f\Big(n_k^{1/2}\Big(Z - \sum_{|u|=n_k} e^{-S(u)} H(e^{S(u)} n_k^{-1/2})\Big)\Big)\Big|\,\mathcal{F}_{n_k}\Big) = \mathbb{E}\big(f(ZL^*)|\,\mathcal{F}_\infty\big) \quad \text{a.s.}$$

which, by a standard argument, entails (7.2).

To obtain distributional convergence (7.3) just observe that (7.2) and the Lebesgue dominated convergence theorem guarantee

$$\lim_{n\to\infty} \mathbb{E} f\Big(n^{1/2}\Big(Z - \sum_{|u|=n} e^{-S(u)} H(e^{S(u)} n^{-1/2})\Big)\Big|\,\mathcal{F}_n\Big) = \mathbb{E} f(ZL^*)$$

which is equivalent to (7.3).

Thus, we are left with proving (7.7) through (7.11). As a preparation, denote by $F(x) := \mathbb{P}\{Z\le x\}$ for $x\in\mathbb{R}$, the distribution function of Z, and recall that Z is a nonnegative random variable, whence $F(x)=0$ for $x<0$. Condition (2.7) reads

$$\lim_{t\to\infty} t(1-F(t)) = 1. \tag{7.13}$$

Further, by Lemma A.1, relation (7.13) is equivalent to the following: for each $\lambda>0$,

$$\lim_{t\to\infty} (H(\lambda t) - H(t)) = \log\lambda \tag{7.14}$$

and implies that

$$H(t)\sim\log t, \quad t\to\infty \tag{7.15}$$

(alternatively, (7.15) also holds by Theorem 2.1).

For any $z\in\mathbb{R}$ and u with $|u|=n_k$, put

$$a(z,u,k) := z e^{S(u)} n_k^{-1/2} + H(e^{S(u)} n_k^{-1/2}),$$

so that $\{T_{u,k}>z\} = \{Z(u) > a(z,u,k)\}$. As a consequence of $\lim_{x\to\infty} x^{-1}H(x)=0$ (using (7.15)) and (7.4), the first term of $a(z,u,k)$ dominates which entails

$$\lim_{k\to\infty} \operatorname{sgn}(z) a(z,u,k) = +\infty \quad \text{a.s.}$$

for $z \neq 0$. Using (7.13) and independence of $Z(u)$ for u with $|u| = n_k$ and $\mathcal{F}_{n_k}$ we obtain, for $z > 0$,

$$\mathbb{P}_{n_k}\big\{T_{u,k} > z\big\} = 1 - F(a(z,u,k)) \sim a(z,u,k)^{-1} \sim z^{-1} n_k^{1/2} e^{-S(u)} \quad \text{a.s.} \tag{7.16}$$

as $k \to \infty$. By a similar reasoning, for $z > 0$, u with $|u| = n_k$ and large enough k,

$$F(a(-z,\varepsilon,k)) = 0 \quad \text{a.s.} \quad \text{and} \quad \mathbb{1}_{\{|T_{u,k}| \leq z\}} = \mathbb{1}_{\{T_{u,k} \leq z\}} \quad \text{a.s.}$$

We shall repeatedly use these observations, without further notice.

PROOF OF (7.7). In view of (7.16), for each $\delta > 0$,

$$\begin{aligned}\sup_{|u|=n_k} \mathbb{P}_{n_k}\big\{|T_{u,k}| > \delta\big\} &\sim \sup_{|u|=n_k} \delta^{-1} e^{-S(u)} n_k^{1/2} \\ &\sim \delta^{-1} \exp(-\inf_{|u|=n_k}(S(u) - 2^{-1}\log n_k)) \quad \text{a.s.}\end{aligned}$$

as $k \to \infty$. By (7.4), the right-hand side converges to 0 a.s. as $k \to \infty$ which proves (7.7).

PROOFS OF (7.8) AND (7.9). By another appeal to (7.16), for any $x > 0$,

$$\sum_{|u|=n_k} \mathbb{P}_{n_k}\{T_{u,k} > x\} \sim x^{-1} n_k^{1/2} \sum_{|u|=n_k} e^{-S(u)} \to (2/(\pi\sigma^2))^{1/2} Z x^{-1} \quad \text{a.s.}$$

as $k \to \infty$ which proves (7.8). Here, the limit relation is a consequence of (7.5). The proof of (7.9) is easy: for large enough k,

$$\sum_{|u|=n_k} \mathbb{P}_{n_k}\big\{T_{u,k} \leq -x\big\} = \sum_{|u|=n_k} F(a(-x,u,k)) = 0 \quad \text{a.s.}$$

PROOF OF (7.10). First, note that according to Theorem 1.6.4 in [8], relation (7.13) entails

$$H_2(t) := \mathbb{E} Z^2 \mathbb{1}_{\{Z \leq t\}} \sim t, \quad t \to \infty. \tag{7.17}$$

For $\varepsilon > 0$ and large enough k,

$$\begin{aligned}&\sum_{|u|=n_k} \mathrm{Var}_{n_k}\big[T_{u,k} \mathbb{1}_{\{|T_{u,k}| \leq \varepsilon\}}\big] \\ &\leq \sum_{|u|=n_k} \mathbb{E}_{n_k}\big[n_k e^{-2S(u)} (Z(u) - H(e^{S(u)} n_k^{-1/2}))^2 \mathbb{1}_{\{Z(u) \leq a(\varepsilon,u,k)\}}\big] \\ &\leq n_k \sum_{|u|=n} e^{-2S(u)} \Big(H_2\big(a(\varepsilon,u,k)\big) - 2H\big(e^{S(u)} n_k^{-1/2}\big) H\big(a(\varepsilon,u,k)\big) + H^2(e^{S(u)} n_k^{-1/2})\Big) \\ &=: I_1(n_k) - 2I_2(n_k) + I_3(n_k).\end{aligned}$$

For the first inequality we have used the fact that the conditional variance does not exceed the conditional second moment. Further, we investigate each term $I_j(n_k)$, $j = 1,2,3$ separately. By (7.17), (7.4) and (7.15), as $k \to \infty$,

$$I_1(n_k) \sim n_k \sum_{|u|=n_k} e^{-2S(u)} a(\varepsilon,u,k) \sim \varepsilon n_k^{1/2} \sum_{|u|=n_k} e^{-S(u)} \quad \text{a.s.}$$

According to (7.5), the last expression converges to $\varepsilon(2/(\pi\sigma^2))^{1/2} Z$ a.s. as $k \to \infty$ which, in its turn, converges to 0 a.s. as $\varepsilon \to 0+$. By (7.4) and (7.15), as $k \to \infty$,

$$I_2(n_k) \sim n_k \sum_{|u|=n} e^{-2S(u)} (S(u) - 2^{-1}\log n_k)(\log\varepsilon + S(u) - 2^{-1}\log n_k) \quad \text{a.s.}$$

In view of (7.6), this and $I_3(n_k)$ converge to 0 a.s. as $k\to\infty$. The proof of (7.10) is complete.
PROOF OF (7.11). For each $\tau>0$ and large k,

$$\sum_{|u|=n_k}\mathbb{E}_{n_k}\big[T_{u,k}\,\mathbb{1}_{|T_{u,k}|\leq\tau\}}\big]=n_k^{1/2}\sum_{|u|=n_k}e^{-S(u)}\big(H(a(\tau,u,k))-H(e^{S(u)}n_k^{-1/2})\big)$$
$$+n_k^{1/2}\sum_{|u|=n_k}e^{-S(u)}H(e^{S(u)}n_k^{-1/2})\big(1-F(a(\tau,u,k))\big):=J_1(n_k)+J_2(n_k).$$

Arguing as in the proof of (7.8) we conclude that, as $k\to\infty$,

$$J_2(n_k)\ \sim\ \tau^{-1}n_k\sum_{|u|=n_k}e^{-2S(u)}H(e^{S(u)}n_k^{-1/2})$$
$$\sim\ \tau^{-1}n_k\sum_{|u|=n_k}e^{-2S(u)}(S(u)-2^{-1}\log n_k)\quad\text{a.s.}$$

The second equivalence is a consequence of (7.15). In view of (7.6), $\lim_{k\to\infty}J_2(n_k)=0$ a.s. Passing to the analysis of $J_1(n_k)$ we first note that, for each $\tau>0$ and u with $|u|=n_k$,

$$\lim_{k\to\infty}\big(H\big(\tau e^{S(u)}n_k^{-1/2}+H(e^{S(u)}n_k^{-1/2})\big)-H\big(e^{S(u)}n_k^{-1/2}\big)\big)=\log\tau\quad\text{a.s.}\tag{7.18}$$

Indeed,

$$H\big(\tau e^{S(u)}n_k^{-1/2}+H(e^{S(u)}n_k^{-1/2})\big)-H\big(e^{S(u)}n_k^{-1/2}\big)\geq H\big(\tau e^{S(u)}n_k^{-1/2}\big)-H\big(e^{S(u)}n_k^{-1/2}\big),$$

and recalling (7.4), the right-hand side converges to $\log\tau$ a.s. as $k\to\infty$ by (7.14). In the converse direction, observe that, in view of (7.15), $\lim_{t\to\infty}t^{-1}H(t)=0$. This in combination with (7.4) ensures that given $\delta>0$ we have, for large enough k, that

$$H\big(\tau e^{S(u)}n_k^{-1/2}+H(e^{S(u)}n_k^{-1/2})\big)-H\big(e^{S(u)}n_k^{-1/2}\big)$$
$$\leq H\big((\tau+\delta)e^{S(u)}n_k^{-1/2}\big)-H\big(e^{S(u)}n_k^{-1/2}\big).$$

Using (7.14) and sending first $k\to\infty$ and then $\delta\to0+$ we conclude that the limit superior in (7.18) does not exceed $\log\tau$. This completes the proof of (7.18). Invoking now (7.18) and (7.4) yields, as $k\to\infty$

$$J_1(n_k)\ \sim\ (\log\tau)n_k^{1/2}\sum_{|u|=n_k}e^{-S(u)}\quad\text{a.s.}$$

Hence, by (7.5), $\lim_{k\to\infty}J_1(n_k)=(\log\tau)(2/(\pi\sigma^2))^{1/2}Z$ a.s. The proof of (7.2) and (7.3) is complete.

Assume now that Conditions $\mathcal{S}_{\rm na}$ and $\mathcal{S}^*$ hold. Put $\tilde H(x):=H(x)-\log x$ for $x>0$. By Theorem 2.2, $\lim_{x\to\infty}\tilde H(x)=c$. This in combination with Lemma 7.1 (a,b) ensures that

$$n^{1/2}\sum_{|u|=n}e^{-S(u)}\tilde H(e^{S(u)}n^{-1/2})\ \overset{\mathbb{P}}{\to}\ (2/(\pi\sigma^2))^{1/2}cZ,\quad n\to\infty.$$

A minor modification of the proof of (7.11) which takes into account the last limit relation justifies (2.9) and (2.10).

APPENDIX A

A.1. A link between a distribution tail and the Laplace transform. In this section we give two results which connect the asymptotic behavior of a distribution tail at ∞ with that of the corresponding Laplace-Stieltjes transform at 0.

LEMMA A.1. *Let $b > 0$ and X be a nonnegative random variable with Laplace transform $\phi^*(s) := \mathbb{E}e^{-sX}$ for $s \geq 0$. For $s > 0$, set $\psi^*(s) := s^{-1}(1 - \varphi^*(s))$,*

$$G^*(s) := \int_0^s \mathbb{P}\{X > y\}\mathrm{d}y \quad \textit{and} \quad H^*(s) := \mathbb{E}X\,\mathbb{1}_{\{X \leq s\}}\,.$$

The following assertions are equivalent:
(i) $\lim_{t\to\infty} t\mathbb{P}\{X > t\} = b$*;*
(ii) for each $\lambda > 0$, $\lim_{s\to 0+}(\psi^*(s/\lambda) - \psi^*(s)) = b\log\lambda$*;*
(iii) for each $\lambda > 0$, $\lim_{t\to\infty}(G^*(\lambda t) - G^*(t)) = b\log\lambda$*;*
(iv) for each $\lambda > 0$, $\lim_{t\to\infty}(H^*(\lambda t) - H^*(t)) = b\log\lambda$*.*
Either of these entails

$$\psi^*(1/t) \;\sim\; G^*(t) \;\sim\; H^*(t) \;\sim\; b\log t, \quad t \to \infty. \tag{A.1}$$

REMARK A.2. Recall that functions ψ^*, G^* and H^* satisfying the assumptions (ii), (iii) and (iv) of Lemma A.1 belong to the de Haan class. In particular, these functions are slowly varying (ψ^* at zero, G^* and H^* at ∞). Relation (A.1) makes the last statement even more precise, showing that all these functions are asymptotically equivalent to the logarithm.

PROOF. The equivalence (i) $\Leftrightarrow$ (iii) follows from Theorem 3.6.8 in [8]. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 3.9.1 in [8] after noting that

$$\psi^*(s) = \int_{[0,\,\infty)} e^{-sy}\mathrm{d}G^*(y), \quad s > 0. \tag{A.2}$$

PROOF OF (iii) $\Rightarrow$ (iv). Integration by parts yields

$$H^*(t) = \int_{[0,\,t]} y\mathrm{d}\mathbb{P}\{X \leq y\} = \int_0^t \mathbb{P}\{X > y\}\mathrm{d}y - t\mathbb{P}\{X > t\} = G^*(t) - t\mathbb{P}\{X > t\}. \tag{A.3}$$

According to the equivalence (i) $\Leftrightarrow$ (iii), $\lim_{t\to\infty} t\mathbb{P}\{X > t\} = b$. Hence, invoking (iii) we arrive at (iv).
PROOF OF (iv) $\Rightarrow$ (i). Write, for any $\delta > 1$,

$$\begin{aligned} t\mathbb{P}\{X > t\} = t\sum_{n\geq 1}\mathbb{P}\{t\delta^{n-1} < X \leq t\delta^n\} &\geq \sum_{n\geq 1}\delta^{-n}\int_{(t\delta^{n-1},\,t\delta^n]} y\mathrm{d}\mathbb{P}\{X \leq y\} \\ &= \sum_{n\geq 1}\delta^{-n}(H^*(t\delta^n) - H^*(t\delta^{n-1})). \end{aligned}$$

Relation (iv) entails that given $\varepsilon > 0$

$$H^*(t\delta^n) - H^*(t\delta^{n-1}) \geq b\log\delta - \varepsilon$$

for large enough t, whence, for such t,

$$t\mathbb{P}\{X > t\} \geq (b\log\delta - \varepsilon))\sum_{n\geq 1}\delta^{-n} = (\delta - 1)^{-1}(b\log\delta - \varepsilon).$$

Sending first $\varepsilon \to 0+$ and then $\delta \to 1-$ we obtain $\liminf_{t\to\infty} t\mathbb{P}\{X > t\} \geq b$. A symmetric argument proves the converse inequality for the limit superior.

Further, it is trivial that (i) entails $G^*(t) \sim b\log t$ as $t \to \infty$. With this at hand, $H^*(t) \sim b\log t$ as $t \to \infty$ is a consequence of (A.3) and (i). Finally, $\psi^*(1/t) \sim G^*(t)$ as $t \to \infty$ follows from (ii) (or (iii)) and Theorem 3.9.1 in [8]. □

LEMMA A.3. *Let $b > 0$, $c \in \mathbb{R}$ and X be a nonnegative random variable with Laplace transform ϕ^*. The following assertions are equivalent:*
(I) $\psi^*(s) = s^{-1}(1-\phi^*(s)) = -b\log s - \gamma + c + o(1)$ as $s \to 0+$, where γ is the Euler-Mascheroni constant;
(II) $G^*(t) = \int_0^t \mathbb{P}\{X > y\}\mathrm{d}y = b\log t + c + o(1)$ as $t \to \infty$;
(III) $H^*(t) = \mathbb{E}X\,\mathbb{1}_{\{X\leq t\}} = b\log t + c - b + o(1)$ as $t \to \infty$.

PROOF. PROOF OF (I) ⇔ (II). Let $\lambda > 0$. Condition (I) ensures that $\lim_{s\to 0+}(\psi^*(s/\lambda) - \psi^*(s)) = b\log\lambda$. Recalling (A.2) and invoking Theorem 3.9.1 in [8] we infer $G^*(t) = \psi^*(1/t) + \gamma + o(1)$ as $t \to \infty$. This in conjunction with (I) proves (II). In the converse direction, condition (II) guarantees that $\lim_{t\to\infty}(G^*(\lambda t) - G^*(t)) = b\log\lambda$. Another appeal to Theorem 3.9.1 in [8] allows us to conclude that $\psi^*(s) = G^*(1/s) - \gamma + o(1)$ as $s \to 0+$, whence (I).
PROOF OF (II) ⇒ (III). As a consequence of (II), for each $\lambda > 0$, $\lim_{t\to\infty}(G^*(\lambda t) - G^*(t)) = b\log\lambda$. Hence, $\lim_{t\to\infty} t\mathbb{P}\{X > t\} = b$ by the implication (iii) ⇒ (i) of Lemma A.1. With this, (A.3) and (II) at hand we obtain

$$H^*(t) = G^*(t) - t\mathbb{P}\{X > t\} = b\log t + c - b + o(1), \quad t \to \infty.$$

PROOF OF (III) ⇒ (II). Relation (III) implies that, for each $\lambda > 0$, $\lim_{t\to\infty}(H^*(\lambda t) - H^*(t)) = b\log\lambda$. Hence, by the implication (iv) ⇒ (i) of Lemma A.1, $\lim_{t\to\infty} t\mathbb{P}\{X > t\} = b$. Now (III) together with (A.3) ensures (II). □

A.2. Results on standard random walks. Here are some general results on the renewal functions associated to the ascending or descending ladder height processes of a centered random walk with finite variance.

LEMMA A.4. *Let $(T_n)_{n\in\mathbb{N}_0}$ be a standard random walk with $T_0 = 0$, $\mathbb{E}T_1 = 0$ and $\mathbb{E}T_1^2 \in (0,\infty)$. Further, let τ'_- and τ'_+ denote a strictly or weakly descending and a strictly or weakly ascending ladder epoch for $(T_n)_{n\in\mathbb{N}_0}$ and $\bar{U}$ the renewal function for the standard random walk with jumps having the same distribution as $|T_{\tau'_-}|$ or $T_{\tau'_+}$. Then*
(a) $\mathbb{E}|T_{\tau'_\pm}| < \infty$; for $\beta > 2$, $\mathbb{E}(T_1)_-^\beta < \infty$ is equivalent to $\mathbb{E}|T_{\tau'_-}|^{\beta-1} < \infty$ and $\mathbb{E}(T_1)_+^\beta < \infty$ is equivalent to $\mathbb{E}T_{\tau'_+}^{\beta-1} < \infty$;
(b) $\lim_{x\to\infty} x^{-1}\bar{U}(x) = (\mathbb{E}|T_{\tau'_\pm}|)^{-1}$.
(c) If the distribution of T_1 is nonarithmetic/d-arithmetic for $d > 0$, then so is the distribution of $T_{\tau'_\pm}$.
(d) Assume that the distribution of T_1 is nonarithmetic. Then

$$\lim_{x\to\infty}((\mathbb{E}|T_{\tau'_\pm}|)\bar{U}(x) - x) = c$$

for a finite constant c if, and only if, $\mathbb{E}(T_1)_\pm^3 < \infty$. If it is the case, then $c = (2\mathbb{E}|T_{\tau'_\pm}|)^{-1}\mathbb{E}T_{\tau'_\pm}^2$.

PROOF. Part (a) is formula (4a) and Corollary 1 in [16]. Part (b) is the elementary renewal theorem. For part (c), see, for instance, p. 2156 in [5]. For part (d), first observe that $\mathbb{E}(T_1)_\pm^3 < \infty$ is equivalent to $\mathbb{E}(T_{\tau'_\pm})^2 < \infty$. Now the result can be derived directly from the Blackwell theorem. Alternatively, while sufficiency of $\mathbb{E}(T_{\tau'_\pm})^2 < \infty$ follows from Example 3.10.3 on p. 242 in [39], necessity of that condition can be obtained along the lines of the aforementioned example with the help of Theorem 4 in [40]. □

A.3. Results on Lebesgue integrable and directly Riemann integrable functions. A function $t:\mathbb{R}^+\to\mathbb{R}^+$ is called *directly Riemann integrable* (dRi) on $\mathbb{R}^+$, if
(a) $\overline{\sigma}(h)<\infty$ for each $h>0$ and
(b) $\lim_{h\to 0+}\big(\overline{\sigma}(h)-\underline{\sigma}(h)\big)=0$, where

$$\overline{\sigma}(h):=h\sum_{n\geq 1}\sup_{(n-1)h\leq y<nh}t(y)\quad\text{and}\quad \underline{\sigma}(h):=h\sum_{n\geq 1}\inf_{(n-1)h\leq y<nh}t(y).$$

If t is dRi, then $\lim_{h\to 0+}\overline{\sigma}(h)=\int_0^\infty t(y)\mathrm{d}y<\infty$, where the integral is an improper Riemann integral.

Lemma A.5 is concerned with an important step in the proof of Theorem 6.5.

LEMMA A.5. *Assume that $\underline{\sigma}(h_0)<\infty$ for some $h_0>0$ and that, for some $a\geq 0$, $x\mapsto e^{-ax}t(x)$ is a nonincreasing function on $\mathbb{R}^+$. Then t is dRi on $\mathbb{R}^+$.*

REMARK A.6. Lemma A.5 is a strengthening of the well-known fact (see, for instance, Corollary 2.17 in [19]) that t is dRi provided that t is Lebesgue integrable and $x\mapsto e^{-ax}t(x)$ is a nonincreasing function. In Lemma A.5 we require less, namely that $\underline{\sigma}(h_0)<\infty$ for some $h_0>0$ which is of course true if t is Lebesgue integrable.

PROOF. Using twice the assumed monotonicity we obtain

$$\infty>e^{2ah_0}\underline{\sigma}(h_0)=e^{2ah_0}h_0\sum_{n\geq 1}\inf_{(n-1)h_0\leq y<nh_0}\big(e^{ay}e^{-ay}t(y)\big)$$
$$\geq e^{ah_0}h_0\sum_{n\geq 1}t(nh_0)\geq\overline{\sigma}(h_0)-h_0\sup_{0\leq y<h_0}t(y).$$

This shows that $\overline{\sigma}(h_0)<\infty$. Remark 2.9 in [44] states that, for $h>0$,

$$\overline{\sigma}(h)\leq(1+2h/h_0)\overline{\sigma}(h_0)$$

which implies that $\overline{\sigma}(h)<\infty$ for each $h>0$. Hence, also $\underline{\sigma}(h)<\infty$ for each $h>0$ because $\underline{\sigma}(h)\leq\overline{\sigma}(h)$. Repeating now, for each $h>0$, the argument based on monotonicity we conclude that, for each $h>0$,

$$e^{2ah}\underline{\sigma}(h)\geq\overline{\sigma}(h)-h\sup_{0\leq y<h}t(y).\tag{A.4}$$

In view of

$$\underline{\sigma}(h)\leq I:=\int_0^\infty t(y)\mathrm{d}y\leq\overline{\sigma}(h)<\infty,\quad h>0,$$

we conclude that t is Lebesgue integrable and that $\limsup_{h\to 0+}\underline{\sigma}(h)\leq I$, whence

$$\lim_{h\to 0+}\underline{\sigma}(h)(e^{2ah}-1)=0.$$

Noting that $\lim_{h\to 0+}h\sup_{0\leq y<h}t(y)=0$, we obtain $\limsup_{h\to 0+}(\overline{\sigma}(h)-\underline{\sigma}(h))\leq 0$ by an appeal to (A.4), which completes the proof. □

Lemma A.7 is needed to justify statements made in Remark 5.5.

LEMMA A.7. *Let $t:\mathbb{R}^+\to\mathbb{R}^+$ and V^* be the right-continuous renewal function of a standard random walk $(S_n^*)_{n\in\mathbb{N}_0}$ with nonnegative jumps of finite mean μ^* which have a nonarithmetic distribution.*
(a) There exist improperly Riemann integrable t and the renewal functions V^* such that

$$\int_{[0,\infty)} t(x+y)\mathrm{d}V^*(y)<\infty \tag{A.5}$$

fails to hold for each $x\geq 0$.
(b) If t is dRi on $\mathbb{R}^+$, then (A.5) holds for each $x\geq 0$.
(c) There exist continuous t and the renewal functions V^* such that (A.5) holds for some $x\geq 0$, yet $\int_0^\infty t(y)\mathrm{d}y=\infty$.
(d) Assume that

$$\sup_{x\geq 0}\int_{[0,\infty)} t(x+y)\mathrm{d}V^*(y)<\infty. \tag{A.6}$$

Then $\int_0^\infty t(y)\mathrm{d}y<\infty$.

PROOF. (a) We only consider the case $x=0$. A modification needed to treat the case $x>0$ is obvious. We use the same t and V^* as in Example 3.10.2 on p. 233 of [39] designed to demonstrate that the key renewal theorem can fail for integrands which are not dRi.

Let a random variable S_1^* take two values α and $1-\alpha$ for some *irrational* $\alpha\in(0,1)$. Then the distribution of S_1^* is nonlattice, and the renewal function V^* is piecewise constant with jumps at the points of the form $k_1\alpha+k_2(1-\alpha)$, $k_1,k_2\in\mathbb{N}_0$, $k_1+k_2>0$. Arrange these points in increasing order and denote the resulting configuration by $b_1<b_2<\ldots$ Consider an infinite sequence of isosceles triangles which do not overlap. They are located in $\mathbb{R}^+\times\mathbb{R}^+$ and have bases situated on the x-axis. The triangles are enumerated $1,2,\ldots$ from left to right. The base of the nth triangle is centered at b_n and has length s_n; the height of the nth triangle is equal to 1. The sequence $(s_n)_{n\in\mathbb{N}}$ is assumed to satisfy $\sum_{n\geq 1}s_n<\infty$. Define the function t as follows: while $t(x)=0$ for x which do not belong to the bases of the triangles, its graph passes through the equal sides of the triangles for all the other x. Plainly, $\int_0^\infty t(x)\mathrm{d}x=2^{-1}\sum_{n\geq 1}s_n<\infty$, that is, the area of the region between the graph $y=t(x)$ and the x-axis is equal to the sum of the areas of all the triangles. Thus, f is improperly Riemann integrable on $\mathbb{R}^+$. Finally, since $t(b_n)=1$ for $n\in\mathbb{N}$,

$$\int_{[0,\infty)} t(x)\mathrm{d}V^*(x)=\sum_{n\geq 1}t(b_n)\sum_{k\geq 1}\mathbb{P}\{S_k^*=b_n\}=\sum_{n\geq 1}\sum_{k\geq 1}\mathbb{P}\{S_k^*=b_n\}=\infty.$$

(b) This follows from

$$\int_{[0,\infty)} t(x+y)\mathrm{d}V^*(y)\leq V^*(1)\sum_{n\geq\lfloor x\rfloor+1}\sup_{n-1\leq y<n}t(y)<\infty,\quad x\geq 0$$

which is just (5.14) with t and V^* replacing g and V, respectively.
(c) We use the same V^* as in part (a). To construct t, consider an infinite sequence of isosceles triangles enumerated $1,2,\ldots$ from left to right. They are located in $\mathbb{R}^+\times\mathbb{R}^+$, have heights 1, and the endpoints of the base of the nth triangle are b_n and b_{n+1}. Now we put $t(x)=0$ for $x<b_1$ and require that the graph of $y=t(x)$ passes through the equal sides of the triangles for all the other positive x. Then $\int_0^\infty t(x)\mathrm{d}x=(1/2)\lim_{n\to\infty}(b_n-b_1)=\infty$. On the other hand, $\int_{[0,\infty)}t(y)\mathrm{d}V^*(y)=0$, so that (A.5) holds with $x=0$.

(d) Let ξ_0^* be a random variable independent of $(S_n^*)_{n\geq 0}$ with distribution function $\mathbb{P}\{\xi_0^* \leq x\} = (1/\mu^*)\int_0^x \mathbb{P}\{S_1^* > y\}\mathrm{d}y$. On the one hand,

$$\mathbb{E}\sum_{n\geq 0} t(\xi_0^* + S_n^*) = \int_{[0,\infty)} \mathbb{E}\sum_{n\geq 0} t(x + S_n^*)\mathrm{d}\mathbb{P}\{\xi_0^* \leq x\}$$
$$= \int_{[0,\infty)}\int_{[0,\infty)} t(x+y)\mathrm{d}V^*(y)\mathrm{d}\mathbb{P}\{\xi_0^* \leq x\} < \infty,$$

where the finiteness is secured by (A.6). On the other hand, the random process $(\tilde{N}^*(x))_{x\geq 0}$ defined by

$$\tilde{N}^*(x) := \#\{n \in \mathbb{N}_0 : \xi_0^* + S_n^* \leq x\}, \quad x \geq 0$$

is a stationary renewal process (the term is standard but misleading; actually the process has stationary increments) which particularly implies that $\tilde{N}^*(x) = x/\mu^*$ for $x \geq 0$. Hence, $\int_0^\infty t(y)\mathrm{d}y = \mathbb{E}\sum_{n\geq 0} t(\xi_0^* + S_n^*) < \infty$. □

Acknowledgements. The authors thank the organizers and participants of the conference 'Branching in Innsbruck' (September 2019). In particular, the talk '1-stable fluctuations of branching Brownian motion at critical temperature' delivered by Pascal Maillard during that conference has given a major impetus to the research reported here. Also, D.B. and A.I. thank Alexander Marynych for a useful discussion.

This project was initiated while D.B. and A.I. were visiting LAGA - Institut Galilée, Université Sorbonne Paris Nord in October 2019. Both gratefully acknowledge hospitality and the financial support.

D.B. was partially supported by the National Science Center, Poland (Sonata Bis, grant number DEC-2014/14/E/ST1/00588).

A major part of A.I.'s work was done under the support of Ulam programme funded by the Polish national agency for academic exchange (NAWA), project no. PPN/ULM/2019/1/00010/DEC/1.

B.M. acknowledges financial support from the PEPS JCJC CNRS grant 2019, as well as from the ANR MALIN (ANR-16-CE93-0003).

Finally, the authors warmly thank the Associate Editor and the two anonymous referees for several helpful comments and pointers to additional relevant literature.

REFERENCES

[1] E. Aïdékon, *Convergence in law of the minimum of a branching random walk*. Ann. Probab. **41** (2013), 1362–1426.

[2] E. Aidekon and Z. Shi, *The Seneta-Heyde scaling for the branching random walk*. Ann. Probab. **42** (2014), 959–993.

[3] G. Alsmeyer and B. Mallein, *A simple method to find all solutions to the functional equation of the smoothing transform*. Preprint (2019) available at https://arxiv.org/abs/1907.04111

[4] G. Alsmeyer and M. Meiners, *Fixed points of the smoothing transform: two-sided solutions*. Probab. Theory Related Fields. **155** (2013), 165–199.

[5] J. Bertoin and R. A. Doney, *On conditioning a random walk to stay nonnegative*. Ann. Probab. **22** (1994), 2152–2167.

[6] J. D. Biggins and A. E. Kyprianou, *Seneta-Heyde norming in the branching random walk*. Ann. Probab. **25** (1997), 337–360.

[7] J. D. Biggins and A. E. Kyprianou, *Measure change in multitype branching*. Adv. Appl. Probab. **36** (2004), 544–581.

[8] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation*. Cambridge University Press, 1989.

[9] S. Brofferio, D. Buraczewski and E. Damek, *On the invariant measure of the random difference equation $X_n = A_nX_{n-1} + B_n$ in the critical case*. Ann. Inst. Henri Poincaré Probab. Stat. **48** (2012), 377–395.

[10] D. Buraczewski, *On invariant measures of stochastic recursions in a critical case*. Ann. Appl. Probab. **17** (2007), 1245–1272.

[11] D. Buraczewski, *On tails of fixed points of the smoothing transform in the boundary case*. Stoch. Proc. Appl. **119** (2009), 3955–3961.

[12] X. Chen, *A necessary and sufficient condition for the nontrivial limit of the derivative martingale in a branching random walk*. Adv. Appl. Probab. **47** (2015), 741–760.

[13] R.M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey and D. E. Knuth, *On the Lambert W function*. Adv. Comp. Math. **5** (1996), 329–359.

[14] D. Denisov and V. Wachtel, *Random walks in cones*. Ann. Probab. **43** (2015), 992–1044.

[15] D. Denisov and V. Wachtel, *Universality of local times of killed and reflected random walks*. Electron. Commun. Probab. **21** (2016), 11 pp.

[16] R. A. Doney, *Moments of ladder heights in random walks*. J. Appl. Probab. **17** (1980), 248–252.

[17] R. A. Doney, *A note on conditioned random walk*. J. Appl. Probab. **20** (1983), 409–412.

[18] R. A. Doney, *The Martin boundary and ratio limit theorems for killed random walks*. J. Lond. Math. Soc., II. Ser. **58** (1998), 761–768.

[19] R. Durrett and T. Liggett, *Fixed points of the smoothing transformation*. Z. Wahrscheinlichkeitstheorie Verw. Geb. **64** (1983), 275–301.

[20] W. Feller, *An introduction to probability theory and its applications*. Vol II, 2nd Edition, Wiley, 1971.

[21] B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, revised edition. Addison-Wesley Publishing Co., 1968.

[22] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 7th edition. Academic Press, 2007.

[23] A. Iksanov, *Renewal theory for perturbed random walks and similar processes*. Birkhäuser, 2016.

[24] A. Iksanov, K. Kolesko and M. Meiners, *Fluctuations of Biggins' martingale at complex parameters*. Ann. Inst. Henri Poincaré Probab. Stat. (2020+), to appear.

[25] A. Iksanov and M. Meiners, *Fixed points of multivariate smoothing transforms with scalar weights*. ALEA, Lat. Am. J. Probab. Math. Stat. **12** (2015), 69–114.

[26] J.-P. Kahane and J. Peyrière, *Sur certaines martingales de Benoit Mandelbrot*. Adv. Math. **22** (1976), 131–145.

[27] Q. Liu, *Fixed points of a generalized smoothing transformation and applications to the branching random walk*. Adv. Appl. Probab. **30** (1998), 85–112.

[28] R. Lyons, *A simple path to BigginsâĂŹ martingale convergence for branching random walk.* In: Classical and Modern Branching Processes. IMA Volumes in Mathematics and its Applications **84** (1997), 217–221. Springer, New York.

[29] T. Madaule, *The tail distribution of the derivative martingale and the global minimum of the branching random walk*. Preprint (2016) available at `https://arxiv.org/abs/1606.03211`.

[30] T. Madaule, *Convergence in law for the branching random walk seen from its tip*. J. Theor. Probab. **30** (2017), 27–63.

[31] P. Maillard, *Branching Brownian motion with selection*. PhD thesis (2012) available at `https://arxiv.org/abs/1210.3500`.

[32] P. Maillard and M. Pain, *1-stable fluctuations in branching Brownian motion at critical temperature I: The derivative martingale*. Ann. Probab. **47** (2019), 2953–3002.

[33] B. Mallein, *Asymptotic of the maximal displacement in a branching random walk*. Graduate J. Math. **1** (2016), 92–104.

[34] B. Mallein, *Genealogy of the extremal process of the branching random walk*. ALEA, Lat. Am. J. Probab. Math. Stat. **15** (2018), 1065–1087.

[35] M. Meiners and S. Mentemeier, *Solutions to complex smoothing equations*. Probab. Theory Related Fields. **168** (2017), 199–268.

[36] J. Peyrière, *Turbulence et dimension de Hausdorff*. C. R. Acad. Sci. Paris. **278** (1974), 567–569.

[37] S. C. Port and C. J. Stone, *Potential theory of random walks on Abelian groups.* Acta Math. **122** (1969), 19–114.

[38] Y.-X. Ren, R. Song, Z. Sun and J. Zhao, *Stable central limit theorems for super Ornstein-Uhlenbeck processes*. Electron. J. Probab. **24**, paper no. 141, 42 pp.

[39] S. I. Resnick, *Adventures in stochastic processes*. 4rd printing, Birkhäuser, 2005.

[40] M. S. Sgibnev, *Renewal theorem in the case of an infinite variance*. Sib. Math. J. **22** (1982), 787–796.

[41] Z. Shi, *Branching random walks. Lecture notes from the 42nd Probability Summer School held in Saint Flour, 2012*. Lecture Notes in Mathematics, 2151. École d'Été de Probabilités de Saint-Flour. Springer, Cham, 2015. x+133 pp.

[42] F. Spitzer, *Principles of random walk*. 2nd edition, 2nd printing, Springer, 2001.

[43] H. Tanaka, *Time reversal of random walks in one-dimension*. Tokyo J. Math. **12** (1989), 159–174.

[44] B. Tsirelson, *From uniform renewal theorem to uniform large and moderate deviations for renewal-reward processes*. Electron. Commun. Probab. **18** (2013), paper no. 52, 13 pp.