Branching Brownian motion conditioned on small maximum

Xinxin Chen* Hui He† Bastien Mallein‡

December 19, 2023

Abstract

For a standard binary branching Brownian motion on the real line, it is known that the typical value of the maximal position M_t among all particles alive at time t is $m_t + \Theta(1)$ with $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$. Further, it is proved independently in [1] and [2] that the branching Brownian motion shifted by m_t (or M_t) converges in law to some decorated Poisson point process. The goal of this work is to study the branching Brownian motion conditioned on $M_t \ll m_t$. We give a complete description of the limiting extremal process conditioned on $\{M_t \leq \sqrt{2}\alpha t\}$ with $\alpha < 1$, which reveals a phase transition at $\alpha = 1 - \sqrt{2}$. We also verify the conjecture of Derrida and Shi [21] on the precise asymptotic behaviour of $\mathbf{P}(M_t \leq \sqrt{2}\alpha t)$ for $\alpha < 1$.

1 Introduction

Branching Brownian motion (BBM) is a spatial branching process that has been the subject of a large literature in the recent years. On the one hand, its duality relationship with the F-KPP reaction-diffusion equation has been the subject of several articles at the interface of probability and analysis of PDE, see for instance [26], [11], [25] and [2]. On the other hand, it is the fundamental model to understand the BBM-universality class which includes the 2-dim Gaussian free field [7], [12] and 2-dim cover times [18], etc.

We consider a one-dimensional standard binary branching Brownian motion. It is a continuous-time particle system on the real line which is constructed as follows. It starts with one individual located at the origin at time 0 that moves according to a standard Brownian motion. After an independent exponential time of parameter 1, the initial particle dies and gives birth to 2 children that start at the position their parent occupied at its death. These 2 children then move according to independent Brownian motions and give birth independently to their own children at rate 1. The particle system keeps evolving in this fashion for all time.

For all $t \geq 0$, we denote by $N(t)$ the collection of the individuals alive at time t. For any $u \in N(t)$ and $s \leq t$, let $X_u(s)$ denote the position at time s of the individual u or that of its ancestor alive at that time. The maximum of the branching Brownian motion at time t is defined as

$$M_t := \max\{X_u(t) : u \in N(t)\}.$$

*School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China, xinxin.chen@bnu.edu.cn

†School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China, hui.he@bnu.edu.cn

‡Université Sorbonne Paris Nord, LAGA, UMR 7539, F-93430, Villetaneuse, France

The asymptotic behaviour of M_t as $t \to \infty$ has been subjected to intense study, partly due to its link with the F-KPP reaction-diffusion equation, defined as

$$\partial_t u = \frac{1}{2}\Delta u - u(1-u). \tag{1.1}$$

Precisely, McKean [26] showed that the function $(z,t) \mapsto u(z,t) = \mathbf{P}(M_t \le z)$ is the unique solution of (1.1) with initial condition $u(z,0) = \mathbb{1}_{\{z>0\}}$ for $z \in \mathbb{R}$.

With the help of (1.1), Bramson [11] proved that uniformly in $z \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbf{P}(M_t \le m_t + z) = \lim_{t\to\infty} u(m_t + z, t) = w(z), \tag{1.2}$$

where $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$ and w is the unique –up to translation–travelling wave solution of the F-KPP equation at speed $\sqrt{2}$, which satisfies $\frac{1}{2}w'' + \sqrt{2}w' - w(1-w) = 0$.

Later, Lalley and Sellke [25] showed that the limiting distribution function w can be written as

$$w(z) := \mathbf{E}[e^{-C_0 e^{-\sqrt{2}z} Z_\infty}], \tag{1.3}$$

where $C_0 > 0$ is a constant and Z_∞ is an a.s. positive random variable, constructed as the almost sure limit of the so-called derivative martingale, defined for all $t \ge 0$ by

$$Z_t := \sum_{u\in N(t)} (\sqrt{2}t - X_u(t)) e^{\sqrt{2}X_u(t) - 2t}.$$

Further, the extremal process $\sum_{u\in N(t)} \delta_{X_u(t)-m_t}$ of the BBM, that describes the relative positions of particles around the tip of the BBM has gained much interest. It is conjectured since the work of [25] that it converges in law to some invariant point process. Then, this convergence has been verified independently by [1] and [2]. Let $\mathcal{E}_t := \sum_{u\in N(t)} \delta_{X_u(t)-m_t}$. They showed that as $t \to \infty$,

$$(\mathcal{E}_t, M_t - m_t) \Longrightarrow (\mathcal{E}, \max_{x\in\mathcal{E}} x),$$

where $\Longrightarrow$ denotes convergence in distribution. In the rest of the article, we always consider the convergence of random measures with the vague topology, i.e., $\mathcal{E}_t \Rightarrow \mathcal{E}$ if $\langle \mathcal{E}_t, \varphi\rangle \Rightarrow \langle \mathcal{E}, \varphi\rangle$ for all continuous compactly supported function φ.

Precisely, the limiting extremal point process $\mathcal{E}$ can be constructed as

$$\mathcal{E} := \sum_{x\in\mathcal{P}} \sum_{y\in\mathcal{D}_x} \delta_{x+y}, \tag{1.4}$$

where conditioned on Z_∞ defined in (1.3), $\mathcal{P}$ is a Poisson point process with intensity $C_0\sqrt{2}Z_\infty e^{-\sqrt{2}x}\mathrm{d}x$ (C_0 is the same as in (1.3)) and conditioned on $\mathcal{P}$, $(\mathcal{D}_x, x \in \mathcal{P})$ are i.i.d. decorated point processes in $(-\infty, 0]$ with an atom at 0, which we refer to as the decoration of the branching Brownian motion. Several properties of this limiting point process have been described in the recent articles [16], [4], [5].

We are interested in the behaviour of the branching Brownian motion conditioned on $\{M_t \le \sqrt{2}\alpha t\}$ with $\alpha < 1$. We consider the shifted point process

$$\mathcal{E}_t(\alpha) := \sum_{u\in N(t)} \delta_{X_u(t)-\sqrt{2}\alpha t}, \quad t \ge 0,$$

under the probability $\mathbf{P}(\cdot|M_t \le \sqrt{2}\alpha t)$. We show that $\mathcal{E}_t(\alpha)$ converges in law in the vague topology to some limiting point process and obtain that the limiting point process exhibits a phase transition at

$$\alpha_c := -\gamma := 1 - \sqrt{2}.$$

The corresponding lower deviation probability $\mathbf{P}(M_t \leq \sqrt{2}\alpha t)$ also exhibits a phase transition at α_c, as previously detected in [20] and conjectured in [21].

Before stating precisely our main result, let us briefly stress a few links with other works in the literature.

1. It is natural to consider also the branching Brownian motion conditioned on large maximum. Conditioned on $\{M_t \geq \sqrt{2}\alpha t\}$ with $\alpha > 1$, this question has been studied in [13] and a Yaglom-type theorem was obtained. It is more subtle when $\alpha = 1$ and Chauvin-Rouault [14] also conjectured a Yaglom theorem in this case. In particular, it is proved in [2] that conditioned on $\{M_t \geq \sqrt{2}t + a\sqrt{t}\}$ with any $a \in (0, \infty)$, the point process $\sum_{u \in N(t)} \delta_{X_u(t) - M_t}$ converges in law to some point process $\mathcal{D}$, which serves as the decoration process appeared in the limiting extremal process of BBM in their work.

2. BBM is usually viewed as a continuous-time analogue of branching random walks (BRW) in discrete time. In fact, the BBM should be considered as a BRW in Schröder case where the offspring could be less than or equal to 1. There exist also BRWs in Böttcher case where the offspring is at least 2. It has been detected in [15] that the atypically small maximum comes from different mechanisms in the two cases. One could expect that the conditioned BRW in Schröder case behaves similarly as conditioned BBM, yet conditioned BRW in Böttcher case would be more complicated and be of different nature. It is an interesting question to understand the conditioned structure in Böttcher case.

3. As 2-dimensional discrete Gaussian free field (2d DGFF) belongs to the BBM-universality class, our result naturally leads to thinking about the description of 2d DGFF conditioned to stay negative/positive, which is connected with the so-called entropic repulsion. However, the entropic repulsion is more challenging because of the following two aspects. First, the interior bulk estimates of DGFF is comparable with the lower deviation estimates of BRW in Böttcher case, which are totally different from that of BBM, see for example [29]. Secondly, the boundary estimates show that only the spins close to the boundary are responsible for the hard wall condition that the whole DGFF stays positive. One can refer to [9], [22] and [8] for more details.

4. Our work on conditioned BBM could lead to further research on conditioned BBM in presence of selection or coalescence which are closely related to the noisy FKPP equation. The appearance of atypically large maximum for BBM with selection is discussed in [19]. The lower deviation for maximum is studied in [27] and [28] for BRW with coalescence. It is intriguing to understand how the conditioned process and the large deviation probabilities are modified by the selection or coalescence.

5. In view of (1.2) and (1.3), the event that M_t is atypically small is related to the event that Z_∞ is atypically small. Inspired by this observation, one may consider the BBM or BRW conditioned on $\{0 < Z_\infty < \varepsilon\}$ which was asked in [24]. In the literature, [6] showed that the law of a Galton-Watson process conditioned on the limiting martingale being small is described as a Galton-Watson process with minimal branching until a given generation, which then behaves as typical process after that generation. This phenomena occurs also in our result. The heuristic will be discussed at the end of this section.

1.1 Main theorems

To state the main theorems, let us introduce the first branching time, defined by

$$\tau := \inf\{t \geq 0 : \#N(t) \geq 2\}.$$

The corresponding position of the initial ancestor is $X_\emptyset(\tau)$. Then we obtain the following result on conditioned BBM.

Theorem 1.1. *Conditioned on $\{M_t \le \sqrt{2}\alpha t\}$, the following convergences in law hold.*

1. *If $\alpha \in (-\gamma, 1)$, then, as $t \to \infty$, we have*

$$\left(\frac{\tau - \frac{(1-\alpha)}{\sqrt{2}}t}{\sqrt{t\frac{(1-\alpha)}{4\sqrt{2}}}}, X_\emptyset(\tau) - (\sqrt{2}\alpha t - m_{t-\tau}), M_t - \sqrt{2}\alpha t\right) \Longrightarrow (\xi, -\chi, -E), \tag{1.5}$$

where ξ and (χ, E) are independent, with ξ a standard Gaussian random variable and E an exponential random variable with parameter $\sqrt{2}\gamma$. The joint distribution of (χ, E) is given by

$$\mathbf{P}(\chi \le x, E \ge y) = \frac{1}{2C^{(1)}} e^{-\sqrt{2}\gamma y} \int_{-\infty}^{x-y} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z, \quad x \in \mathbb{R},\ y \in \mathbb{R}_+,$$

where

$$C^{(1)} := \frac{1}{2}\int_{\mathbb{R}} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z \in (0, \infty). \tag{1.6}$$

Moreover, jointly with the convergence in (1.5), as $t \to \infty$, we have

$$\mathcal{E}_t(\alpha) \Longrightarrow \mathcal{E}_\infty(\alpha) = \mathcal{E}^- := \sum_{x \in \mathcal{E}_1 \cup \mathcal{E}_2} \delta_{x-\chi}, \tag{1.7}$$

where given χ, $\mathcal{E}_1$ and $\mathcal{E}_2$ are i.i.d. point processes distributed as $\mathcal{E}$, defined in (1.4), conditioned on $\{\max \mathcal{E} \le \chi\}$.

2. *If $\alpha = -\gamma$, then as $t \to \infty$, we have*

$$\left(\frac{t-\tau}{\sqrt{t}}, X_\emptyset(\tau) - (\sqrt{2}\alpha t - m_{t-\tau}), M_t - \sqrt{2}\alpha t\right) \Longrightarrow (\xi_\alpha, -\chi, -E), \tag{1.8}$$

where ξ_α and (χ, E) are independent, (χ, E) are the same as in (1.5) and ξ_α is a positive random variable with density $2^{-3(\sqrt{2}+1)/4}\Gamma((3\sqrt{2}-1)/4)u^{3\gamma/2}e^{-2u^2}\mathrm{d}u$. Further, as $t \to \infty$, we have jointly

$$\mathcal{E}_t(\alpha) \Longrightarrow \mathcal{E}_\infty(\alpha) = \mathcal{E}^-, \tag{1.9}$$

where $\mathcal{E}^-$ is the same as in (1.7).

3. *If $\alpha < -\gamma$, then, as $t \to \infty$, we have*

$$\left(t - t \wedge \tau, \sqrt{2}\alpha t - X_\emptyset(t \wedge \tau), M_t - \sqrt{2}\alpha t\right) \Longrightarrow (\xi_\alpha, -\chi_\alpha, -E_\alpha), \tag{1.10}$$

where ξ_α is distributed as

$$\frac{1}{-\alpha\Phi(\alpha)}\delta_0(\mathrm{d}s) + \frac{1}{\Phi(\alpha)}\int_{\mathbb{R}} e^{\sqrt{2}\alpha z + (1-\alpha^2)s} u(z,s)^2 \mathrm{d}z\mathrm{d}s,$$

with $\Phi(\alpha) := -\frac{1}{\alpha} + \sqrt{2}\int_0^\infty \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \in (0,\infty)$*,* E_α *is distributed as an exponential random variable with parameter* $-\sqrt{2}\alpha$*, and the joint distribution of* $(\xi_\alpha, \chi_\alpha, E_\alpha)$ *is given by*

$$\mathbf{P}(\xi_\alpha \le x_1, \chi_\alpha \le x_2, E_\alpha \ge x_3) = \frac{1}{\Phi(\alpha)}\left(\mathbb{1}_{\{x_3<x_2\}} \int_{x_3}^{x_2} \sqrt{2} e^{\sqrt{2}\alpha z} \mathrm{d}z + \sqrt{2}\int_0^{x_1} \mathrm{d}s \int_{-\infty}^{x_2-x_3} e^{\sqrt{2}\alpha(x_3+z)+(1-\alpha^2)s} u(z,s)^2 \mathrm{d}z \right),$$

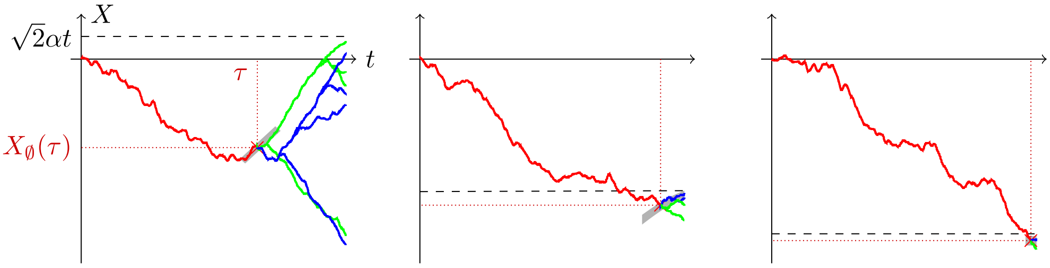
for any $x_1, x_3 \in \mathbb{R}_+$ *and* $x_2 \in \mathbb{R}$*. Further, as* $t \to \infty$*, we have jointly*

$$\mathcal{E}_t(\alpha) \Longrightarrow \mathcal{E}_\infty(\alpha) := \delta_{-\chi_\alpha} \mathbb{1}_{\{\xi_\alpha=0\}} + \mathbb{1}_{\{\xi_\alpha>0\}} \sum_{x\in\mathcal{B}_1\cup\mathcal{B}_2} \delta_{x-\chi_\alpha}, \tag{1.11}$$

where given $(\xi_\alpha, \chi_\alpha)$*,* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are i.i.d. copies of* $\sum_{u\in N(\xi_\alpha)} \delta_{X_u(\xi_\alpha)}$ *conditioned on* $\{M_{\xi_\alpha} \le \chi_\alpha\}$*.*

Remark 1.2. One could see in the limiting point process $\mathcal{E}^-$ the union of two independent copies of conditioned $\mathcal{E}$. That is because for the conditioned BBM, the initial ancestor gives birth at time τ to two children, which independently produce typically behaved BBMs and thus give rise to independent copies of $\mathcal{E}$ conditioned to remain below some appropriate level. In the regime where $\alpha < -\gamma$, we note that the limiting distribution of $t - t\wedge\tau$ as $t \to \infty$ has a Dirac mass at 0, which corresponds to the probability that no branching occurs before time t.

To illustrate this phenomena, we draw, in Figure 1, schemes of the expected behaviour of the branching Brownian motion conditioned to stay below $\sqrt{2}\alpha t$ in the three regimes.



(a) $\alpha \in (-\gamma, 1)$ (b) $\alpha = -\gamma$ (c) $\alpha < -\gamma$

Figure 1: Scheme of the first branching time in different conditioning scenarios. The initial particle is drawn in red, its two offspring giving birth to the green and blue subtrees respectively. The typical branching zone is drawn as a grey area. Its width is of order $t^{1/2}$ and its height of order 1 in cases (a) and (b).

In this work, we also study the precise asymptotic of $\mathbf{P}(M_t \le \sqrt{2}\alpha t)$ in the three regimes. Derrida and Shi [20] obtained the exponential decay as follows,

$$\mathbf{P}(M_t \le \sqrt{2}\alpha t) = \begin{cases} e^{-2(\sqrt{2}-1)(1-\alpha)t+o(t)}, & \text{for } -\gamma < \alpha < 1, \\ e^{-(1+\alpha^2)t+o(t)}, & \text{for } \alpha \le -\gamma, \end{cases} \quad \text{as } t \to \infty.$$

A phase transition occurs at $-\gamma = 1-\sqrt{2} \approx -0.414$. Later, Derrida and Shi [21] conjectured the second order as below

$$\mathbf{P}(M_t \le \sqrt{2}\alpha t) \sim_{t\to\infty} \begin{cases} C^{(1)}(\frac{\alpha+\gamma}{\sqrt{2}}t)^{\frac{3\gamma}{2}}e^{-2\gamma(1-\alpha)t}, & \text{if } \alpha > -\gamma, \\ \frac{\Phi(\alpha)}{\sqrt{4\pi}}t^{-\frac{1}{2}}e^{-(1+\alpha^2)t}, & \text{if } \alpha < -\gamma, \end{cases} \tag{1.12}$$

with $C^{(1)}$ define in (1.6) and

$$\Phi(\alpha) = -\frac{1}{\alpha} + \sqrt{2}\int_0^\infty \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y e^{(1-\alpha^2)s+\sqrt{2}\alpha y}u(y,s)^2 \in (0,\infty). \tag{1.13}$$

Here, we write $f(t) \sim g(t)$ if $\lim_{t\to\infty}\frac{f(t)}{g(t)} = 1$.

We verify this conjecture and obtain the asymptotic for $\alpha_c = -\gamma = 1-\sqrt{2}$ in the next theorem. Recall

$$C^{(1)} := \frac{1}{2}\int_{\mathbb{R}} e^{-\sqrt{2}\gamma z}w(z)^2\mathrm{d}z \in (0,\infty).$$

Theorem 1.3. *As $t \to \infty$, the following convergences hold.*

1. *If $-\gamma < \alpha < 1$,*
$$\mathbf{P}(M_t \le \sqrt{2}\alpha t) \sim C^{(1)}(v_\alpha t)^{\frac{3\gamma}{2}}e^{-2\gamma(1-\alpha)t}, \tag{1.14}$$
where $v_\alpha := \frac{\gamma+\alpha}{\sqrt{2}} \in (0,1)$.

2. *If $\alpha = -\gamma$,*
$$\mathbf{P}(M_t \le \sqrt{2}\alpha t) \sim C^{(2)}t^{3\gamma/4}e^{-(1+\gamma^2)t}, \tag{1.15}$$
where
$$C^{(2)} := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_+} u^{3\gamma/2}e^{-2u^2}\mathrm{d}u \int_{\mathbb{R}} e^{-\sqrt{2}\gamma z}w(z)^2\mathrm{d}z = \frac{C^{(1)}\Gamma(\frac{3\sqrt{2}-1}{4})}{\sqrt{2\pi}2^{\frac{3\sqrt{2}-1}{4}}}.$$

3. *If $\alpha < -\gamma$,*
$$\mathbf{P}(M_t \le \sqrt{2}\alpha t) \sim \frac{\Phi(\alpha)}{\sqrt{4\pi}}t^{-\frac{1}{2}}e^{-(1+\alpha^2)t}. \tag{1.16}$$

Remark 1.4. In fact, we could look closer around the phase transition point $-\gamma$ and obtain the following results by a straightforward adaptation of the reasoning used in Section 5. We leave the proof to interested readers. Let $a : \mathbb{R}_+ \to \mathbb{R}$ with $a_t = o(t)$.

1. If $a_t = o(\sqrt{t})$, then
$$\mathbf{P}(M_t \le -\sqrt{2}\gamma t + a_t) \sim C^{(2)}t^{3\gamma/4}e^{-2\sqrt{2}\gamma t+\sqrt{2}\gamma a_t}. \tag{1.17}$$

2. If $a_t = a\sqrt{t}$ with $a \in \mathbb{R}$, there exists a positive function $a \mapsto C(a)$ such that
$$\mathbf{P}(M_t \le -\sqrt{2}\gamma t + a_t) \sim C(a)t^{3\gamma/4}e^{-2\sqrt{2}\gamma t+\sqrt{2}\gamma a_t}. \tag{1.18}$$

3. If $\lim_{t\to\infty}\frac{a_t}{\sqrt{t}} = \infty$ and $a_t = o(t)$, then there exist $C^{(3)}, C^{(4)} > 0$ such that
$$\mathbf{P}(M_t \le -\sqrt{2}\gamma t + a_t) \sim C^{(3)}a_t^{3\gamma/2}e^{-2\sqrt{2}\gamma t+\sqrt{2}\gamma a_t}, \tag{1.19}$$
$$\mathbf{P}(M_t \le -\sqrt{2}\gamma t - a_t) \sim C^{(4)}(t/a_t)^{3\gamma/2+1}t^{-1/2}e^{-2\sqrt{2}t-\sqrt{2}\gamma a_t-\frac{a_t^2}{4t}}. \tag{1.20}$$

Finally, we could consider the BBM conditioned on the event $\{M_t \le m_t - a_t\}$ where $\lim_{t\to\infty} a_t = \infty$ and $a_t = o(t)$. In that case the first branching time happens at a time of order a_t, and the process after that first branching time is a branching Brownian motion conditioned on an event of positive probability. So the eventual limiting process is the same as in the regime with $\alpha \in (-\gamma, 1)$.

Remark 1.5. If $a_t = o(t)$ and $\lim_{t\to\infty} a_t = \infty$, then as $t \to \infty$,

$$\mathbf{P}(M_t \le m_t - a_t) \sim C^{(1)} e^{-\sqrt{2}\gamma a_t}. \tag{1.21}$$

Moreover, conditioned on $\{M_t \le m_t - a_t\}$,

$$\left(\frac{\tau - \frac{1}{2}a_t}{\sqrt{a_t/8}}, X_\emptyset(\tau) - (\sqrt{2}\tau - a_t), M_t - (m_t - a_t) \right) \Longrightarrow (\xi, -\chi, -E), \tag{1.22}$$

and jointly,

$$\sum_{u\in N(t)} \delta_{X_u(t)-(m_t-a_t)} \Longrightarrow \mathcal{E}^-, \tag{1.23}$$

where $(\xi, \chi, E, \mathcal{E}^-)$ is the same as in Theorem 1.1.

Note that (1.21) is already known in the literature (see [2]); the joint convergence in distribution described in (1.22–1.23) is new. One could follow the arguments in Section 3 to complete the proof. We shall omit the details.

Remark 1.6. In this article, we choose to focus on branching Brownian motions with binary branching, to keep the proofs as simple as possible. Up to minor changes, one can assume the number of children made by an individual at death to be i.i.d. integer-valued random variables of law ν. As long as $\nu(0) = 0$ (i.e. the process never gets extinct) and $\sum_{k=1}^{\infty} k(\log k)^2 \nu(k) < \infty$ (an integrability condition guaranteeing the non-degeneracy of the limit Z_∞), we expect similar results to hold.

Remark 1.7. Bai and Hartung in [3] considered the following joint deviation probability

$$\mathbf{P}(M_t \le \sqrt{2}\alpha t, (\tau, X_\emptyset(\tau)) \in \cdot), \quad \alpha < 1.$$

They obtained the first order of the decay rate for various cases by imposing constraints on the first branching time and location. Based on our work, it is possible to further refine their results to obtain precise prefactors.

Before proving our main result, we quickly recall the heuristics given in [20] to explain the asymptotic decay of $\mathbf{P}(M_t \le \sqrt{2}\alpha t)$ in (1.12) in the next section.

1.2 Heuristics behind Theorem 1.3

Recall that τ is the first branching time of the process and $X_\emptyset(\tau)$ the position of the particle at that first branching time. As particles behave independently after they are born, the probability of observing an unusually low maximum decays sharply after each branching event. Therefore, to maximise the probability of $M_t \le \sqrt{2}\alpha t$, a good strategy is to control the first branching time τ and the position $X_\emptyset(\tau)$. Since we expect exponential decay in t of $\mathbf{P}\{M_t \le \sqrt{2}\alpha t\}$, then it is reasonable to conjecture that $\tau \approx \lambda_\alpha t$ conditioned on $\{M_t \le \sqrt{2}\alpha t\}$, for some $\lambda_\alpha \in [0, 1]$, by noticing the fact that $\mathbf{P}(\tau > s) = e^{-s}$. Additionally, after that branching time, the offsprings should led to two independent regular branching Brownian motions with length $t - \tau$, which is an event happening with probability $O(1)$. Therefore the maximal position at time t should be around level $X_\emptyset(\tau)+\sqrt{2}(t-\tau)$, which has to be lower than $\sqrt{2}\alpha t$. This yields another condition $X_\emptyset(\tau) \leq \sqrt{2}\alpha t + \sqrt{2}(\tau - t)$.

Then, with B a standard Brownian motion, observe that

$$\mathbf{P}(\tau \approx \lambda t, X_\emptyset(\tau) \leq \sqrt{2}\alpha t + \sqrt{2}(\tau - t)) \approx e^{-\lambda t}\mathbf{P}(B_{\lambda t} \leq \sqrt{2}\alpha t + \sqrt{2}(\lambda - 1)t) \\ \approx \exp\left(-t\left(\lambda + \frac{(\alpha + (\lambda-1))^2}{\lambda}\right)\right).$$

Thus, to maximize this probability, one has to choose the parameter $\lambda_\alpha \in [0,1]$ that minimizes the quantity

$$\lambda + \frac{(\alpha - (1-\lambda))^2}{\lambda}.$$

Note that if $\alpha > -\gamma = 1-\sqrt{2}$, this minimum is attained at $\lambda_\alpha = \frac{(1-\alpha)}{\sqrt{2}} \in [0,1]$, whereas if $\alpha \leq \gamma$, this minimum is attained at $\lambda_\alpha = 1$.

As a result, we expect three different behaviours for the branching Brownian motion conditioned on having a maximum smaller than $\sqrt{2}\alpha t$, depending on whether α is larger than, smaller than, or equal to $-\gamma$. In the first case, the branching time should happen at some intermediate time in the process, and the branching Brownian motion after this first branching time should behave as a regular process, conditioned on an event of positive probability. If $\alpha < -\gamma$, then one expects the process not to branch until the very end of the process, which allows an explicit description of the extremal process in that case. In the intermediate case $\alpha = -\gamma$, the branching time should be such that $t - \tau$ is large, but negligible with respect to t. In this setting, the behaviour of the process after that time should not be too different from the case $\alpha > -\gamma$.

Organization. The rest of the paper is organised as follows. In Section 2, we state some well-known results on branching Brownian motion and show some rough bounds of $u(z,t)$. In Section 3, we treat the case where $\alpha \in (-\gamma, 1)$. Section 4 is devoted to the study of the regime that $\alpha < -\gamma$. In Section 5, we consider the critical case $\alpha = \gamma$. The proofs of some technical lemmas are postponed to Appendix A.

In the following, we write $f(t) \sim g(t)$ as $t \to \infty$ to denote $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$. As usual, $f(t) = o_t(g(t))$ means $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$. The quantities $(C_i)_{i\in\mathbb{N}}$ and $(c_i)_{i\in\mathbb{N}}$ represent positive constants, and c, C are non-specified positive constants, that might change from line to line, taken respectively small enough and large enough.

2 Preliminary results

In this section, we recall previously known results on the maximum and extremal process of branching Brownian motions. We source most of the results stated here from the book of Bovier [10] for convenience, and refer the reader to it for the origins of these results. Using these results, we obtain first order estimates on $u(z,t) = \mathbf{P}(M_t \leq z)$.

Write $\mathbf{E}\left[X; A\right]$ for $\mathbf{E}\left[X\mathbb{1}_A\right]$, with X a random variable and A a measurable event. For all $\varphi \in \mathcal{C}_c^+(\mathbb{R})$, the set of non-negative, continuous and compactly supported functions, we denote by

$$u_\varphi : (z,t) \in \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbf{E}\left[e^{-\sum_{u\in N(t)} \varphi(X_u(t)-z)}; M_t \leq z\right] = \mathbf{E}\left[\prod_{u\in N(t)} f_\varphi(z - X_u(t))\right], \quad (2.1)$$

where

$$f_\varphi : y \mapsto e^{-\varphi(-y)}\mathbb{1}_{\{y\geq 0\}}.$$

Then u_φ is the unique solution of the F-KPP equation (1.1) with initial condition f_φ, i.e.

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u - u(1-u), \\ u_\varphi(z,0) = f_\varphi(z), \text{ for all } z \in \mathbb{R}. \end{cases} \tag{2.2}$$

We remark that the cumulative distribution function of M_t is given by $u(z,t) = u_0(z,t)$.

By [10, Proposition 2.22], the joint convergence in law of the centred extremal process and maximal displacement of the branching Brownian motion can be rewritten as the following point-wise convergence

$$\lim_{t\to\infty} u_\varphi(m_t + z, t) = w_\varphi(z), \quad \forall \varphi \in \mathcal{C}_c^+(\mathbb{R}), \forall z \in \mathbb{R}, \tag{2.3}$$

where

$$w_\varphi(z) := \mathbf{E}\left[e^{-\int \varphi(\cdot - z)\mathrm{d}\mathcal{E}}; \max \mathcal{E} \le z\right].$$

Moreover, convergence (2.3) in fact holds uniformly on compact sets, by [10, Lemma 5.5 and Theorem 5.9]. For any $K > 0$, as $\min_{z\in[-K,K]} w_\varphi(z) > 0$, this uniform convergence result implies that

$$\lim_{t\to\infty} \sup_{|z|\le K} \frac{|u_\varphi(m_t + z, t) - w_\varphi(z)|}{w_\varphi(z)} = 0. \tag{2.4}$$

Hence, uniformly for $z \in [-K, K]$, $u(m_t + z, t) = w(z)(1 + o(1))$ as $t \to \infty$.

This straightforward upper bound, combined with the lower deviation results of Derrida and Shi [20] gives us the following lemma.

Lemma 2.1. *For any $\beta \ge 1$ and $\varepsilon > 0$, there exists $t_{\varepsilon,\beta} > 1$ such that for any $t \ge t_{\varepsilon,\beta}$,*

$$u(\sqrt{2}at, t) \le \begin{cases} 1, & \text{if } a \ge 1; \\ e^{-2\gamma(1-a)t+\varepsilon t}, & \text{if } -\gamma \le a < 1; \\ e^{-(1+a^2)t+\varepsilon t}, & \text{if } -\beta \le a < -\gamma; \\ e^{-a^2 t}, & \text{if } a < -\beta. \end{cases} \tag{2.5}$$

Proof. Let $\beta \ge 1$. We begin by noting that $u(z,t) \le 1$ for any $z \in \mathbb{R}$ and $t \ge 0$. Let $(B_t, t \ge 0)$ be a standard Brownian motion. It follows from [10, Lemma 1.1] that for any $z > 0$ and $t > 0$,

$$\int_z^\infty \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\mathrm{d}z = \int_{-\infty}^{-z} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}}\mathrm{d}z = \mathbf{P}(B_t > z) \le \frac{\sqrt{t}}{z\sqrt{2\pi}} e^{-\frac{z^2}{2t}}. \tag{2.6}$$

At time t, the system contains $\#N(t) \ge 1$ individuals, the positions of which are distributed with the same law as B_t. Therefore, for any $t \ge 0$ and $z \in \mathbb{R}$,

$$u(z,t) = \mathbf{P}(M_t \le z) \le \mathbf{P}(B_t \le z),$$

which using (2.6), yields for $z < 0$:

$$u(z,t) \le \frac{\sqrt{t}}{-z\sqrt{2\pi}} e^{-\frac{z^2}{2t}}. \tag{2.7}$$

Thus, for any $a < -1$, we have

$$u(\sqrt{2}at, t) \le \frac{1}{-2a\sqrt{\pi t}} e^{-a^2 t} \le e^{-a^2 t}, \tag{2.8}$$

for all $t \geq 1$. To complete the proof, it is therefore enough to bound $u(\sqrt{2}at, t)$ for $a \in [-\beta, 1)$. Let us reformulate Derrida and Shi's result [20, Theorem 1] as follows:

$$\lim_{t\to\infty} \frac{1}{t} \log u(\sqrt{2}at, t) = \psi(a) := \begin{cases} 0, & \text{if } a \geq 1; \\ -2\gamma(1-a), & \text{if } -\gamma \leq a < 1; \\ -(1+a^2), & \text{if } a < -\gamma. \end{cases} \tag{2.9}$$

Note that being a cumulative distribution function for any $t \geq 0$, the function $z \mapsto u(z, t)$ is non-decreasing. Thus, both $\frac{\log u(\sqrt{2}at,t)}{t}$ and $\psi(a)$ are non-decreasing in $a \in \mathbb{R}$, and moreover ψ is continuous. By Dini's theorem, the convergence in (2.9) holds uniformly on any compact sets in $\mathbb{R}$, hence in particular on $[-\beta, 1]$. As a result, for all $\varepsilon > 0$, there exists $t_{\varepsilon,\beta} > 1$ such that for all $t \geq t_{\varepsilon,\beta}$, we have

$$\sup_{a\in[-\beta,1]} \left| \frac{1}{t} \log u(\sqrt{2}at, t) - \psi(a) \right| \leq \varepsilon. \tag{2.10}$$

We then deduce (2.5) from (2.10) and (2.8). □

Next, we recall [15, Theorem 1.7], that gives a tight estimate on the moderate lower deviations of the maximal displacement: for any sequence (a_t) such that $\lim_{t\to\infty} a_t = \infty$ and $a_t = o(t)$,

$$\mathbf{P}(M_t \leq m_t - a_t) = e^{-\sqrt{2}(\gamma + o_t(1))a_t}.$$

We strengthen the above estimate into the following non-asymptotic upper bound for $u(m_t - z, t)$.

Lemma 2.2. *For any $\delta \in (0, 1)$, there exist $K_\delta \geq 1$ and $T_\delta \geq 1$, such that for any $t \geq T_\delta$ and any $z \geq K_\delta$,*

$$u(m_t - z, t) = \mathbf{P}(M_t \leq m_t - z) \leq c_\delta e^{-\sqrt{2}\gamma(1-\delta)z}, \tag{2.11}$$

with $c_\delta > 1$ a constant depending on δ.

Proof. The idea of the proof of this result is mainly borrowed from the proof of Theorem 3.2 (Case 2) in [23]. We apply the Markov property at some intermediate time, and observe that either there is an anomalously small number of particles alive at that time, or all of the particles alive at that time must satisfy Lemma 2.1. The detailed proof is postponed to Appendix A.1. □

Next, we present a decomposition of the branching Brownian motion at its first branching point, which severs as the basic idea behind the proofs of all the main results. Recall (2.1).

Lemma 2.3. *Let $\varphi \in \mathcal{C}_c^+(\mathbb{R})$. Then u_φ satisfies*

$$u_\varphi(z, t) = e^{-t}\, \mathbf{E}\left(e^{-\varphi(B_t - z)}; B_t \leq z \right) + \int_0^t e^{-s} \mathrm{d}s \int_{\mathbb{R}} \mathbf{P}(B_s \in \mathrm{d}y) u_\varphi(z - y, t - s)^2 \tag{2.12}$$

$$=: U_1^\varphi(z, t) + U_2^\varphi(z, t), \tag{2.13}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

Proof. In fact, (2.12) is a simple consequence of the Markov property applied at the first branching time of the branching Brownian motion. Indeed, at time t, the original ancestor did not split with probability e^{-t}, in which case its position is distributed as a Gaussian random variable with variance t. Otherwise, the ancestor died at time s with probability $e^{-s}\mathrm{d}s$, in which case the branching Brownian motion at time t has the same law as the sum of two independent branching Brownian motions at time $t - s$, shifted by the position of the ancestor at time s, which is distributed as B_s. □

Taking $\varphi = 0$ yields that for $u(z, s) = \mathbf{P}(M_s \le z)$,

$$u(z,t) = e^{-t}\mathbf{P}(B_t \le z) + \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathbf{P}(B_s \in \mathrm{d}y) e^{-s} u(z-y, t-s)^2. \tag{2.14}$$

As we write u for u_0, we denote by U_1 and U_2 the quantities defined above with $\varphi \equiv 0$. (2.14) allows us to bootstrap close to optimal bounds on $u(\sqrt{2}t - a_t, t)$ from a priori bounds, using Laplace's method (see e.g. [17, Chapter 4]). This allows us to obtain equivalents for different regimes as $t, a_t \to \infty$.

To complete this section, we an uniform estimate of $U_1^\varphi(\sqrt{2}\alpha t, t)$ for $\alpha \in [0, 1)$, as well as an exact asymptotic for $\alpha < 0$.

Lemma 2.4. *Let $\varphi \in \mathcal{C}_c^+(\mathbb{R})$. For $\alpha \ge 0$, we have*

$$\frac{e^{-\|\varphi\|_\infty}}{2} e^{-t} \le U_1^\varphi(\sqrt{2}\alpha t, t) \le e^{-t}, \tag{2.15}$$

where $\|\varphi\|_\infty := \sup\{\varphi(x) : x \in \mathbb{R}\}$. Moreover, for $\alpha < 0$, we have

$$\lim_{t\to\infty} \sqrt{t} e^{(1+\alpha^2)t} U_1^\varphi(\sqrt{2}\alpha t, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\varphi(y) - \sqrt{2}\alpha y} \mathrm{d}y. \tag{2.16}$$

In particular, for $\alpha < 0$ we have $U_1(\sqrt{2}\alpha t, t) \sim \frac{e^{-(1+\alpha^2)t}}{\sqrt{4\pi t}|\alpha|}$ as $t \to \infty$.

Proof. We have $U_1^\varphi(\sqrt{2}\alpha t, t) = e^{-t}\,\mathbf{E}(e^{-\varphi(B_t - \sqrt{2}\alpha t)}; B_t \le \sqrt{2}\alpha t)$. For all $\alpha \ge 0$, as φ is non-negative, we have $1 \ge \mathbf{E}(e^{-\varphi(B_t - \sqrt{2}\alpha t)}; B_t \le \sqrt{2}\alpha t) \ge e^{-\|\varphi\|_\infty}\mathbf{P}(B_t \le 0)$, which is enough to prove (2.15).

Additionally, for $\alpha < 0$, by Girsanov transform, we then have

$$\begin{aligned} U_1^\varphi(\sqrt{2}\alpha t, t) &= e^{-(1+\alpha^2)t}\,\mathbf{E}\left(e^{-\varphi(B_t) - \sqrt{2}\alpha B_t}; B_t \le 0\right) \\ &= \frac{e^{-(1+\alpha^2)t}}{\sqrt{2\pi t}} \int_{-\infty}^0 e^{-y^2/2t - \varphi(y) - \sqrt{2}\alpha y} \mathrm{d}y. \end{aligned} \tag{2.17}$$

The dominated convergence theorem yields

$$\lim_{t\to\infty} \int_{-\infty}^0 e^{-y^2/2t - \varphi(y) - \sqrt{2}\alpha y} \mathrm{d}y = \int_{-\infty}^0 e^{-\varphi(y) - \sqrt{2}\alpha y} \mathrm{d}y,$$

which completes the proof. □

3 The case $-\gamma < \alpha < 1$

In this section, we treat the case when $1 - \sqrt{2} < \alpha < 1$ and prove (1.5), (1.7) and (1.14). Recall that $\gamma = \sqrt{2} - 1$ and if $-\gamma < \alpha < 1$, we expect the existence of $\lambda_\alpha \in (0, 1)$ such that with high probability the first branching time of the branching Brownian motion conditioned on the event $\{M_t \le \sqrt{2}\alpha t\}$ is close to $\lambda_\alpha t$. In this situation, we have $\lambda_\alpha = \frac{1-\alpha}{\sqrt{2}}$.

For all $\alpha \in (-\gamma, 1)$, we have $u(\sqrt{2}\alpha t, t) = e^{-2\gamma(1-\alpha)t(1+o_t(1))}$ by [20]. Moreover, observe that

$$2\gamma(1-\alpha) < \begin{cases} 1, & \text{if } 0 \le \alpha < 1; \\ 1 + \alpha^2, & \text{if } -\gamma < \alpha < 0. \end{cases}$$

As a result, Lemma 2.4 shows that $U_1(\sqrt{2}\alpha t,t) = o_t(1)u(\sqrt{2}\alpha t,t)$ for all $\alpha \in (-\gamma, 1)$, which by (2.13) implies that $u(\sqrt{2}\alpha t,t) \sim U_2(\sqrt{2}\alpha t,t)$ as $t\to\infty$. We thus turn to study $U_2^\varphi(\sqrt{2}\alpha t,t)$, which is by definition

$$U_2^\varphi(\sqrt{2}\alpha t,t) = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}z \frac{e^{-(t-s)-\frac{(m_s+z-\sqrt{2}\alpha t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u_\varphi(m_s+z,s)^2.$$

Recalling that $v_\alpha = \frac{\gamma+\alpha}{\sqrt{2}} = 1-\frac{1-\alpha}{\sqrt{2}} \in (0,1)$, our next lemma consists in the observation that most of the mass on this double integral is carried by $\{(s,z) : |s-v_\alpha t| \leq A\sqrt{t}, |z-m_s| \leq K\}$, with A, K large enough constants. This is consistent with (1.14), and can be thought of as a proof of the tightness of the family of variables

$$\left\{(t^{-1/2}(\tau-(1-v_\alpha)t), X_\emptyset(\tau) - (\sqrt{2}\alpha t - m_{t-\tau}), M_t - \sqrt{2}\alpha t, \mathcal{E}_t), t\geq 0\right\}.$$

For any Borel sets $I\subset[0,t]$ and $B\subset\mathbb{R}$, let

$$U_2^\varphi(\sqrt{2}\alpha t,t,I,B) := \int_I \mathrm{d}s \int_B \mathrm{d}z \frac{e^{-(t-s)-\frac{(m_s+z-\sqrt{2}\alpha t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u_\varphi(m_s+z,s)^2.$$

Lemma 3.1. *Let $\alpha\in(-\gamma,1)$, we set $I_{t,A} = \left[v_\alpha t - A\sqrt{t}, v_\alpha t + A\sqrt{t}\right]\cap[0,t]$ for all $A,t>0$. For all $\varphi\in\mathcal{C}_c^+(\mathbb{R})$, we have*

$$\limsup_{K\to\infty}\limsup_{t\to\infty} \frac{e^{2\gamma(1-\alpha)t}}{t^{3\gamma/2}}\left[U_2^\varphi(\sqrt{2}\alpha t,t) - U_2^\varphi(\sqrt{2}\alpha t,\, t,\, I_{t,A},\, [-K,K])\right] = o_A(1). \tag{3.1}$$

The proof of Lemma 3.1 is postponed to Appendix A.2. A consequence of this result is that the asymptotic behaviour of $U_2^\varphi(\sqrt{2}\alpha t,t)$ as $t\to\infty$ is captured by the following lemma.

Lemma 3.2. *Let $\alpha\in(-\gamma,1)$, we set $I_{t,a,b} = \left[v_\alpha t + a\sqrt{t}, v_\alpha t + b\sqrt{t}\right]\cap[0,t]$ for all $a<b\in\mathbb{R}$. Then for all $a<b$ and $a'<b'$, we have*

$$\lim_{t\to\infty} \frac{e^{2\gamma(1-\alpha)t}}{(v_\alpha t)^{3\gamma/2}} U_2(\sqrt{2}\alpha t, t, I_{t,a,b}, [a',b']) = \int_a^b \frac{e^{-\frac{2\sqrt{2}}{1-\alpha}r^2}}{\sqrt{\sqrt{2}\pi 1-\alpha}}\mathrm{d}r \int_{a'}^{b'} e^{-\sqrt{2}\gamma} w_\varphi(z)^2\mathrm{d}z. \tag{3.2}$$

Proof. Recall that we can write

$$U_2^\varphi(\sqrt{2}\alpha t, t, I_{t,a,b}, [a',b']) = \int_{v_\alpha t + a\sqrt{t}}^{v_\alpha t+b\sqrt{t}} \mathrm{d}s \frac{e^{-(t-s)}}{\sqrt{2\pi(t-s)}} \int_{a'}^{b'} e^{-\frac{(\sqrt{2}\alpha t - m_s - z)^2}{2(t-s)}} u_\varphi(m_s+z,s)^2\mathrm{d}z.$$

By the uniform convergence (2.4), we observe that uniformly in $s\in I_{t,a,b}$,

$$\int_{a'}^{b'} e^{-\frac{(\sqrt{2}\alpha t - m_s - z)^2}{2(t-s)}} u_\varphi(m_s+z,s)^2\mathrm{d}z \sim \int_{a'}^{b'} e^{-\frac{(\sqrt{2}\alpha t - m_s - z)^2}{2(t-s)}} w_\varphi(z)^2\mathrm{d}z,$$

as $t\to\infty$. Then, with the change of variable $s=ut$, we have

$$U_2(\sqrt{2}\alpha t, t, I_{t,a,b}, [a',b'])$$
$$\sim \int_{v_\alpha + \frac{a}{\sqrt{t}}}^{v_\alpha+\frac{b}{\sqrt{t}}} \mathrm{d}u \frac{te^{-tg_\alpha(u)}}{\sqrt{2\pi t(1-u)}} \int_{a'}^{b'} e^{\frac{u-\alpha}{1-u}(\frac{3}{2}\log(ut) - \sqrt{2}z) - \frac{\left(\frac{3}{2\sqrt{2}}\log(ut)-z\right)^2}{2t(1-u)}} w_\varphi(z)^2\mathrm{d}z,$$

as $t \to \infty$, by setting

$$g_\alpha(u) : u \in (0,1) \mapsto (1-u) + \frac{(\alpha - u)^2}{1-u}. \tag{3.3}$$

Note that uniformly in $z \in [a', b']$ and in $u \in [v_\alpha + \frac{a}{\sqrt{t}}, v_\alpha + \frac{b}{\sqrt{t}}]$, as $t \to \infty$ we have

$$\frac{\left(\frac{3}{2\sqrt{t}} \log(ut) - z\right)^2}{2t(1-u)} = o_t(1)$$

$$\text{and} \quad \frac{u - a'}{1-u}\left(\frac{3}{2}\log(ut) - \sqrt{2}z\right) = \frac{3\gamma}{2}\log(v_\alpha t) - \sqrt{2}\gamma z + o_t(1).$$

It then follows that as $t \to \infty$,

$$U_2(\sqrt{2}\alpha t, t, I_{t,a,b}, [a', b'])$$
$$\sim \frac{(v_\alpha t)^{3\gamma/2}}{\sqrt{2\pi(1-v_\alpha)}} \int_{v_\alpha + \frac{a}{\sqrt{t}}}^{v_\alpha + \frac{b}{\sqrt{t}}} e^{-tg_\alpha(u)}\sqrt{t}\mathrm{d}u \int_{a'}^{b'} e^{-\sqrt{2}\gamma z} w_\varphi(z)^2 \mathrm{d}z. \tag{3.4}$$

We only need to estimate the first integration by doing an asymptotic expansion of g_α around v_α. By change of variable $r = \sqrt{t}(u - v_\alpha)$, we have

$$\int_{v_\alpha + \frac{a}{\sqrt{t}}}^{v_\alpha + \frac{b}{\sqrt{t}}} e^{-tg_\alpha(u)}\sqrt{t}\mathrm{d}u = \int_a^b e^{-tg_\alpha(v_\alpha + \frac{r}{\sqrt{t}})}\mathrm{d}r.$$

Note that g_α is smooth and strictly convex, and attains its minimum of $2\gamma(1-\alpha)$ at $u = v_\alpha$. By Taylor's expansion at v_α, we have as $|h| \downarrow 0$,

$$g_\alpha(v_\alpha + h) - g_\alpha(v_\alpha) = \frac{2\sqrt{2}}{1-\alpha}h^2 + o(h^2). \tag{3.5}$$

Hence,

$$\int_{v_\alpha + \frac{a}{\sqrt{t}}}^{v_\alpha + \frac{b}{\sqrt{t}}} e^{-tg_\alpha(u)}\sqrt{t}\mathrm{d}u = e^{-2\gamma(1-\alpha)t} \int_a^b e^{-\frac{2\sqrt{2}}{1-\alpha}r^2 + o_t(1)}\mathrm{d}r$$
$$\sim e^{-2\gamma(1-\alpha)t} \int_a^b e^{-\frac{2\sqrt{2}}{1-\alpha}r^2}\mathrm{d}r,$$

as $t \to \infty$ by dominated convergence. In view of (3.4), this is enough to conclude (3.2). □

To prove (1.5), (1.7) and (1.14), first, for all $\varphi \in \mathcal{C}_c^+(\mathbb{R})$, $x_1, x_2 \in \mathbb{R}$ and $x_3 \geq 0$, we set

$$F_t(\varphi; x_1, x_2, x_3)$$
$$:= \mathbf{E}\left(e^{-\int \varphi \mathrm{d}\mathcal{E}_t(\alpha)}; \frac{\tau - (1-v_\alpha)t}{\sqrt{t}} \leq x_1, X_\emptyset(\tau) \geq \sqrt{2}\alpha t - m_{t-\tau} - x_2, M_t \leq \sqrt{2}\alpha t - x_3\right),$$

and we shall study the asymptotic behaviour of this quantity as $t \to \infty$. Applying the Markov property at time τ, we have

$$F_t(\varphi; x_1, x_2, x_3) = U_2^{\tau_{x_3}\varphi}\left(\sqrt{2}\alpha t, t, \left[v_\alpha t - x_1\sqrt{t}, t\right], (-\infty, x_2]\right),$$

with $\tau_{x_3}\varphi : y \mapsto \varphi(y-x_3)$. Therefore, using Lemma 3.1 gives

$$\lim_{t\to\infty} e^{2\gamma(1-\alpha)t}(v_\alpha t)^{-3\gamma/2} F_t(\varphi; x_1, x_2, x_3)$$
$$= \int_{-x_1}^{A} \frac{e^{-\frac{2\sqrt{2}}{1-\alpha}r^2}}{\sqrt{\sqrt{2}\pi 1-\alpha}}\mathrm{d}r \int_{-K}^{x_2} e^{-\sqrt{2}\gamma z} w_\varphi(z-x_3)^2 \mathrm{d}z + o_A(1) + o_K(1),$$

with the $o_A(1)$ term being uniform in K, using that by definition, $w_{\tau_x\varphi} = w_\varphi(\cdot - x)$. Hence, letting $K\to\infty$ then $A\to\infty$, we conclude that

$$\lim_{t\to\infty} \frac{e^{2\gamma(1-\alpha)t}}{(v_\alpha t)^{3\gamma/2}} F_t(\varphi; x_1, x_2, x_3) = \int_{-x_1}^{\infty} \frac{e^{-\frac{2\sqrt{2}}{1-\alpha}r^2}}{\sqrt{\sqrt{2}\pi 1-\alpha}}\mathrm{d}r \int_{-\infty}^{x_2} e^{-\sqrt{2}\gamma z} w_\varphi(z-x_3)^2 \mathrm{d}z. \tag{3.6}$$

With (3.6) in hand, we are now ready to prove (1.5), (1.7) and (1.14).

Proof of (1.14). We first prove (1.14). By (2.14), we have

$$\mathbf{P}(M_t \le \sqrt{2}\alpha t) = U_1(\sqrt{2}\alpha t, t) + U_2(\sqrt{2}\alpha t, t) = U_2(\sqrt{2}\alpha t, t) + o(t^{3\gamma/2}e^{-(1+\alpha^2)t}),$$

using Lemma 2.4. Applying then Lemma 3.1, for all $A, K > 0$ we have

$$\lim_{t\to\infty} (v_\alpha t)^{-3\gamma/2} e^{(1+\alpha^2)t}\mathbf{P}(M_t \le \sqrt{2}\alpha t)$$
$$= \lim_{t\to\infty} (v_\alpha t)^{-3\gamma/2} e^{(1+\alpha^2)t} F_t(0; A, K, 0) + o_A(1) + o_K(1).$$

Hence, letting $K\to\infty$ then $A\to\infty$, by the monotone convergence theorem, (3.6) yields

$$\lim_{t\to\infty} (v_\alpha t)^{-3\gamma/2} e^{(1+\alpha^2)t}\mathbf{P}(M_t \le \sqrt{2}\alpha t) = \int_{\mathbb{R}} \frac{e^{-\frac{2\sqrt{2}}{1-\alpha}r^2}}{\sqrt{2\pi\frac{1-\alpha}{\sqrt{2}}}}\mathrm{d}r \int_{\mathbb{R}} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z$$
$$= \frac{1}{2}\int_{\mathbb{R}} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z = C^{(1)}.$$

□

Proof of (1.5). By (3.6), for all $x_1, x_2 \in \mathbb{R}$ and $x_3 \in \mathbb{R}_+$, we have

$$\mathbf{P}\left(\frac{\tau - \frac{(1-\alpha)}{\sqrt{2}}t}{\sqrt{t\frac{(1-\alpha)}{4\sqrt{2}}}} \le x_1, X_\emptyset(\tau) - (\sqrt{2}\alpha t - m_{t-\tau}) \ge -x_2, M_t - \sqrt{2}\alpha t \le -x_3\right)$$
$$= F_t\left(0; x_1\sqrt{\tfrac{(1-\alpha)}{4\sqrt{2}}},\, x_2,\, x_3\right)$$
$$\sim (v_\alpha t)^{3\gamma/2} e^{-(1+\alpha^2)t} \int_{-x_1}^{\infty} \frac{e^{-\frac{r^2}{2}}}{\sqrt{8\pi}}\mathrm{d}r \int_{-\infty}^{x_2} e^{-\sqrt{2}\gamma z} w(z-x_3)^2 \mathrm{d}z, \quad \text{as } t\to\infty.$$

Thus

$$\lim_{t\to\infty} \frac{F_t(0; x_1\sqrt{\frac{(1-\alpha)}{4\sqrt{2}}},\, x_2,\, x_3)}{\mathbf{P}(M_t \le \sqrt{2}\alpha t)}$$
$$= \frac{1}{2C^{(1)}} \int_{-\infty}^{x_1} \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}}\mathrm{d}r \cdot e^{-\sqrt{2}\gamma x_3} \int_{-\infty}^{x_2 - x_3} e^{-\sqrt{2}\gamma z} w(z)^2 \mathrm{d}z,$$

which yields (1.5). □

Proof of (1.7). We finally turn to the proof of (1.7), i.e., the joint convergence in distribution of the extremal process seen from $\sqrt{2}\alpha t$, conditioned on $\{M_t \leq \sqrt{2}\alpha t\}$. By a straightforward adaptation of [10, Proposition 2.2], to obtain this weak convergence, it is enough to obtain for all $\varphi \in \mathcal{C}_c^+(\mathbb{R})$ and $x_1, x_2 \in \mathbb{R}$, $x_3 \geq 0$ the convergence

$$\lim_{t\to\infty} \mathbf{E}\left[e^{-\int \varphi \mathrm{d}\mathcal{E}_t(\alpha)}; \tau \leq (1-v_\alpha)t + x_1\sqrt{t}, X_\emptyset(\tau) \geq \sqrt{2}\alpha t - m_{t-\tau} - x_2 \middle| M_t \leq \sqrt{2}\alpha t\right] \\ = \mathbf{E}\left[e^{-\int \varphi \mathrm{d}\mathcal{E}^-}; \chi \leq x_2\right] \mathbf{P}\left(\xi \leq x_1 \sqrt{\tfrac{4\sqrt{2}}{1-\alpha}}\right).$$

By (3.6) and (1.7), we have immediately that

$$\lim_{t\to\infty} \frac{F(\varphi, x_1, x_2, 0)}{\mathbf{P}(M_t \leq \sqrt{2}\alpha t)} = \frac{1}{2C^{(1)}} \int_{-\infty}^{x_1} \mathrm{d}r \frac{e^{-\frac{2\sqrt{2}r^2}{1-\alpha}}}{\sqrt{2\pi \frac{1-\alpha}{4\sqrt{2}}}} \times \int_{-\infty}^{x_2} e^{-\sqrt{2}\gamma z} w_\varphi(z)^2 \mathrm{d}z.$$

Observe that according to the definition of $\mathcal{E}^-$, writing $\mathcal{E}$ for the limiting extremal process of the unconditioned branching Brownian motion, we have

$$\mathbf{E}\left[e^{-\int \varphi \mathrm{d}\mathcal{E}^-}; \chi \leq x_2\right] = \int_{-\infty}^{x_2} \mathbf{E}\left[e^{-\sum_{x\in\mathcal{E}} \varphi(x-z)} \middle| \max \mathcal{E} \leq z\right]^2 \mathbf{P}(\chi \in \mathrm{d}z) \\ = \frac{1}{2C^{(1)}} \int_{-\infty}^{x_2} e^{-\sqrt{2}\gamma z} w_\varphi(z)^2 \mathrm{d}z,$$

which is therefore enough to end the proof. □

Remark 3.3. Theorem 1.5, which can be thought of as $\alpha = 1 - o_t(1)$, could be obtained following a similar line of proof as above. The principal difference is that the Laplace method in the proof of Lemma 3.2 has to be applied with a maximum obtained on the boundary of the interval of definition. All other estimates follow with straightforward modifications, by replacing $1-\alpha$ by $a_t/\sqrt{2}t$. We thus feel free to omit the proof of Theorem 1.5.

4 The case $\alpha < -\gamma$

We now treat the case of $\alpha < -\gamma$. We use in this section that, conditioned on the event $\{M_t \leq \sqrt{2}\alpha t\}$, with high probability no branching occurs before time $t - O(1)$. We use this observation to prove (1.10), (1.11) and (1.16), by using the decomposition presented in Lemma 2.4. Contrarily to the previous case, U_1^φ and U_2^φ are of the same order of magnitude.

Note that the asymptotic behaviour of $U_1^\varphi(\sqrt{2}\alpha t, t)$ is given by Lemma 2.4. To study the asymptotic behaviour of $U_2^\varphi(\sqrt{2}\alpha t, t)$, we begin by showing that $t - \tau$ is tight conditioned on $\{M_t \leq \sqrt{2}\alpha t\}$.

Lemma 4.1. *Assume that $\alpha < -\gamma$, then for all $\varphi \in \mathcal{C}_c^+(\mathbb{R})$ we have*

$$\lim_{A\to\infty} \lim_{t\to\infty} \sqrt{t} e^{(1+\alpha^2)t} U_2^\varphi(\sqrt{2}\alpha t, t, [A, t], \mathbb{R}) = 0. \tag{4.1}$$

The proof of Lemma 4.1 is postponed to Appendix A.3. The next lemma completes the description of the asymptotic of U_2^φ.

Lemma 4.2. *If $\alpha < -\gamma$, then for any $x > 0$ and $-\infty \le c < d \le \infty$ we have*

$$\lim_{t\to\infty} \sqrt{t}e^{(1+\alpha^2)t}U_2^{\varphi}(\sqrt{2}\alpha t, t, [0,x], (c,d)) = \int_0^x \int_c^d e^{\sqrt{2}\alpha y} u_\varphi(y,s)^2 e^{(1-\alpha^2)s} \mathrm{d}y\mathrm{d}s. \tag{4.2}$$

Moreover, we have

$$\int_0^\infty \int_{\mathbb{R}} e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}y\mathrm{d}s < \infty. \tag{4.3}$$

Proof. Fix $x > 0$ and $-\infty \le c < d \le \infty$. Observe that we can rewrite

$$\begin{aligned}
U_2^{\varphi}(\sqrt{2}\alpha t, t, [0,x], [c,d]) &= \int_0^x \mathrm{d}s \int_c^d \mathrm{d}y \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - y)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u_\varphi(y,s)^2 \\
&= \frac{e^{-(1+\alpha^2)t}}{\sqrt{2\pi t}} \int_0^x \mathrm{d}s \int_c^d \mathrm{d}y \sqrt{\frac{t}{t-s}} e^{(1-\alpha^2)s} e^{\sqrt{2}\alpha y - \frac{(\sqrt{2}\alpha s - y)^2}{2(t-s)}} u_\varphi(y,s)^2 \\
&\sim \frac{e^{-(1+\alpha^2)t}}{\sqrt{2\pi t}} \int_0^x \mathrm{d}s \int_c^d \mathrm{d}y e^{(1-\alpha^2)s} e^{\sqrt{2}\alpha y - \frac{(\sqrt{2}\alpha s - y)^2}{2(t-s)}} u_\varphi(y,s)^2,
\end{aligned}$$

as $t \to \infty$. Then, by the monotone convergence theorem, as $t \to \infty$ we have

$$\int_0^x \int_c^d e^{\sqrt{2}\alpha y - \frac{(\sqrt{2}\alpha s - y)^2}{2(t-s)}} u_\varphi(y,s)^2 e^{(1-\alpha^2)s} \mathrm{d}y\mathrm{d}s \to \int_0^x \int_c^d e^{\sqrt{2}\alpha y} u_\varphi(y,s)^2 e^{(1-\alpha^2)s} \mathrm{d}y\mathrm{d}s,$$

which completes the proof of (4.2).

The rest of the proof is devoted to show that $\int_0^\infty \int_{\mathbb{R}} e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}s\mathrm{d}y < \infty$. As a first step, we bound for any $s \ge 0$ the quantity $I_s := \int_{\mathbb{R}} e^{\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}y$. First, by (2.7), for all $y \le 0$ we have $0 \le u(y,s) \le \frac{s^{1/2}}{|y|} e^{-y^2/2s} \wedge 1$, therefore

$$I_s \le \int_{-\infty}^1 e^{\sqrt{2}\alpha y} \mathrm{d}y + \int_1^\infty e^{\sqrt{2}\alpha y} e^{-y^2/s} \mathrm{d}y \le \frac{e^{\sqrt{2}\alpha}}{-\sqrt{2}\alpha} + s\sqrt{\pi s} e^{\alpha^2 s},$$

by (2.6). As a result, for all $A > 0$, we have

$$\int_0^A \int_{\mathbb{R}} e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}s\mathrm{d}y = \int_0^A e^{(1-\alpha^2)s} I_s \mathrm{d}s < \infty. \tag{4.4}$$

To complete the proof of (4.3), it is enough to bound $\int_A^\infty \int_{\mathbb{R}} e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}y\mathrm{d}s$ for $A \ge 1$ large enough. Recall that $u(y,s) = \mathbf{P}(M_s \le y)$ is close to 1 for $y \gg \sqrt{2}s$ and to 0 for $y \ll \sqrt{2}s$. Observe that

$$\begin{aligned}
\int_A^\infty \int_{\sqrt{2}s}^\infty e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}y\mathrm{d}s &\le \int_A^\infty \int_{\sqrt{2}s}^\infty e^{(1-\alpha^2)s+\sqrt{2}\alpha y} \mathrm{d}y\mathrm{d}s \\
&= \frac{1}{-\sqrt{2}\alpha} \int_A^\infty e^{(1-\alpha^2)s+2\alpha s} \mathrm{d}s < \infty,
\end{aligned} \tag{4.5}$$

using that for all $\alpha < -\gamma$, $1 - \alpha^2 + 2\alpha < 0$. Therefore, we only need to bound

$$\int_A^\infty \int_{-\infty}^{\sqrt{2}s} e^{(1-\alpha^2)s+\sqrt{2}\alpha y} u(y,s)^2 \mathrm{d}y\mathrm{d}s = \int_A^\infty \int_{-\infty}^1 e^{(1-\alpha^2)s+2x\alpha s} u(\sqrt{2}xs, s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s,$$

by change of variable $y = \sqrt{2}sx$. We now apply Lemma 2.1 to bound $u(\sqrt{2}xs, s)^2$ for s large enough, depending on the region to which x belongs.

Let $\varepsilon > 0$, that will be taken small enough later on, and $\beta > 1 - \alpha/2 > 1$. We assume that $A > t_{\varepsilon,\beta}$, and we bound the above integral using (2.5). First, for x in the interval $[-\gamma, 1]$, we have

$$\begin{aligned}
&\int_A^\infty \int_{-\gamma}^1 e^{(1-\alpha^2)s+2\alpha xs} u(\sqrt{2}xs, s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s \\
&\leq \int_A^\infty \int_{-\gamma}^1 e^{(1-\alpha^2)s+2\alpha xs} e^{-4\gamma(1-x)s+2\varepsilon s} \sqrt{2}s \mathrm{d}x\mathrm{d}s \\
&\leq \int_A^\infty \sqrt{2}se^{(1-\alpha^2-4\gamma+2\varepsilon)s} \int_{-\gamma}^1 e^{(2\alpha+4\gamma)xs} \mathrm{d}x\mathrm{d}s \\
&\leq \begin{cases} \frac{1}{\sqrt{2}(\alpha+2\gamma)} \int_A^\infty e^{(1-\alpha^2+2\alpha)s+2\varepsilon s}\mathrm{d}s, & \text{if } -2\gamma < \alpha < -\gamma; \\ \sqrt{2}(1+\gamma) \int_A^\infty se^{(1-\alpha^2+2\alpha)s+2\varepsilon s}\mathrm{d}s, & \text{if } \alpha = -2\gamma; \\ \frac{1}{-\sqrt{2}(\alpha+2\gamma)} \int_A^\infty e^{(1-\alpha^2-4\gamma-2\alpha\gamma-4\gamma^2)s+2\varepsilon s}\mathrm{d}s, & \text{if } \alpha < -2\gamma. \end{cases}
\end{aligned}$$

As $1 - \alpha^2 - 4\gamma - 2\alpha\gamma - 4\gamma^2 < 0$ for all $\alpha < -2\gamma$, we conclude that for all $\varepsilon > 0$ small enough,

$$\int_A^\infty \int_{-\gamma}^1 e^{(1-\alpha^2)s+2\alpha xs} u(\sqrt{2}xs, s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s < \infty. \tag{4.6}$$

We then consider the case $x \in [-\beta, -\gamma]$. In fact

$$\begin{aligned}
&\int_A^\infty \int_{-\beta}^{-\gamma} e^{(1-\alpha^2)s+2\alpha xs} u(\sqrt{2}xs, s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s \\
&\leq \int_A^\infty \int_{-\beta}^{-\gamma} e^{(1-\alpha^2)s+2\alpha xs} e^{-2(1+x^2)s+2\varepsilon s} \sqrt{2}s \mathrm{d}x\mathrm{d}s \\
&\leq \int_A^\infty \sqrt{2}se^{-(1+\alpha^2/2-2\varepsilon)s} \int_{-\beta}^{-\gamma} e^{-2(x-\alpha/2)^2 s} \mathrm{d}x\mathrm{d}s \\
&\leq \int_A^\infty \sqrt{\frac{\pi s}{2}} e^{-(1+\alpha^2/2-2\varepsilon)s} \mathrm{d}s < \infty,
\end{aligned} \tag{4.7}$$

for all $\varepsilon < 1/2$. Similarly, for $x < -\beta$:

$$\begin{aligned}
\int_A^\infty \int_{-\infty}^{-\beta} e^{(1-\alpha^2)s+2\alpha xs} u(\sqrt{2}xs, s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s &\leq \int_A^\infty \sqrt{2}se^{(1-\alpha^2/2)s} \int_{-\infty}^{-\beta} e^{-2(x-\alpha/2)^2 s} \mathrm{d}x\mathrm{d}s \\
&\leq \int_A^\infty \sqrt{2}se^{(1-\alpha^2/2)s} \int_{-\infty}^{-\beta-\alpha/2} e^{-2y^2 s} \mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Using that $-\beta - \alpha/2 < -1$, we have for all $s > 0$:

$$\int_{-\infty}^{-\beta-\alpha/2} e^{-2y^2 s} \mathrm{d}y \leq \int_1^\infty e^{-2y^2 s} \mathrm{d}y \leq \frac{1}{4s} e^{-2s},$$

by (2.6), yielding

$$\int_A^\infty \int_{-\beta}^{-\gamma} e^{(1-\alpha^2)s+2\alpha xs} u(\sqrt{2}xs, s)^2 \sqrt{2}s \mathrm{d}x\mathrm{d}s \leq \frac{1}{2\sqrt{2}} \int_A^\infty e^{(-1-\alpha^2/2)s} < \infty. \tag{4.8}$$

Consequently, using (4.5–4.8), for any $A > 0$ large enough

$$\int_A^\infty \int_{-\infty}^1 e^{(1-\alpha^2)s+2x\alpha s} u^2(\sqrt{2}xs, s)\sqrt{2}s\mathrm{d}x\mathrm{d}s < \infty,$$

which, with (4.4), completes the proof of (4.3). □

We now first prove the joint convergence in law of the first branching time and position, and the shifted extremal process, conditioned on $\{M_t \le \sqrt{2}\alpha t\}$, when $\alpha < -\gamma$.

Observe that by same reasonings as (2.17), we have

$$\begin{aligned}\mathbf{E}\left(e^{-\int \varphi \mathrm{d}\mathcal{E}_t(\alpha)}; \tau \ge t, M_t \le \sqrt{2}\alpha t - z\right) \\ = \mathbf{E}\left(e^{-\varphi(B_t - \sqrt{2}\alpha t)}; B_t \le \sqrt{2}\alpha t - z\right) \sim \frac{e^{-(1+\alpha^2)t}}{\sqrt{2\pi t}} \int_{-\infty}^{-z} e^{-\varphi(y)-\sqrt{2}\alpha y}\mathrm{d}y, \quad (4.9)\end{aligned}$$

as $t \to \infty$. Similarly to the case $\alpha \in (-\gamma, 1)$, the key to the proof of this theorem is the determination of the asymptotic behavior of

$$\begin{aligned}&G_t(\varphi;\ x_1, x_2, x_3) \\ &:= \mathbf{E}\left(e^{-\int \varphi \mathrm{d}\mathcal{E}_t(\alpha)}; t - t\wedge\tau \le x_1, x_2 \le \sqrt{2}\alpha t - X_\emptyset(\tau \wedge t), M_t - \sqrt{2}\alpha t \le -x_3\right),\end{aligned}$$

as $t \to \infty$, for $x_1 \ge 0, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}$.

Using the branching property at time τ, we observe that

$$G_t(\varphi;\ x_1, x_2, x_3) = U_2^{\tau_{x_3}\varphi}(\sqrt{2}\alpha t, t, [0, x_1], [x_2, \infty)),$$

and therefore, by Lemma 4.2 we have

$$\lim_{t\to\infty} t^{1/2} e^{(1+\alpha^2)t} G_t(\varphi;\ x_1, x_2, x_3) = \int_0^{x_1} e^{(1-\alpha^2)s} \int_{x_2}^\infty e^{\sqrt{2}\alpha y} u_\varphi(y - x_3, s)^2 \mathrm{d}y\mathrm{d}s. \quad (4.10)$$

We are now in the position to prove (1.10), (1.11) and (1.16). We begin with proving (1.16).

Proof of (1.16). Observe that by (2.13), we have

$$\mathbf{P}(M_t \le \sqrt{2}\alpha t) = U_1(\sqrt{2}\alpha t, t) + U_2(\sqrt{2}\alpha t, t).$$

Using (4.10) with $x_2 = -\infty$ and $x_1 = A$ together with Lemma 4.1, we have letting $t \to \infty$ then $A \to \infty$:

$$\lim_{t\to\infty} t^{1/2} e^{(1+\alpha^2)t} U_2(\sqrt{2}\alpha t, t) = \int_0^\infty e^{(1-\alpha^2)s} \int_{\mathbb{R}} e^{\sqrt{2}\alpha y} u(y - x_3, s)^2 \mathrm{d}y\mathrm{d}s,$$

which, together with (4.9), implies $\mathbf{P}(M_t \le \sqrt{2}\alpha t) \sim \frac{\Phi(\alpha)}{\sqrt{4\pi t}} e^{-(1+\alpha^2)t}$ as $t \to \infty$.

□

Proof of (1.10) *and* (1.11). To prove (1.10), it is enough to observe that

$$\lim_{t\to\infty} \frac{G_t(\varphi;\ x_1, x_2, x_3)}{\mathbf{P}(M_t \le \sqrt{2}\alpha t)} = \frac{\sqrt{4\pi}}{\Phi(\alpha)} \int_0^{x_1} e^{(1-\alpha^2)s} e^{\sqrt{2}\alpha x_3} \int_{-\infty}^{x_2 - x_3} e^{\sqrt{2}\alpha y} u_\varphi(y, s)^2 \mathrm{d}y\mathrm{d}s,$$

by (4.10). This proves that $(t-t\wedge\tau, \sqrt{2}\alpha t-X_\emptyset(t\wedge\tau), M_t-\sqrt{2}\alpha t)$ jointly converge in distribution as $t\to\infty$.

We now prove the convergence of the extremal process $\mathcal{E}_t(\alpha)$. For any $\varphi\in\mathcal{C}_c^+(\mathbb{R})$, using again the decomposition at first branching time of the branching Brownian motion, we have

$$\mathbf{E}\left[e^{-\int\varphi \mathrm{d}\mathcal{E}_t(\alpha)}; M_t\le\sqrt{2}\alpha t\right]=U_1^\varphi(\sqrt{2}\alpha t,t)+U_2^\varphi(\sqrt{2}\alpha t,t),$$

which, by (4.10) and (1.16), yields

$$\lim_{t\to\infty}\mathbf{E}\left[e^{-\int\varphi \mathrm{d}\mathcal{E}_t(\alpha)}\Big|M_t\le\sqrt{2}\alpha t\right]$$
$$=\frac{\sqrt{2}}{\Phi(\alpha)}\Big(\int_{-\infty}^0 e^{-\varphi(z)-\sqrt{2}\alpha z}\mathrm{d}z+\int_0^\infty \mathrm{d}s\int_{\mathbb{R}}e^{\sqrt{2}\alpha z+(1-\alpha^2)s}u_\varphi(z,s)^2\mathrm{d}z\Big).$$

As $u(z,s)=\mathbf{E}\left(e^{-\sum_{u\in N(t)}\varphi(X_u(s)-z)};M_s\le z\right)$, we observe that we can rewrite this limit as

$$\int_{\mathbb{R}_+\times\mathbb{R}}\mathbf{E}\left[\exp\Big(-\sum_{u\in N(s)}\varphi(X_u(s)-z)\Big)\Big|M_s\le z\right]^2\mathbf{P}(\xi_\alpha\in\mathrm{d}s,\chi_\alpha\in\mathrm{d}z)$$
$$=\mathbf{E}[e^{-\int\varphi\mathrm{d}\mathcal{E}_\infty(\alpha)}],$$

proving that $\mathcal{E}_t(\alpha)$ converges weakly to $\mathcal{E}_\infty(\alpha)$ in $\mathbf{P}(\cdot|M_t\le\sqrt{2}\alpha t)$-distribution. In the same spirit as in the proof of case $\alpha>-\gamma$, one could obtain the joint convergence in distribution of $\mathcal{E}_t(\alpha)$ with $\Big(t-t\wedge\tau, X_\emptyset(t\wedge\tau)\Big)$, thus completing the proof of (1.11). □

5 The critical case $\alpha=-\gamma$

We consider in this section the case $\alpha=1-\sqrt{2}=-\gamma$ and prove (1.8), (1.9) and (1.15). In this case, the first branching time should occur around time $t-O(t^{1/2})$ with high probability. We use again (2.14) to compute the asymptotic behaviour of $\mathbf{P}(M_t\le-\sqrt{2}\gamma t)$, and decompose the integral (2.12) onto sub-intervals of interest to prove the joint convergence in distribution of the first branching time and position and the extremal process of the branching Brownian motion.

Recall that, by (2.13), we have $u(-\sqrt{2}\gamma t,t)=U_1(-\sqrt{2}\gamma t,t)+U_2(-\sqrt{2}\gamma t,t)$, and by Lemma 2.4, we have, as $t\to\infty$,

$$U_1(-\sqrt{2}\gamma t,t)\sim\frac{1}{\sqrt{4\pi t}}e^{-(1+\gamma^2)t}\ll t^{3\gamma/4}e^{-(1+\gamma^2)t}. \tag{5.1}$$

Therefore, to complete the proof of (1.15), it is enough to show that

$$U_2(-\sqrt{2}\gamma t,t)\sim C^{(2)}t^{3\gamma/4}e^{-(1+\gamma^2)t},\ \text{as } t\to\infty. \tag{5.2}$$

Moreover, for any $0<a<b<t$ and $a'<b'$, we set

$$U_2(-\sqrt{2}\gamma t,t,[a,b],[a',b']):=\int_a^b e^{-(t-s)}ds\int_{a'}^{b'}\frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t-a_t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}u(m_s+z,s)^2\mathrm{d}z.$$

Equation (5.2) follows from the next two lemmas.

Lemma 5.1. *For all $A, K > 0$, we have*

$$\limsup_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} \left| U_2(-\sqrt{2}\gamma t, t) - U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [-K, K]) \right| = o_A(1) + o_K(1).$$

This lemma allows to localise the first branching time and position, conditioned on $\{M_t \leq -\sqrt{2}\gamma t\}$. We note that it is similar to the proof of Lemma 3.1, and postpone its proof to Appendix A.4. The next lemma gives a more detailed estimate of the time at which this branching event occurs.

Lemma 5.2. *For any $0 < a < b$ and $c < d$ fixed, for all $\varphi \in \mathcal{C}_c^+(\mathbb{R})$, we have*

$$\lim_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} U_2^\varphi(-\sqrt{2}\gamma t, t, [a\sqrt{t}, b\sqrt{t}], [a', b']) = \frac{1}{\sqrt{2\pi}} \int_a^b r^{3\gamma/2} e^{-2r^2} \mathrm{d}r \int_{a'}^{b'} e^{-\sqrt{2}\gamma z} w_\varphi(z)^2 \mathrm{d}z.$$

Proof. This proof is similar to the proofs of Lemmas 3.2 and 4.2. By (2.4), we have

$$\begin{aligned} U_2^\varphi(-\sqrt{2}\gamma t, t, [a\sqrt{t}, b\sqrt{t}], [a', b']) &= \int_{a\sqrt{t}}^{b\sqrt{t}} e^{-(t-s)} \mathrm{d}s \int_{a'}^{b'} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u_\varphi(m_s + z, s)^2 \mathrm{d}z \\ &\sim \int_{a\sqrt{t}}^{b\sqrt{t}} e^{-(t-s)} \mathrm{d}s \int_{a'}^{b'} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} w_\varphi(z)^2 \mathrm{d}z, \end{aligned}$$

as $t \to \infty$. Note that $1 + \gamma^2 = 2\sqrt{2}\gamma$. Hence by simple calculations we obtain

$$U_2^\varphi(-\sqrt{2}\gamma t, t, [a\sqrt{t}, b\sqrt{t}], [a', b']) \sim e^{-(1+\gamma^2)t} \int_{a\sqrt{t}}^{b\sqrt{t}} s^{3\gamma/2} e^{-\frac{2s^2}{t}} \frac{\mathrm{d}s}{\sqrt{2\pi t}} \int_{a'}^{b'} e^{-\sqrt{2}\gamma z} w_\varphi^2(z) \mathrm{d}z, \quad \text{as } t \to \infty.$$

With a change of variable $s = r\sqrt{t}$, the proof is now complete. ∎

Now we are ready to prove (1.10), (1.11) and (1.16).

Proof of (1.15). Recall that it is enough to prove (5.2). For all $t > 0$ and $A, K > 0$, we have

$$\begin{aligned} U_2(-\sqrt{2}\gamma t, t) = &\Big(U_2(-\sqrt{2}\gamma t, t) - U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [-K, K]) \Big) \\ &+ U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [-K, K]). \end{aligned}$$

Therefore, using Lemmas 5.1 and 5.2, we obtain

$$\lim_{t\to\infty} t^{-3\gamma/4} e^{(1+\gamma^2)t} U_2(-\sqrt{2}\gamma t, t) = o_A(1) + o_K(1) + \frac{1}{\sqrt{2\pi}} \int_{1/A}^{A} r^{3\gamma/2} e^{-2r^2} \mathrm{d}r \int_{-K}^{K} e^{-\sqrt{2}\gamma z} w^2(z) \mathrm{d}z,$$

which converges to $C^{(2)}$ as $A, K \to \infty$. ∎

Proof of (1.8) *and* (1.9). For any $x_1, x_3 \in \mathbb{R}_+$ and $x_2 \in \mathbb{R}$, applying the Markov property at time τ, and using (5.1), we have

$$\begin{aligned}&\mathbf{P}\Big(\tau \geq t - x_1\sqrt{t}, X_\emptyset(\tau) \geq (-\sqrt{2}\gamma t + m_{t-\tau}) - x_2, M_t \leq -\sqrt{2}\gamma t - x_3\Big)\\ =&\int_{t-x_1\sqrt{t}}^t e^{-r}\mathrm{d}r \int_{(-\sqrt{2}\gamma t - m_{t-r})-x_2}^{\infty} \mathbf{P}(B_r \in \mathrm{d}y)u^2(-\sqrt{2}\gamma t - x_3 - y, t-r)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + o_t(1)t^{3\gamma/4}e^{-(1+\gamma^2)t}\\ =&\int_0^{x_1\sqrt{t}} \mathrm{d}s \int_{-\infty}^{x_2-x_3} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t+x_3)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u^2(m_s+z,s)\mathrm{d}z + o_t(1)t^{3\gamma/4}e^{-(1+\gamma^2)t},\end{aligned}$$

using the change of variables $s = t - r$ and $z = -\sqrt{2}\gamma t - x_3 - y - m_s$. We now apply Lemma 5.1 to obtain

$$\begin{aligned}&\mathbf{P}\Big(\tau \geq t - x_1\sqrt{t}, X_\emptyset(\tau) \geq (-\sqrt{2}\gamma t + m_{t-\tau}) - x_2, M_t \leq -\sqrt{2}\gamma t - x_3\Big)\\ =&(o_{A,t}(1) + o_{K,t}(1) + o_t(1))t^{3\gamma/4}e^{-(1+\gamma^2)t}\\ &\qquad + \int_{\sqrt{t}/A}^{x_1\sqrt{t}} \mathrm{d}s \int_{-K}^{x_2-x_3} \frac{e^{-\frac{(z+m_s+\sqrt{2}\gamma t+x_3)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u^2(m_s+z,s)\mathrm{d}z.\end{aligned}$$

Then Lemma 5.2 gives

$$\begin{aligned}&\mathbf{P}\Big(\tau \geq t - x_1\sqrt{t}, X_\emptyset(\tau) \geq (-\sqrt{2}\gamma t + m_{t-\tau}) - x_2, M_t \leq -\sqrt{2}\gamma t - x_3\Big)\\ &\qquad\sim \frac{1}{\sqrt{2\pi}} t^{3\gamma/4}e^{-(1+\gamma^2)t} \int_0^{x_1} r^{3\gamma/2}e^{-2r^2}\mathrm{d}r \int_{-\infty}^{x_2-x_3} e^{-\sqrt{2}\gamma(z+x_3)}w^2(z)\mathrm{d}z \text{ as } t \to \infty,\end{aligned}$$

which, together with (1.15) proves (1.8).

We now turn to the proof of (1.9). For any $\varphi \in \mathcal{C}_c^+(\mathbb{R})$, using again the Markov property at time τ, we have

$$\begin{aligned}&\mathbf{E}[e^{-\int \varphi \mathrm{d}\mathcal{E}_t(-\gamma)}; M_t \leq -\sqrt{2}\gamma t]\\ =&e^{-t}\,\mathbf{E}[e^{-\varphi(B_t+\sqrt{2}\gamma t)}; B_t \leq -\sqrt{2}\gamma t]\\ &+ \int_0^t e^{-r}\mathrm{d}r \int_{\mathbb{R}} \mathbf{P}(B_r \in \mathrm{d}y)\,\mathbf{E}\left[e^{-\sum_{u\in N(t-r)}\varphi(X_u(t-r)+y+\sqrt{2}\gamma t)}; M_{t-r} \leq -\sqrt{2}\gamma t - y\right]^2.\end{aligned}$$

On the one hand, by (2.16),

$$e^{-t}\,\mathbf{E}[e^{-\varphi(B_t+\sqrt{2}\gamma t)}; B_t \leq -\sqrt{2}\gamma t] \leq e^{-t}\mathbf{P}(B_t \leq -\sqrt{2}\gamma t) = o_t(1)t^{3\gamma/4}e^{-(1+\alpha^2)t}.$$

On the other hand, using again Lemmas 5.1 and 5.2, we obtain

$$\begin{aligned}&\int_0^t e^{-r}\mathrm{d}r \int_{\mathbb{R}} \mathbf{P}(B_r \in \mathrm{d}y)\,\mathbf{E}[e^{-\sum_{u\in N(t-r)}\varphi(X_u(t-r)+y+\sqrt{2}\gamma t)}; M_{t-r} \leq -\sqrt{2}\gamma t - y]^2\\ \sim& t^{3\gamma/4}e^{-(1+\gamma^2)t} \int_0^\infty r^{3\gamma/2}e^{-2r^2}\frac{dr}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\sqrt{2}\gamma z}w_\varphi^2(z)\mathrm{d}z,\end{aligned}$$

as $t \to \infty$. It thus follows, using (1.15), that

$$\lim_{t\to\infty} \mathbf{E}\left[e^{-\int \varphi d\mathcal{E}_t(-\gamma)} | M_t \leq -\sqrt{2}\gamma t\right] = \frac{1}{2C^{(1)}} \int_{\mathbb{R}} e^{-\sqrt{2}\gamma z}w_\varphi^2(z)\mathrm{d}z = \mathbf{E}\left[e^{-\int \varphi d\mathcal{E}^-}\right],$$

which, by [10, Proposition 2.2] is enough to conclude (1.9). □

A Proof of Lemmas

We prove in this section some of technical lemmas, that are needed to complete the proofs in the previous sections.

A.1 Proof of Lemma 2.2

Recall that Lemma 2.2 consists in the following non-asymptotic estimate: for all $\delta > 0$, $\mathbf{P}(M_t \leq m_t - z) \leq c_\delta e^{-\sqrt{2}\gamma(1-\delta)z}$ for all $t, z \geq 1$.

Proof of Lemma 2.2. We begin by bounding $\mathbf{P}(M_t \leq m_t - z)$ for $1 < z \leq t$. Denote by $n(t) := \# N(t)$ the total number of particles alive at time t. As every individual gives birth at exponential rate to two children, the process $(n(t), t \geq 0)$ is a standard Yule process. Hence $\mathbf{P}(n(t) = k) = e^{-t}(1 - e^{-t})^{k-1}$ for any $k \in \mathbb{N}$. Let $\eta \in (0, \delta)$ small enough, such that $J := \lfloor \frac{\sqrt{2}\gamma}{\eta} \rfloor \geq 1$. Observe that

$$
\begin{aligned}
\mathbf{P}(M_t \leq m_t - z) \leq & \mathbf{P}(n(Jz\eta) \leq z^3) + \mathbf{P}(n(Jz\eta) > z^3; M_t \leq m_t - z) \\
\leq & z^3 e^{-Jz\eta} + \sum_{k=1}^{J} \mathbf{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); M_t \leq m_t - z). \qquad (A.1)
\end{aligned}
$$

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, independent of the branching Brownian motion. Using [23, Lemma 5.1], for any $0 < s < t$ and $x \in \mathbb{R}$, we have

$$
\mathbf{P}(M_t \leq x) \leq \mathbf{P}\Big(\max_{u \in N(s)} (B_s + M^u_{t-s}) \leq x \Big),
$$

where $M^u_y := \max_{v \in N(y+s), u \preccurlyeq v} X_v(y+s) - X_u(s)$, and $u \preccurlyeq v$ means that v is a descendant of u. For any $1 \leq k \leq J$, one has

$$
\begin{aligned}
& \mathbf{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); M_t \leq m_t - z) \\
\leq & \mathbf{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); \max_{u \in N(kz\eta)} (B_{kz\eta} + M^u_{t-kz\eta}) \leq m_t - z) \\
\leq & \mathbf{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); B_{kz\eta} \leq m_t - m_{t-kz\eta} - z) + \mathbf{P}(M_{t-kz\eta} \leq m_{t-kz\eta})^{z^3}. \qquad (A.2)
\end{aligned}
$$

On the one hand, for $1 \leq k \leq J$, by (2.6),

$$
\begin{aligned}
& \mathbf{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); B_{kz\eta} \leq m_t - m_{t-kz\eta} - z) \\
\leq & \mathbf{P}(n((k-1)z\eta) \leq z^3)\mathbf{P}(B_{kz\eta} \leq -(1 - \sqrt{2}k\eta)z) \\
\leq & z^3 e^{-(k-1)z\eta} \frac{\sqrt{kz\eta}}{(1 - \sqrt{2}k\eta)z} e^{-\frac{(1-\sqrt{2}k\eta)^2}{2k\eta} z}.
\end{aligned}
$$

As $(1 - \sqrt{2}k\eta)\sqrt{z} \geq (1 - 2\gamma)\sqrt{z} > \sqrt{J\eta}$ for $z > 100$, we deduce that

$$
\begin{aligned}
\mathbf{P}(n((k-1)z\eta) \leq z^3 < n(kz\eta); B_{kz\eta} \leq m_t - m_{t-kz\eta} - z) \\
\leq z^3 e^{\eta z} \exp\Big\{ - \Big[k\eta + \frac{(1 - \sqrt{2}k\eta)^2}{2k\eta}\Big] z \Big\}. \qquad (A.3)
\end{aligned}
$$

On the other hand, as $t - J\eta z \geq (1-\sqrt{2}\gamma)t \to \infty$ as $t \to \infty$, using that $z \leq t$ and the convergence (2.3), there exist $t_0 \geq 1$ and $c_0 > 0$ such that for all $t \geq t_0$ and $1 \leq z \leq t$, one has $\mathbf{P}(M_{t-kz\eta} \leq m_{t-kz\eta}) \leq e^{-c_0} < 1$. Then,

$$\mathbf{P}(M_{t-kz\eta} \leq m_{t-kz\eta})^{z^3} \leq e^{-c_0 z^3}. \tag{A.4}$$

As a result, using (A.2), (A.3) and (A.4), for $t > t_0$ and $100 < z \leq t$, (A.1) becomes that

$$\begin{aligned}
&\mathbf{P}(M_t \leq m_t - z)\\
\leq & z^3 e^{\eta z} e^{-\sqrt{2}\gamma z} + J \sup_{1\leq k\leq J}\Big(z^3 e^{\eta z}\exp\Big(-\Big[k\eta + \frac{(1-\sqrt{2}k\eta)^2}{2k\eta}\Big]z\Big) + e^{-c_0 z^3}\Big)\\
\leq & z^3 e^{\eta z} e^{-\sqrt{2}\gamma z} + \frac{\sqrt{2}\gamma}{\eta}\sup_{0<s<\sqrt{2}\gamma}\Big(z^3 e^{\eta z}\exp\Big(-\Big[s + \frac{(1-\sqrt{2}s)^2}{2s}\Big]z\Big) + e^{-c_0 z^3}\Big)\\
= & z^3 e^{\eta z} e^{-\sqrt{2}\gamma z} + \frac{\sqrt{2}\gamma}{\eta}\Big(z^3 e^{\eta z} e^{-\sqrt{2}\gamma z} + e^{-c_0 z^3}\Big).
\end{aligned}$$

For $\delta \in (0,1)$ small enough, we could take $\eta = \sqrt{2}\gamma\delta/2$, $t \geq t_0$ and $z \in [K_\delta, t]$ such that

$$\mathbf{P}(M_t \leq m_t - z) \leq c_\delta e^{-\sqrt{2}\gamma(1-\delta)z}.$$

Up to enlarging the constant c_δ, this equation will hold for all $1 \leq z \leq t$.

We now bound $\mathbf{P}(M_t \leq m_t - z)$ with $z \geq t$. We apply (2.5) and obtain that for $z \geq t \geq t_{\varepsilon,\beta}$,

$$\begin{aligned}
\mathbf{P}(M_t \leq m_t - z) &\leq u\Big(\sqrt{2}\Big(1-\tfrac{z}{\sqrt{2}t}\Big)t, t\Big)\\
&\leq \begin{cases} e^{-\sqrt{2}\gamma z + \varepsilon t}, & \text{if } t \leq z < 2t;\\ e^{-(1+(1-\frac{z}{\sqrt{2}t})^2)t+\varepsilon t}, & \text{if } 2t \leq z \leq \sqrt{2}(1+\beta)t;\\ e^{-(1-\frac{z}{\sqrt{2}t})^2 t}, & \text{if } z \geq \sqrt{2}(1+\beta)t.\end{cases}
\end{aligned}$$

Note that $1 + a^2 \geq 2\gamma(1-a)$ for $a < 1-\sqrt{2}$. So, $(1+(1-\frac{z}{\sqrt{2}t})^2)t \geq \sqrt{2}\gamma z$ if $z \geq 2t$. We also have $\Big(1-\frac{z}{\sqrt{2}t}\Big)^2 t \geq \sqrt{2}\gamma z$ if $z \geq \sqrt{2}(1+\sqrt{2})t$. By taking $\beta = \sqrt{2}$ and $\varepsilon = \sqrt{2}\gamma\delta$, we thus get that for $t \geq t_{\varepsilon,\beta}$ and $z \geq t$,

$$\mathbf{P}(M_t \leq m_t - z) \leq e^{-\sqrt{2}\gamma z + \varepsilon t} \leq e^{-\sqrt{2}\gamma(1-\delta)z}.$$

We hence conclude that for any $\delta \in (0,1)$, there exist $T_\delta = t_{\varepsilon,\beta} \vee t_0$ and $K_\delta \geq 1$ such that for any $t \geq t_\delta$ and $z \geq K_\delta$, (2.11) holds. Thus, up to enlarging again constant c_δ, the proof is now complete. □

A.2 Proof of Lemma 3.1

We assume here that $\alpha \in (-\gamma, 1)$. The aim of this section is to prove that for all $\varphi \in \mathcal{C}_c^+(\mathbb{R})$, setting $I_{t,A} = [v_\alpha t - At^{1/2}, v_\alpha t + At^{1/2}]$, we have

$$\lim_{A,K\to\infty}\limsup_{t\to\infty}\frac{e^{2\gamma(1-\alpha)t}}{t^{3\gamma/2}}\int_{(I_{t,A}\times[-K,K])^c}\mathrm{d}s\mathrm{d}z\frac{e^{-(t-s)-\frac{(m_s+z-\sqrt{2}\alpha t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}u_\varphi(m_s+z,s)^2 = 0. \tag{A.5}$$

As φ is non-negative, we observe that

$$u_\varphi(z,t) = \mathbf{E}\left(e^{-\sum_{u\in N(t)}\varphi(X_u(t)-z)}; M_t \le z\right) \le \mathbf{P}(M_t \le z) = u(z,t).$$

It is enough to prove that (A.5) holds for $\varphi \equiv 0$.

Therefore, the objective of the section can be restated as follows: conditioned on $\{M_t \le \sqrt{2}\alpha t\}$, we show that the first branching time τ is with high probability located around $(1-v_\alpha)t + O(\sqrt{t})$, and the position at which that particle branches satisfies $\sqrt{2}\alpha t - m_{t-\tau} + O(1)$ with high probability.

The idea of the proof is the following: we use (2.14) to rewrite u as the sum of U_1 and U_2. By Lemma 2.4, U_1 add a negligible contribution to u, so that

$$u(\sqrt{2}\alpha t, t) \approx \int_{[0,t]\times\mathbb{R}} \mathrm{d}s\mathrm{d}z \frac{e^{-(t-s)-\frac{(m_s+z-\sqrt{2}\alpha t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u(m_s+z,s)^2.$$

Moreover, by Lemma 3.2, a large contribution to u is carried by the regions of the form $I_{A,t}\times[-K,K]$, with $A>0$ and K large enough. We now use a priori domination estimates for u (e.g. Lemma 2.1) and methods similar to the proof of Laplace's method.

We decompose the proof of Lemma 3.1 into three parts, by considering the contribution of various domains of $[0,t]\times\mathbb{R}$.

A.2.1 Linear bounds on the first splitting time

As a first step towards the proof of Lemma 3.1, we show that for all $\varepsilon > 0$,

$$\mathbf{P}\Big(|\tau - (1-v_\alpha)t| > \varepsilon t, M_t \le \sqrt{2}\alpha t\Big) \ll u(\sqrt{2}\alpha t, t).$$

Lemma A.1. *Let $\alpha \in (-\gamma, 1)$. For all $\varepsilon > 0$ small enough, we have*

$$\limsup_{t\to\infty} \frac{1}{t}\log U_2(\sqrt{2}\alpha t, t, [0,(v_\alpha-\varepsilon)t]) < -2\gamma(1-\alpha), \tag{A.6}$$

$$\limsup_{t\to\infty} \frac{1}{t}\log U_2(\sqrt{2}\alpha t, t, [(v_\alpha+\varepsilon)t, t]) < -2\gamma(1-\alpha). \tag{A.7}$$

To prove this result, we begin by bounding the probability that a split occurs at the very end of the process.

Lemma A.2. *Let $\alpha \in (-\gamma, 1)$. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\limsup_{t\to\infty} \frac{1}{t}\log U_2(\sqrt{2}\alpha t, t, [0,\varepsilon t]) < -2\gamma(1-\alpha). \tag{A.8}$$

Proof. Equation (A.8) can be rewritten as

$$\mathbf{P}(0 \le t-\tau \le \varepsilon t, M_t \le \sqrt{2}\alpha t) \ll t^{3\gamma/2}e^{-2\gamma(1-\alpha)t}.$$

First, note that $\mathbf{P}(0 \le t-\tau \le \varepsilon t, M_t \le \sqrt{2}\alpha t) \le \mathbf{P}(\tau \ge (1-\varepsilon)t) = e^{-(1-\varepsilon)t}$. Thus (A.8) holds for all α such that $2\gamma(1-\alpha) < 1$, i.e. $\alpha > -\gamma/2$, for all $\varepsilon > 0$ small enough. We now assume that $\alpha \le -\gamma/2 < 0$, and decompose the above probability as

$$\begin{aligned}\mathbf{P}(0 \le t-\tau \le \varepsilon t, M_t \le \sqrt{2}\alpha t) \le \\ &\mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \le \sqrt{2}(\alpha+2\varepsilon)t, M_t \le \sqrt{2}\alpha t) \\ &+ \mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \ge \sqrt{2}(\alpha+2\varepsilon)t, M_t \le \sqrt{2}\alpha t),\end{aligned} \tag{A.9}$$

and bound these two quantities separately. Notice that

$$\begin{aligned}\mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \le \sqrt{2}(\alpha+2\varepsilon)t, M_t \le \sqrt{2}\alpha t)\\ \le \mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \le \sqrt{2}(\alpha+2\varepsilon)t)\\ \le \int_{(1-\varepsilon)t}^{t} e^{-s}\mathbf{P}(B_s \le \sqrt{2}(\alpha+2\varepsilon)t)\mathrm{d}s \le Ct^{-1/2}e^{-(1-\varepsilon)t}e^{-\frac{(\alpha+2\varepsilon)^2}{2(1-\varepsilon)}t},\end{aligned}$$

using (2.6). As $1+\frac{\alpha^2}{2} > 2\gamma(1-\alpha)$ for all $\alpha \in (-\gamma, -\gamma/2]$, we deduce that for all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \le \sqrt{2}(\alpha+2\varepsilon)t) \le Ce^{-2\gamma(1-\alpha)t-\delta t}. \tag{A.10}$$

We now turn to bounding the second probability in (A.9). Using the Markov property at time τ, we bound it as

$$\begin{aligned}\mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \ge \sqrt{2}(\alpha+2\varepsilon)t, M_t \le \sqrt{2}\alpha t)\\ \le \int_{(1-\varepsilon)t}^{t} e^{-s}\,\mathbf{E}\left(u(t-s,\sqrt{2}\alpha t - B_s)^2 \mathbb{1}_{\{B_s \ge \sqrt{2}(\alpha+2\varepsilon)t\}}\right)\mathrm{d}s.\end{aligned}$$

By Lemma 2.1, for all $s < \varepsilon t$ and $y \le -2\varepsilon t$, we have $u(s,y) \le e^{-y^2/2s}$, yielding

$$\begin{aligned}\mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \ge \sqrt{2}(\alpha+2\varepsilon)t, M_t \le \sqrt{2}\alpha t)\\ \le e^{-(1-\varepsilon)t}\int_0^{\varepsilon t}\int_{\sqrt{2}(\alpha+2\varepsilon)t}^{\infty} e^{-\frac{(\sqrt{2}\alpha t-y)^2}{s}}e^{-\frac{y^2}{2(t-s)}}\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Using that

$$-\frac{y^2}{2(t-s)} - \frac{(y-\sqrt{2}\alpha t)^2}{s} = -\frac{2\alpha^2 t^2}{2t-s} - \frac{(y-\sqrt{2}\alpha\frac{2(t-s)}{2t-s})^2}{\frac{2(t-s)s}{2t-s}},$$

we obtain

$$\begin{aligned}\mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \ge \sqrt{2}(\alpha+2\varepsilon)t, M_t \le \sqrt{2}\alpha t)\\ \le e^{-(1+\alpha^2-\varepsilon)t}\int_0^{\varepsilon t}\int_{\sqrt{2}(\alpha+2\varepsilon-\frac{2(t-s)}{2t-s})t}^{\infty} e^{-\frac{z^2}{\frac{2(t-s)s}{2t-s}}}\mathrm{d}z\mathrm{d}s \le \frac{\sqrt{2\pi}\varepsilon^{3/2}}{\sqrt{1-\varepsilon/2}}t^{3/2}e^{-(1+\alpha^2-\varepsilon)t}.\end{aligned}$$

Therefore, as $1+\alpha^2 > 2\gamma(1-\alpha)$ for all $\alpha \in (-\gamma, 1)$, we conclude that for all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\mathbf{P}(0 \le t-\tau \le \varepsilon t, X_\emptyset(\tau) \ge \sqrt{2}(\alpha+2\varepsilon)t, M_t \le \sqrt{2}\alpha t) \le Ce^{-2\gamma(1-\alpha)t-\delta t}. \tag{A.11}$$

In view of (A.9), equations (A.10) and (A.11) show that there exists ε_0 so that for all $0 < \varepsilon < \varepsilon_0$, (A.8) holds. □

We now bound the probability that the first splitting time in the branching Brownian motion occurs after time at a distance at least εt from the expected time $(1-v_\alpha)t$.

Lemma A.3. *Let $\alpha \in (-\gamma, 1)$. There exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$,*

$$\limsup_{t\to\infty} \frac{1}{t}\log U_2(\sqrt{2}\alpha t, t, [\varepsilon t, (v_\alpha - \varepsilon)t]) < -2\gamma(1-\alpha), \tag{A.12}$$

$$\limsup_{t\to\infty} \frac{1}{t}\log U_2(\sqrt{2}\alpha t, t, [(v_\alpha + \varepsilon)t, t]) < -2\gamma(1-\alpha). \tag{A.13}$$

Proof. Let $a < b$ such that $[a,b] \subset (0, v_\alpha) \cup (v_\alpha, 1]$. By definition of U_2, we have

$$\begin{aligned} U_2(\sqrt{2}\alpha t, t, [at, bt]) &\leq \int_{at}^{bt} \int_{\mathbb{R}} \frac{\mathrm{d}z}{\sqrt{2\pi t}} e^{-(t-s) - \frac{(z-\sqrt{2}\alpha t)^2}{2(t-s)}} u(z,s)^2 \mathrm{d}z\mathrm{d}s \\ &\leq \int_a^b \int_{\mathbb{R}} e^{-t(1-r) - t\frac{(\sqrt{2}hr - \sqrt{2}\alpha)^2}{2(1-r)}} u(\sqrt{2}htr, tr)^2 \sqrt{2} t^{3/2} r \mathrm{d}r \frac{\mathrm{d}h}{\sqrt{2\pi}}, \end{aligned}$$

by change of variables $r = s/t$ and $h = z/\sqrt{2}s$. We then use Lemma 2.1 to bound $u(\sqrt{2}htr, r)$ uniformly in (h, r) for t large enough. For all $\delta > 0$ and $\beta \geq 1$, for all t large enough we have

$$U_2(\sqrt{2}\alpha t, t, [at, bt]) \leq \frac{t^{3/2}}{\sqrt{\pi}} \int_a^b \int_{\mathbb{R}} e^{-t\left(1 - r + \frac{(hr-\alpha)^2}{1-r} + 2\overline{\Psi}_\beta(h) - \delta\right)} \mathrm{d}h\mathrm{d}r, \tag{A.14}$$

where we set

$$\overline{\Psi}_\beta(a) := \begin{cases} 0, & \text{if } a \geq 1; \\ \sqrt{2}\gamma(1-a), & \text{if } -\gamma \leq a < 1; \\ (1 + a^2), & \text{if } -\beta \leq a < -\gamma; \\ a^2, & \text{if } a < -\beta. \end{cases} \tag{A.15}$$

To complete this proof, it is therefore enough to prove that the right-hand side of (A.14) decays exponentially fast, at a rate larger than $2\gamma(1-\alpha)$. To do so, we decompose the integral over $\mathbb{R}$ into thee subsets : $(-\infty, -\beta)$, $[-\beta, 1]$ and $(1, \infty)$.

We first observe that on the interval $[1, \infty)$, by change of variable $v = hr - \alpha$, we have

$$\begin{aligned} \int_a^b \int_1^\infty e^{-t\left(1 - r + \frac{(hr-\alpha)^2}{1-r} - \delta\right)} \mathrm{d}h\mathrm{d}r &\leq b \int_a^b \int_{r-\alpha}^\infty e^{-t(1-r+\frac{v^2}{1-r} - \delta)} \mathrm{d}v\mathrm{d}r \\ &\leq Ct^{-1/2} \Big(\int_a^\alpha e^{-t(1-r)} \mathrm{d}r + \int_\alpha^b e^{-t(1-r+\frac{(r-\alpha)^2}{1-r})} \Big) \mathrm{d}r, \end{aligned}$$

using (2.6) to bound the integrals over v. Hence, one straightforwardly obtains that

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log \int_a^b \int_1^\infty e^{-t\left(1 - r + \frac{(hr-\alpha)^2}{1-r} - \delta\right)} \mathrm{d}h\mathrm{d}r \\ &\qquad\qquad \leq \delta - \min(1-\alpha, g_\alpha(b)) < -2\gamma(1-\alpha), \end{aligned} \tag{A.16}$$

for $\delta > 0$ small enough, where g_α is the function defined in (3.3), which attains its maximum at v_α with value $2\gamma(1-\alpha)$.

Similarly, as $[a,b] \times [-\beta, 1]$ is compact, we also have

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log \int_a^b \int_{-\gamma}^1 e^{-t\left(1 - r + \frac{(hr-\alpha)^2}{1-r} + 2\overline{\Psi}_\beta(h) - \delta\right)} \mathrm{d}h\mathrm{d}r \\ &\leq \delta - \inf_{\substack{r\in[a,b] \\ h\in[-\beta,1]}} 1 - r + \frac{(hr-\alpha)^2}{1-r} + 2\overline{\Psi}_\beta(h) \leq \delta - \inf_{\substack{r\in[a,b] \\ h\in[-\beta,1]}} 1 - r + \frac{(hr-\alpha)^2}{1-r} + 2\sqrt{2}\gamma(1-h). \end{aligned}$$

The function $(h,r) \in (-\infty, 1] \times [\varepsilon, 1] \mapsto 1 - r + \frac{(hr-\alpha)^2}{1-r} + 2\sqrt{2}\gamma(1-h)$ attaining its unique minimum at $(v_\alpha, 1)$, we conclude again that, choosing $\delta > 0$ small enough, we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \int_a^b \int_{-\gamma}^1 e^{-t\left(1 - r + \frac{(hr-\alpha)^2}{1-r} + 2\overline{\Psi}_\beta(h) - \delta\right)} \mathrm{d}h\mathrm{d}r < -2\gamma(1-\alpha). \tag{A.17}$$

Finally, choosing $\beta > 0$ large enough so that the function $h \mapsto \frac{(hr-\alpha)^2}{1-r} + 2h^2$ is strictly decreasing on $(-\infty, -\beta]$, we have

$$\int_a^b \int_{-\infty}^{-\beta} e^{-t\left(1-r+\frac{(hr-\alpha)^2}{1-r}+2h^2-\delta\right)} \mathrm{d}h\mathrm{d}r \leq \int_a^b e^{-t\left(1-r+\frac{(-\beta r-\alpha)^2}{1-r}-\delta\right)} \mathrm{d}r \int_{-\infty}^{-\beta} e^{-2h^2 t}\mathrm{d}h$$
$$\leq Ct^{-1/2} e^{-2\beta^2 t} \int_a^b e^{-t\left(1-r+\frac{(-\beta r-\alpha)^2}{1-r}-\delta\right)} \mathrm{d}r \leq Ct^{-1/2} e^{-t(2\beta^2-\delta)},$$

which, if we choose β large enough, will be smaller than $e^{-t(2\gamma(1-\alpha)+\eta)}$ for some $\eta > 0$, for all t large enough. Using this estimate in combination with (A.16) and (A.17) allows us, by (A.14), to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log U_2(\sqrt{2}\alpha t, t, [at, bt]) < -2\gamma(1-\alpha),$$

which completes the proof of (A.12) and (A.13). □

The proof of Lemma A.1 is then a combination of Lemmas A.2 and A.3.

A.2.2 Tightness of the normalized first splitting time

We now precise the estimates on

$$\mathbf{P}\Big(|\tau - v_\alpha t| > A\sqrt{t}, M_t \leq \sqrt{2}\alpha t\Big),$$

bounding this quantity as $t \to \infty$ then $A \to \infty$.

Lemma A.4. *Given $\alpha \in (-\gamma, 1)$, we have*

$$\lim_{A\to\infty} \limsup_{t\to\infty} e^{2\gamma(1-\alpha)t} t^{-3\gamma/2} U_2(\sqrt{2}\alpha t, t, [0, v_\alpha t - A\sqrt{t}]) = 0; \tag{A.18}$$
$$\lim_{A\to\infty} \limsup_{t\to\infty} e^{2\gamma(1-\alpha)t} t^{-3\gamma/2} U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t}, t]) = 0. \tag{A.19}$$

As a first step, we show that with high probability, $|\tau - v_\alpha t| = o(t^{1/2}\log t)$ conditioned on the maximal displacement being small.

Lemma A.5. *Let $\alpha \in (-\gamma, 1)$. There exists $\varepsilon_\alpha > 0$ such that for all $\varepsilon \in (0, \varepsilon_\alpha)$, for all $\rho > 0$ we have*

$$\limsup_{t\to\infty} t^\rho e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [(v_\alpha - \varepsilon)t, v_\alpha t - \sqrt{t}\log t]) = 0; \tag{A.20}$$
$$\limsup_{t\to\infty} t^\rho e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [v_\alpha t + \sqrt{t}\log t, (v_\alpha + \varepsilon)t]) = 0. \tag{A.21}$$

Proof. The two formulas being proved in a very similar way, we only prove the first one. Note that without loss of generality, one can choose $\varepsilon > 0$ small enough that $v_\alpha - 2\varepsilon > \min(\alpha, 0)$. By definition of U_2, we have

$$
\begin{aligned}
&U_2(\sqrt{2}\alpha t, t, [(v_\alpha-\varepsilon)t, v_\alpha t-\sqrt{t}\log t])\\
=&\int_{(v_\alpha-\varepsilon)t}^{v_\alpha t-\sqrt{t}\log t} \mathrm{d}s \int_{\mathbb{R}} \frac{\mathrm{d}z}{\sqrt{2\pi t}} e^{-(t-s)-\frac{(\sqrt{2}s+z-\sqrt{2}\alpha t)^2}{2(t-s)}} u(\sqrt{2}s+z,s)^2\\
\leq & t^{1/2}\int_{v_\alpha-\varepsilon}^{v_\alpha-\frac{\log t}{\sqrt{t}}} \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}z e^{-t(1-u)+\frac{\left(z+\sqrt{2}t(u-\alpha)\right)^2}{2t(1-u)}} u(\sqrt{2}ut+z,ut)^2\\
\leq & t^{1/2}\int_{v_\alpha-\varepsilon}^{v_\alpha-\frac{\log t}{\sqrt{t}}} \mathrm{d}u e^{-tg_\alpha(u)} \int_{\mathbb{R}} \mathrm{d}z e^{-\frac{z(2\sqrt{2}t(u-\alpha)+z)}{2t(1-u)}} u(\sqrt{2}ut+z,ut)^2,
\end{aligned}
$$

with g_α the function defined in (3.3). Using (3.5), there exists $c>0$ such that for all $r\in[v_\alpha-\varepsilon, v_\alpha]$, $g_\alpha(r)\leq g_\alpha(v_\alpha)-c(r-v_\alpha)^2$. Thus

$$
\begin{aligned}
&e^{2\gamma(1-\alpha)t}U_2(\sqrt{2}\alpha t, t, [(v_\alpha-\varepsilon)t, v_\alpha t-\sqrt{t}\log t])\\
&\qquad \leq t\int_{-\varepsilon}^{-\frac{\log t}{t^{1/2}}} \mathrm{d}v e^{-ctv^2}\int_{\mathbb{R}} \mathrm{d}z e^{-\frac{z(2\sqrt{2}t(v_\alpha+v-\alpha)+z)}{2t(1-v_\alpha-v)}} u(\sqrt{2}(v_\alpha+v)t+z,(v_\alpha+v)t)^2.
\end{aligned}
$$

We now use Lemma 2.2, i.e. that for all $\delta>0$ there exists $c_\delta>0$ such that for all $t\geq 1$ and $z\in\mathbb{R}$, we have $u(m_t-z,t)\leq c_\delta e^{-\sqrt{2}\gamma(1-\delta)z_+}$. Therefore, up to a change of variables, for all $v\in[-\varepsilon,0]$, writing $a_t(v)=\frac{3}{2\sqrt{2}}\log((v_\alpha+v)t)$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}z e^{-\frac{z(2\sqrt{2}t(v_\alpha+v-\alpha)+z)}{2t(1-v_\alpha-v)}} u(\sqrt{2}(v_\alpha+v)t+z,(v_\alpha+v)t)^2\\
\leq &\int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} u(m_{(v_\alpha+v)t}-y,(v_\alpha+v)t)^2\\
\leq & c_\delta\int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+}.
\end{aligned}
$$

As a result, we get

$$
\begin{aligned}
&e^{2\gamma(1-\alpha)t}U_2(\sqrt{2}\alpha t, t, [(v_\alpha-\varepsilon)t, v_\alpha t-\sqrt{t}\log t])\\
&\qquad \leq Ct^{1/2}\int_{-\varepsilon}^{-\frac{\log t}{t^{1/2}}} \mathrm{d}v e^{-ctv^2}\int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+}. \qquad (A.22)
\end{aligned}
$$

We now bound this quantity in two different ways for $y\geq 0$ and $y\leq 0$.

We first observe that for all $v\in[-\varepsilon,0]$, using that $v_\alpha>\alpha+2\varepsilon$,

$$
\begin{aligned}
&\int_{-\infty}^{0} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} \leq \int_{-\infty}^{0} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha))}{2t(1-v_\alpha-v)}}\\
&\qquad\qquad \leq \frac{1-v_\alpha-v}{\sqrt{2}(v_\alpha+v-\alpha)}\Big((v+v_\alpha)t\Big)^{\frac{3}{2}\frac{v_\alpha+v-\alpha}{1-v_\alpha-v}} \leq \frac{1-v_\alpha+\varepsilon}{2\sqrt{2}\varepsilon}\Big(v_\alpha t\Big)^{\frac{3}{2}\frac{v_\alpha-\alpha}{(1-v_\alpha)}}. \qquad (A.23)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{0}^{\infty} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t(v)))}{2t(1-v_\alpha-v)}} e^{-2\sqrt{2}\gamma(1-\delta)y}\\
&\quad \leq e^{2\sqrt{2}\gamma(1-\delta)a_t(v)}\int_{a_t(v)}^{\infty} \mathrm{d}x e^{x\left(\sqrt{2}\frac{(v_\alpha-\alpha)}{(1-v_\alpha)}-2\sqrt{2}\gamma(1-\delta)\right)} \leq \frac{1}{\sqrt{2}\gamma(1-2\delta)}(v_\alpha t)^{3\gamma(1-2\delta)/2}, \qquad (A.24)
\end{aligned}
$$

for all $\delta > 0$ small enough, using that $v_\alpha - \alpha = \frac{\gamma}{\sqrt{2}}(1-\alpha) = \gamma(1-v_\alpha)$.

Hence, plugging (A.23) and (A.24) into (A.22), we deduce that there exist $C > 0$ and $\rho > 0$ so that for all $t \geq 1$ large enough,

$$e^{2\gamma(1-\alpha)t} U_2(\sqrt{2}\alpha t, t, [(v_\alpha - \varepsilon)t, v_\alpha t - \sqrt{t}\log t]) \leq C t^\rho \int_{-\varepsilon}^{-\frac{\log t}{t^{1/2}}} \mathrm{d}v e^{-ctv^2} \leq C t^\rho e^{-c(\log t)^2},$$

which concludes the proof of (A.20). □

Proof of Lemma A.4. By Lemmas A.1 and A.5, to prove Lemma A.4, it is enough to bound for all t large enough, the quantities $U_2(\sqrt{2}\alpha t, t, [v_\alpha t - \sqrt{t}\log t, v_\alpha t - A\sqrt{t}])$ and $U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t} v_\alpha t - \sqrt{t}\log t])$ by $M(A)e^{-2\gamma(1-\alpha)t} t^{-3\gamma/2}$, with $A \mapsto M(A)$ a positive function converging to 0 as $A \to \infty$.

The proofs of (A.18) and (A.19) being very similar and symmetric, we only prove the second one. We write

$$\begin{aligned} &U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t}, v_\alpha t + \sqrt{t}\log t]) \\ \leq & t^{-1/2} \int_{v_\alpha t + A\sqrt{t}}^{v_\alpha t + \sqrt{t}\log t} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}z e^{-(t-s) + \frac{(z-m_s)^2}{2(t-s)}} u(m_s + z, s)^2 \\ \leq & t^{1/2} e^{-2\gamma(1-\alpha)t} \int_{At^{-1/2}}^{t^{-1/2}\log t} \mathrm{d}v e^{-ctv^2} \int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t(v))(2\sqrt{2}t(v_\alpha + v - \alpha) - (y + a_t(v)))}{2t(1-v_\alpha - v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+}, \end{aligned}$$

with the same computations as the ones used to obtain (A.22), using Lemma 2.2.

Using that $|v| \leq t^{-1/2}\log t$, hence that $a_t(v) = \frac{3}{2\sqrt{2}}\log((v_\alpha + v)t) = a_t(0) + o_t(1)$, we obtain, for all t large enough:

$$\begin{aligned} &U_2(\sqrt{2}\alpha t, t, [v_\alpha t + A\sqrt{t}, v_\alpha t + \sqrt{t}\log t]) \\ \leq & 2c_\delta t^{1/2} e^{-2\gamma(1-\alpha)t} \int_{At^{-1/2}}^{t^{-1/2}\log t} \mathrm{d}v e^{-ctv^2} \int_{\mathbb{R}} \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha + v - \alpha) - (y + a_t)}{2t(1-v_\alpha - v)}} e^{-2\sqrt{2}\gamma(1-\delta)y_+}, \end{aligned} \tag{A.25}$$

where $a_t = a_t(0) = \frac{3}{2}\log(v_\alpha t)$.

We then compute for all $|v| < t^{-1/2}\log t$,

$$\begin{aligned} \int_{-\infty}^0 \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha + v - \alpha) - (y+a_t))}{2t(1-v_\alpha - v)}} &\leq \int_{-\infty}^0 \mathrm{d}y e^{\sqrt{2}\frac{(y+a_t)(v_\alpha + v - \alpha)}{(1-v_\alpha - v)}} \\ &\leq \exp\left(\sqrt{2}a_t \frac{v_\alpha + v - \alpha}{1 - v_\alpha - v}\right) \leq \exp\left(\sqrt{2}a_t\left(\frac{v_\alpha - \alpha}{1 - v_\alpha} + Cv\right)\right), \end{aligned}$$

for all t large enough, using Taylor's expansion. Hence, with $(v_\alpha - \alpha)/(1 - v_\alpha) = \gamma$, there exists $C > 0$ such that for all t large enough,

$$\int_{-\infty}^0 \mathrm{d}y e^{\frac{(y+a_t)(2t(v_\alpha + v - \alpha) - (y+a_t))}{t(1-v_\alpha - v)}} \leq C(v_\alpha t)^{3\gamma/2}. \tag{A.26}$$

Similarly, we have

$$\begin{aligned}&\int_0^\infty \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}}e^{-2\sqrt{2}\gamma(1-\delta)y}\\ \leq& e^{2\sqrt{2}\gamma(1-\delta)a_t}\int_{a_t}^\infty \mathrm{d}x e^{\frac{x(2\sqrt{2}t(v_\alpha+v-\alpha)-x)}{2t(1-v_\alpha-v)}-2\sqrt{2}\gamma(1-\delta)x}\\ \leq&(v_\alpha t)^{3\gamma(1-\delta)}\int_{a_t}^\infty \mathrm{d}x e^{\sqrt{2}x\left(\frac{v_\alpha+v-\alpha}{1-v_\alpha-v}-2\gamma(1-\delta)\right)}\leq C(v_\alpha t)^{3\gamma(1-\delta)}(v_\alpha t)^{\frac{3}{2}\frac{v_\alpha-v-\alpha}{1-v_\alpha}-3\gamma(1-\delta)}.\end{aligned}$$

Hence, using that $\frac{v_\alpha-v-\alpha}{1-v_\alpha}=\gamma+O(t^{-1/2}\log t)$, we obtain that for all t large enough

$$\int_0^\infty \mathrm{d}y e^{\frac{(y+a_t)(2t(v_\alpha+v-\alpha)-(y+a_t))}{t(1-v_\alpha-v)}}\leq C(v_\alpha t)^{3\gamma/2}. \tag{A.27}$$

As a result, with (A.26) and (A.27), (A.25) becomes

$$U_2(\sqrt{2}\alpha t,t,[v_\alpha t+A\sqrt{t},v_\alpha t+\sqrt{t}\log t])\leq Ct^{3\gamma/2}e^{-2\gamma(1-\alpha)t}\int_A^\infty e^{-cw^2}\mathrm{d}w.$$

By dominated convergence, the proof of (A.19) is now complete. □

A.2.3 Tightness of the centred splitting position

To complete the proof of Lemma 3.1, we prove that the position at which the first splitting occurs $X_\emptyset(\tau)$ is tightly concentrated around the position $\sqrt{2}\alpha t-m_{t-\tau}$, on the event $|\tau-v_\alpha t|\leq A\sqrt{t}$.

Lemma A.6. *Let $\alpha\in(-\gamma,1)$. For any fixed $A>0$,*

$$\lim_{K\to\infty}\lim_{t\to\infty}\frac{e^{2\gamma(1-\alpha)t}}{t^{3\gamma/2}}U_2(\sqrt{2}\alpha t,t,[v_\alpha t-A\sqrt{t},v_\alpha+A\sqrt{t}],[-K,K]^c)=0. \tag{A.28}$$

Proof. Let $K>0$ and $A>0$. We observe that with similar computations as in the proof of Lemma A.4, setting $a_t=\frac{3}{2\sqrt{2}}\log(v_\alpha t)$, we have

$$\begin{aligned}&e^{2\gamma(1-\alpha)t}U_2(\sqrt{2}\alpha t,t,[v_\alpha t-A\sqrt{t},v_\alpha t+A\sqrt{t}],[-K,K]^c)\\ \leq& Ct^{1/2}\int_{-At^{-1/2}}^{At^{-1/2}}\mathrm{d}v e^{-ctv^2}\int_{[-K,K]^c}\mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}}u(m_{(v_\alpha+v)t}-y,(v_\alpha+v)t)\\ \leq& Ct^{1/2}\int_{-At^{-1/2}}^{At^{-1/2}}\mathrm{d}v e^{-ctv^2}\int_{[-K,K]^c}\mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}}e^{-2\sqrt{2}\gamma(1-\delta)y_+},\end{aligned}$$

where we used again Lemma 2.2.

We then observe, with similar computations as in the proof of Lemma A.4 again that

$$\begin{aligned}\int_{-\infty}^{-K}\mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}}&\leq t^{3\gamma/2}e^{-(\gamma-\delta)K},\\ \int_K^\infty \mathrm{d}y e^{\frac{(y+a_t)(2\sqrt{2}t(v_\alpha+v-\alpha)-(y+a_t))}{2t(1-v_\alpha-v)}}e^{-2\sqrt{2}\gamma(1-\delta)y}&\leq t^{3\gamma/2}e^{-(\gamma-\delta)K},\end{aligned}$$

using that for all t large enough, $\left|\frac{v_\alpha-v-\alpha}{1-v_\alpha}-\gamma\right|\leq\delta$. Therefore, letting $t\to\infty$ then $K\to\infty$, we obtain, for all $A>0$, that (A.28) holds. □

Lemma 3.1 is then a consequence of Lemmas A.1, A.4 and A.6.

A.3 Proof of Lemma 4.1

Similarly to the previous section, it is enough to prove Lemma 4.1 for $\varphi \equiv 0$ by a straightforward domination argument. The proof is obtained in a similar, but slightly simple fashion.

Proof. Let $\alpha < -\gamma$ here. Note that by change of variable $y = \sqrt{2}as$ and (2.5), for any $\varepsilon > 0$ and $A \geq t_{\varepsilon,\beta}$, with $\beta = K\alpha$,

$$\begin{aligned} U_2(\sqrt{2}\alpha t, t, [A,t]) = & \int_A^t \mathrm{d}s \int_{\mathbb{R}} \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - \sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u^2(\sqrt{2}as, s)\sqrt{2}s\mathrm{d}a \\ \leq & \Sigma_1(A,t) + \Sigma_2(A,t) + \Sigma_3(A,t) + \Sigma_4(A,t), \end{aligned}$$

where

$$\Sigma_1(A,t) := \int_A^t \mathrm{d}s \int_1^\infty \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - \sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \sqrt{2}s\mathrm{d}a,$$

$$\Sigma_2(A,t) := \int_A^t \mathrm{d}s \int_{-\gamma}^1 \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - \sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} e^{-4\gamma(1-a)s+2\varepsilon s} \sqrt{2}s\mathrm{d}a,$$

$$\Sigma_3(A,t) := \int_A^t \mathrm{d}s \int_{K\alpha}^{-\gamma} \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - \sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} e^{-2(1+a^2)s+2\varepsilon s} \sqrt{2}s\mathrm{d}a,$$

$$\Sigma_4(A,t) := \int_A^t \mathrm{d}s \int_{-\infty}^{K\alpha} \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t - \sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} e^{-2a^2 s} \sqrt{2}s\mathrm{d}a.$$

Recall g_α from (3.3). By change of variables $z = \sqrt{2}as - \sqrt{2}\alpha t$ and $s = ut$ and by (2.6), one gets that

$$\begin{aligned} \Sigma_1(A,t) = & \int_{A/t}^1 te^{-t(1-u)}\mathrm{d}u \int_{\sqrt{2}ut - \sqrt{2}\alpha t}^\infty \frac{e^{-\frac{z^2}{2t(1-u)}}}{\sqrt{2\pi(1-u)t}}\mathrm{d}z \\ \leq & \int_{A/t}^1 te^{-t(1-u)} \frac{\sqrt{t(1-u)}}{\sqrt{2}(u-\alpha)t} e^{-\frac{(u-\alpha)^2}{1-u}t}\mathrm{d}u \leq \frac{\sqrt{t}}{|\alpha|}\int_{A/t}^1 e^{-tg_\alpha(u)}\mathrm{d}u. \end{aligned}$$

Clearly, $g_\alpha(h) = g_\alpha(0) + g'_\alpha(0)h + o(h)$ as $|h| \to 0$. Note that $g_\alpha(0) = 1 + \alpha^2$ and $g'_\alpha(0) = (\alpha-1)^2 - 2 > 0$ for $\alpha < 1 - \sqrt{2}$. This, together with Taylor's expansion, gives that, for any $u \in [\frac{A}{\sqrt{t}}, 1]$,

$$g_\alpha(u) \geq g_\alpha\Big(\frac{A}{\sqrt{t}}\Big) = g_\alpha(0) + (g'_\alpha(0) + o_t(1))\frac{A}{\sqrt{t}},$$

which implies that, for t sufficiently large,

$$\frac{\sqrt{t}}{|\alpha|}\int_{\frac{A}{\sqrt{t}}}^1 e^{-tg_\alpha(u)}\mathrm{d}u \leq \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|} te^{-\frac{Ag'_\alpha(0)}{2}\sqrt{t}} = o_t(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}.$$

On the other hand, since $g_\alpha(h) = g_\alpha(0) + g'_\alpha(0)h + o(h)$ as $|h| \to 0$, then

$$\begin{aligned} \frac{\sqrt{t}}{|\alpha|}\int_{\frac{A}{t}}^{\frac{A}{\sqrt{t}}} e^{-tg_\alpha(u)}\mathrm{d}u = & \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|}\int_{\frac{A}{t}}^{\frac{A}{\sqrt{t}}} te^{-t(g'_\alpha(0)+o_t(1))u}\mathrm{d}u \\ = & \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|}\int_1^{\sqrt{t}} Ae^{-(g'_\alpha(0)+o_t(1))uA}\mathrm{d}u = o_{t,A}(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|}. \end{aligned}$$

Thus $\Sigma_1(A,t) \le o_{t,A}(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}|\alpha|}$. Next, we shall handle $\Sigma_2(A,t)$. If $\alpha < -2\gamma$, then $\gamma s - (\alpha + 2\gamma)t > 0$. So, by change of variable $z = as - \alpha t + 2\gamma(s-t)$ and (2.6),

$$\begin{aligned}
\Sigma_2(A,t) &= \int_A^t e^{-(t-s)-4\gamma s+4\gamma\alpha t+4\gamma^2(t-s)+2\varepsilon s}\mathrm{d}s \int_{\gamma s-(\alpha+2\gamma)t}^{(1+2\gamma)s-(\alpha+2\gamma)t} \frac{e^{-\frac{z^2}{t-s}}}{\sqrt{2\pi(t-s)}}\sqrt{2}\mathrm{d}z \\
&\le \int_A^t e^{-(t-s)-4\gamma s+4\gamma\alpha t+4\gamma^2(t-s)+2\varepsilon s} \frac{\sqrt{t-s}}{\sqrt{2}\Big(\gamma s-(\alpha+2\gamma)t\Big)} e^{-\frac{(\gamma s-(\alpha+2\gamma)t)^2}{t-s}}\mathrm{d}s \\
&= \int_{A/t}^1 \frac{\sqrt{t(1-u)}}{\sqrt{2}\Big(\gamma u-(\alpha+2\gamma)\Big)} e^{-t[\frac{(\alpha+\gamma)^2}{1-u}-2\gamma(1+\gamma)(1-u)+2-2\alpha\gamma-2\varepsilon u]}\mathrm{d}u \\
&\le \frac{\sqrt{t}}{|\alpha|-2\gamma}\int_{A/t}^1 e^{-tg_{\alpha,\varepsilon}(u)}\mathrm{d}u,
\end{aligned}$$

where $g_{\alpha,\varepsilon}(u) = \frac{(\alpha+\gamma)^2}{1-u} - (1+\gamma^2)(1-u) + 2 - 2\alpha\gamma - 2\varepsilon u$. Observe that for $\varepsilon \in (0,1/2)$ and $u \in (0,1)$,

$$g'_{\alpha,\varepsilon}(u) = \frac{(\alpha+\gamma)^2}{(1-u)^2} + (1+\gamma^2) - 2\varepsilon \ge L_\varepsilon := (\alpha+\gamma)^2 + (1+\gamma^2) - 2\varepsilon,$$

and that $g_{\alpha,\varepsilon}(0) = \alpha^2+1$. Then, for any $h \in (0,1)$,

$$\min_{u\in[h,1]} g_{\alpha,\varepsilon}(u) \ge g_{\alpha,\varepsilon}(h) \ge \alpha^2+1+L_\varepsilon h.$$

This implies that if $\alpha < -2\gamma$, then

$$\Sigma_2(A,t) \le \frac{\sqrt{t}}{|\alpha|-2\gamma}\int_{A/t}^1 e^{-t(\alpha^2+1+L_\varepsilon u)}\mathrm{d}u = \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}(|\alpha|-2\gamma)}\int_{A/t}^1 e^{-L_\varepsilon ut}t\mathrm{d}u,$$

which is $o_A(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}$. If $-2\gamma \le \alpha < -\gamma$, then

$$\begin{aligned}
\Sigma_2(A,t) &\le \int_A^t \Big(\int_{-\gamma}^1 \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t-\sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} e^{-2\gamma(1-a)s+2\varepsilon s}\sqrt{2}s\mathrm{d}a\Big)\mathrm{d}s \\
&= \int_A^t \Big(\int_{-(\alpha+\gamma)t}^{(1+\gamma)s-(\alpha+\gamma)t} \frac{e^{-\frac{z^2}{t-s}}}{\sqrt{2\pi(t-s)}}\sqrt{2}\mathrm{d}z\Big) e^{\gamma^2(t-s)+2\gamma\alpha t-(t-s)-2\gamma s+2\varepsilon s}\mathrm{d}s \\
&\le \int_A^t \frac{\sqrt{t-s}}{-(\alpha+\gamma)t} e^{-\frac{(\alpha+\gamma)^2t^2}{t-s}+\gamma^2(t-s)+2\gamma\alpha t-(t-s)-2\gamma s+2\varepsilon s}\mathrm{d}s \\
&= \int_{A/t}^1 \frac{\sqrt{t(1-u)}}{|\alpha|-\gamma} e^{-th(u)}\mathrm{d}u,
\end{aligned}$$

where in the first equality, we change variable $z = sa - \alpha t + \gamma(s-t)$, the second inequality holds by (2.6) and $h(u) = \frac{(\alpha+\gamma)^2}{1-u} - 2\varepsilon u + (1+\alpha^2) - (\alpha+\gamma)^2$. Note that for any $\varepsilon \in \Big(0, \frac{(\alpha+\gamma)^2}{2}\Big)$ and $u \in (0,1)$,

$$h'(u) = \frac{(\alpha+\gamma)^2}{(1-u)^2} - 2\delta \ge \widetilde{L}_\varepsilon := (\alpha+\gamma)^2 - 2\varepsilon > 0,$$

with $h(0) = \alpha^2 + 1$. It hence follows that if $-2\gamma \le \alpha < -\gamma$, then

$$\Sigma_2(A,t) \le \int_{A/t}^{1} \frac{\sqrt{t(1-u)}}{|\alpha|-\gamma} e^{-t(\alpha^2+1+\widetilde{L}_\varepsilon u)} \mathrm{d}u = o_A(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}.$$

For $\Sigma_3(A,t)$, one sees that

$$\begin{aligned}\Sigma_3(A,t) &= \int_A^t e^{-t-s+2\varepsilon s-\frac{2\alpha^2t^2}{2t-s}} \mathrm{d}s \int_{K\alpha}^{-\gamma} e^{-\frac{s(2t-s)}{t-s}(a-\frac{\alpha t}{2t-s})^2} \frac{s}{\sqrt{\pi(t-s)}} da \\ &\le \int_A^t \frac{\sqrt{s}}{\sqrt{2t-s}} e^{-t-s+2\varepsilon s-\frac{2\alpha^2t^2}{2t-s}} \mathrm{d}s \le \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}} \int_A^t \sqrt{s} e^{-(1-2\varepsilon)s} \mathrm{d}s,\end{aligned}$$

which is $o_A(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}$ as long as $\varepsilon \in (0,1/2)$. On the other hand,

$$\begin{aligned}\Sigma_4(A,t) &= \int_A^t \mathrm{d}s \int_{-\infty}^{K\alpha} \frac{e^{-(t-s)-\frac{(\sqrt{2}\alpha t-\sqrt{2}as)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} e^{-2a^2s}\sqrt{2}s da \\ &= \int_A^t e^{-t+s-\frac{2\alpha^2t^2}{2t-s}} \mathrm{d}s \int_{-\infty}^{K\alpha} e^{-\frac{s(2t-s)}{t-s}(a-\frac{\alpha t}{2t-s})^2} \frac{s}{\sqrt{\pi(t-s)}} da \\ &= \int_A^t e^{-t+s-\frac{2\alpha^2t^2}{2t-s}} \sqrt{\frac{s}{2t-s}} \mathrm{d}s \int_{-\infty}^{K\alpha-\frac{\alpha t}{2t-s}} e^{-\frac{s(2t-s)}{t-s}z^2} \frac{\mathrm{d}z}{\sqrt{\pi\frac{t-s}{s(2t-s)}}}.\end{aligned}$$

Choose $K > 1$ such that $(K-1)|\alpha| > 1$ and $K\alpha - \frac{\alpha t}{2t-s} < -1$. Then by (2.6),

$$\begin{aligned}\Sigma_4(A,t) &\le \int_A^t e^{-t+s-\frac{2\alpha^2t^2}{2t-s}} \sqrt{\frac{s}{2t-s}} \frac{\sqrt{t-s}}{\sqrt{s(2t-s)}} e^{-\frac{s(2t-s)}{t-s}} \mathrm{d}s \\ &= \int_A^t \frac{\sqrt{t-s}}{2t-s} e^{-t-\frac{2\alpha^2t^2}{2t-s}} e^{-s-\frac{s^2}{t-s}} \mathrm{d}s \le \frac{e^{-(1+\alpha^2)t}}{\sqrt{t}} \int_A^t e^{-s} \mathrm{d}s,\end{aligned}$$

as $\frac{1}{2t} \le \frac{1}{2t-s} \le \frac{1}{t}$ and $\sqrt{t-s} \le \sqrt{t}$. Therefore, $\Sigma_4(A,t) = o_A(1)\frac{e^{-(1+\alpha^2)t}}{\sqrt{t}}$. □

A.4 Proof of Lemma 5.1

Using again a domination argument, it is enough to prove Lemma 5.1 for $\varphi \equiv 0$. We decompose it into the two following lemmas, that we prove one by one.

Lemma A.7.

$$\lim_{A\to\infty}\lim_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} U_2(-\sqrt{2}\gamma t, t, [A\sqrt{t}, t]) = 0; \tag{A.29}$$

$$\lim_{A\to\infty}\lim_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} U_2(-\sqrt{2}\gamma t, t, [0, \sqrt{t}/A]) = 0. \tag{A.30}$$

Lemma A.8. *For any $A > 0$ fixed,*

$$\lim_{K\to\infty}\lim_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} U_2(-\sqrt{2}\gamma t, t, [\frac{1}{A}\sqrt{t}, A\sqrt{t}], (-\infty, -K]) = 0 \tag{A.31}$$

and

$$\lim_{K\to\infty}\lim_{t\to\infty} \frac{e^{(1+\gamma^2)t}}{t^{3\gamma/4}} U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [K, \infty)) = 0. \tag{A.32}$$

Proof of Lemma A.7. Proof of (A.29): Observe that

$$\begin{aligned}
&U_2(-\sqrt{2}\gamma t, t, [A\sqrt{t}, t]) \\
&\quad = U_2(-\sqrt{2}\gamma t, t, [A\sqrt{t}, t], [-K, \infty)]) + U_2(-\sqrt{2}\gamma t, t, [A\sqrt{t}, t], (-\infty, -K)) \\
&\quad =: U_{(A.33)a} + U_{(A.33)b}.
\end{aligned} \tag{A.33}$$

As $u(m_s + z, s) \leq 1$, one sees that

$$\begin{aligned}
U_{(A.33)a} &\leq \int_{A\sqrt{t}}^{t} \mathrm{d}s \int_{-K}^{\infty} \mathrm{d}z \frac{e^{-(t-s)-\frac{(z+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \\
&= \int_{A\sqrt{t}}^{t} e^{-(t-s)} \mathrm{d}s \int_{-K+m_s+\sqrt{2}\gamma t}^{\infty} \frac{e^{-\frac{z^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}},
\end{aligned}$$

which by (2.6) is bounded by

$$\begin{aligned}
&\int_{A\sqrt{t}}^{t} e^{-(t-s)} \frac{\sqrt{t-s}}{-K+m_s+\sqrt{2}\gamma t} e^{-\frac{(K+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}} \mathrm{d}s \\
\leq & c_4 \frac{e^{-(1+\gamma^2)t}}{\sqrt{t}} \int_{A\sqrt{t}}^{t} e^{-\frac{2s^2}{t-s}+(\sqrt{2}K+\frac{3}{2}\log s)\frac{s+\gamma t}{t-s}} \mathrm{d}s.
\end{aligned}$$

For t large enough, one has

$$\Big(\sqrt{2}K + \frac{3}{2}\log s\Big)\frac{s+\gamma t}{t-s} \leq \begin{cases} \frac{s^2}{t-s}, & \text{if } s \in [\sqrt{t}\log t, t]; \\ \frac{3\gamma}{2}\log s + 2\sqrt{2}K + \frac{3\sqrt{2}(\log t)^2}{\sqrt{t}}, & \text{if } s \in [A\sqrt{t}, \sqrt{t}\log t], \end{cases}$$

which implies that

$$U_{(A.33)a} \leq c_5 \frac{e^{-(1+\gamma^2)t}}{\sqrt{t}} \Big(\int_{\sqrt{t}\log t}^{t} e^{-\frac{s^2}{t-s}} \mathrm{d}s + e^{2\sqrt{2}K} \int_{A\sqrt{t}}^{\sqrt{t}\log t} s^{3\gamma/2} e^{-\frac{2s^2}{t}} \mathrm{d}s \Big) = o_t\Big(\frac{e^{-(1+\gamma^2)t}}{\sqrt{t}}\Big).$$

On the other hand, for s sufficiently large and $z < -K$, by similar reasonings as in Lemma A.4, we have for $\delta \in (0, 1/2]$, $\eta = 1 - 2\delta$, $\varepsilon < \frac{\gamma\eta}{1+2\gamma(1-\delta)}$,

$$\begin{aligned}
U_{(A.33)b} &\leq c_\delta^2 \int_{A\sqrt{t}}^{\varepsilon t} \frac{\sqrt{t-s}e^{-(t-s)-\frac{(-\sqrt{2}\gamma t - m_s + K)^2}{2(t-s)}}}{-2s+\sqrt{2}\gamma\eta(t-s)} \mathrm{d}s \\
&\quad + c_\delta^2 \int_{\varepsilon t}^{t} e^{-(t-s)(1-\gamma^2(1+\eta)^2)-\sqrt{2}\gamma(1+\eta)(m_s-\sqrt{2}\gamma t)} \mathrm{d}s \\
&=: U_{(A.33)b1} + U_{(A.33)b2},
\end{aligned}$$

that we bound separately.

Note that

$$\begin{aligned}
U_{(A.33)b2} &= \int_{\varepsilon}^{1} (ut)^{3\gamma(1+\eta/2)} t e^{-t[(1-u)(1-\gamma^2(1+\eta)^2)+2\gamma(1+\eta)(u+\gamma)]} \mathrm{d}u \\
&\leq t^{\frac{3\gamma}{2}(1+\eta)+1} e^{-t\min_{u\in[\varepsilon,1]}[(1-u)(1-\gamma^2(1+\eta)^2)+2\gamma(1+\eta)(u+\gamma)]}.
\end{aligned}$$

One can check that

$$\min_{u\in[\varepsilon,1]}[(1-u)(1-\gamma^2(1+\eta)^2)+2\gamma(1+\eta)(u+\gamma)]$$
$$=(1+\gamma^2)+\varepsilon(1+\gamma^2)\eta-\eta^2\gamma^2(1-\varepsilon).$$

Take $\varepsilon\in(\frac{\eta\gamma^2}{1+\gamma^2},\frac{\gamma\eta}{1+2\gamma(1-\delta)})$ as $\eta=1-2\delta\in(0,1)$. Then,

$$U_{(\mathrm{A.33})b2}\le t^{\frac{3\gamma}{2}(1+\eta)+1}e^{-t(1+\gamma^2+\varepsilon\eta^2\gamma^2)}=o_t(1)t^{3\gamma/4}e^{-t(1+\gamma^2)}.$$

It remains to bound $U_{(\mathrm{A.33})b1}$. Recalling (3.3), we observe that

$$\begin{aligned}U_{(\mathrm{A.33})b1}\le&\frac{C^{(7)}_{\delta,\varepsilon}}{\sqrt{t}}\int_{A\sqrt{t}}^{\varepsilon t}e^{-(t-s)-\frac{(-\sqrt{2}\gamma t-m_s+K)^2}{2(t-s)}}\mathrm{d}s\\ \le&C^{(7)}_{\delta,\varepsilon}e^{C^{(8)}_{\delta,\varepsilon}K}\sqrt{t}\int_{\frac{A}{\sqrt{t}}}^{\varepsilon}e^{-tg_{-\gamma}(u)+\frac{3}{2}\log(ut)\frac{u+\gamma}{1-u}}\mathrm{d}u,\end{aligned}$$

where we use the fact that for $s\in[A\sqrt{t},\varepsilon t]$,

$$\begin{aligned}\frac{(-\sqrt{2}\gamma t-m_s+K)^2}{2(t-s)}&=\frac{2(\gamma t+s)^2+(\frac{3}{2\sqrt{2}}\log s+K)^2-2\sqrt{2}(\gamma t+s)(\frac{3}{2\sqrt{2}}\log s+K)}{2(t-s)}\\ &\ge\frac{(\gamma t+s)^2}{t-s}-\frac{3(\gamma t+s)\log s}{2(t-s)}-\frac{\sqrt{2}\gamma K}{(1-\varepsilon)}.\end{aligned}\tag{A.34}$$

Since $g_{-\gamma}(u)=1+\gamma^2+2u^2+o(u^2)$, as $u\downarrow 0$, then

$$\sqrt{t}\int_{\frac{\log t}{\sqrt{t}}}^{\varepsilon}e^{-tg_{-\gamma}(u)+\frac{3}{2}\log(ut)\frac{u+\gamma}{1-u}}\mathrm{d}u\le\sqrt{t}\int_{\frac{\log t}{\sqrt{t}}}^{\varepsilon}(ut)^{3(\varepsilon+\gamma)/2}e^{-u^2t-(1+\gamma^2)t}\mathrm{d}u,$$

which is $o_t(1)t^{3\gamma/4}e^{-(1+\gamma^2)t}$. For $u\in[\frac{A}{\sqrt{t}},\frac{\log t}{\sqrt{t}}]$, $\log(ut)\frac{u+\gamma}{1-u}=\gamma\log(ut)+o_t(1)$. Therefore,

$$\begin{aligned}\sqrt{t}\int_{\frac{A}{\sqrt{t}}}^{\frac{\log t}{\sqrt{t}}}e^{-tg_{-\gamma}(u)+\frac{3}{2}\log(ut)\frac{u+\gamma}{1-u}}\mathrm{d}u&\le e\int_{\frac{A}{\sqrt{t}}}^{\frac{\log t}{\sqrt{t}}}(ut)^{3\gamma/2}e^{-t(1+\gamma^2)-u^2t}\mathrm{d}u\\ &\le et^{3\gamma/4}e^{-(1+\gamma^2)t}\int_A^{\log t}x^{3\gamma/2}e^{-2x^2}\mathrm{d}x=o_A(1)t^{3\gamma/4}e^{-(1+\gamma^2)t}.\end{aligned}$$

We have completed the proof of (A.29).

Proof of (A.30): We have

$$\begin{aligned}&U_2(-\sqrt{2}\gamma t,t,[0,\sqrt{t}/A])\\ &\quad=U_2(-\sqrt{2}\gamma t,t,[0,\sqrt{t}/A],[-K,\infty))+U_2(-\sqrt{2}\gamma t,t,[0,\sqrt{t}/A],[-\infty,-K])\\ &\quad=:U_{(\mathrm{A.35})a}+U_{(\mathrm{A.35})b}.\end{aligned}\tag{A.35}$$

As $u(m_s+a,s)\le 1$, applying (2.6) gives that for t large enough,

$$\begin{aligned}U_{(\mathrm{A.35})a}&\le\int_0^{\sqrt{t}/A}\frac{\sqrt{t-s}}{-K+m_s+\sqrt{2}\gamma t}e^{-(t-s)-\frac{(m_s+\sqrt{2}\gamma t-K)^2}{2(t-s)}}\mathrm{d}s\\ &\le c_7e^{\sqrt{2}K}e^{-(1+\gamma^2)t}\int_0^{\sqrt{t}/A}s^{3\gamma/2}e^{-\frac{2s^2}{t}}\frac{\mathrm{d}s}{\sqrt{t}}\\ &=c_7e^{\sqrt{2}K}t^{3\gamma/4}e^{-(1+\gamma^2)t}\int_0^{1/A}u^{3\gamma/2}e^{-2u^2}\mathrm{d}u,\end{aligned}$$

which is $o_A(1)t^{3\gamma/4}e^{-(1+\gamma^2)t}$. Similarly as $U_{(A.33)b1}$,

$$\begin{aligned}
U_{(A.35)b} \leq & C^{(12)}_{\delta,\varepsilon} e^{C^{(11)}_{\delta,\varepsilon}K}\sqrt{t}\int_0^{\frac{1}{A\sqrt{t}}} e^{-tg_{-\gamma}(u)+\frac{3\gamma}{2}\log(ut)\frac{u+\gamma}{1-u}}\mathrm{d}u \\
\leq & C^{(1)}_{\delta,\varepsilon,K} t^{3\gamma/4}e^{-(1+\gamma^2)t}\int_0^{\frac{1}{A\sqrt{t}}}(u\sqrt{t})^{3\gamma/2}e^{-u^2t}\sqrt{t}\mathrm{d}u \\
= & c_8 t^{3\gamma/4}e^{-(1+\gamma^2)t}\int_0^{1/A} u^{3\gamma/2}e^{-u^2}\mathrm{d}u = o_A(1)t^{3\gamma/4}e^{-(1+\gamma^2)t},
\end{aligned}$$

concluding (A.30). □

Proof of Lemma A.8. Proof of (A.31): Take $\delta \in (0, 1/3)$ and $\eta = 1 - 2\delta$. By similar reasoning as above, we have

$$\begin{aligned}
& \int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} \mathrm{d}s \int_{-\infty}^{-K} \mathrm{d}z \frac{e^{-(t-s)-\frac{(z+m_s-\sqrt{2}ct)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} u^2(m_s+z,s) \\
\leq & c_\delta^2 \int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} \frac{\sqrt{t-s}e^{-(t-s)-\frac{(-\sqrt{2}\gamma t-m_s+K)^2}{2(t-s)}-\sqrt{2}\gamma(1+\eta)K}}{-2s+\sqrt{2}\gamma\eta(t-s)}\mathrm{d}s \\
\leq & C^{(1)}_{\delta,\gamma,A} t^{3\gamma/4}e^{-K\sqrt{2}\gamma(1-3\delta)}\int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} \frac{1}{\sqrt{t}} e^{-(t-s)-\frac{(s+\gamma t)^2}{t-s}}\mathrm{d}s \\
\leq & C^{(2)}_{\delta,\gamma,A} e^{-K\sqrt{2}\gamma(1-2\delta)} t^{3\gamma/4}e^{-(1+\gamma^2)t},
\end{aligned}$$

where for the second inequality, we used the fact that for $s \in [\frac{\sqrt{t}}{A}, A\sqrt{t}]$,

$$\begin{aligned}
\frac{(-\sqrt{2}\gamma t - m_s + K)^2}{2(t-s)} \geq & \frac{(\gamma t+s)^2}{t-s} - \frac{3(\gamma t+s)\log s}{2(t-s)} - \frac{\sqrt{2}(\gamma t+s)\gamma K}{(t-s)} \\
\geq & \frac{(\gamma t+s)^2}{t-s} - \frac{3\gamma}{4}\log t - \sqrt{2}\gamma^2 K + o_t(1).
\end{aligned}$$

and the last inequality follows from the fact that $(t-s) + \frac{(s+\gamma t)^2}{t-s} = (1+\gamma^2)t + \frac{(1+\gamma)^2 s^2}{t-s}$.
Proof of (A.32): For $z \geq K$, using the fact $u(m_s+z,s) \leq 1$, we obtain that

$$\begin{aligned}
U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [K,\infty)) \leq & \int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} \mathrm{d}s \int_K^\infty \mathrm{d}z \frac{e^{-(t-s)-\frac{(z+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \\
= & \int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} e^{-(t-s)}\mathrm{d}s \int_{K+m_s+\sqrt{2}\gamma t}^\infty \mathrm{d}z \frac{e^{-\frac{z^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}},
\end{aligned}$$

which by (2.6) is less than

$$\int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} e^{-(t-s)}\frac{\sqrt{t-s}}{K+m_s+\sqrt{2}\gamma t}e^{-\frac{(K+m_s+\sqrt{2}\gamma t)^2}{2(t-s)}}\mathrm{d}s \leq \frac{C_{\gamma,A}}{\sqrt{t}}\int_{\frac{\sqrt{t}}{A}}^{A\sqrt{t}} e^{-(t-s)-\frac{(m_s+\sqrt{2}\gamma t)^2}{2(t-s)}-K\frac{\sqrt{2}\gamma t+m_s}{t-s}}\mathrm{d}s.$$

Similarly as above, we end up with

$$U_2(-\sqrt{2}\gamma t, t, [\sqrt{t}/A, A\sqrt{t}], [K,\infty)) \leq C_{\gamma,A}e^{-K\sqrt{2}\gamma}t^{3\gamma/4}e^{-(1+\gamma^2)t}.$$

This suffices to conclude (A.32). □

Acknowledgements

The authors would like to thank an anonymous referee for pointing out several mistakes in a preliminary version.

References

[1] E. Aïdékon, J. Berestycki, É. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probability Theory and Related Fields*, 157(1-2):405–451, 2013.

[2] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probability Theory and Related Fields*, 157(3-4):535–574, 2013.

[3] Yanjia Bai and Lisa Hartung. Refined large deviation principle for branching Brownian motion conditioned to have a low maximum. *ALEA Lat. Am. J. Probab. Math. Stat.*, 19(1):859–880, 2022.

[4] Julien Berestycki, Éric Brunet, Aser Cortines, and Bastien Mallein. A simple backward construction of branching Brownian motion with large displacement and applications. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 58(4):2094–2113, 2022.

[5] Julien Berestycki, Éric Brunet, Cole Graham, Leonid Mytnik, Jean-Michel Roquejoffre, and Lenya Ryzhik. The distance between the two BBM leaders. *Nonlinearity*, 35(4):1558, 2022.

[6] N. Berestycki, N. Gantert, P.Mörters, and N. Sidorova. Galton-Watson trees with vanishing martingale limit. *Journal of Statistical Physics*, 155(4):737–762, 2014.

[7] M. Biskup and O. Louidor. Full extremal process, cluster law and freezing for the two-dimensional discrete Gaussian free field. *Advances in Mathematics*, 330:589–687, 2016.

[8] Erwin Bolthausen, Jean-Dominique Deuschel, and Giambattista Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic. *The Annals of Probability*, 29(4):1670–1692, 2001.

[9] Erwin Bolthausen, Jean-Dominique Deuschel, and Ofer Zeitouni. Entropic repulsion of the lattice free field. *Communications in Mathematical Physics*, 170(2):417–443, 1995.

[10] Anton Bovier. *Gaussian processes on trees: From spin glasses to branching Brownian motion*, volume 163. Cambridge University Press, 2017.

[11] M. Bramson. *Convergence of solutions of the Kolmogorov equation to travelling waves*, volume 285. Providence, RI: American Mathematical Society (AMS), 1983.

[12] Maury Bramson, Jian Ding, and Ofer Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Communications on Pure and Applied Mathematics*, 69(1):62–123, 2016.

[13] Brigitte Chauvin and Alain Rouault. KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probability Theory and Related Fields*, 80(2):299–314, 1988.

[14] Brigitte Chauvin and Alain Rouault. Supercritical branching Brownian motion and K-P-P equation in the critical speed-area. *Mathematische Nachrichten*, 149(1):41–59, 1990.

[15] Xinxin Chen and Hui He. Lower deviation and moderate deviation probabilities for maximum of a branching random walk. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 56(4):2507–2539, 2020.

[16] Aser Cortines, Lisa Hartung, and Oren Louidor. The structure of extreme level sets in branching Brownian motion. *The Annals of Probability*, 47(4):2257–2302, 2019.

[17] A. Dembo and O. Zeitouni. *Large deviations techniques and applications.*, volume 38. New York, NY: Springer, 2nd edition, 1998.

[18] Amir Dembo, Yuval Peres, Jay Rosen, and Ofer Zeitouni. Cover times for Brownian motion and random walks in two dimensions. *Annals of Mathematics*, pages 433–464, 2004.

[19] B. Derrida and Z. Shi. Large deviation for the branching Brownian motion in presence of selection or coalescence. *Journal of Statistical Physics*, 163:1285–1311, 2016.

[20] B. Derrida and Z. Shi. Large deviations for the rightmost position in a branching Brownian motion. In *Modern problems of stochastic analysis and statistics. Selected contributions in honor of Valentin Konakov's 70th birthday, Moscow, Russia, May 29 – June 2, 2016*, pages 303–312. Cham: Springer, 2017.

[21] B. Derrida and Z. Shi. Slower deviations of the branching Brownian motion and of branching random walks. *Journal of Physics A: Mathematical and Theoretical*, 50(34):13, 2017.

[22] Jean-Dominique Deuschel and Giambattista Giacomin. Entropic repulsion for massless fields. *Stochastic Processes and Their Applications*, 89(2):333–354, 2000.

[23] Nina Gantert and Thomas Höfelsauer. Large deviations for the maximum of a branching random walk. *Electronic Communications in Probability*, 23:1–12, 2018.

[24] Yueyun Hu. How big is the minimum of a branching random walk? *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 52(1):233–260, 2016.

[25] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *The Annals of Probability*, 15(3):1052–1061, 1987.

[26] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Communications on Pure and Applied Mathematics*, 28:323–331, 1975.

[27] B. Meerson and P. Sasorov. Negative velocity fluctuations of pulled reaction fronts. *Physical Review E*, 84:030101, 2011.

[28] B. Meerson, A. Vilenkin, and P. Sasorov. Emergence of fluctuating traveling front solutions in macroscopic theory of noisy invasion fronts. *Physical Review E.*, 87:012117, 2013.

[29] Rishideep Roy. Branching random walk in the presence of a hard wall. *arXiv preprint arXiv:1806.02565*, 2018.