PRESCRIPTION OF GAUSS CURVATURE ON COMPACT HYPERBOLIC ORBIFOLDS

JÉRÔME BERTRAND

Abstract. In this paper, we generalize a result by Alexandrov on the Gauss curvature prescription for Euclidean convex bodies. We prove an analogous result for hyperbolic orbifolds. In addition to the duality theory for convex sets, our main tool comes from optimal mass transport.

Keywords: Convex bodies, Gauss curvature, optimal mass transport

1. Introduction

In this paper, we consider the problem of prescribing the Gauss curvature (in a generalised measure-theoretic sense) of convex sets in the Minkowski spacetime. This problem is a generalisation of a similar problem on Euclidean convex sets, raised and solved by Alexandrov in the 40's. Our proof builds on the fact (also valid for Alexandrov's problem) that this problem is equivalent to a strong form of a well-known statement in optimal mass transport: the Kantorovich duality. We mean that the solutions of both problems are in one-to-one correspondence if the cost function is appropriate. Let us first recall the Euclidean result which was proved by A.D. Alexandrov in [1, 2].

In order to state the result, we recall the notion of Gauss curvature measure introduced by Alexandrov. Consider a convex body Ω in $\mathbb{R}^{m+1}$ and assume that the origin of $\mathbb{R}^{m+1}$ is located within Ω. Under these assumptions, the map

$$\overrightarrow{\rho}: \begin{array}{ccc} \mathbb{S}^m & \longrightarrow & \partial\Omega \\ x & \longmapsto & \rho(x)x \end{array} \tag{1}$$

is a homeomorphism (where $\rho(x) = \sup\{s; sx \in \Omega\}$).

The Gauss curvature measure is the Borel probability measure

$$\mu := \sigma(\mathcal{G} \circ \overrightarrow{\rho}(\cdot))$$

where σ stands for the uniform Borel probability measure on $\mathbb{S}^m$ (considered as the unit sphere centered at the origin) and $\mathcal{G} : \partial\Omega \rightrightarrows \mathbb{S}^m$ is the Gauss map. In other terms, the Gauss curvature measure is the *pull-back* of the uniform measure through the map $\mathcal{G} \circ \overrightarrow{\rho}$. We point out that this definition makes sense for *general* convex bodies as a consequence of [3, Lemma 5.2]. Note also that the curvature measure depends on the location of the origin within the convex body and is invariant under homotheties about that point.

Alexandrov found a necessary and sufficient condition for μ arising from the construction above.

Theorem 1.1 (Alexandrov). *Let σ be the uniform probability measure on $\mathbb{S}^m$ and μ be a Borel probability measure on $\mathbb{S}^m$ satisfying for any Borel set $\omega \subset \mathbb{S}^m$,*

$$\mu(\omega) < \sigma(\{x \in \mathbb{S}^m; \inf_{w\in\omega} \langle x, w\rangle > 0\})$$

Then, there exists a unique convex body in $\mathbb{R}^{m+1}$ containing 0 in its interior (up to homotheties) whose μ is the Gauss curvature measure.

Mathematics Subject Classification (2010): Primary 49Q20; Secondary 52C07, 53C42.

The author is partially supported by the A.N.R. through the project GMT.

The goal of this paper is to prove an analogue of Alexandrov's theorem for hyperbolic orbifolds, this includes in particular all hyperbolic manifolds as explained in Section 2.1. Here, we consider Alexandrov's result as an embedding result for manifolds homeorphic to the sphere like in (1) with a given Gauss curvature. For hyperbolic manifolds, the Gauss formula prevents from embedding such a manifold as the boundary of Euclidean convex set. Following [13] where Labourie and Schlenker considered smooth surfaces, the idea is then to embed its universal cover in an equivariant way in the Minkowski spacetime $(\mathbb{R}^m_1, q)$ (where the Gauss formula is reversed for space-like planes, see [18, 20]). In order to propose a general definition, we first have to define Fuchsian convex set.

Definition 1.2 (Fuchsian convex set)**.** A Fuchsian convex set in the Minkowski spacetime is a non-empty (Euclidean) closed convex set which lies on the (open) future cone at the origin C_f and which is invariant under the action of a given cocompact discrete subgroup Γ of the isometry group of $\mathbb{R}^m_1$ fixing the cone C_f, namely

$$\Gamma.\mathcal{C} = \mathcal{C}.$$

The set of isometries of $\mathbb{R}^m_1$ fixing C_f is denoted by $Isom^+(\mathbb{R}^m_1)$.

It is proved further in the paper that any inward normal vector (with respect to q) at a point $x \in \partial\mathcal{C}$ where $\mathcal{C}$ is a Fuchsian convex set, belongs to $\mathbb{H}^m$. Here, we identify the hyperbolic space $\mathbb{H}^m$ with

$$\{x \in \mathbb{R}^m_1 ; q(x) = -1 \text{ and } x_{m+1} > 0\}.$$

As a consequence, we define the Gaus curvature measure as the Γ-invariant measure induced by

$$\mu = \sigma(\mathcal{G} \circ p^{-1}(\cdot))$$

where σ is now the Riemannian measure on $\mathbb{H}^m$ and $p : \partial\mathcal{C} \longrightarrow \mathbb{H}^m$ is the projection onto $\mathbb{H}^m$ (namely $p(x) = \frac{x}{\sqrt{-q(x)}}$). A more detailed exposition on the Gauss curvature measure is given in Section 2.2.

The main result of this paper is the following.

Theorem 1.3. *Let $M = \mathbb{H}^m/\Gamma$ be a compact hyperbolic orbifold and μ be a Borel probability measure on M. Then, there exists a unique (up to homotheties) Γ-invariant homeomorphism $\Phi : \mathbb{H}^m \longrightarrow \mathbb{R}^m_1$ such that $\Phi(\mathbb{H}^m)$ is the boundary of a Γ-Fuchsian convex set[1] and $Vol(M)\mu$ is the Gauss curvature measure of M. In particular, if μ is a finite sum of Dirac masses then the corresponding convex set is an equivariant polyhedron.*

Remark 1.4. This result generalizes [11] where a similar result is proved when the embedded manifold is the boundary of an equivariant, spacelike, and two-dimensional convex polyhedron.

Our strategy to prove this result is to show that the solutions of Alexandrov's problem in the Minkowski spacetime are in one-to-one correspondence with the solutions of Kantorovich's dual problem on functions recalled below

Theorem 1.5. *Let σ_M be the uniform measure on a compact hyperbolic orbifold M and μ be an other measure on M whose total mass is identically equal to that of σ_M. We set $c(n,x) = \ln(\cosh(d(n,x)))$ where d is the distance on M. Then, the following Kantorovich duality holds*

$$\max_{\mathcal{A}} \left\{ \int_M \phi(n) d\sigma_M(n) + \int_M \psi(x) d\mu(x) \right\} = \min_{\Pi \in \Gamma(\sigma_M, \mu)} \int_{M \times M} c(n,x)\, d\,\Pi(n,x) \tag{2}$$

where $\mathcal{A}$ denotes the set of pairs (ϕ, ψ) of Lipschitz functions on M such that for all $x, n \in M$,

$$\phi(n) + \psi(x) \leq c(n,x)$$

and $\Gamma(\sigma, \mu)$ the set of plans whose marginals are σ_M and μ respectively.

[1]In a previous draft of the paper, we give another *equivalent* definition of "Fuchsian" embedding; we switched to this easier-to-state definition after discussions with F. Fillastre.

Moreover, there exists a unique pair of maximisers $(\phi, \phi^c) \in \mathcal{A}$ *(up to adding a constant to* ϕ*) in the left-hand side problem. The pair of functions* (ϕ, ϕ^c) *satisfies for all* $n, x \in M$

$$\begin{aligned} \phi^c(x) &= \min_{n\in M} c(n,x) - \phi(n) \\ \phi(n) &= \min_{x\in M} c(n,x) - \phi^c(x). \end{aligned} \tag{3}$$

Note that the function $f = \ln\circ\cosh$ is a strictly convex and C^1 function of a real variable such that $f(0) = 0$ and $f'(0) = 0$. Therefore, Theorem 1.5 is a particular instance of a more general result due to McCann [15, Sections 2 and 5] on Riemannian manifolds and the author on Alexandrov spaces [5], applied with $c(n, x) = \ln\circ\cosh(d(n, x))$ as a cost function. This result gives the existence and the *uniqueness*[2] of Lipschitz solutions (ϕ, ϕ^c) to the left-hand side problem.

1.1. Comments and related results.

Our motivation to study this generalization of Iskhakov's result [11] comes from a paper by Oliker [16]. In this paper, Oliker proves Alexandrov's theorem through the study of Kantorovich's primal problem (the left hand-side in (2)). Moreover his proof, as Alexandrov's one, consists in establishing the result for convex polyhedra first and then "passing to the limit". Oliker also studied the regularity of the convex body in terms of the density when the Gauss curvature measure is absolutely continuous [17], generalizing work of Pogorelov [19] on surfaces.

Let us also mention the Minkowski problem which is, somehow, dual to Alexandrov's problem in Euclidean space. In the polyhedron case, it consists in prescribing the normal vectors and the area of each face; in the smooth case, it is the problem of finding which functions can be realised as the Gauss curvature of a convex set (see for instance [2, 24]). It is worth noticing that this problem can also be treated by methods involving or related to optimal mass transport [14, 7]. A study of other curvature measures in the smooth case has been carried out in [10] (see also the references therein). Recently, Barbot, Béguin and Zéghib [4] proved by analytical methods, a result similar to Theorem 1.3 for the Minkowski problem in $\mathbb{R}^2_1$.

Acknowledgements. The author wishes to thank F. Fillastre, C. Lecuire, and J.-M. Schlenker for useful discussions on Alexandrov's problem in the Minkowski spacetime.

2. Duality of star-shaped sets in the Minkowski spacetime

2.1. Background on the Minkowski spacetime and groups of hyperbolic isometries.

2.1.1. *The Minkowski spacetime.* In this part, we recall the results we need in Lorentzian geometry to prove our main theorem. We refer to the textbooks [18, 20] for (much) more on the Lorentzian geometry. The Minkowski spacetime in $m + 1$ dimensions, denoted by $\mathbb{R}^m_1$, is $\mathbb{R}^{m+1}$ endowed with the quadratic form

$$q(x) = \sum_{i=1}^{m} x_i^2 - x_{m+1}^2.$$

For simplicity, we use the same notation for the associated bilinear form. This quadratic form is non-degenerate and admits isotropic vectors. The isotropic cone is divided into the future cone and the past cone. For later use, let us define the future cone through the origin as

$$C_f = \{z \in \mathbb{R}^m_1;\, q(z) < 0 \text{ and } z_{m+1} > 0\}.$$

At some places, we shall also consider the future cone through a general point $x \in \mathbb{R}^n_1$, this cone will be denoted $C_{x,f}$.

[2]In McCann's paper, uniqueness of ϕ (up to adding a constant) is stated in an equivalent way as the uniqueness of the optimal transport map

Definition 2.1 (time-like, space-like, and light-like subspaces)**.** A vector $x \in \mathbb{R}^m_1$ is said to be time-like if $q(x) < 0$, space-like if $q(x) > 0$, and light-like if $q(x) = 0$. We set $C_{x,t} = \{z \in \mathbb{R}^m_1;\ q(\overrightarrow{xz}) < 0\}$, $C_{x,s} = \{z \in \mathbb{R}^m_1;\ q(\overrightarrow{xz}) > 0\}$, and $C_{x,l} = \{z \in \mathbb{R}^m_1;\ q(\overrightarrow{xz}) = 0\}$ the set of time-like vectors, space-like vectors, and light-like vectors centered at x. When x is omitted means that we consider a cone through the origin of $\mathbb{R}^m_1$.

Similarly, a vector subspace V of $\mathbb{R}^m_1$ is said to be time-like if it has a time-like vector, space-like if every nonzero vector is space-like and light-like otherwise.

Definition 2.2 (Hyperbolic space)**.** We denote by $\mathbb{H}^m$ the hyperbolic space endowed with its canonical metric of curvature -1. Throughout the paper, we shall use that $\mathbb{H}^m$ is isometric to the following subset of C_f:

$$\{x \in \mathbb{R}^m_1; q(x) = -1 \text{ and } x_{m+1} > 0\}.$$

The hyperbolic space is to be considered as the unit sphere of C_f relative to q. More generally, we denote by $S(r) = \left\{x \in \mathbb{R}^m_1; q(x) = -r^2 \text{ and } x_{m+1} > 0\right\}$ the sphere of radius r. Besides, it is worth noticing that the distance d induced by the Riemannian metric of $\mathbb{H}^m$ is related with q in the following way:

$$\forall x, n \in \mathbb{H}^m \quad q(n,x) = -\cosh(d(n,x)). \tag{4}$$

We conclude this review section on the Minkowski space with a useful result on the linear isometry group.

Definition 2.3 (Isometry groups)**.** We denote by $Isom(\mathbb{R}^m_1)$ the group of isometries of $\mathbb{R}^{m+1}$, namely the linear maps which preserve the quadratic form q. The subset of positive isometry group denoted by $Isom^+(\mathbb{R}^m_1)$ is the subset of $Isom(\mathbb{R}^m_1)$ whose elements preserve the future cone C_f. This group is homeomorphic to the (Riemmanian) isometry group of $\mathbb{H}^m$.

Moreover, $Isom^+(\mathbb{R}^m_1)$ acts transitively on the set of vector subspaces of given dimension $n \leq m$ and type (namely either time-like, space-like, or light-like).

Last, we set $||\cdot||$ the Euclidean norm in $\mathbb{R}^{m+1}$.

2.1.2. *Discrete cocompact subgroups of Isometries.* In this part, we recall classical fact about isometry subgroups of the hyperbolic space. Our main reference is [20]. We start with the definition of several kinds of subgroups to be considered in the sequel.

Definition 2.4 (Discrete, cocompact, and torsion-free subgroups of $Isom(\mathbb{H}^m)$)**.** Let Γ be a subgroup of $Isom(\mathbb{H}^m)$. Γ is said to be

- *discrete* if the induced topology (of $Isom(R^m_1)$) is the discrete topology. It can be shown that it is equivalent (in this setting) to require that Γ is
- *discontinuous* if for any compact subset $K \subset \mathbb{H}^m$, the set $K \cap gK$ is nonempty for finitely many $g \in \Gamma$.
- *torsion-free* if for each $x \in \mathbb{H}^m$, the stabilizer $\{g \in \Gamma; g.\,x = x\} = \{1\}$ is trivial.
- *cocompact* if $\mathbb{H}^m/\Gamma$ endowed with the quotient topology is a compact space.

We also have

Proposition 2.5. *Let Γ be a discrete subgroup of $Isom(\mathbb{H}^m)$. Then, each orbit $\Gamma.\,x = \{\gamma.\,x, \gamma \in \Gamma\}$ is a discrete closed subset of $\mathbb{H}^m$. Moreover, the formula*

$$d_\Gamma(\Gamma.\,x, \Gamma.\,y) = \inf_{\gamma \in \Gamma} d(x, \gamma.\,y)$$

defines a distance on $\mathbb{H}^m/\Gamma$ whose induced topology is the quotient topology. Moreover, the canonical map $p_\Gamma : \mathbb{H}^m \longrightarrow \mathbb{H}^m/\Gamma$ is an open map.

Definition 2.6 (Fundamental domains)**.** A connected subset D of $\mathbb{H}^m$ is a *fundamental domain* for a group of isometries Γ if

(1) D is an open set

(2) the elements of $\{g.\, D; g \in \Gamma\}$ are pairwise disjoint
(3) $\mathbb{H}^m = \cup_{g\in\Gamma} g.\, \overline{D}$.

Moreover, a fundamental domain is said to be *locally finite* if $\{g.\, \overline{D}; g \in \Gamma\}$ is locally finite, namely for each $x \in \mathbb{H}^m$, there exists an open neighborhood of x which intersects only finitely many $g.\, \overline{D}$ with $g \in \Gamma$.

Theorem 2.7 (Existence of fundamental domain)**.** *Let Γ be a non-trivial discrete subgroup of $Isom(\mathbb{H}^m)$. Then, there exists a convex, locally finite, fundamental domain D for Γ. Moreover, D is bounded whenever Γ is cocompact.*

We also need this general property of fundamental domains for discrete groups.

Theorem 2.8. *Let D be a convex, locally finite fundamental domain for a non-trivial discrete subgroup of $Isom(\mathbb{H}^m)$. Then, the boundary of D is negligible with respect to the Riemannian measure, $\sigma(\partial D) = 0$.*

An easy consequence of this result is the

Corollary 2.9. *Let Γ be a discrete cocompact subgroup of $Isom(\mathbb{H}^m)$ and $f : \mathbb{H}^m/\Gamma \longrightarrow \mathbb{R}$ be a Lipschitz map. Then f admits a unique Lipschitz Γ-invariant lifting $\widetilde{f} : \mathbb{H}^m \longrightarrow \mathbb{R}$, namely*

$$\forall x \in \mathbb{H}^m,\, \forall \gamma \in \Gamma, \quad \widetilde{f}(\gamma.\, x) = f([x]).$$

Proof. Let D be a convex, locally finite fundamental domain and $\widetilde{f}$ be defined on $\cup_{\gamma\in\Gamma}\gamma.\, D$ by the formula $\widetilde{f}(\gamma.\, x) = f([x])$. $\widetilde{f}$ is well-defined since $\gamma.\, D$ are pairwise disjoint. Moreover, $\widetilde{f}$ is Lipschitz on $\cup_{\gamma\in\Gamma}\gamma.\, D$ with the same Lipschitz constant $Lip(f)$ as f. Indeed,

$$\widetilde{f}(\gamma.\, x, \gamma'.\, y) = f([x], [y]) \leq Lip(f)\, d_\Gamma([x], [y]) \leq Lip(f)\, d(\gamma.\, x, \gamma'.\, y)$$

by definition of d_Γ. Now, since $\cup_{\gamma\in\Gamma}\gamma.\, D$ is dense in $\mathbb{H}^m$ thanks to Theorem 2.8, $\widetilde{f}$ admits a unique Lipschitz Γ-invariant extension to $\mathbb{H}^m$. ☐

Now, we deal with the geometric objects we consider in this paper.

Definition 2.10 (Hyperbolic orbifolds and hyperbolic manifolds)**.** We call *hyperbolic orbifold* the quotient space $\mathbb{H}^m/\Gamma$ where Γ is a discrete subgroup of isometries of $\mathbb{H}^m$. This space is a metric space according to Proposition 2.5, moreover it is an Alexandrov space of finite dimension and curvature $K \geq -1$ according to the proposition below. Last, if Γ is further assumed to be torsion-free, then $\mathbb{H}^m/\Gamma$ is actually a smooth hyperbolic manifold.

Remark 2.11*.* Our definition of hyperbolic orbifold is non-standard, however as explained in [20, Chapter13], this definition is equivalent to the usual one and is easier to state.

Proposition 2.12. *Let X be a hyperbolic orbifold. Then, X has curvature bounded from below by -1 in the sense of Alexandrov.*

Proof. This is a particular case of [6, Proposition 10.2.4]. ☐

2.2. **Properties of Fuchsian convex sets.** In this part, we establish the properties of Fuchsian convex sets required in the rest of this paper.

Lemma 2.13. *Let $\mathcal{C}$ be a Fuchsian convex set and $\Gamma \subset Isom^+(\mathbb{R}^m_1)$ be its related subgroup. Then,*
a) $\mathcal{C}$ is not contained in a plane of positive codimension.
b) all supporting planes to $\mathcal{C}$ are space-like.
c) $\mathcal{C}$ is contained in the future half-space delimited by any supporting plane to $\mathcal{C}$.

Proof. Suppose $\mathcal{C}$ is contained in a plane P of positive codimension. Without loss of generality, we can assume that P is of codimension 1. Now, fix a point $x \in \mathcal{C}$ and consider its orbit under the action of Γ, $\Gamma.\, x$. Since Γ is a subgroup of isometries fixing C_f, $\Gamma.\, x$ is contained in a sphere $S(r)$ of radius $r = \sqrt{-q(x)}$ in C_f. Recall that such a sphere is homothetic to the hyperbolic space. Now, elementary considerations lead to the following alternatives. $S(r)\cap P$ is either a hyperellipse when P is space-like, or a totally geodesic hyperplane of $S(r)$ when P is time-like, or a horosphere of $S(r)$ when P is light-like (see [20, p 127] and use the stereographic projection from the conformal ball model [20, p 122]). Note that by assumption, Γ is a cocompact discrete subgroup, therefore $\mathbb{H}^m$ admits a *bounded* fundamental domain D for Γ (see Section 2.1). This gives

$$\mathbb{H}^m = \cup_{\gamma\in\Gamma}\gamma.\overline{D}.$$

This property is in contradiction with all the previous alternatives since in all cases, $\mathbb{H}^m$ is not at finite distance of $S(r)\cap P$. Thus a) is proved.

Let P be a supporting plane at a point $x \in \partial\mathcal{C}$ and $r = \sqrt{-q(x)}$. Then, $\Gamma.\,x \subset \mathcal{C}\cap S(r)$. As before, $P\cap S(r)$ is either a hyperplane, a hyperellipse or a horosphere. Moreover, each connected component of $S(r)\setminus(P\cap S(r))$ corresponds to the set of points of $S(r)$ on a given side of P. Therefore, since $\Gamma.\,x$ is a subset of the convex set $\mathcal{C}$ whose P is a supporting plane, $\Gamma.\,x$ is contained in exactly one connected component of $S(r)\setminus(P\cap S(r))$ up to points in P. Using again that fundamental domains are bounded under our assumptions, we get that the other connected component of $S(r)\setminus(P\cap S(r))$ has to be bounded. Thus, P is necessarily a space-like plane and Γ is contained in the future half-space of P. □

A class of examples is given by

Definition 2.14 (Fuchsian convex polyhedron)**.** F is a Γ-Fuchsian convex polyhedron if there exist $x_1,\cdots,x_k \in \mathbb{H}^m$ pairwise non-collinear and $\lambda_1,\cdots,\lambda_k > 0$ such that

$$F = \{z \in C_f; q(z-(1/\lambda_i)\gamma.\,x_i, \gamma.\,x_i) \le 0 \ \ \forall i = 1,\cdots,k,\, \forall\gamma\in\Gamma\}.$$

F is obviously Γ-invariant. The fact that F is a Fuchsian convex set follows from Lemma 2.30.

A useful property of Fuchsian convex set is given by the following proposition.

Proposition 2.15. *Let $\mathcal{C}$ be a Fuchsian convex set. The projection*

$$\begin{array}{rccc} p: & \partial\mathcal{C} & \longrightarrow & \mathbb{H}^m \\ & x & \longmapsto & \frac{x}{\sqrt{-q(x)}} \end{array}$$

is a homeomorphism.

Proof. The proof follows from the fact, proved below, that if $x,y \in \partial\mathcal{C}$ then $\overrightarrow{xy}$ is space-like. Consequently, the map $p:\partial\mathcal{C}\longrightarrow\mathbb{H}^m$ is continuous and injective. Now, this map is also onto. Indeed, by assumption $\mathcal{C}$ is in C_f, thus in particular $0\notin\mathcal{C}$. Since Γ is a cocompact subgroup, there exists a fundamental domain F which is bounded. By combining these two properties, we get that the distance between 0 and $\mathcal{C}$ is positive. Therefore, for any vector $n\in\mathbb{H}^m$, there exists a hyperplane P q-orthogonal to n separating $\{0\}$ and $\mathcal{C}$. By moving P orthogonally along $\{tn, t>0\}$ till it meets $\mathcal{C}$, we get the existence of a supporting plane to $\mathcal{C}$ orthogonal to n. To get the continuity of p^{-1}, we prove that p is a proper map. To this aim, we fix K a compact subset of $\mathbb{H}^m$. Let us proved that $p^{-1}(K)$ is bounded. If not, we claim that there exists a sequence $x_k\in p^{-1}(K)$ such that $||x_k||\to+\infty$, and $n_k\in\mathcal{G}(x_k)$ such that $n_k\to\infty$ in $\mathbb{H}^m$. Inded if n_k were bounded, we would get by combining the fact that $\mathcal{C}$ is future-convex (Lemma 2.13) and $||x_k||\to+\infty$ that $\mathcal{C}$ is empty, therefore n_k is unbounded. But then by compactness of K, we can find a converging subsequence x_{n_k} to $x_\infty\in K$ where n_{n_k} is unbounded in $\mathbb{H}^m$. Moreover, we can also assume that $n_{n_k}/||n_{n_k}||$ converges to a vector n_∞. By assumption on n_{n_k}, we get that $n_\infty\in\partial C_f$, in other terms n_∞ is light-like. We conclude by noticing that the set of normal vectors to $\mathcal{C}$ is closed and thus we get a contradiction with Lemma 2.13 b). □

Remark 2.16*.* This proposition was proved for Fuchsian embedding in the 2-dimensional smooth case in [13].

Now we prove the lemma used in the proof of Proposition 2.15.

Lemma 2.17. *Let $\mathcal{C}$ be a Fuchsian convex set and $x,y\in\partial\mathcal{C}$ then the vector $\overrightarrow{xy}$ is space-like.*

Proof. We shall argue by contradiction and assume that $\overrightarrow{xy}$ is not space-like. Without loss of generality, we can assume that y belongs to the closure $\overline{C_{x,f}}$ of the future cone through x. We note that the segment $[x, y]$ cannot lie on $\partial\mathcal{C}$ otherwise there would exist (thanks to the Hahn-Banach theorem) a supporting plane which is not space-like. Thus, there exists a point $c \in (x, y) \cap \overset{\circ}{\mathcal{C}}$. Consider a supporting plane P of $\mathcal{C}$ through y. By assumption on $\overrightarrow{xy}$, $x \notin P$, hence the point c clearly belongs to the past halfspace delimited by P. This contradicts Lemma 2.13.c). ☐

2.3. **Gauss curvature measure.** Thanks to Proposition 2.15, we are now in position to properly define the Gauss curvature measure of a Fuchsian convex set. We start with a technical fact.

Lemma 2.18. *Let $\mathcal{C}$ be a Fuchsian convex set. The Gauss map of $\mathcal{C}$ is defined as the multivalued map $\mathcal{G} : \partial\mathcal{C} \rightrightarrows \mathbb{H}^m$ which maps each point $x \in \partial\mathcal{C}$ onto the set $\mathcal{G}(x)$ of inward unit normal vectors (with respect to q) to supporting hyperplane to $\mathcal{C}$ at x. Moreover, for any Borel set $U \subset \partial\mathcal{C}$ (for the induced topology), the set $\mathcal{G}(U)$ is a Borel subset of $\mathbb{H}^m$.*

Proof. A proof for Euclidean convex sets is given in [24, Lemma 2.2.10]. To get the proof in our setting, note that the Gauss map relative to q is nothing but the Euclidean Gauss map of $\mathcal{C}$ composed with a symmetry with respect to the plane orthogonal to the vertical axis. ☐

As a consequence, we get

Lemma 2.19. *Given Γ a discrete cocompact subgroup of $Isom^+(\mathbb{H}^m)$, we define*

$$p_\Gamma : \mathbb{H}^m \longrightarrow \mathbb{H}^m/\Gamma$$

the canonical map. Then, the mapping

$$p_\Gamma \circ \mathcal{G} \circ p^{-1} : \mathbb{H}^m \longrightarrow \mathbb{H}^m/\Gamma$$

is well-defined and Γ-invariant. Therefore, it factorizes to a map

$$\mathcal{G}_\Gamma : \mathbb{H}^m/\Gamma \longrightarrow \mathbb{H}^m/\Gamma \tag{5}$$

such that $\mathcal{G}_\Gamma(U)$ is a Borel set whenever $U \subset \mathbb{H}^m/\Gamma$ is Borel.

Proof. First note that $\Gamma.\,\mathcal{C} = \mathcal{C}$ yields $\Gamma.\,\partial\mathcal{C} = \partial\mathcal{C}$ since Γ is made of isometries. To get the result, we shall prove that for any $x \in \partial\mathcal{C}$, $\mathcal{G}(\gamma\,.x) = \gamma.\,\mathcal{G}(x)$. To this aim and given $x_0 \in \partial\mathcal{C}$, note that $n \in \mathcal{G}(x_0)$ if and only if $q(x - x_0, n) \leq 0$ for any $x \in \mathcal{C}$ (this follows from Lemma 2.13). Now, using that $\Gamma \subset Isom^+(\mathbb{R}^m_1)$, it is obvious that

$$q(x - x_0, n) \leq 0 \text{ iff } q(\gamma\,.x - \gamma\,.x_0, \gamma\,.n) \leq 0$$

therefore $\gamma\,.n$ is a normal vector to $\mathcal{C}$ at $\gamma\,.x_0$ and the result is proved since p_Γ maps Borel set on Borel set. ☐

In the same vein, we also need the

Lemma 2.20. *Let $\mathcal{C}$ be a Fuchsian convex set. Then*

$$\sigma(\{n \in \mathbb{H}^m; \exists x \neq x' \in \partial\mathcal{C}, n \in \mathcal{G}(x) \cap \mathcal{G}(x')\}) = 0. \tag{6}$$

Proof. In the Euclidean case, this is proved for possibly unbounded convex hypersurfaces in [3, Lemma 5.2]. We conclude in our case by using a symmetry as in the proof of Lemma 2.18. ☐

The last technical tool we need in this part is the following

Lemma 2.21. *Let $\mathcal{C}$ be a Fuchsian convex set and Γ its related subgroup of isometries. Then, there exists a unique canonical Borel measure $\sigma_{\mathbb{H}^m/\Gamma}$ on $\mathbb{H}^m/\Gamma$, its total mass equals $\sigma_{\mathbb{H}^m}(D)$ where $\sigma_{\mathbb{H}^m}$ is the Riemannian measure on $\mathbb{H}^m$ and D is any convex, locally finite, fundamental domain for Γ. In the sequel, $\sigma_{\mathbb{H}^m}(D)$ is denoted by $Vol(\mathbb{H}^m/\Gamma)$.*

Proof. Since Γ is a discrete cocompact sugroup of isometries, it admits according to Theorem 2.7 a convex, locally finite, fundamental domain $D \subset \mathbb{H}^m$. Moreover, any such fundamental domain D^* has the same volume V (V is finite since we assume Γ is cocompact) since Theorem 2.8 indicates that $\sigma_{\mathbb{H}^m}(\partial D^*) = 0$ under these hypotheses. Therefore, we can define $\sigma_{\mathbb{H}^m/\Gamma}$ by means of any D^* as above, the condition on the boundary ∂D^* gives us uniqueness of $\sigma_{\mathbb{H}^m/\Gamma}$. □

Definition 2.22 (Gauss curvature measure)**.** Let $\mathcal{C}$ be a Fuchsian convex set and Γ its related subgroup of isometries. We define the Gauss curvature measure μ as the Borel measure on $\mathbb{H}^m/\Gamma$ defined by the formula

$$\mu(U) = \sigma_{\mathbb{H}^m/\Gamma}(\mathcal{G}_\Gamma(U))$$

where U is a Borel subset of $\mathbb{H}^m/\Gamma$. Note that $\mu(\mathbb{H}^m/\Gamma) = \sigma_{\mathbb{H}^m/\Gamma}(\mathbb{H}^m/\Gamma) = Vol(\mathbb{H}^m/\Gamma)$.

Remark 2.23. Note that the Gauss curvature measure remains the same if $\mathcal{C}$ is replaced by $Ho(\mathcal{C})$ with Ho any homothety about the origin of $\mathbb{R}^m_1$.

2.4. **Radial and support functions of Fuchsian convex sets.** We now define the radial and support function of a Fuchsian convex set. These functions are completely analogous to those relative to Euclidean convex body and are related to each other as in the Euclidean case. This relation is the starting point of the proof of Theorem 1.3.

Thanks to Proposition 2.15, we can define the radial function of a Fuchsian convex set in the following way.

Definition 2.24. Let $\mathcal{C}$ be a Fuchsian convex set. The radial function $\theta : \mathbb{H}^m \longrightarrow (0, +\infty)$ of $\partial\mathcal{C}$ is defined by the following formula

$$\forall x \in \mathbb{H}^m,\ \ p^{-1}(x) = \theta(x)x.$$

Equivalently,

$$\theta(x) = \sup\{s > 0; sx \in \mathcal{C}^c\}.$$

The radial function satisfies the following property.

Lemma 2.25. *Let $\mathcal{C}$ be a Fuchsian convex set. For any supporting plane P to $\mathcal{C}$ through $z \in \partial\mathcal{C}$ (namely $P = z + n^\perp$ with $n \in \mathbb{H}^m$), the following inequality holds true*

$$\forall x \in \mathbb{H}^m,\ \theta(x)q(x,n) \leq \theta(x_0)q(x_0,n) \tag{7}$$

where $x_0 = p(z)$. Conversely, if the inequality above holds true then the plane $p^{-1}(x_0) + n^\perp$ is a supporting plane to $\mathcal{C}$ at $p^{-1}(x_0)$, in other terms

$$(7) \Leftrightarrow n \in \mathcal{G}(\theta(x_0)x_0). \tag{8}$$

Last, the function θ is invariant under the action of Γ:

$$\forall x \in \mathbb{H}^m, \forall \gamma \in \Gamma\ \ \theta(\gamma.x) = \theta(x).$$

Proof. Let P be a supporting plane to $\mathcal{C}$ through $p^{-1}(x_0)$. Since by assumption P is space-like, $P = p^{-1}(x_0) + n^\perp$ with $n \in \mathbb{H}^m$. We can parametrize $P \cap C_f$ thanks to the map

$$\begin{array}{rccc} f: & \mathbb{H}^m & \longrightarrow & P \cap C_f \\ & x & \longmapsto & \theta_P(x)x \end{array}$$

where $\theta_P(x)$ is defined by the equality

$$q(\theta_P(x)x - \theta(x_0)x_0, n) = 0.$$

Now, thanks to Lemma 2.13c), the convexity of $\mathcal{C}$ is equivalent to

$$\forall x \in \mathbb{H}^m,\ \theta(x) \geq \theta_P(x)$$

for any supporting plane P. This gives the inequality since $q(x,n) < 0$. The rest of the proof follows easily. □

By analogy with the Euclidean case, we define the support function in the following way.

Definition 2.26. We denote by $h : \mathbb{H}^m \to (-\infty, 0)$ the support function defined by the formula

$$h(n) = \sup_{x \in \mathbb{H}^m} \theta(x) q(x, n). \tag{9}$$

2.5. **Duality on Fuchsian convex sets.** Now, we want to define a mapping anologous to the polar transform in Euclidean space. Despite this mapping being valid for any subset of C_f, we mainly concentrate on the polar transform of certain star-shaped subsets of the future cone trough the origin.

Definition 2.27 (Future star-shaped sets)**.** A future star-shaped set is a closed set S contained in C_f, whose *complement* S^c is star-shaped with respect to the origin and whose radial function $\theta_S : \mathbb{H}^m \longrightarrow (0, +\infty)$ defined by the formula

$$\theta_S(x) = \sup\{s \in (0, +\infty); sx \in S^c\}$$

is a continuous function, bounded away from 0 and ∞. By analogy with Fuchsian convex sets, we define the support function h_S of any $S \in \mathcal{E}$ by the formula (9).

The set of all future star-shaped sets is denoted by $\mathcal{E}$.

Remark 2.28. Thanks to Proposition 2.15, the projection $p : \partial\mathcal{C} \to \mathbb{H}^m$ is a homeomorphism thus the radial function of a Fuchsian convex set $\mathcal{C}$, $\theta_{\mathcal{C}} = \sqrt{-q(p^{-1}(x))}$ is continuous. It is also bounded away from 0 and ∞ since it is invariant under the action of a cocompact group Γ. In short, any Fuchsian convex set belongs to $\mathcal{E}$.

Now, we define the polar transform for any non-empty $S \subset C_f$.

Definition 2.29 (Polar transform of a future set)**.** Given $S \subset C_f$ a nonempty set, the polar transform of S is defined by

$$S^\circ = \{x \in C_f; \forall n \in S,\, q(x, n) \leq -1\}.$$

The basic properties of this polar transform, similar to the Euclidean polar transform, are summarized in the following

Lemma 2.30. *Let $S \subset C_f$ be a nonempty set. Then, S° is a closed convex set. Moreover, $(S^\circ)^\circ \supset S$ and, if we further assume that S^c is a star-shaped set with respect to 0, then equality holds if and only if S is a convex set. When $S \in \mathcal{E}$, we have relations between the radial and support function similar to those in the Euclidean case, namely*

$$-\frac{1}{\theta_{S^\circ}} = h_S \tag{10}$$

and

$$\frac{1}{\theta_S} \leq \inf_{n \in \mathbb{H}^m} \frac{q(\cdot, n)}{h_S(n)}$$

where the two functions are identically equal if and only if S is a convex set.

Proof. The polar transform of S can be rewritten in the following way

$$S^\circ = \cap_{n \in S}\{x \in \mathbb{R}^{m+1}; q(n, x) \leq -1\} \cap C_f.$$

Therefore, since $\{x \in \mathbb{R}^{m+1}; q(n, x) \leq -1\}$ is a half-space, S° is a closed convex set. Now, the symmetry of q obviously implies that $(S^\circ)^\circ \supset S$. Let us show that if S is convex and S^c is star-shaped then equality holds. Pick $x \in C_f \setminus S$ and consider P a supporting space-like plane to S at $p^{-1}(x)$. We denote by $n \in \mathbb{H}^m$ a normal vector to P. To get the existence of such a plane, it is sufficient to prove that there is plan orthogonal to n that separates S and the origin. Now if there was no such plan, we would get that $0 \in \mathcal{C}$ by considering a sequence of points which belong to S and a plane orthogonal to n where the distance between the origin and the corresponding sequence of planes goes to 0, a contradiction. Therefore, Lemma 2.25 yields that for any $z \in S$, $q(z, n) \leq q(p^{-1}(x), n) < 0$ and $q(x, n) > q(p^{-1}(x), n)$ since $x \in C_f \setminus S$ by assumption. Therefore the first inequality implies $n/\sqrt{-q(p^{-1}(x),n)} \in S^\circ$ while the second gives $q(x, n/\sqrt{-q(p^{-1}(x),n)}) > -1$. By combining these properties, we get $x \notin (S^\circ)^\circ$.

Now, let us prove the inequalities involving radial and support functions. By assumption on S, we have

$$S^\circ = \{n \in C_f; \forall x \in \partial S, \theta_S(x)q(x,n) \leq -1\}.$$

Now, fixing n and considering $\lambda\theta_S(x)q(x,n) \leq -1$ for $\lambda > 0$, we get using $q(x,n) < 0$, that

$$\theta_{S^\circ}(n) = \sup_{x\in\mathbb{H}^m} \frac{-1}{\theta_S(x)q(x,n)}$$

and (10) is proved. To prove the last formula, we first use that $(S^\circ)^\circ \supset S$ which can be restated as $\theta_{(S^\circ)^\circ} \leq \theta_S$. Now applying (10) to $(S^\circ)^\circ$ yields

$$\frac{1}{\theta_{(S^\circ)^\circ}} = -h_{S^\circ} = -\sup_{n\in\mathbb{H}^m} \theta_{S^\circ}(n)q(n,\cdot).$$

Applying (10) to S now gives

$$\frac{1}{\theta_S} \leq \frac{1}{\theta_{(S^\circ)^\circ}} \leq \inf_{n\in\mathbb{H}^m} \frac{q(\cdot,n)}{h_S(n)}.$$

To conclude, note that equality holds if and only if S is convex since by assumption S^c is star-shaped with respect to the origin.

□

Now, we restrict our attention to Fuchsian convex sets. Recall that for these sets, the functions $\theta_{\mathcal{C}}$ and $h_{\mathcal{C}}$ are also invariant under the action of a cocompact subgroup of isometries Γ. Therefore, we can consider $\theta_{\mathcal{C}}$ and $h_{\mathcal{C}}$ as functions on the compact space $\mathbb{H}^m/\Gamma$.

Lemma 2.31. *The radial and support functions of $\mathcal{C}$ considered as functions on $\mathbb{H}^m/\Gamma$, are Lipschitz regular.*

Proof. First, note that thanks to Corollary 2.9, a Γ-invariant function on $\mathbb{H}^m$ is Lipchitz if and only if the function induced on $\mathbb{H}^m/\Gamma$ is Lipschitz. Thus, we shall consider the radial and support function as functions on the compact space $\mathbb{H}^m/\Gamma$. By compactness, it is sufficient to prove that $\theta_{\mathcal{C}}$ and $h_{\mathcal{C}}$ are *locally* Lipschitz functions. Note that since $\mathcal{C}$ is a convex set, $\theta_{\mathcal{C}}$ is a locally Lipschitz function viewed (locally) as a function of $X \in \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+1}$. Combining this together with the fact that the inverse of the Riemannian exponential map from $\mathbb{H}^m$ to $T_x\mathbb{H}^m$ (for an arbitrary x) is locally Lipschitz as well as the Euclidean orthogonal projection from $T_x\mathbb{H}^m$ to $\mathbb{R}^m \times \{0\}$, we get that $\theta_{\mathcal{C}}$ is Lipchitz on a neighborhood of any point. We conclude that $\theta_{\mathcal{C}}$ is locally Lipschitz on $\mathbb{H}^m/\Gamma$ by using that the canonical map $p_\Gamma : \mathbb{H}^m \longrightarrow \mathbb{H}^m/\Gamma$ is open (see Proposition 2.5). It remains to prove that $h_{\mathcal{C}}$ is a Lipschitz function. Recall that for any $n \in \mathbb{H}^m$, $h_{\mathcal{C}}(n) = \sup_{x\in\mathbb{H}^m} \theta_{\mathcal{C}}(x)q(x,n)$. Using that $q(x,n) = -\cosh(d_{\mathbb{H}^m}(x,n))$ together with the Γ-invariance of $\theta_{\mathcal{C}}$ and $h_{\mathcal{C}}$, we get that for any $[n] \in \mathbb{H}^m/\Gamma$,

$$h_{\mathcal{C}}([n]) = \sup_{[x]\in\mathbb{H}^m/\Gamma} -\theta_{\mathcal{C}}([x])\cosh(d_\Gamma([x],[n])).$$

Finally, using that $\theta_{\mathcal{C}}$ is bounded away from zero and the compactness of $\mathbb{H}^m/\Gamma$, we get the existence of $[x_0]$ such that

$$h_{\mathcal{C}}([n]) = \theta_{\mathcal{C}}([x_0])\cosh(d_\Gamma([x_0],[n]))$$

which in turn implies that $h_{\mathcal{C}} : \mathbb{H}^m/\Gamma \longrightarrow (-\infty, 0)$ is Lipschitz (with the Lipschitz constant depending on the diameter of $\mathbb{H}^m/\Gamma$). □

To summarize the results of this section, the radial function $\theta_{\mathcal{C}}$ and support function $h_{\mathcal{C}}$ of a Fuchsian convex set can be seen as functions on the compact space $\mathbb{H}^m/\Gamma$. Moreover, the induced functions, still denoted by $\theta_{\mathcal{C}}$ and $h_{\mathcal{C}}$, are Lipschitz regular; besides the first one is positive while the second is negative. As a consequence, the functions $\ln(1/\theta_{\mathcal{C}})$ and $\ln(-h_{\mathcal{C}})$ are well-defined and thanks to Lemma 2.30, satisfy the relations

$$\ln(-h_{\mathcal{C}}(n)) = \inf_{x\in\mathbb{H}^m/\Gamma}\left\{\ln(-q(x,n)) - \ln\left(\frac{1}{\theta_{\mathcal{C}}(x)}\right)\right\}$$

and

$$\ln\left(\frac{1}{\theta_{\mathcal{C}}(x)}\right) = \inf_{n\in\mathbb{H}^m/\Gamma}\{\ln(-q(x,n)) - \ln(-h_{\mathcal{C}}(n))\}$$

for any $x, n \in \mathbb{H}^m/\Gamma$. For simplicity, let us denote by c the function $c(n,x) = \ln(-q(x,n))$. In optimal mass transport theory, the above relations are well-known: $\ln(-h_{\mathcal{C}}(n))$ and $\ln\left(\frac{1}{\theta_{\mathcal{C}}(x)}\right)$ are obtained one from the other through the *c-transform* as recalled in the next section. This simple remark is the starting point of the proof of our main theorem.

3. Equivalence of Theorem 1.3 and Theorem 1.5

In this part, we show that the solutions of both problems are in one-to-one correspondence. Therefore, thanks to McCann's theorem and its generalization to Alexandrov spaces, this will give us a proof of Alexandrov's theorem for hyperbolic orbifolds. We use in this part some elementary properties of c-concave functions to be defined below.

3.1. **c-concave functions.** Throughout this part $c(n,x) = \ln(-q(n,x)) = \ln\circ\cosh(d(n,x))$.

Definition 3.1 (c-concave function and c-subdifferential)**.** Let $\mathbb{H}^m/\Gamma$ be a hyperbolic orbifold and $\phi : \mathbb{H}^m/\Gamma \longrightarrow \mathbb{R}\cup\{-\infty\}$ be a function. We define $\phi^c : \mathbb{H}^m/\Gamma \longrightarrow \mathbb{R}\cup\{-\infty, +\infty\}$, the c-transform of ϕ by the formula

$$\phi^c(x) = \inf_{n\in\mathbb{H}^m} c(n,x) - \phi(n).$$

Such a function ϕ is said to be c-concave if for all $x \in \mathbb{H}^m/\Gamma$, $\phi^c(x) < +\infty$ (so that $(\phi^c)^c$ is well-defined) and $(\phi^c)^c = \phi$ (in the rest of the paper, we will write ϕ^{cc}). The c-subdifferential of a function ϕ is defined by the formula

$$\partial_c\phi = \{(n,x) \in \mathbb{H}^m/\Gamma \times \mathbb{H}^m/\Gamma; \phi(n) + \phi^c(x) = c(n,x)\}.$$

Under our assumptions, it is not difficult to prove that c-concave function is Lipschitz (for instance, a proof is given in [15, Lemma 2]).

Lemma 3.2. *Let $\mathbb{H}^m/\Gamma$ be a hyperbolic orbifold and $\phi : \mathbb{H}^m/\Gamma \longrightarrow \mathbb{R}\cup\{-\infty\}$ be a c-concave map. Then, ϕ is Lipschitz map.*

3.2. **Proof of the equivalence.** The first step of the proof is to check that to a Fuchsian convex set corresponds a unique pair (ϕ,ψ) of c-concave functions (with $\psi = \phi^c$) where uniqueness is to be understood up to homotheties on Fuchsian convex sets and up to adding/subtracting a constant to the pair of c-concave maps.

Now, a Fuchsian convex set (as its Euclidean analogue) is clearly determined by the data of its support h and radial θ functions on $\mathbb{H}^m/\Gamma$ (Lipschitz lifting to $\mathbb{H}^m$ exists thanks to Corollary 2.9). Moreover, as explained in Sections 2.4 and 2.5, the functions θ and h are Lipschitz regular, positive and negative respectively, and $\ln(-h)$ and $\ln\left(\frac{1}{\theta}\right)$ are c-concave.

Conversely, given two functions ϕ and ϕ^c where ϕ is c-concave, Lemma 3.2 insures that ϕ and ϕ^c are Lipschitz. Easy computations then show that $\theta = \exp(-\phi^c) > 0$ and $h = -\exp(\phi) < 0$ are Lipschitz regular and satisfy

$$h = \sup_{x\in\mathbb{H}^m/\Gamma}\theta(x)q(x,\cdot) \text{ and } \frac{1}{\theta} = \inf_{n\in\mathbb{H}^m/\Gamma}\frac{q(\cdot,n)}{h(n)}.$$

Thus, Lemma 2.30 implies that the liftings of h and θ on $\mathbb{H}^m$ determine a unique *Fuchsian* convex set $\mathcal{C}$ (indeed θ is Γ-invariant yields $\partial\mathcal{C}$ is Γ-invariant then we conclude by using that Γ is made of linear isometries).

So we are left with proving that, on one hand, the pair of c-concave maps (ϕ,ψ) we get (as explained above) from the datum of a Fuchsian convex set is indeed a solution of Kantorovitch's dual problem. On the other hand, we also have to prove that the Fuchsian convex set determined by a solution of Kantorovitch's dual problem (ϕ,ψ) admits μ as Gauss curvature measure. We start with the proof of the first point.

To this aim, we just have to find a transport plan $\Pi\in\Gamma(\sigma_{\mathbb{H}^m/\Gamma},\mu)$ with the property

$$\Pi(\{(n,x)\in\mathbb{H}^m/\Gamma\times\mathbb{H}^m/\Gamma;\,\phi(n)+\psi(x)=c(n,x)\})=Vol(\mathbb{H}^m/\Gamma)$$

Indeed, the existence of such a plan yields the equality

$$\int_{\mathbb{H}^m/\Gamma}\phi(n)d\sigma_{\mathbb{H}^m/\Gamma}(n)+\int_{\mathbb{H}^m/\Gamma}\psi(x)d\mu(x)=\int_{\mathbb{H}^m/\Gamma\times\mathbb{H}^m/\Gamma}c(n,x)\,d\Pi(n,x)$$

which in turns gives that (ϕ,ψ) is a solution of Kantorovitch's dual problem. Recall that we write $\partial_c\phi=\{(n,x)\in\mathbb{H}^m/\Gamma\times\mathbb{H}^m/\Gamma;\,\phi(n)+\psi(x)=c(n,x)\}$. We claim that the plan $\Pi(A)=\sigma_{\mathbb{H}^m/\Gamma}(p_n(A\cap\partial_c\phi))$ (where $A\subset\mathbb{H}^m/\Gamma\times\mathbb{H}^m/\Gamma$ is a Borel set, and p_n stands for the projection onto the n coordinate) defines a measure in $\Gamma(\sigma_{\mathbb{H}^m/\Gamma},\mu)$. First, since $\partial_c\phi$ is a compact set, $p_n(\partial\phi\cap A)$ is a Suslin set hence measurable. Now, Π is σ-additive since, given two disjoint sets A and B,

$$p_n(\partial_c\phi\cap A)\cap p_n(\partial_c\phi\cap B)\subset\{n\in\mathbb{H}^m/\Gamma;\exists\,x\neq x'\in\mathbb{H}^m/\Gamma,n\in\mathcal{G}_\Gamma(x)\cap\mathcal{G}_\Gamma(x')\}$$

thanks to Lemma 2.25. Now, let D be a fundamental domain for Γ. Recall that $\gamma.\,D\cap D=\emptyset$ for any non-trivial $\gamma\in\Gamma$. By definition of $\sigma_{\mathbb{H}^m/\Gamma}$, we have

$$\begin{aligned}(11)\quad &\sigma_{\mathbb{H}^m/\Gamma}(\{n\in\mathbb{H}^m/\Gamma;\exists\,x\neq x'\in\mathbb{H}^m/\Gamma,n\in\mathcal{G}_\Gamma(x)\cap\mathcal{G}_\Gamma(x')\})=\\ &\qquad\sigma(\{n\in D;\exists\,x\neq x'\in D,n\in\mathcal{G}(p^{-1}(x))\cap\mathcal{G}(p^{-1}(x'))\})=0\end{aligned}$$

where the last equality follows from Lemma 2.20. Moreover, note that $\Pi(\mathbb{H}^m/\Gamma\times\mathbb{H}^m/\Gamma)=\mu(\mathbb{H}^m/\Gamma)=Vol(\mathbb{H}^m/\Gamma)$ since $\mathcal{G}_\Gamma$ is onto. With these properties in hand, it is easy to check that $\Pi\in\Gamma(\sigma_{\mathbb{H}^m/\Gamma},\mu)$.

It remains to prove that the Fuchsian convex set determined by a solution of Kantorovitch's dual problem (ϕ,ψ) admits μ as Gauss curvature measure. This follows from the sequence of identities below where U is an arbitrary Borel subset of $\mathbb{H}^m/\Gamma$ and $\Pi_0\in\Gamma(\sigma_{\mathbb{H}^m/\Gamma},\mu)$ an optimal plan .

$$\begin{aligned}\mu(U)&=\Pi_0(\mathbb{H}^m/\Gamma\times U)\\&=\Pi_0(\mathbb{H}^m/\Gamma\times U\cap\{(n,x)\in(\mathbb{H}^m/\Gamma)^2;\phi(n)+\psi(x)=c(n,x)\})\\&=\Pi_0(\mathbb{H}^m/\Gamma\times U\cap\{(n,x)\in(\mathbb{H}^m/\Gamma)^2;n\in\mathcal{G}_\Gamma(x)\})\\&=\Pi_0(\mathcal{G}_\Gamma(U)\times U\cap\{(n,x);n\in\mathcal{G}_\Gamma(x)\})\\&=\Pi_0(\mathcal{G}_\Gamma(U)\times\mathbb{H}^m/\Gamma\cap\{(n,x);n\in\mathcal{G}_\Gamma(x)\})\\&=\Pi_0(\mathcal{G}_\Gamma(U)\times\mathbb{H}^m/\Gamma)\\&=\sigma_{\mathbb{H}^m/\Gamma}(\mathcal{G}_\Gamma(U))\end{aligned}$$

where we used several times that $\Pi_0(\{(n,x)\in(\mathbb{H}^m/\Gamma)^2;\phi(n)+\psi(x)=c(n,x)\})=Vol(\mathbb{H}^m/\Gamma)$ and, to get the equality in line 5, the fact that

$$\begin{aligned}&\mathcal{G}_\Gamma(U)\times U^c\cap\{(n,x)\in(\mathbb{H}^m/\Gamma)^2;n\in\mathcal{G}_\Gamma(x))\}\subset\\&\qquad\qquad\{(n,x)\in(\mathbb{H}^m/\Gamma)^2;\exists\,x'\neq x,n\in\mathcal{G}_\Gamma(x)\cap\mathcal{G}_\Gamma(x')\}\end{aligned}$$

which yields

$$\begin{aligned}&\Pi_0(\mathcal{G}_\Gamma(U)\times U^c\cap\{(n,x)\in(\mathbb{H}^m/\Gamma)^2;n\in\mathcal{G}_\Gamma(x))\})\leq\\&\qquad\qquad\sigma_{\mathbb{H}^m/\Gamma}(\{n\in\mathbb{H}^m/\Gamma;\exists\,x'\neq x,n\in\mathcal{G}_\Gamma(x)\cap\mathcal{G}_\Gamma(x')\})\end{aligned}$$

and finally gives us the result thanks to (11).

It remains to prove that the Fuchsian convex set is a polyhedron when the Gauss curvature measure is a finite sum of Dirac masses. To this aim, we need the intermediate result below.

Lemma 3.3. *Let* Γ *be a discrete cocompact group and* F *be a* Γ*-Fuchsian convex polyhedron. Then, the polar set* F° *is also a* Γ*-Fuchsian convex polyhedron.*

Proof. According to Lemma 2.30, F° is a convex subset of C_f. Moreover, F° is Γ-invariant since F is Γ-invariant.The only thing to prove is that F° is a polyhedron in the sense of Definition 2.14. Recall that F is a Fuchsian convex polyhedron if and only if there exist $x_1, \cdots, x_s \in \mathbb{H}^m$ and $\lambda_1, \cdots, \lambda_s$, s positive numbers, such that

$$F = \{\lambda_i \gamma.\, x_i;\, i \in \{1, \cdots, s\}, \gamma \in \Gamma\}^\circ.$$

Note that using $(S_1 \cup S_2)^\circ = S_1^\circ \cap S_2^\circ$, we can assume $s = 1$ in the definition above. Moreover, up to a homothety about the origin, we can assume that $\lambda_1 = 1$. Now, let D be a convex, locally finite fundamental domain for Γ. Since Γ is discrete then for any $x \in C_f$, the set $\Gamma.\, x$ is a closed discrete subset of C_f (see Lemma 2.5). Therefore, $\Gamma.\, x_1$ (where x_1 is such that $F = \{x_1\}^\circ$) intersects finitely many points in $\overline{D}$. Let $y_1, \cdots, y_s \in \mathbb{H}^m$ be those points (pairwise distinct). An easy computation gives $q(y_i - y_j) > 0$ when $i \neq j$, therefore the convex hull of $\{y_1, \cdots, y_s\}$ in $\mathbb{R}^{m+1}$ is a polyhedron whoses faces are space-like. Therefore, each face F_i can be written as $F_i = \{z \in C_f; q(z - \mu_i n_i, n_i) = 0\}$ with $n_i \in \mathbb{H}^m$ and $\mu_i > 0$. In other terms, each face F_i is contained in the boundary of $\{(1/\mu_i) n_i\}^\circ$. We conclude by using that $\mathbb{H}^m = \cup_{\gamma \in \Gamma} \gamma.\, \overline{D}$ by definition of a fundamental domain. □

Now, let μ be a finite combination of Dirac masses and $\mathcal{C}$ be "the" Fuchsian convex set whose μ is the Gauss curvature measure. Let D be a convex, locally finite, fundamental domain. We set $x_1, \cdots, x_k \in \overline{D}$ such that $supp\, \mu = \{[x_1], \cdots, [x_k]\}$. Let $h : \mathbb{H}^m \longrightarrow (-\infty, 0)$ and $\theta : \mathbb{H}^m :\longrightarrow (0, +\infty)$ be the radial and support functions of $\mathcal{C}$. Recall that μ is supported on the compact set $\{x \in \mathbb{H}^m; \exists n;\, h(n) = \theta(x) q(n, x)\}/\Gamma$. Note that using Lemma 2.30, we have $\theta_{\mathcal{C}^\circ} = -1/h_{\mathcal{C}}$ and $h_{\mathcal{C}^\circ} = -1/\theta_{\mathcal{C}}$. Therefore, using that $n \in \mathcal{G}(p^{-1}(x))$ iff $h(n) = \theta(x) q(n, x)$, we get that for any $n, x \in \mathbb{H}^m$,

$$n \in \mathcal{G}^{\mathcal{C}}(p_{\mathcal{C}}^{-1}(x)) \Longleftrightarrow x \in \mathcal{G}^{\mathcal{C}^\circ}(p_{\mathcal{C}^\circ}^{-1}(n))$$

where we add ${}^{\mathcal{C}}$ and ${}^{\mathcal{C}^\circ}$ to $\mathcal{G}$ in order to distinguish between the two Gauss maps. Therefore, combining this together with the assumption on μ, we get that $\mathcal{C}^\circ$ has finitely many normal vectors in $\overline{D}$, namely the vectors $x_1, \cdots, x_k$. This yields that $\mathcal{C}^\circ$ is a Fuchsian convex polyhedron. Thanks to Lemma 3.3, the proof is complete.

References

1. A.D. Aleksandrov, *Existence and uniqueness of a convex surface with a given integral curvature.*, C. R. (Dokl.) Acad. Sci. URSS, n. Ser. **35** (1942), 131–134 (English).
2. A. D. Alexandrov, *Convex polyhedra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005, Translated from the 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze and A. B. Sossinsky, With comments and bibliography by V. A. Zalgaller and appendices by L. A. Shor and Yu. A. Volkov. MR MR2127379 (2005j:52002)
3. Ilya J. Bakelman, *Convex analysis and nonlinear geometric elliptic equations*, Springer-Verlag, Berlin, 1994, With an obituary for the author by William Rundell, Edited by Steven D. Taliaferro. MR MR1305147 (95k:35063)
4. T. Barbot, F. Béguin, and A. Zeghib, *Prescribing Gauss curvature of surfaces in 3-dimensional spacetimes, application to the minkowski problem in the minkowski space*, Ann. Inst. Fourier 61 (2011), no 2, 511–591.
5. J. Bertrand. Existence and uniqueness of optimal maps on Alexandrov spaces, Adv. Math. **219**, (2008), no. 3, 838–851. MR 2442054 (2009j:49094)
6. Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001.
7. G. Carlier, *On a theorem of Alexandrov*, J. Nonlinear Convex Anal. **5** (2004), no. 1, 49–58. MR MR2049056 (2005a:53129)
8. F. Fillastre *Fuchsian convex bodies: basics of Brunn-Minkowski theory*, To appear in G.A.F.A.
9. Wilfrid Gangbo and Robert J. McCann, *The geometry of optimal transportation*, Acta Math. **177** (1996), no. 2, 113–161. MR MR1440931 (98e:49102)

10. Pengfei Guan, Changshou Lin, and Xi-Nan Ma, *The existence of convex body with prescribed curvature measures*, Int. Math. Res. Not. IMRN (2009), no. 11, 1947–1975. MR MR2507106 (2010e:53129)
11. I. Iskhakov, *On hyperbolic surfaces tesselations and equivariant spacelike convex polyhedra*, PhD thesis, Ohio State University, 2000.
12. L. Kantorovitch, *On the translocation of masses*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **37** (1942), 199–201. MR MR0009619 (5,174d)
13. François Labourie and Jean-Marc Schlenker, *Surfaces convexes fuchsiennes dans les espaces lorentziens à courbure constante*, Math. Ann. **316** (2000), no. 3, 465–483. MR MR1752780 (2001g:53127)
14. Robert J. McCann, *Existence and uniqueness of monotone measure-preserving maps*, Duke Math. J. **80** (1995), no. 2, 309–323. MR MR1369395 (97d:49045)
15. ———, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. **11** (2001), no. 3, 589–608. MR MR1844080 (2002g:58017)
16. Vladimir Oliker, *Embedding* $\mathbf{S}^n$ *into* $\mathbf{R}^{n+1}$ *with given integral Gauss curvature and optimal mass transport on* $\mathbf{S}^n$, Adv. Math. **213** (2007), no. 2, 600–620. MR MR2332603 (2008f:49049)
17. Vladimir I. Oliker, *The Gauss curvature and Minkowski problems in space forms*, Recent developments in geometry (Los Angeles, CA, 1987), Contemp. Math., vol. 101, Amer. Math. Soc., Providence, RI, 1989, pp. 107–123. MR MR1034975 (91e:53058)
18. Barrett O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983, With applications to relativity. MR MR719023 (85f:53002)
19. A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, American Mathematical Society, Providence, R.I., 1973, Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 35. MR MR0346714 (49 #11439)
20. John G. Ratcliffe, *Foundations of hyperbolic manifolds*, second ed., Graduate Texts in Mathematics, vol. 149, Springer, New York, 2006. MR MR2249478 (2007d:57029)
21. R. Tyrrell Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR MR0274683 (43 #445)
22. R. Tyrrell Rockafellar and Roger J.-B. Wets, *Variational analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 317, Springer-Verlag, Berlin, 1998. MR MR1491362 (98m:49001)
23. Ludger Rüschendorf, *On c-optimal random variables*, Statist. Probab. Lett. **27** (1996), no. 3, 267–270. MR 1395577 (97h:62051)
24. Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993. MR MR1216521 (94d:52007)
25. C. S. Smith and M. Knott, *Note on the optimal transportation of distributions*, J. Optim. Theory Appl. **52** (1987), no. 2, 323–329. MR 879207 (88d:90076)
26. Cédric Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003. MR MR1964483 (2004e:90003)
27. ———, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009, Old and new. MR MR2459454

Institut de Mathématiques de Toulouse, UMR CNRS 5219, Université Toulouse III, 31062 Toulouse cedex 9, France

E-mail address: jerome.bertrand@math.univ-toulouse.fr