EULER CHARACTERISTIC OF REAL NON DEGENERATE TROPICAL COMPLETE INTERSECTIONS

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Abstract. We show that the Euler characteristic of a real non degenerate tropical complete intersection depends only on the Newton polytopes defining it. It is equal to the usual signature of a generic complex complete intersection with same Newton polytopes when this signature is defined.

Introduction

Tropical geometry appeared recently in various fields of mathematics (See [28], [13], [27], [21]). Tropical varieties can be defined as the topological closure of the image under the valuation of algebraic varieties over the field of Puiseux Series $\mathbb{K}$. For example T is a tropical hypersurface if there exists an algebraic hypersurface $Z_{\mathbb{K}}$ in $(\mathbb{K}^*)^n$ such that $T = \overline{V(Z)}$ where V is the coordinatewise valuation (we rather take minus the valuation). By a theorem due to Kapranov (see Theorem 3.2) tropical hypersurfaces are nonlinearity loci of piecewise linear convex functions on $\mathbb{R}^n$ of the form $f^{\text{trop}}(x) = \max_{\omega \in \Omega}(< x, \omega > -a_\omega)$ where Ω is a finite subset of $\mathbb{Z}^n$ and a_ω a real number. One of the important application of tropical geometry is due to Mikhalkin [21] who gave a combinatorial way to count the number of curves of given degree and genus passing through the appropriate number of given generic points. Mikhalkin's proof uses a complexification of tropical curves and a patchworking principle. The algorithm exposed in [21] has a real counterpart for which it is necessary to introduce the real part of complexified tropical curves. The relation between these real tropical objects and objects appearing in Viro combinatorial patchworking method is very deep (see [35], for example). Actually nonsingular real tropical hypersurfaces are equivalent from the topological point of view to the so called primitive T-hypersurfaces appearing in the combinatorial Viro method.

In [30], Bernd Sturmfels generalized the combinatorial patchworking method to complete intersections (see Section 2). The above definition also applies for tropical varieties. Namely, one can define a tropical variety to be the image of an algebraic variety over $\mathbb{K}$ under the valuation map (see Section 3). This leads also to the notion of complex tropical variety and real tropical variety (see Section 6). In Section 5 we give a definition for the notion of nondegenerate tropical complete intersection which builds on the definition of a nonsingular tropical hypersurface in a manner similar to the classical complex situation, recalled in Section 1. We extend in Section 4 the definition of tropical intersection multiplicity numbers which was introduced by Mikhalkin in [25] and show that our definition is consistent with the classical situation. In particular, all intersection multiplicity numbers which occur in a nondegenerate tropical intersection are equal to 1 (or 0). We think that our definition of tropical intersection multiplicity numbers can be of independent interest. The goal of this paper was to extend a previous result of the first author (see [3]) from the case of hypersurfaces to the case of complete intersections. Roughly speaking,

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we prove that if $f_1, \dots, f_k$ are polynomials in $\mathbb{K}[z_1, \dots, z_n]$ which define a nondegenerate tropical intersection Y^{trop}, then the Euler characteristic of the corresponding real tropical intersection $\mathbb{R}Y^{\text{trop}}$ depend only on the Newton polytopes $\Delta_1, \dots, \Delta_k$ of the polynomials and is equal to the mixed signature $\tilde{\sigma}(Y)$ of a generic complex intersection Y defined by complex polynomials with the same Newton polytopes. The precise statement is given in Theorem 8.1. The notion of mixed signature is defined by means of the so called E-polynomials (see Section 7). When Y is a projective complete intersection of even dimension (over $\mathbb{C}$), then the mixed signature $\tilde{\sigma}(Y)$ is equal to the usual signature $\sigma(Y) = \sum_{p+q=0\,[2]} (-1)^p h^{p,q}(Y)$, where the $h^{p,q}(Y)$ are Hodge numbers. One advantage of the mixed signature is that it is defined for a non projective variety and is additive, as it is the case for the Euler characteristic. With the help of this additivity property, we are able to reduce the proof of the main result to a proof of the toric hypersurface case. The proof of the toric hypersurface case uses heavily results obtained by V. Batyrev and L. Borisov in the paper [1].

1. Toric geometry

We fix some notations and recall some standard properties of toric geometry. We refer to [14] for more details. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n and $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. The associated complex torus is $\mathbb{T}_N := \text{Spec}(\mathbb{C}[M]) = \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^n$. Let $f \in \mathbb{C}[M]$ be a Laurent polynomial in the group algebra associated with M

$$f(x) = \sum c_m x^m,$$

where each m belongs to M and only a finite number of c_m are non zero. We will usually have $M = \mathbb{Z}^n$, so that $\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The *support* of f is the subset of M consisting of all m such that the coefficient c_m is non zero. The convex hull of this support in the real affine space generated by M is called *the Newton polytope* of f. This is a *lattice* polytope, or a polytope *with integer vertices*, which means that all the vertices of Δ belong to M. In this paper all polytopes will be lattice polytopes and the ambient lattice M will be clear from the context. We denote by $M(\Delta)$ the saturated sublattice of M which consists of all integer vectors parallel to Δ and by $N(\Delta)$ the dual lattice. The dimension of Δ is the rank of $M(\Delta)$, or alternatively the dimension of the real vector space $M(\Delta)_{\mathbb{R}}$ generated by Δ. The polynomial f (or rather $x^{-m} f \in \mathbb{C}[M(\Delta)]$ for any choice of m in the support of f) defines an hypersurface Z_f in the torus $\mathbb{T}_{N(\Delta)}$. Let X_Δ denote the projective toric variety associated with Δ. The variety X_Δ contains $\mathbb{T}_{N(\Delta)}$ as a dense Zarisky open subset and we denote by $\bar{Z}_f$ the Zarisky closure of Z_f in X_Δ. Let Γ be any face of Δ. If f^Γ is *the truncation* of f to Γ, that is, the polynomial obtained from f by keeping only those monomials whose exponents belong to Γ, then $\bar{Z}_f \cap \mathbb{T}_{N(\Gamma)} = Z_{f^\Gamma}$ and $\bar{Z}_f \cap X_\Gamma = \bar{Z}_{f^\Gamma}$. We have the classical notion of non degenerate Laurent polynomial.

Definition 1.1. *A polynomial f with Newton polytope Δ is called non degenerate if for any face Γ of Δ of positive dimension (including Δ itself), the toric hypersurface Z_{f^Γ} is a nonsingular hypersurface.*

Note that if Γ is a vertex of Δ, then Z_{f^Γ} is empty. In the previous definition, one may alternatively consider f^Γ as a polynomial in $\mathbb{C}[M]$ and thus look at the corresponding hypersurface of the whole torus $\mathbb{T}_N$. Indeed, this hypersurface of $\mathbb{T}_N$ is the product of $Z_{f^\Gamma} \subset \mathbb{T}_{N(\Gamma)}$ with the subtorus of $\mathbb{T}_N$ corresponding to a complement of $M(\Gamma)$ in M. If Δ is the Newton polytope of f, then the projective hypersurface $\bar{Z}_f \subset X_\Delta$ is non singular if and only if f is non degenerate and X_Δ has eventually a finite number of singularities which are zero-dimensional $\mathbb{T}_{N(\Delta)}$-orbits corresponding to vertices of Δ.

Consider polynomials $f_1, \dots, f_k \in \mathbb{C}[M]$ and denote by Δ_i the Newton polytope of f_i. Let Δ be the *Minkowsky sum* of these polytopes

$$\Delta = \Delta_1 + \cdots + \Delta_k.$$

Each polynomial f_i seen as a polynomial in $\mathbb{C}[M(\Delta)]$ defines a toric hypersurface $Z_{f_i,\Delta}$ in $\mathbb{T}_{N(\Delta)}$ and it makes sense to consider the toric intersection

$$Z_{f_1,\Delta} \cap \cdots \cap Z_{f_k,\Delta} \subset \mathbb{T}_{N(\Delta)}. \tag{1.1}$$

Denote by $\bar{Z}_{f_i,\Delta}$ the Zarisky closure in X_Δ of $Z_{f_i,\Delta}$. For each $i = 1, \dots, k$ there is a toric surjective map $\rho_i : X_\Delta \to X_{\Delta_i}$ such that $Z_{f_i,\Delta} = \rho_i^{-1}(Z_{f_i})$ and $\bar{Z}_{f_i,\Delta} = \rho_i^{-1}(\bar{Z}_{f_i})$. This leads to

$$\bar{Z}_{f_1,\Delta} \cap \cdots \cap \bar{Z}_{f_k,\Delta} \subset X_\Delta. \tag{1.2}$$

Each face Γ of Δ can be uniquely written as a Minkowsky sum

$$\Gamma = \Gamma_1 + \cdots + \Gamma_k \tag{1.3}$$

where Γ_i is a face of Δ_i. Substituting the truncation $g_i := f_i^{\Gamma_i}$ to f_i and Γ_i to Δ_i gives the toric intersection

$$Z_{g_1,\Gamma} \cap \cdots \cap Z_{g_k,\Gamma} \subset \mathbb{T}_{N(\Gamma)}. \tag{1.4}$$

which leads to

$$\bar{Z}_{g_1,\Gamma} \cap \cdots \cap \bar{Z}_{g_k,\Gamma} \subset X_\Gamma. \tag{1.5}$$

Similarly to the hypersurface case the intersection of (1.2) with $\mathbb{T}_{N(\Gamma)}$ (resp., X_Γ) coincides with (1.4) (resp., (1.5)). Moreover, the intersection (1.2) is the union over all faces Γ of Δ of the toric intersections (1.4).

The *Cayley polynomial* associated with $f_1, \dots, f_k$ is the polynomial $F \in \mathbb{C}[M \oplus \mathbb{Z}^k]$ defined by

$$F(x, y) = \sum_{i=1}^{k} y_i f_i(x). \tag{1.6}$$

Its Newton polytope is the *Cayley polytope* associated with $\Delta_1, \dots, \Delta_k$ and will be denoted by

$$C(\Delta_1, \dots, \Delta_k) \subset M_\mathbb{R} \times \mathbb{R}^k. \tag{1.7}$$

Since F is a homogeneous (of degree 1) with respect to the variable y, the polytope $C(\Delta_1, \dots, \Delta_k)$ lies on a hyperplane and has thus dimension at most $n + k - 1$. In fact, the dimension of $C(\Delta_1, \dots, \Delta_k)$ is $\dim(\Delta) + k - 1$. The faces of $C(\Delta_1, \dots, \Delta_k)$ are themselves Cayley polytopes. Namely, the faces of $C(\Delta_1, \dots, \Delta_k)$ are the Newton polytopes of all polynomials

$$\sum_{i \in I} y_i f_i^{\Gamma_i}(x)$$

such that $\emptyset \neq I \subset \{1, \dots, k\}$ and $\Gamma = \sum_{i \in I} \Gamma_i$ is a face of $\sum_{i \in I} \Delta_i$ with Γ_i a face of Δ_i for each i. We will call *admissible* such a collection $(\Gamma_i)_{i \in I}$. Note that by face we do not mean proper face. In particular $(\Delta_i)_{i \in I}$ is admissible for any non empty subset I of $\{1, \dots, k\}$. If $(\Gamma_i)_{i \in I}$ is admissible, we also call admissible the collection of polynomials $(f_i^{\Gamma_i})_{i\in I}$ and the corresponding toric intersection

$$\bigcap_{i\in I} Z_{f_i^{\Gamma_i},\Gamma} \subset \mathbb{T}_{N(\Gamma)}. \tag{1.8}$$

Definition 1.2. *The k-uple $(f_1,\dots,f_k)$ is non degenerate if the associated Cayley polynomial $F(x,y)=\sum_{i=1}^k y_i f_i(x)$ is non degenerate.*

The following result is based on the classical *Cayley trick* (see, for example, [16]).

Proposition 1.3. *The k-uple $(f_1,\dots,f_k)$ is non degenerate if and only if any admissible toric intersection* (1.8) *is a complete intersection.*

Proof. As mentionned earlier, we can consider the polynomials $f_i^{\Gamma_i}$ occurring in (1.8) as polynomials in $\mathbb{C}[M]$ and thus look at the corresponding intersection in the whole torus $\mathbb{T}_N$. An easy computation shows that if hypersurfaces defined by polynomials $g_i\in\mathbb{C}[M]$, $i\in I$, do not intersect transversally at a point $X\in\mathbb{T}_N$, then there exists $\lambda=(\lambda_j)_{j\in J}\in(\mathbb{C}^*)^{|J|}$ with $J\subset I$ so that $\sum_{j\in J} y_j g_j(x)$ defines an hypersurface with a singular point at $(X,\lambda)\in\mathbb{T}_N\times(\mathbb{C}^*)^{|J|}$. Similarly, if a truncation $\sum_{i\in I} y_i g_i(x)$ of F to a face of $C(\Delta_1,\dots,\Delta_k)$ defines an hypersurface with a singular point (X,λ) in the corresponding torus, then the hypersurfaces defined by g_i for $i\in I$ will not intersect transversally at $X\in\mathbb{T}_N$. □

2. Combinatorial patchworking

The *combinatorial patchworking*, also called *T-construction*, is a particular case of the Viro method. The general Viro method starts with a *convex* polyhedral subdivision of a polytope Δ contained in the positive orthant $(\mathbb{R}_+)^n$ of $\mathbb{R}^n$. Recall that in this paper all polytopes, including those of a polyhedral subdivision, are lattice polytopes. Here, the ambient lattice is $\mathbb{Z}^n$.

Definition 2.1. *A polyhedral subdivision of a polytope Δ of dimension n is called convex (or coherent) if there exists a convex piecewise-linear function $\nu:\Delta\to\mathbb{R}$ whose maximal domains of linearity coincide with the n-polytopes of the subdivision.*

We begin with a brief description of the combinatorial patchworking in the hypersurface case (see, for example, [19], [34] or [16]). Let $\Delta\subset(\mathbb{R}_+)^n$ be a polytope of maximal dimension n. Start with a convex triangulation $\mathcal{S}$ of Δ and a sign distribution $\delta:\mathrm{vert}(\mathcal{S})\to\{\pm1\}$ at the vertices of $\mathcal{S}$. Let $\nu:\Delta\to\mathbb{R}$ be any function which certifies the convexity of $\mathcal{S}$ and consider the polynomial

$$f_t(x)=\sum_{\mathrm{vert}(\mathcal{S})}\delta(w)t^{\nu(w)}x^w$$

where the sum is taken over the set of vertices of $\mathcal{S}$. Such a polynomial is called a *T-polynomial.*

Denote by $s_{(i)}$ the reflection about the i-th coordinate hyperplane in $\mathbb{R}^n$. Let Δ^* be the union of the 2^n symmetric copies of Δ via compositions of these reflections and extend $\mathcal{S}$ uniquely to a triangulation $\mathcal{S}^*$ which is symmetric with respect to the coordinate hyperplanes. Extend the sign distribution δ to a sign distribution δ^* at the vertices of $\mathcal{S}^*$ so that a vertex of $\mathcal{S}^*$ and its image under a reflection $s_{(i)}$ have the same sign if and only if the i-th coordinate of the vertex is even. If σ is an n-simplex of $\mathcal{S}^*$ whose vertices have different signs, select the hyperplane piece which is the convex hull of the middle points of the edges of σ with endpoints of opposite signs. The union of all these selected pieces produces a piecewise-linear hypersurface H^* in Δ^*.

We perform identifications on the boundary of Δ^* in the following way. Let Γ be any proper face of Δ and consider the cone generated by all outward real vectors which are orthogonal to the facets of Δ incident to Γ. The integer vectors in this cone form a finitely generated semi-goup. Identify two points lying on two symmetric copies of Γ whenever they are symmetric via ${s_{(1)}}^{v_1} \circ {s_{(2)}}^{v_2} \circ \cdots \circ {s_{(n)}}^{v_n}$ for some $v = (v_1, \ldots, v_n)$ in this semi-group. Denote by $\widetilde{\Delta}$ the result of these identifications. By a classical result (see, for example, [16] Theorem 5.4 p. 383 [30] Proposition 2), there is an homeomorphism between the real part $\mathbb{R}X_\Delta$ of X_Δ and $\widetilde{\Delta}$. Moreover, this homeomorphism can be choosen so that it respect the stratification by torus orbits in the sense that the real torus orbit corresponding to a face Γ of Δ is sent to the image under the previous identifications of the union of the symmetric copies of the interior of Γ. In particular, the dense real torus $(\mathbb{R}^*)^n \subset \mathbb{R}X_\Delta$ is sent to the union of the symmetric copies of the interior of Δ. Denote by $\widetilde{H}$ the image of H^* in $\widetilde{\Delta}$.

Theorem 2.2 (T-construction, O. Viro)**.** *For $t > 0$ sufficiently small, the polynomial f_t is non degenerate. Moreover, there exists an homeomorphism $\mathbb{R}X_\Delta \to \widetilde{\Delta}$ which respects the stratification by torus orbits and induces an homeomorphism between the real part of the hypersurface $\bar{Z}_f \subset X(\Delta)$ and $\widetilde{H}$.*

We now describe the extension of the combinatorial patchworking to the case of complete intersections due to B. Sturmfels [31]. Start with $k \geq 2$ polytopes $\Delta_1, \ldots, \Delta_k$ in $(\mathbb{R}_+)^n$. Assume that each Δ_i comes with a convex polyhedral subdivision $\mathcal{S}_i$ induced by a convex piecewise-linear map $\nu_i : \Delta_i \to \mathbb{R}$. These functions $\nu_1, \ldots, \nu_k$ define a convex polyhedral subdivision of the Minkowsky sum $\Delta = \Delta_1 + \cdots + \Delta_k$ in the following way (see [30], [29] or [5]). Let $\hat{\Delta}_i$ be the convex hull of the set $\{(x, \nu_i(x)), x \in \Delta_i\}$ in $\mathbb{R}^n \times \mathbb{R}$. Let $\hat{\Delta} \subset \mathbb{R}^n \times \mathbb{R}$ be the Minkowski sum $\hat{\Delta}_1 + \cdots + \hat{\Delta}_k$. Let $\mathcal{MS}$ be the convex polyhedral subdivision of Δ induced by ν. Each lower face $\hat{\Gamma}$ of $\hat{\Delta}$ can be uniquely written as a Minkowsky sum $\hat{\Gamma}_1 + \cdots + \hat{\Gamma}_k$ of lower faces of $\hat{\Delta}_1, \ldots, \hat{\Delta}_k$. Projecting to Δ, this gives a representation of each polytope Γ of $\mathcal{MS}$ as $\Gamma = \Gamma_1 + \cdots + \Gamma_k$ with $\Gamma_i \in \mathcal{S}_i$ for $i = 1, \ldots, k$. Such a representation is not unique in general, and we shall always use the one obtained by projecting lower faces of $\hat{\Delta}$. The polyhedral subdision $\mathcal{MS}$ together with the associated representation of each of its polytopes is called a convex or coherent *mixed subdivision*. Sturmfels' theorem requires the following genericity condition. Namely, assume that each subdivision $\mathcal{S}_i$ is a triangulation and that

$$\dim \Gamma = \dim \Gamma_1 + \cdots + \dim \Gamma_k$$

for any $\Gamma = \Gamma_1 + \cdots + \Gamma_k \in \mathcal{MS}$ with $\Gamma_i \in \mathcal{S}_i$. We call such a mixed subdivision a convex *tight mixed subdivision.* (See [5, 6] for another versions of the Viro method for complete intersections).

Suppose now that for $i = 1, \ldots, k$ a sign distribution $\delta_i : \mathrm{vert}(\mathcal{S}_i) \to \pm 1$ is given. Consider the T-polynomials associated with these data

$$f_{i,t}(x) = \sum_{\mathrm{vert}(\mathcal{S}_i)} \delta_i(w) t^{\nu_i(w)} x^w \, .$$

Extend $\mathcal{MS}$ to a subdivision $\mathcal{MS}^*$ of Δ^* by means of the reflections about coordinate hyperplanes. Hence $\mathcal{MS}^*$ consists of the polytopes $s(\Gamma) = s(\Gamma_1) + \cdots + s(\Gamma_k)$ where s is a composition of coordinate hyperplane reflections and $\Gamma = \Gamma_1 + \cdots + \Gamma_k \in \mathcal{MS}$ $(\Gamma_i \in \mathcal{S}_i)$. Extend δ_i to a sign distribution δ_i^* at the vertices of $\mathcal{S}_i^*$ using the rule described above. Define a sign distribution

$\delta : \mathrm{vert}(\mathcal{MS}) \to \{\pm 1\}^k$ by assigning $(\delta_1(v_1), \ldots, \delta_k(v_k))$ to each vertex v of $\mathcal{MS}$ with representation $v = v_1 + \cdots + v_k$. Extend δ to a sign distribution δ^* at the vertices of $\mathcal{MS}^*$ so that the i-th sign of a symmetric copy $s(v)$ of $v = v_1 + \ldots + v_k$ is $\delta_i^*(s(v_i))$.

For any $i = 1, 2, \ldots, k$, let $H_i^* \subset \Delta_i^*$ be the piecewise-linear hypersurface constructed via the combinatorial patchworking from $\mathcal{S}_i$ and δ_i. Let $H_i^{\Delta,*} \subset \Delta^*$ be the union over all polytopes $s(\Gamma) = s(\Gamma_1) + \ldots + s(\Gamma_k) \in \mathcal{MS}^*$ of $\oplus_{j \neq i} s(\Gamma_j) + H_i^* \cap s(\Gamma_i) \subset s(\Gamma)$. Let $\widetilde{H_i^\Delta}$ denote the image of H_i^Δ in $\widetilde{\Delta}$.

Theorem 2.3 (B. Sturmfels)**.** *For $t > 0$ sufficiently small the k-uple $(f_{1,t}, \ldots, f_{k,t})$ is non degenerate. Moreover, there exists an homeomorphism $\mathbb{R}X_\Delta \to \widetilde{\Delta}$ which respects the stratification by torus orbits and induces for each i an homeomorphism between the real part of the hypersurface $\bar{Z}_f^\Delta \subset X(\Delta)$ and $\widetilde{H_i^\Delta}$.*

These two versions – for hypersurfaces and complete intersections – of the combinatorial patchworking are related by the so-called *combinatorial Cayley trick*. Consider the Cayley polynomial $F_t \in \mathbb{R}[x, y]$ associated with $(f_{1,t}, \ldots, f_{k,t})$

$$F_t(x, y) = \sum_{i=1}^{k} y_i \, f_{i,t}(x) \, .$$

Its Newton polytope is the Cayley polytope $C(\Delta_1, \ldots, \Delta_k) \subset \mathbb{R}_+^{n+k}$. Let (a, b) be coordinates on $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$. Consider the subspace B of $\mathbb{R}^{n+k}$ defined by $b_1 = b_2 = \cdots = b_k = 1/k$ and identify it with $\mathbb{R}^n$ via the projection $(a, b) \mapsto a$. This identifies $B \cap C(\Delta_1, \ldots, \Delta_k)$ with $\Delta = \Delta_1 + \cdots + \Delta_k$ dilated by $1/k$. Note that the space defined by $b_i = 1$ and $b_j = 0$ for $j \neq i$ intersects $C(\Delta_1, \ldots, \Delta_k)$ along a face which can be identified with Δ_i via the projection. Consider a polyhedral subdivision of $C(\Delta_1, \ldots, \Delta_k)$. If F is a polytope of maximal dimension $\dim \Delta + k - 1$ in this subdivision, then it intersects the space defined by $b_i = 1$ and $b_j = 0$ for $j \neq i$ along a nonempty face F_i, which projects to a (nonempty) subpolytope Γ_i of Δ_i. Then $F \cap B$ is identified via the projection with the polytope $\Gamma = \Gamma_1 + \cdots + \Gamma_k \subset \Delta$ dilated by $1/k$. This gives a correspondence between polyhedral subdivisions of $C(\Delta_1, \ldots, \Delta_k)$ and mixed subdivisions of $\Delta = \Delta_1 + \cdots + \Delta_k$. It is easily seen that triangulations of $C(\Delta_1, \ldots, \Delta_k)$ are sent to tight mixed subdivision via this correspondence. The following result can be found, for example, in [29].

Proposition 2.4. *The correspondence described above is a bijection between the set of convex polyhedral subdivision of $C(\Delta_1, \ldots, \Delta_k)$ and the set of mixed subdivisions of $\Delta = \Delta_1 + \cdots + \Delta_k$. More pecisely, any function $\nu : C(\Delta_1, \ldots, \Delta_k) \to \mathbb{R}$ restricts to functions $\nu_i : \Delta_i \to \mathbb{R}$ if we identify Δ_i with a face of the Cayley polytope via the projection. The bijection sends the coherent polyhedral subdivision of $C(\Delta_1, \ldots, \Delta_k)$ defined by ν to the coherent mixed subdivision of Δ defined by $(\nu_1, \ldots, \nu_k)$.*

Note that in the situation of Theorem 2.3, the Cayley polynomial F_t is a T-polynomial. The non degeneracy of $(f_{1,t}, \ldots, f_{k,t})$ in Theorem 2.3 follows from Proposition 1.3 and Theorem 2.2 applied to F_t.

3. Standard definitions and properties in tropical geometry

The setting and notation here are the same as in [3]. A detailed exposition can be found in [21] and in [18], for example. Let $\mathbb{K}$ be the field of Puiseux series. An element of $\mathbb{K}$ is a series

$g(t) = \sum_{r \in R} b_r t^r$ where each b_r is a complex number and $R \subset \mathbb{Q}$ is bounded from below and contained in an arithmetic sequence. Consider the valuation $\mathrm{val}(g(t)) := \min\{r \mid b_r \neq 0\}$. Using Mikhalkin's conventions, we rather use minus the valuation $v(g) := -\mathrm{val}(g)$. Define

$$\begin{array}{rccc} V: & (\mathbb{K}^*)^n & \longrightarrow & \mathbb{R}^n \\ & z & \longmapsto & (v(z_1), \ldots, v(z_n)). \end{array}$$

Let f be a polynomial in $\mathbb{K}[z_1, \ldots, z_n] = \mathbb{K}[z]$. It is of the form $f(z) = \sum_{\omega \in A} c_\omega z^\omega$ with A a finite subset of $\mathbb{Z}^n$ and $c_\omega \in \mathbb{K}^*$. Let $Z_f = \{z \in (\mathbb{K}^*)^n \mid f(z) = 0\}$ be the zero set of f in $(\mathbb{K}^*)^n$.

Definition 3.1. *The tropical hypersurface Z_f^{trop} associated to f is the closure (in the usual topology) of the image under V of Z_f:*

$$Z_f^{\mathrm{trop}} = \overline{V(Z_f)} \subset \mathbb{R}^n.$$

There are other equivalent definitions of a tropical hypersurface. Namely, define

$$\begin{array}{rccc} \nu: & A & \longrightarrow & \mathbb{R} \\ & \omega & \longmapsto & -v(c_\omega) \end{array}$$

Its Legendre transform is the piecewise-linear convex function

$$\begin{array}{rccc} \mathcal{L}(\nu): & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \max_{\omega \in A}(x \cdot \omega - \nu(\omega)) \end{array}$$

Theorem 3.2 (Kapranov)**.** *The tropical hypersurface Z_f^{trop} is the corner locus of $\mathcal{L}(\nu)$.*

The corner locus of $\mathcal{L}(\nu)$ is the set of points at which it is not differentiable. Another way to define a tropical hypersurface is to use the *tropical semiring* $\mathbb{R}_{\mathrm{trop}}$, which is $\mathbb{R} \cup \{-\infty\}$ endowed with the following tropical operations. The tropical addition of two numbers is the maximum of them, and thus its neutral element is $-\infty$. The tropical multiplication is the ordinary addition with the convention that $x + (-\infty) = -\infty + x = -\infty$. Removing the neutral element for the tropical addition, we get the *one dimensional tropical torus* $\mathbb{T}_{\mathrm{rop}} := \mathbb{R} = \mathbb{R}_{\mathrm{trop}} \setminus \{-\infty\}$. A multivariate tropical polynomial is a polynomial in $\mathbb{R}[x_1, \ldots, x_n]$ where the addition and multiplication are the tropical ones (strictly speaking, the coefficients are in $\mathbb{R}_{\mathrm{trop}}$, but as usual we omit the monomials whose coefficients are the neutral element for the addition). Hence, a tropical polynomial is given by a maximum of finitely many affine functions whose linear parts have integer coefficients and constant parts are real numbers. The tropicalization of a polynomial

$$f(z) = \sum_{\omega \in A} c_\omega z^\omega \in \mathbb{K}[z]$$

where the coefficients $c_\omega \in \mathbb{K}$ are all non zero is the tropical polynomial

$$\mathrm{Trop}(f)(z) = \sum_{\omega \in A} v(c_\omega) z^\omega \in \mathbb{R}[z].$$

This tropical polynomial coincides with the piecewise-linear convex function $\mathcal{L}(\nu)$ defined above. Therefore, Theorem 3.2 asserts that Z_f^{trop} is the corner locus of $\mathrm{Trop}(f)$. Conversely, the corner locus of any tropical polynomial is a tropical hypersurface (just take a polynomial in $\mathbb{K}[z]$ whose coefficients have the right valuations). For these reasons, we will sometimes speak about the tropical hypersurface defined by a polynomial f without specifying if f is in $\mathbb{K}[z]$ or if f is a tropical polynomial (the tropicalization of the latter).

The Newton polytope of the tropical hypersurface Z_f^{trop} is the convex hull of A and will be denoted by Δ. One can associate to Z_f^{trop} a polyhedral subdivision $\mathcal{S}$ of Δ in the following way. Let $\hat{\Delta} \subset \mathbb{R}^n \times \mathbb{R}$ be the convex hull of all points $(\omega, v(c_\omega))$ with $\omega \in A$. Define

$$\begin{array}{rccl} \hat{\nu}: & \Delta & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \min\{y \mid (x,y) \in \hat{\Delta}\}. \end{array} \tag{3.1}$$

The domains of linearity of $\hat{\nu}$ form a convex polyhedral subdivision $\mathcal{S}$ of Δ. The hypersurface Z_f^{trop} is an $(n-1)$-dimensional piecewise linear complex which induces a polyhedral subdivision Ξ of $\mathbb{R}^n$. We will call *cells* the elements of Ξ. Note that these cells have rational slopes. The n-dimensional cells of Ξ are the closures of the connected components of the complementary part of Z_f^{trop}. The lower dimensional cells of Ξ are contained in Z_f^{trop} and we will just say that they are cells of Z_f^{trop}. Both subdivisions $\mathcal{S}$ and Ξ are dual in the following sense. There is a one-to-one correspondence between Ξ and $\mathcal{S}$, which reverses the incidence relations, and such that if $\sigma \in \mathcal{S}$ corresponds to $\xi \in \Xi$ then

(1) $\dim \xi + \dim \sigma = n$,
(2) the cell ξ and the polytope σ span orthonogonal real affine spaces,
(3) the cell ξ is unbounded if and only if σ lies on a proper face of Δ.

Note that under this correspondence the cells of Z_f^{trop} correspond to positive dimensional polytopes of $\mathcal{S}$. We now underline some similarities between complex toric hypersurfaces and tropical hypersurfaces. As in the complex case, we can start with a polynomial f whose exponent vectors belong to a lattice $M \simeq \mathbb{Z}^n$. Then, in view of the definition of $\mathcal{L}(\nu)$ and Theorem 3.2, the tropical hypersurface lies in the real vector space $N_{\mathbb{R}} \simeq \mathbb{R}^n$ generated by the lattice N dual to M. This real vector space $N_{\mathbb{R}}$ can be interpreted as the tropical torus $\mathbb{T}\mathrm{rop}_N$ associated with the lattice N, so that $Z_f^{\mathrm{trop}} \subset \mathbb{T}\mathrm{rop}_N = N_{\mathbb{R}}$ is in fact a toric tropical hypersurface. The polynomial f also defines a toric tropical hypersurface in $N(\Delta)_{\mathbb{R}} \simeq \mathbb{R}^{\dim \Delta}$ and $Z_f^{\mathrm{trop}} \subset N_{\mathbb{R}}$ is the product of this hypersurface with the tropical torus $\simeq \mathbb{R}^{n-\dim \Delta}$ associated with (the dual of) a complement of $M(\Delta)$ in M. The unbounded cells of Z_f^{trop} gives rise to toric tropical hypersurfaces defined by truncations of f to faces of Δ. Namely, consider a face Γ of Δ and let $\gamma \subset N$ be the semigroup formed by all elements of N which are identically zero on $M(\Gamma)$ and are negative on any vector $w = m' - m \in M(\Delta)$ with $m' \in \Delta \setminus \Gamma$ and $m \in \Gamma$ (in other words, γ consists of all integer vectors of N orthogonal to Γ and going outside Δ). Note that $N(\Gamma)$ is the quotient $\frac{N}{\gamma+(-\gamma)}$, where $\gamma + (-\gamma)$ is the subgroup of N generated by the semigroup γ. Consider the unbounded cells of Z_f^{trop} which intersect any hyperplane $\{w \in M_{\mathbb{R}} \mid v \cdot w = c\}$ with c big enough and v in γ. The cells of the tropical hypersurface $Z_{f^\Gamma}^{\mathrm{trop}} \subset N(\Gamma)_{\mathbb{R}}$ are exactly the images of these cells under the quotient map $N_{\mathbb{R}} \to N(\Gamma)_{\mathbb{R}}$. Comparing with the classical complex situation, this leads to the notion of *tropical variety* $\mathbb{T}\mathrm{rop}_\Delta$ associated with Δ with properties analogous to those of the complex projective toric variety X_Δ. Geometrically, one can think about $\mathbb{T}\mathrm{rop}_\Delta$ as being the image of Δ by the composition of a translation and a dilatation, so that $\bar{Z}_f^{\mathrm{trop}} \subset \mathbb{T}\mathrm{rop}_\Delta$ can be obtained from Z_f^{trop} by cutting the the unbounded cells of Z_f^{trop} along the faces of $\mathbb{T}\mathrm{rop}_\Delta$.

We finish with the definition of a tropical variety in $N_{\mathbb{R}}$. We use the one which seems to be the most commonly accepted (see, for example, [15] for a discussion about possible other definitions).

Definition 3.3. *A tropical variety in $N_{\mathbb{R}}$ is the closure of the image under V of the zero set of an ideal $I \subset \mathbb{K}[z_1, \dots, z_n] = \mathbb{K}[z]$. We will denote this tropical variety by Z_I^{trop}.*

In other words, Z_I^{trop} is the common intersection of all tropical hypersurfaces Z_f^{trop} for $f \in I$. There exists a finite number of polynomials $f_1, \dots, f_k \in I \subset \mathbb{K}[z]$ so that Z_I^{trop} is the common intersection of the corresponding tropical hypersurfaces (see [7]). Such a collection of polynomials is called a *tropical basis* of Z_I^{trop}. On the other hand, it is known that the common intersection of tropical hypersurfaces is not always a tropical variety.

4. Intersection multiplicity numbers between tropical hypersurfaces

Recall that all polytopes under consideration have vertices in the underlying lattice $M \simeq \mathbb{Z}^n$. A k-dimensional simplex σ with vertices $m_0, m_1, \dots, m_k$ is called *primitive* if the vectors $m_1 - m_0, \dots, m_k - m_0$ form a basis of the lattice $M(\sigma)$, or equivalently, if these vectors can be completed to form a basis of M. Obviously, the faces of a primitive simplex are themselves primitives simplices.

Consider a k-dimensional vector subspace of $M_{\mathbb{R}}$ with rational slopes. It intersects M in a saturated subgroup γ of rank k and coincides with the real vector space $\gamma_{\mathbb{R}}$ generated by γ. Any basis of γ produces an isomorphism between γ and $\mathbb{Z}^k$, and then by extension an isomorphism between $\gamma_{\mathbb{R}}$ and $\mathbb{R}^k$. Let Vol_γ be the volume form on $\gamma_{\mathbb{R}}$ obtained as the pull-back via such an isomorphism of the usual Euclidian k-volume on $\mathbb{R}^k$. For simplicity, we will write Vol_k instead of Vol_γ since the lattice γ will be clear from the context. Note that Vol_k does not depend on the isomorphism $\gamma \simeq \mathbb{Z}^k$ since two basis of γ are obtained from each other by integer invertible linear map which has determinant ± 1. Any basis $(\gamma_1, \dots, \gamma_k)$ of γ generate a k-dimensional parallelotope $P \subset \gamma_{\mathbb{R}}$ (isomorphic to the cube $[0,1]^k \subset \mathbb{R}^k$) called *fundamental parallelotope* of γ and which verifies $\mathrm{Vol}_k(P) = 1$. Two primitive k-simplices on $\gamma_{\mathbb{R}}$ have the same volume under Vol_k (they are interchanged by an invertible integer linear map), and this volume is $\frac{1}{k!}$ since a fundamental parallelotope of γ can be subdivided into $k!$ primitive k-simplices. We will often use the normalized volume

$$\mathrm{vol}_k(\,\cdot\,) := k! \cdot \mathrm{Vol}_k(\,\cdot\,)$$

on $\gamma_{\mathbb{R}}$. This normalized volume takes all nonnegative integer values on polytopes (with vertices in γ), and we have $\mathrm{vol}_k(\sigma) = 1$ for a polytope σ if and only if σ is a k-dimensional primitive simplex. We will use the following elementary fact.

Remark 4.1. Let γ be a subgroup of a free group Λ. Assume that Λ and γ have the same rank k, so that the index $[\Lambda : \gamma]$ of γ in Λ is well-defined. Then, for any basis $(\gamma_1, \dots, \gamma_k)$ of γ and any basis $e = (e_1, \dots, e_k)$ of Λ we have

$$[\Lambda : \gamma] = \mathrm{Vol}_k(G) = \mathrm{vol}_k(g) = |\det(G_{ij})|,$$

where G (resp., g) is the k-dimensional parallelotope (resp., k-dimensional simplex) generated by $\gamma_1, \dots, \gamma_k$ and (G_{ij}) is the $k \times k$-matrix whose j-th column is the vector of coordinates of γ_j with respect to $(e_1, \dots, e_k)$.

Consider now tropical polynomials $f_1, \dots, f_k$ in $\mathbb{R}[x_1, \dots, x_n]$ or more generally in $\mathbb{R}[M]$ with $M \simeq \mathbb{Z}^n$. Denote by Δ_i the Newton polytope of f_i. Recall that each tropical hypersurface $Z_{f_i}^{\mathrm{trop}}$ defines a piecewise linear polyhedral subdivision Ξ_i of $N_{\mathbb{R}}$ which is dual to a convex polyhedral subdivision $\mathcal{S}_i$ of Δ_i. The union of these tropical hypersurfaces defines a piecewise linear polyhedral subdivision Ξ of $N_{\mathbb{R}}$. Any non-empty cell of Ξ can be written as

$$\xi = \bigcap_{i=1}^{k} \xi_i \tag{4.1}$$

with $\xi_i \in \Xi_i$ for $i = 1, \ldots, k$. Any cell $\xi \in \Xi$ can be uniquely written in this way if one requires that ξ lies *in the relative interior* of each ξ_i. We shall always refer to this unique form. Denote by $\mathcal{MS}$ the mixed subdivision of $\Delta = \Delta_1 + \cdots + \Delta_k$ induced by the tropical polynomials $f_1, \ldots, f_k$. Recall that any polytope $\sigma \in \mathcal{MS}$ comes with a privilegied representation

$$\sigma = \sigma_1 + \cdots + \sigma_k$$

with $\sigma_i \in \mathcal{S}_i$. The above duality-correspondence applied to the (tropical) product of the tropical polynomials gives rise to the following result.

Proposition 4.2. *There is a one-to-one duality correspondence between Ξ and $\mathcal{S}$, which reverses the incidence relations, and such that if $\sigma \in \mathcal{MS}$ corresponds to $\xi \in \Xi$ then*

1. *if $\xi = \bigcap_{i=1}^{k} \xi_i$ with $\xi_i \in \Xi_i$ for $i = 1, \ldots, k$, then σ has representation $\sigma = \sigma_1 + \cdots + \sigma_k$ where each σ_i is the polytope dual to ξ_i.*
2. $\dim \xi + \dim \sigma = n$,
3. *the cell ξ and the polytope σ span orthonogonal real affine spaces,*
4. *the cell ξ is unbounded if and only if σ lies on a proper face of Δ.*

We put weights on the cells of each subdivision Ξ_i in the following way. If $\xi_i \in \Xi_i$ is a cell of maximal dimension n (which means that ξ_i is not a cell of the tropical hypersurface $Z_{f_i}^{\text{trop}}$), then its weight is defined by $w(\xi_i) := 0$. If $\xi_i \in \Xi_i$ is a cell of positive codimension d_i, then

$$w(\xi_i) := \text{vol}_{d_i}(\sigma_i)$$

where $\sigma_i \in \mathcal{S}_i$ is the polytope corresponding to ξ_i. We now define weights on the cells of Ξ in the following way. Consider a cell $\xi \in \Xi$

$$\xi = \bigcap_{i=1}^{k} \xi_i$$

where $\xi_i \in \Xi_i$ for $i = 1, \ldots, k$ (and ξ lies in the relative interior of each ξ_i). Let $\sigma_i \in \mathcal{S}_i$ be the polytope corresponding to ξ_i. Set $d_i := \text{codim}\, \xi_i = \dim \sigma_i$ and $d := \text{codim}\, \xi = \dim \sigma$. Recall that for a polytope $P \subset M_{\mathbb{R}}$, we denote by $M(P)$ the subgroup of M consisting of all integer vectors which are parallel to P.

Definition 4.3. *The weight of ξ is defined as follows.*

- *(Tranversal case.) If $d_1 + \cdots + d_k = d$, then*

$$\begin{aligned} w(\xi) &= \textstyle\prod_{i=1}^{k} w(\xi_i) \cdot [M(\sigma) : M(\sigma_1) + \cdots + M(\sigma_k)] \\ &= \textstyle\prod_{i=1}^{k} vol_{d_i}(\sigma_i) \cdot [M(\sigma) : M(\sigma_1) + \cdots + M(\sigma_k)] \end{aligned}$$

- *(General case.) Translate the tropical hypersurfaces by small generic vectors so that all intersections emerging from ξ are transversal intersections. Define $w(\xi)$ as the sum of the weights at the transversal intersections emerging from ξ and which are cells of codimension d.*

Our weights are similar to those used by Mikhalkin in [25] in order to define tropical cycles. Note however that in [25] only top-dimensional cells are equipped with weights. In our situation, the top-dimensional cells of the cycle corresponding to the intersection of our tropical hypersurfaces are cells $\xi \in \Xi$ of codimension $d = k$. It follows straightforwardly from the definitions and Lemma 4.4 below that on these top-dimensional cells both weights coincide. We will show in Theorem 4.5 that our weight does not depend (in the non transversal case) on the translation vectors. It is then natural to interpret $w(\xi)$ as being the *intersection multiplicity number* between the tropical hypersurfaces $Z_{f_1}^{\text{trop}}, \ldots, Z_{f_k}^{\text{trop}}$ along the cell ξ.

Lemma 4.4. *Let γ_1 and γ_2 be saturated subgroups of a free group N such that $\gamma_1 + \gamma_2$ and N have same rank. Then the index of $\gamma_1 + \gamma_2$ in N satisfies to*

$$[N : \gamma_1 + \gamma_2] = [(\gamma_1 \cap \gamma_2)^{\perp} : \gamma_1^{\perp} + \gamma_2^{\perp}],$$

where $\gamma^{\perp}$ denotes the subgroup of the dual lattice $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ consisting of all elements of M which vanish on a subgroup γ of N.

Proof. If $\gamma_1 \cap \gamma_2 = \{0\}$ then $(\gamma_1 \cap \gamma_2)^{\perp} = M$ and the corresponding equality has been proven in [23]. The general case reduces to this case in the following way. Let n be the rank of N. If γ_1 and γ_2 are saturated then so is $\gamma_1 \cap \gamma_2$. This implies that the quotient $N/(\gamma_1 \cap \gamma_2)$ is a free group of rank $n - \mathrm{rk}\,(\gamma_1 \cap \gamma_2)$. We have a group isomorphism

$$\frac{N}{\gamma_1 + \gamma_2} \simeq \frac{N/(\gamma_1 \cap \gamma_2)}{(\gamma_1 + \gamma_2)/(\gamma_1 \cap \gamma_2)}$$

and also

$$(\gamma_1 + \gamma_2)/(\gamma_1 \cap \gamma_2) = \frac{\gamma_1}{\gamma_1 \cap \gamma_2} + \frac{\gamma_2}{\gamma_1 \cap \gamma_2}.$$

The group dual to $N/(\gamma_1 \cap \gamma_2)$ is isomorphic to $(\gamma_1 \cap \gamma_2)^{\perp} \subset M$. It remains to note that if γ_1 and γ_2 are saturated subgroups of N, then for $i = 1, 2$ the subgroup $\frac{\gamma_i}{\gamma_1 \cap \gamma_2}$ of $\frac{N}{\gamma_1 + \gamma_2}$ is also saturated. $\square$

Let $P_1, \ldots, P_\ell$ be polytopes with vertices in a saturated lattice γ of rank ℓ. The map $(\lambda_1, \ldots, \lambda_\ell) \mapsto \mathrm{Vol}_\ell(\lambda_1 P_1 + \cdots + \lambda_\ell P_\ell)$ is a homogeneous polynomial map of degree ℓ. The coefficient of the monomial $\lambda_1 \cdots \lambda_\ell$ is called the *mixed volume* of $P_1, \ldots, P_\ell$ and is denoted by

$$MV_\ell(P_1, \ldots, P_\ell).$$

A famous theorem due to Bernstein states that this mixed volume is the number of solutions in the torus associated with the lattice γ of a generic polynomial system $f_1 = \ldots = f_\ell = 0$ where each f_i has P_i as Newton polytope. Note that $MV_\ell(P_1, \ldots, P_\ell) = 0$ if $P = P_1 + \cdots + P_\ell$ has not full dimension ℓ or if at least one P_i has dimension zero. We may also consider mixed volumes associated with any number $m \leq \ell$ of polytopes among $P_1, \ldots, P_\ell$ (see [9]). Namely, let $P_1, \ldots, P_m$ be $m \leq \ell$ polytopes with vertices in a lattice of rank ℓ and let $\underline{t} = (t_1, \ldots, t_m)$ be a m-uple of positive integer numbers such that $\sum_{i=1}^{m} t_i = \ell$. Then define

$$MV_\ell(P_1, \ldots, P_m\,;\, \underline{t}) := MV_\ell(\underbrace{P_1, \ldots, P_1}_{t_1}, \ldots\ldots, \underbrace{P_m, \ldots, P_m}_{t_m}),$$

where on the right each P_i is repeated t_i times. These mixed volumes appear in the coefficients of the homogeneous degree ℓ polynomial map $\mathrm{Vol}_\ell(\lambda_1 P_1 + \cdots + \lambda_m P_m)$ (see [9], page 327).

We are now able to state a formula for the weight $w(\xi)$ defined above which shows in particular that this weight does not depend on the chosen translation vector.

Theorem 4.5. *Let ξ be a cell of Ξ and $w(\xi)$ be its weight as defined in Definition 4.3. If ξ is not a cell of $\cap_{i=1}^{k} Z_{f_i}^{\text{trop}}$, or equivalently if at least one d_i is zero, then $w(\xi) = 0$. Assume now that all d_i are positive integer numbers.*

- *If the tropical hypersurfaces intersect transversally along ξ, which means that $d = d_1 + \cdots + d_k$, then letting $\underline{d} := (d_1, \dots, d_k)$ we have*

$$w(\xi) \;=\; MV_d(\sigma_1, \dots, \sigma_k; \underline{d}) \tag{4.2}$$

- *In the general case, we have $d \geq d_1 + \cdots + d_k$ and*

$$w(\xi) = \sum_{\underline{t}\,,\, t_1 + \cdots + t_k = d} MV_d(\sigma_1, \dots, \sigma_k; \underline{t}) \tag{4.3}$$

As a particular case, if $d = \operatorname{codim} \xi = k$ then

$$w(\xi) = MV_k(\sigma_1, \dots, \sigma_k). \tag{4.4}$$

We first note that this is consistent with the hypersurface case. Indeed, for $k = 1$ we are automatically in the transversal case $\xi = \xi_1$ and $w(\xi) = MV_{d_1}(\sigma_1, \dots, \sigma_1) = d_1! \cdot \mathrm{Vol}_{d_1}(\sigma_1) = \mathrm{vol}_{d_1}(\sigma_1)$ as required. Before giving a proof of Theorem 4.5, we need some intermediate results.

A Minkowsky sum $Q_1 + \cdots + Q_\ell$ of polytopes such that $\dim(Q_1 + \cdots + Q_\ell) = \dim Q_1 + \cdots + \dim Q_\ell$ is called *a direct Minkowsky sum* and is denoted by $Q_1 \oplus \cdots \oplus Q_\ell$. A convex mixed subdivision $\mathcal{MS}$ of a polytope $P = P_1 + \cdots + P_\ell$ is called *pure* if for any polytope $Q \in \mathcal{MS}$ with representation $Q = Q_1 + \cdots + Q_\ell$ we have $Q = Q_1 \oplus \cdots \oplus Q_\ell$. Note that a convex tight mixed subdivision is a convex pure mixed subdivision with the additional property that each Q_i is a simplex. Tight and pure convex mixed subdivisions are generic within all convex mixed subdivisions of given polytopes.

Lemma 4.6. *Let $P = P_1 + \cdots + P_\ell \subset M_{\mathbb{R}} \simeq \mathbb{R}^\ell$.*

(1) *For any convex pure mixed subdivision of $P = P_1 + \cdots + P_\ell$, we have*

$$MV_\ell(P_1, \dots, P_\ell) = \sum Vol_\ell(Q_1 \oplus \cdots \oplus Q_\ell) \tag{4.5}$$

where the sum is taken over all polytopes $Q = Q_1 \oplus \cdots \oplus Q_\ell$ of the mixed subdivision such that each Q_i is an edge (in particular Q is a n-parallelotope).

(2) *More generally, for any convex mixed subdivision of $P = P_1 + \cdots + P_\ell$, we have*

$$MV_\ell(P_1, \dots, P_\ell) = \sum MV_\ell(Q_1, \dots, Q_\ell) \tag{4.6}$$

where the sum is taken over all polytopes $Q = Q_1 + \cdots + Q_\ell$ of the mixed subdivision.

Proof. Formula (4.5) is well-known (see, for example, [9], Chapt. 7) and not difficult to prove from the definition of mixed volume given above. Formula (4.6) is a simple consequence of (4.5). Indeed, we may perturb slightly functions $\nu_1, \dots, \nu_\ell$ determining a given convex mixed subdivision of $P = P_1 + \cdots + P_\ell$ so that the new functions induce pure mixed subdivisions of each polytope $Q = Q_1 + \cdots + Q_\ell$ of the initial mixed subdivision. Then these functions define a pure mixed subdivision of $P = P_1 + \cdots + P_\ell$ and it remains to apply (4.5) simultaneously to all these pure mixed subdivisions. $\square$

Proof of Theorem 4.5. Assume first that $d = d_1 + \cdots + d_k$ (transversal case). If some d_i is zero, then it follows directly from Definition 4.3 that $w(\xi) = 0$. Assume that $d_i \geq 1$ for $i = 1, \ldots, k$. We prove Formula (4.2) with the help of Bernstein's theorem. Consider a generic polynomial system

$$f_{i,1} = \cdots = f_{i,d_i} = 0 \ , \quad i = 1, \ldots, k, \tag{4.7}$$

of $d = d_1 + \cdots + d_k$ equations where each $f_{i,j}$ has σ_i as Newton polytope. By Bernstein's theorem, the number of solutions to (4.7) in the torus associated with $M(\sigma)$ is exactly $MV_d(\sigma_1, \ldots, \sigma_k; \underline{d})$ where $\underline{d} = (d_1, \ldots, d_k)$. Since $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_k$, the number of solutions to (4.7) in the torus associated with $M(\sigma_1) + \cdots + M(\sigma_k)$ is the product $N = \prod_{i=1}^k N_i$ where N_i is the number of solutions to

$$f_{i,1} = \ldots = f_{i,d_i} = 0$$

in the torus associated with $M(\sigma_i)$. By Bernstein's theorem $N_i = \mathrm{MV}_{d_i}(\sigma_i, \ldots, \sigma_i) = d_i! \cdot \mathrm{Vol}_{d_i}(\sigma_i) = \mathrm{vol}_{d_i}(\sigma_i)$. Let $(e_1, \ldots, e_d)$ be a basis of $M(\sigma)$ and identify the associated torus with $(\mathbb{C}^*)^d$ via this basis. Let $z_1, \ldots, z_N$ be the solutions to (4.7) in the subtorus of $(\mathbb{C}^*)^d$ associated with $M(\sigma_1) + \cdots + M(\sigma_k)$. Then the solutions to (4.7) in $(\mathbb{C}^*)^d$ are obtained by solving for each z_l a system

$$x^{m_i} = z_{l,i} \ , \quad i = 1, \ldots, d,$$

where $m_1, \ldots, m_d$ are the vectors of coordinates of a basis of $M(\sigma_1) + \cdots + M(\sigma_k)$ with respect to $(e_1, \ldots, e_d)$ and $z_l = (z_{l,1}, \ldots, z_{l,d}) \in (\mathbb{C}^*)^d$. The number of solutions to such a system is the absolute value of the $(d \times d)$-determinant $|m_{i,j}|$ which is equal to $[M(\sigma) : M(\sigma_1) + \cdots + M(\sigma)]$. This proves Formula (4.2).

Consider now the general case. We have obviously $d \geq d_1 + \cdots + d_k$. let $\nu_i : \Delta_i \to \mathbb{R}$ be functions given by the tropical hypersurfaces and which induce the corresponding mixed subdivision $\mathcal{MS}$ of $\Delta = \Delta_1 + \cdots + \Delta_k$. Denote by $\mathcal{S}_i$ the convex polyhedral subdivision of Δ_i induced by ν_i. Translations of the tropical hypersurfaces by a small generic vector correspond to small perturbations of the functions ν_i so that for each $i = 1, \ldots, k$ the polyhedral subdivision of Δ_i induced by the resulting function $\tilde{\nu_i}$ coincide with $\mathcal{S}_i$. The intersections between the tropical hypersurfaces which emerge from ξ after such a small perturbation are all transversal intersections if and only if the mixed subdivision of $\sigma = \sigma_1 + \cdots + \sigma_k$ induced by $\tilde{\nu_1}, \ldots, \tilde{\nu_k}$ is a pure mixed subdivision $\mathcal{MS}(\sigma)$. Then each polytope $\Gamma \in \mathcal{MS}(\sigma)$ has a priviliegied representation

$$\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k \tag{4.8}$$

where each $\Gamma_i \in \mathcal{S}_i$ and the weight of ξ is by definition the sum of weights of the cells corresponding to polytopes $\Gamma \in \mathcal{MS}(\sigma)$ such that $\dim \Gamma = d$. Suppose that $d_i = 0$ for some $i = 1, \ldots, k$. Then for each $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k \in \mathcal{MS}(\sigma)$ we have $\dim \Gamma_i = 0$, hence $w(\xi) = 0$. Assume now that $d_i \geq 1$ for $i = 1, \ldots, k$. The previous reasonning shows that we may forget about the polytopes $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k \mathcal{MS}(\sigma)$ such that $\dim \Gamma_i = 0$ for some $i \in \{1, \ldots, k\}$. Hence, we have

$$w(\xi) = \sum_{\Gamma \in \mathcal{MS}(\sigma) \,:\, \dim \Gamma = d} MV_d(\Gamma_1, \ldots, \Gamma_k; (\dim \Gamma_1, \ldots, \dim \Gamma_k)) \tag{4.9}$$

Let $\underline{t} = (t_1, \ldots, t_k)$ be any k-uple of positive integer numbers such that $t_1 + \cdots + t_k = d$. Consider the polytope

$$\underbrace{\sigma_1 + \cdots + \sigma_1}_{t_1} + \cdots + \underbrace{\sigma_k + \cdots + \sigma_k}_{t_k}$$

together with the convex mixed subdivision induced by the functions $\tilde{\nu_1},\dots,\tilde{\nu_k}$, where $\tilde{\nu_i}$ is used for each copy of σ_i. This mixed subdivision consists of the polytopes

$$\underbrace{\Gamma_1+\cdots+\Gamma_1}_{t_1}+\cdots+\underbrace{\Gamma_k+\cdots+\Gamma_k}_{t_k}$$

for $\Gamma=\Gamma_1\oplus\cdots\oplus\Gamma_k\in\mathcal{MS}(\sigma)$. By Formula (4.6) in Lemma 4.6, we get

$$MV_d(\sigma_1,\dots,\sigma_k;\underline{t})=\sum_{\Gamma\in\mathcal{MS}(\sigma)} MV_d(\Gamma_1,\dots,\Gamma_k;\underline{t}).$$

This sum can actually be taken over all $\Gamma\in\mathcal{MS}(\sigma)$ such that $\dim\Gamma=d$ and $\dim\Gamma_i\geq 1$ for $i=1,\dots,d$ since otherwise $MV_d(\Gamma_1,\dots,\Gamma_k;\underline{t})=0$. But now if $\underline{t}\neq(\dim\Gamma_1,\dots,\Gamma_k)$ and $t_1+\cdots+t_k=\dim\Gamma_1+\cdots+\dim\Gamma_k$, then there exists $i\in\{1,\dots,k\}$ such that $\dim\Gamma_i<t_i$ and thus $\mathrm{vol}_{t_i}(\Gamma_i)=0$, which implies that $MV_d(\Gamma_1,\dots,\Gamma_k;\underline{t})=0$. Therefore,

$$MV_d(\sigma_1,\dots,\sigma_k;\underline{t})=\sum_{\Gamma\in\mathcal{MS}(\sigma)\,:\,\dim\Gamma_i=t_i\text{ for all }i} MV_d(\Gamma_1,\dots,\Gamma_k;\underline{t}).$$

and thus

$$\sum_{\underline{t}\,:\,t_1+\cdots+t_k=d} MV_d(\sigma_1,\dots,\sigma_k;\underline{t})=\sum_{\Gamma\in\mathcal{MS}(\sigma)\,:\,\dim\Gamma=d} MV_d(\Gamma_1,\dots,\Gamma_k;(\dim\Gamma_1,\dots,\dim\Gamma_k)).$$

This proves Formula (4.3). Finally, If $k=d$, then there is only one d-uple $\underline{t}=(t_1,\dots,t_d)$ of positive integer numbers such that $t_1+\cdots+t_d=d$, namely $\underline{t}=(1,1,\dots,1)$. Hence, we get $w(\xi)=MV_d(\sigma_1,\dots,\sigma_k;(1,1,\dots,1))=MV_d(\sigma_1,\dots,\sigma_k)$. □

The tropical Bernstein theorem is now a simple consequence of Theorem 4.5.

Corollary 4.7. *Suppose tropical hypersurfaces $Z_1,\dots,Z_n\subset N_{\mathbb{R}}\simeq\mathbb{R}^n$ with Newton polytopes $\Delta_1,\dots,\Delta_n$ intersect in finitely many points. Then the total number of intersection points counted with multiplicities is equal to the mixed volume $MV_n(\Delta_1,\dots,\Delta_n)$.*

Proof. The common intersection points are in one-to-one correspondence with the polytopes $\sigma=\sigma_1+\cdots+\sigma_n$ in the dual mixed subdivision $\mathcal{MS}$ of $\Delta_1+\cdots+\Delta_n$. Each intersection point is a cell of codimension n, hence by Formula (4.4), Theorem 4.5, the intersection multiplicity number of the tropical hypersurfaces at this point is equal to $\mathrm{MV}_n(\sigma_1,\dots,\sigma_n)$, where $\sigma=\sigma_1+\cdots+\sigma_n$ is the corresponding polytope in the mixed subdivision. Hence the total number of intersection points counted with multiplicities is $\sum_{\sigma\in\mathcal{MS}}\mathrm{MV}_n(\sigma_1,\dots,\sigma_n)$. But this sum is equal to $\mathrm{MV}(\Delta_1,\dots,\Delta_n)$ by Formula (4.6), Lemma 4.6. □

5. Non degenenerate tropical complete intersections

All the definitions in this section build on the following definition of a nonsingular tropical hypersurface in the same way as definitions in Section 1 builded on that of a nonsingular complex hypersurface.

Definition 5.1. *A tropical hypersurface is non singular is its dual polyhedral subdivision is a primitive (convex) triangulation, that is, a triangulation whose all simplices are primitive.*

This definition is well-established in the case of tropical plane curves. In the general case, it can be motivated by the fact that around a vertex corresponding to a primitive n-simplex, a tropical hypersurface coincides with a tropical hypersurface with Newton polytope this simplex.

But such a simplex is given by a basis of the ambient lattice M, and identifying M with $\mathbb{Z}^n$ via this basis identifies the simplex with the standard unit simplex in $\mathbb{Z}^n$. Hence, up to a basis change of the ambient lattice, a non singular tropical hypersurface coincides around each vertex with a tropical linear hyperplane. Non singular tropical hypersurfaces with a given Newton polytope do not always exist. The simplest example is given by the non primitive tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,2)$ in $\mathbb{R}^3$ which meets the lattice $\mathbb{Z}^3$ at its vertices and has thus no primitive triangulation (see [4]). Recall that a tropical hypersurface lies in $N_{\mathbb{R}} \simeq \mathbb{R}^n$, which is the tropical torus associated with some lattice N. Hence, at this point, a tropical hypersurface is in fact a toric tropical hypersurface. A primitive (convex) triangulation of a polytope induces a primitive (convex) triangulation of each of its faces. Recall that the truncation f^Γ of a tropical polynomial f to a face Γ of its Newton polytope also defines a tropical hypersurface in the corresponding tropical torus $N(\Gamma)_{\mathbb{R}}$. Hence, in contrast to the complex case, if f defines a nonsingular tropical hypersurface in the corresponding tropical torus, then so do automatically all its truncations. Comparing with the classical definition 1.1 of a non degenerate polynomial, this leads to the following definition.

Definition 5.2. *A tropical polynomial is non degenerate if all its truncations define nonsingular tropical hypersurfaces in the corresponding tropical tori, or equivalently, if its dual polyhedral subdivision is a primitive triangulation.*

Consider now a k-uple $(f_1, \ldots, f_k)$ of tropical polynomials in $\mathbb{R}[x_1, \ldots, x_n]$, or more generally in $\mathbb{R}[M]$ with $M \simeq \mathbb{Z}^n$. Let Δ_i be the Newton polytope of f_i. Define the associated tropical Cayley polynomial $F \in \mathbb{R}[M \oplus \mathbb{Z}^s]$ by

$$F(x,y) = \sum_{i=1}^{s} y_i f_i(x). \tag{5.1}$$

where the operation are the tropical ones. Its Newton polytope is the associated cayley polytope $C(\Delta_1, \ldots, \Delta_k)$. We have the following analogue of the classical definition 1.2.

Definition 5.3. *The k-uple $(f_1, \ldots, f_k)$ of tropical polynomials is non degenerate if the associated Cayley polynomial F is non degenerate which means that the dual polyhedral subdivision of $C(\Delta_1, \ldots, \Delta_k)$ is a primitive triangulation.*

Recall that a collection $(\Gamma_i)_{i \in I}$ of faces of $\Delta_1, \ldots, \Delta_k$ is called admissible if $I \subset \{1, \ldots, k\}$ and $\Gamma_I = \sum_{i \in I} \Gamma_i$ is face a of $\Delta_I = \sum_{i \in I} \Delta_i$. The faces of $C(\Delta_1, \ldots, \Delta_k)$ are exactly the Cayley polytopes of the admissible collections $(\Gamma_i)_{i \in I}$. Since a primitive triangulation of a polytope induces primitive triangulations of its faces, it follows that if $(f_1, \ldots, f_k)$ is non degenerate, then for any admissible collection $(\Gamma_i)_{i \in I}$ of faces of $\Delta_1, \ldots, \Delta_k$, the $|I|$-uple of tropical polynomials $(f_i^{\Gamma_i})_{i \in I}$ is also non degenerate. For simplicity denote by Z_i the hypersurface defined by f_i. If Γ_i is a face of Δ_i, we will denote by Z_{i,Γ_i} the tropical hypersurface in $N(\Gamma_i)_{\mathbb{R}}$, or in $N_{\mathbb{R}}$, defined by the truncation of f_i to Γ_i. The next result is the tropical analogue of Proposition 1.3.

Proposition 5.4. *The collection $(f_1, \ldots, f_k)$ of tropical polynomials is non degenerate if and only if for any admissible collection $(\Gamma_i)_{i \in I}$ of faces of $\Delta_1, \ldots, \Delta_k$ the hypersurfaces Z_{i,Γ_i} have only transversal intersections each with intersection multiplicity number* 1.

Proof. If $(f_1, \ldots, f_k)$ is non degenerate, then the corresponding convex polyhedral subdivision of the Cayley polytope $C(\Delta_1, \ldots, \Delta_k)$ is a primitive triangulation, and thus the corresponding convex mixed subdivision $\mathcal{MS}$ of $\Delta = \Delta_1 + \cdots + \Delta_k$ is tight. In particular, the mixed subdivision

$\mathcal{MS}$ is pure which means that the hypersurfaces $Z_1, \ldots, Z_k$ have only transversal intersections. These intersections are in one-to-one correspondence with polytopes

$$\sigma = \sigma_1 \oplus \cdots \oplus \sigma_k \in \mathcal{MS}$$

such that $d_i := \dim \sigma_i \geq 1$ for $i = 1, \ldots, k$. Letting $d = \dim \sigma = d_1 + \cdots + d_k$, the intersection multiplicity number of $Z_1, \ldots, Z_k$ along the cell ξ dual to σ is

$$w(\xi) = \prod_{i=1}^{k} \mathrm{vol}_{d_i}(\sigma_i) \cdot [M(\sigma) : M(\sigma_1) + \cdots + M(\sigma_k)].$$

We are going to show that we also have

$$w(\xi) = \mathrm{vol}_{d+k-1}\left(C(\sigma_1, \ldots, \sigma_k)\right).$$

Since $C(\sigma_1, \ldots, \sigma_k)$ is a primitive simplex, this will imply that $w(\xi) = 1$. Each σ_i is a simplex as well as $C(\sigma_1, \ldots, \sigma_k)$. Thus the last volume is the absolute value of a $(d+k-1)$-determinant D whose columns are the coordinates with respect to a basis of $M(C(\sigma_1, \ldots, \sigma_k)) = M(\sigma) \times \mathbb{Z}^k$ of vectors edges spanning the simplex. The same determinant but with respect to a basis of $(M(\sigma_1) + \cdots + M(\sigma_k)) \times \mathbb{Z}^k$ is a determinant $\tilde{D}$ which factors into a product of k determinants $D_1, \ldots, D_k$. Each factor D_i has size d_i and is a determinant whose columns are the coordinates with respect to a basis of $M(\sigma_i)$ of vectors edges spanning the simplex σ_i. The absolute value of D_i is just $\mathrm{vol}_{d_i}(\sigma_i)$. To finish, the basis change corresponds to multiply $\tilde{D}$ by the index $[M(\sigma) : M(\sigma_1) + \ldots + M(\sigma_k)]$ (compare with Remark 4.1). Finally, these results also hold for any admissible $(\Gamma_i)_{i\in I}$ since if $(f_1, \ldots, f_k)$ is non degenerate then $(f_i^{\Gamma_i})_{i\in I}$ is non degenerate too.

Let us show the converse implication. Clearly, if for any admissible collection $(\Gamma_i)_{i\in I}$ of faces of $\Delta_1, \ldots, \Delta_k$ the hypersurfaces Z_{i,Γ_i} have only transversal intersections, then the mixed subdivision $\mathcal{MS}$ of $\Delta = \Delta_1 + \cdots + \Delta_k$ is pure. Consider a full dimensional polytope in the polyhedral subdivision of $C(\Delta_1, \ldots, \Delta_k)$. It corresponds via the combinatorial Cayley trick to a polytope $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_k \in \mathcal{MS}$ with $\dim \sigma = \dim \Delta$. Precisely, the starting polytope is just the Cayley polytope $C(\sigma_1, \ldots, \sigma_k)$. Set as above $d_i = \dim \sigma_i$ and $d := \dim \sigma = \dim \Delta$. Let I be the subset of those $i \in \{1, \ldots, k\}$ such that $d_i \neq 0$. Then σ is dual to a cell ξ of the common intersection of the hypersurfaces Z_i for $i \in I$ and the intersection multiplicity of these hypersurfaces along ξ is $\mathrm{vol}_{d_I+k-1}(C(\sigma_i, i \in I))$, where $C(\sigma_i, i \in I)$ is the Cayley polytope associated with σ_i for $i \in I$ and d_I is the dimension of $\sum_{i\in I} \sigma_i$. This Cayley polytope lies on the face $C(\Delta_i, i \in I)$ of $C(\Delta_1, \ldots, \Delta_k)$. One can check that

$$\mathrm{vol}_{d+k-1}\left(C(\sigma_1, \ldots, \sigma_k)\right) = \mathrm{vol}_{d_I+k-1}\left(C(\sigma_i, i \in I)\right).$$

Thus both members are equal to 1 and it follows that $C(\sigma_1, \ldots, \sigma_k)$ is a primitive simplex. $\square$

6. Complex and real tropical varieties

Complex and real tropical varieties were introduced by Mikhalkin in [21]. Here we follow [3] and reproduce the definition and notations for the reader's convenience. Consider a Puiseux series $g = \sum_{r\in R} b_r t^r \in \mathbb{K}^*$. Recall that $\mathrm{val}(g)$ is the smallest exponent appearing in g (the usual valuation of g), and that we defined $v(g)$ to be $-\mathrm{val}(g)$ (see Section 3). Define the argument $\arg(g)$ to be the usual argument of the coefficient $b_{\mathrm{val}(g)}$ of the monomial with smallest exponent. Consider the map

$$\begin{array}{cccc} W: & (\mathbb{K}^*)^n & \longrightarrow & \mathbb{R}^n \times (S^1)^n \\ & z & \longmapsto & (v(z_1), \ldots, v(z_n), \arg(z_1), \ldots, \arg(z_n)). \end{array}$$

or alternatively

$$\begin{array}{rccl} V_{\mathbb{C}}: & (\mathbb{K}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ & z & \longmapsto & (e^{v(z_1)+i\arg(z_1)}, \ldots, e^{v(z_n)+i\arg(z_n)}) \end{array}$$

We will define a complex tropical variety as the topological closure of the image of a variety in $(\mathbb{K}^*)^n$ under either $V_{\mathbb{C}}$ or W. We will call both homeomorphic objects a complex tropical variety and use one and the other in turns depending on the context. For the rest of this section, let $f, f_1, \ldots, f_k$ be polynomials in $\mathbb{K}[z_1, \ldots, z_n]$. We denote by Z_f^{trop} the zero set of f in $(\mathbb{K}^*)^n$ and by Y^{trop} the common zero set of $f_1, \ldots, f_k$ in $(\mathbb{K}^*)^n$.

Definition 6.1. *The complex tropical hypersurface $\mathbb{C}Z_{f,V_{\mathbb{C}}}^{\mathrm{trop}}$ (resp. $\mathbb{C}Z_{f,W}^{\mathrm{trop}}$) associated with f is the topological closure of the image under $V_{\mathbb{C}}$ (resp. W) of the hypersurface Z_f. The complex tropical intersection $\mathbb{C}Y_{V_{\mathbb{C}}}^{\mathrm{trop}}$ (resp. $\mathbb{C}Y_{W}^{\mathrm{trop}}$) associated with $f_1, \ldots, f_k$ is the topological closure of the image under $V_{\mathbb{C}}$ (resp. W) of Y^{trop}.*

A polynomial $\sum c_w z^w \in \mathbb{K}[z_1, \ldots, z_n]$ is a real polynomial if the coefficients a_r of each series $c_\omega = \sum_{r \in R} a_r t^r$ are real. Assume from now on that $f_1, \ldots, f_k$ and f are real polynomials.

Definition 6.2. *The real tropical hypersurface associated with f is the intersection of $\mathbb{C}Z_{f,V_{\mathbb{C}}}^{\mathrm{trop}}$ with $(\mathbb{R}^*)^n$, or alternatively the intersection of $\mathbb{C}Z_{f,W}^{\mathrm{trop}}$ with $\mathbb{R}^n \times \{0, \pi\}^n$. More generally, the real tropical complete intersection associated with $f_1, \ldots, f_k$ is the intersection of $\mathbb{C}Y_{V_{\mathbb{C}}}^{\mathrm{trop}}$ with $(\mathbb{R}^*)^n$, or alternatively the intersection of $\mathbb{C}Y_{W}^{\mathrm{trop}}$ with $\mathbb{R}^n \times \{0, \pi\}^n$.*

See [3] for pictures of real tropical curves. The sign of a Puiseux series $g = \sum_{r \in R} b_r t^r \in \mathbb{K}^*$ is defined to be the sign of the coefficient $b_{\mathrm{val}(g)}$ of the monomial with smallest exponent. For any $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, denote by $\mathbb{R}(\epsilon)$ the connected component of $(\mathbb{R}^*)^n$ (called orthant) which consists of all $(x_1, \ldots, x_n)$ such that $(-1)^{\epsilon_i} x_i > 0$ for $i = 1, \ldots, n$. We keep the notation $(\mathbb{R}_+)^n$ for the positive orthant which corresponds to $\epsilon = (0, \ldots, 0)$. Denote by $\mathbb{R}Z_{f,V_{\mathbb{C}},\epsilon}^{\mathrm{trop}}$ the intersection of $\mathbb{R}Z_{f,V_{\mathbb{C}}}^{\mathrm{trop}}$ with $\mathbb{R}(\epsilon)$. If $\epsilon \in \{0,1\}^n$, let $\tilde{\epsilon}$ be the element of $\{0, \pi\}^n$ defined by $\tilde{\epsilon}_i = \pi \Leftrightarrow \epsilon_i = 1$, and define $\mathbb{R}Z_{f,W,\epsilon}^{\mathrm{trop}} \subset \mathbb{R}^n$ to be the image of $\mathbb{R}Z_{f,W}^{\mathrm{trop}} \cap (\mathbb{R}^n \times \{\tilde{\epsilon}\})$ under the natural identification of $\mathbb{R}^n \times \{\tilde{\epsilon}\}$ with $\mathbb{R}^n$. If Z_f^{trop} is nonsingular one can reconstruct $\mathbb{R}Z_{f,V_{\mathbb{C}},\epsilon}^{\mathrm{trop}}$ only from the data of Z_f^{trop} and the collection of signs of the coefficients of f (see [24] pp. 25 and 37, [35], and [26] Appendix for the case of amoebas). Consider the tropical hypersurface $Z_f^{\mathrm{trop}} \subset \mathbb{R}^n$, the induced subdivision Ξ_f of $\mathbb{R}^n$ and the dual subdivision $\mathcal{S}_f$ of its Newton polytope Δ_f. Let δ_f be the sign distribution at the vertices of $\mathcal{S}_f$ such that a vertex ω is labelled with the sign of the corresponding coefficient c_ω in $f(z) = \sum c_w z^w \in \mathbb{K}[z_1, \ldots, z_n]$.

Lemma 6.3. *Assume Z_f^{trop} is nonsingular. Then its positive part $\mathbb{R}Z_{f,W,(0,\ldots,0)}^{\mathrm{trop}}$ is the closure of the union of the $(n-1)$-cells of Z_f^{trop} which are dual to edges with vertices getting different signs via δ_f. More generally, let $\epsilon \in \{0,1\}^n$ and define the polynomial f_ϵ by $f_\epsilon(x_1, \cdots, x_n) = f((-1)^{\epsilon_1} x_1, \ldots, (-1)^{\epsilon_n} x_n)$. Then, $\mathbb{R}Z_{f,W,\epsilon}^{\mathrm{trop}}$ is the closure of the union of the $(n-1)$-cells of $Z_{f_\epsilon}^{\mathrm{trop}}$ which are dual to edges with vertices getting different signs via δ_{f_ϵ}.*

It is worth noting that $\mathbb{R}Z_{f,W,\epsilon}^{\mathrm{trop}}$ and $\mathbb{R}Z_{f,V_{\mathbb{C}},\epsilon}^{\mathrm{trop}}$ are homeomorphic for each $\epsilon \in \{0,1\}^n$. In particular $\mathbb{R}Z_{f,W}^{\mathrm{trop}}$ and $\mathbb{R}Z_{f,V_{\mathbb{C}}}^{\mathrm{trop}}$ are homeomorphic. We use the notations of Section 2. Let $H \subset \Delta^*$ be the piecewise linear hypersurface which is constructed by means of the combinatorial patchworking out of the data $\mathcal{S}_f$ and δ_f. As a direct consequence of Lemma 6.3, we obtain the following result.

Proposition 6.4. *Assume that Z_f^{trop} is nonsingular, or equivalently, that the subdivision $\mathcal{S}_f$ is a primitive triangulation. Then there exists an homeomorphism $h : (\mathbb{R}^*)^n \to (\Delta \setminus \partial\Delta)^*$ such that $h(\mathbb{R}Z_{f,W}^{\text{trop}}) = H \cap (\Delta \setminus \partial\Delta)^*$. The same property holds for $\mathbb{R}Z_{f,V_{\mathbb{C}}}^{\text{trop}}$.*

Here $(\Delta \setminus \partial\Delta)^*$ is the union of the 2^n symmetric copies of the relative interior of Δ under the hyperplane reflections. Denote by $\Delta_1, \dots, \Delta_k$ the Newton polytopes of $f_1, \dots, f_k$, respectively, and set $\Delta = \Delta_1 + \cdots + \Delta_k$. Each polynomial f_i determines a convex polyhedral subdivision $\mathcal{S}_i$ of Δ_i and a sign distribution δ_i at the vertices of $\mathcal{S}_i$. Consider the piecewise linear hypersurface $H_i^{\Delta,*} \subset \Delta^*$ constructed out of these data by means of the combinatorial patchworking for complete intersections (see Section 2).

Proposition 6.5. *Assume that $f_1, \dots, f_k$ define a non degenerate tropical complete intersection, which means that the corresponding convex polyhedral subdivision of the Cayley polytope $C(\Delta_1, \dots, \Delta_k)$ is a primitive triangulation. Then, there exists an homeomorphism $h : (\mathbb{R}^*)^n \to (\Delta \setminus \partial\Delta)^*$ such that $h(\mathbb{R}Z_{f_i,W}^{\text{trop}}) = H_i^{\Delta,*} \cap (\Delta \setminus \partial\Delta)^*$ for $i = 1, \dots, k$. The similar property holds for the real tropical hypersurfaces $\mathbb{R}Z_{f_i,V_{\mathbb{C}}}^{\text{trop}}$.*

Therefore, $\mathbb{R}Y_W^{\text{trop}}$ (resp., $\mathbb{R}Y_{V_{\mathbb{C}}}^{\text{trop}}$) is homeomorphic to the common intersection inside $(\Delta \setminus \partial\Delta)^*$ of the piecewise linear hypersurfaces $H_i^{\Delta,*}$. Recall that $\mathbb{C}Y_{V_{\mathbb{C}}}^{\text{trop}}$ is a subset of the torus $(\mathbb{C}^*)^n$. We may assume without loss of generality that the polytope Δ has non empty interior. Consider the usual compactification of $(\mathbb{C}^*)^n$ into the toric variety X_Δ associated with Δ, and let $\iota : (\mathbb{C}^*)^n \hookrightarrow X_\Delta$ denote the corresponding inclusion. We define the compactification $\overline{\mathbb{C}Y}_{V_{\mathbb{C}}}^{\text{trop}}$ to be the closure of $\iota(\mathbb{C}Y_{V_{\mathbb{C}}}^{\text{trop}})$ in X_Δ. Note that the stratification of X_Δ into orbits of the action of $(\mathbb{C}^*)^n$ defines a natural stratification of $\overline{\mathbb{C}Y}_{V_{\mathbb{C}}}^{\text{trop}}$.

We sum up natural maps in the following commutative diagram.

$$\begin{tikzcd}
\mathbb{R}Y_W^{\text{trop}} \arrow[r, "\sim"] \arrow[d, hook] & \mathbb{R}Y_{V_{\mathbb{C}}}^{\text{trop}} \arrow[r, hook] \arrow[d, hook] & (\mathbb{R}^*)^n \arrow[r, hook, "\iota_{\mathbb{R}}"] \arrow[d, hook] & \mathbb{R}X_\Delta \arrow[d, hook] \\
\mathbb{C}Y_W^{\text{trop}} \arrow[r, "\sim"] & \mathbb{C}Y_{V_{\mathbb{C}}}^{\text{trop}} \arrow[r, hook] & (\mathbb{C}^*)^n \arrow[r, hook, "\iota"] & X_\Delta
\end{tikzcd}$$

Define $\overline{\mathbb{R}Y}_{V_{\mathbb{C}}}^{\text{trop}}$ to be the intersection of $\overline{\mathbb{C}Y}_{V_{\mathbb{C}}}^{\text{trop}}$ with the real part $\mathbb{R}X_\Delta$ of X_Δ. Clearly $\overline{\mathbb{R}Y}_{V_{\mathbb{C}}}^{\text{trop}}$ is also the closure of $\iota_{\mathbb{R}}(\mathbb{R}Y_{V_{\mathbb{C}}}^{\text{trop}})$ in $\mathbb{R}X_\Delta$. One can see that the natural stratification of $\overline{\mathbb{R}Y}_{V_{\mathbb{C}}}^{\text{trop}}$ induced by the torus action corresponds to the stratification of the T-complete intersection of Theorem 2.3 induced by the face complex of Δ. Consider for $i = 1, \dots, k$ the piecewise linear hypersurface $\widetilde{H_i^\Delta} \subset \widetilde{\Delta}$ (see Section 2).

Proposition 6.6. *Assume that $f_1, \dots, f_k$ define a non degenerate tropical complete intersection. Then, there exists an homeomorphism $h : \mathbb{R}X_\Delta \to \widetilde{\Delta}$ sending $\overline{\mathbb{R}Y}_{V_{\mathbb{C}}}^{\text{trop}}$ to the common intersection of the piecewise linear hypersurfaces $\widetilde{H_i^\Delta}$.*

7. E-polynomials and mixed signature

We recall briefly definitions and some properties of so-called E-polynomials, see [1] and [12]. Let X be a quasi-projective algebraic variety over $\mathbb{C}$. For each pair of integers (p, q), we set

$$e^{p,q}(X) = \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(X)),$$

where $h^{p,q}(H_c^k(X))$ is the dimension of the (p,q)-component of the mixed Hodge structure of the k-th cohomology with compact supports. If X is a nonsingular projective variety then we have $e^{p,q}(X) = (-1)^{p+q}h^{p,q}(X)$ (see [12]).

The *E-polynomial* of X is the sum

$$E(X;u,v) = \sum_{p,q} e^{p,q}(X)u^p v^q, \tag{7.1}$$

(see [1], and [12] where $E(X;u,\bar{u})$ was introduced). We have the following properties.

- If X is a disjoint union of a finite number of locally closed varieties X_i, $i \in I$, then

$$E(X;u,v) = \sum_{i\in I} E(X_i;u,v) \tag{7.2}$$

- $$E(X \times Y;u,v) = E(X;u,v) \cdot E(Y;u,v) \tag{7.3}$$

- If $\pi : Y \to X$ is a locally trivial fibration with respect to the Zarisky topology and F is the fiber over a closed point of X, then

$$E(Y;u,v) = E(X;u,v) \cdot E(F;u,v) \tag{7.4}$$

In particular, we get (see [12]) $E(\mathbb{C}P^1;u,v) = 1 + uv$, $E(\mathbb{C};u,v) = uv$, $E(\mathbb{C}^*;u,v) = uv - 1$ and thus

$$E(\mathbb{C}^k;u,v) = u^k v^k \ , \quad E((\mathbb{C}^*)^k;u,v) = (uv-1)^k$$

Let us define

$$\varphi(u) := \frac{E(X;1,u) + E(X;u,1)}{2} \tag{7.5}$$

and

$$\tilde{\sigma}(X) := \varphi(-1). \tag{7.6}$$

We will call $\tilde{\sigma}(X)$ the **mixed signature** of X. This is justified by the following result.

Proposition 7.1. *If X is a nonsingular projective variety then its mixed signature and usual signature coincide:*

$$\tilde{\sigma}(X) = \sigma(X).$$

Proof. If X is a nonsingular projective variety then

$$\sigma(X) = \sum_{p+q=0 \bmod 2} (-1)^p h^{p,q}(X),$$

where $h^{p,q}(X)$ is the usual Hodge number of type (p,q) of X, and the result follows from the fact that $e^{p,q}(X) = (-1)^{p+q}h^{p,q}(X)$ (see [12]). $\square$

The mixed signature of a torus is given by

$$\tilde{\sigma}((\mathbb{C}^*)^k) = (-2)^k \tag{7.7}$$

The additivity of the E-polynomial implies that of the mixed signature.

Proposition 7.2. *If X is a disjoint union of a finite number of locally closed varieties X_i, $i \in I$, then*

$$\tilde{\sigma}(X) = \sum_{i \in I} \tilde{\sigma}(X_i)$$

Following [12] we show how the mixed signature of a toric complete intersection can be expressed in terms of mixed signatures of toric hypersurfaces.

Consider polynomials $f_1, f_2 \ldots, f_k \in \mathbb{C}[x]$, $x = (x_1, \ldots, x_n)$, which define a toric complete intersection

$$Y = \{f_1 = f_2 = \ldots = f_k = 0\} \subset (\mathbb{C}^*)^n.$$

Introduce auxiliary coordinates $y_1, \ldots, y_k$ and for $I \subset \{1, \ldots, k\}$ define the toric hypersurface X_I by

$$X_I = \left\{ \sum_{i \in I} y_i f_i(x) - 1 = 0 \right\} \subset (\mathbb{C}^*)^{n+|I|}.$$

Proposition 7.3. *We have*

$$\tilde{\sigma}(Y) = (-2)^n + (-1)^k \sum_{I \subset \{1,\ldots,n\}} \tilde{\sigma}(X_I).$$

Proof. Denote by X the hypersurface in $(\mathbb{C}^*)^n \times \mathbb{C}^k$ with equation $\sum_{i=1}^k y_i f_i(x) - 1 = 0$. The restriction to X of the projection $(\mathbb{C}^*)^n \times \mathbb{C}^k \to (\mathbb{C}^*)^n$ is a locally trivial fibration over $(\mathbb{C}^*)^n \setminus Y$ with each fiber a linear subspace of $\mathbb{C}^k$. It follows from the properties of the E-polynomial that

$$\begin{aligned} E(X; u, v) &= E(\mathbb{C}^{k-1}; u, v) \cdot [E((\mathbb{C}^*)^n; u, v) - E(Y; u, v)] \\ &= (uv)^{k-1} \cdot [(uv - 1)^n - E(Y)]. \end{aligned}$$

Passing to the mixed signature yields

$$\tilde{\sigma}(X) = (-1)^{k-1} [(-2)^n - \tilde{\sigma}(Y)].$$

By additivity, we have

$$\tilde{\sigma}(X) = \sum_{I \subset \{1,\ldots,n\}} \tilde{\sigma}(X_I)$$

and the result follows. □

Assume now that the polynomials $f_1, \ldots, f_k$ which define Y are real polynomials so that Y and the hypersurfaces X_I become real algebraic varieties. We are interested in the (topological) Euler characteristic of $\mathbb{R}Y$. Like the E-polynomial, the Euler characteristic is additive and multiplicative. In particular, we have the obvious formula $\chi(\mathbb{R}P^1) = 0$, $\chi(\mathbb{R}) = -1$, $\chi(\mathbb{R}^*) = -2$. Using the multiplicativity, we get $\chi(\mathbb{R}^k) = (-1)^k$ and

$$\chi((\mathbb{R}^*)^k) = (-2)^k \tag{7.8}$$

Comparing with (7.7) this leads to a key observation for the present paper: the mixed signature of a complex torus is equal to the Euler characteristic of its real part. Recall that a toric variety is a real variety (defined by polynomial equations with real coefficients) and is the disjoint union of the torus orbits. Together with the additivity of the mixed signature and the Euler characteristic, the previous observation leads then to the following result, which will be not used after.

Proposition 7.4. *For any toric variety X, we have*

$$\tilde{\sigma}(X) = \chi(\mathbb{R}X)$$

We obtain the following analogue of Proposition 7.3 for the Euler characteristic of the real part in place of the mixed signature.

Proposition 7.5. *We have*

$$\chi(\mathbb{R}Y) = (-2)^n + (-1)^k \sum_{I \subset \{1,\dots,n\}} \chi(\mathbb{R}X_I).$$

Proof. We adapt the proof of Proposition 7.3. Let X the hypersurface in $(\mathbb{C}^*)^n \times \mathbb{C}^k$ with equation $\sum_{i=1}^k y_i f_i(x) - 1 = 0$. Using the projection $(\mathbb{C}^*)^n \times \mathbb{C}^k \to (\mathbb{C}^*)^n$ together with the additivity and multiplicativity properties of the Euler characteristic yields

$$\chi(\mathbb{R}X) = \chi(\mathbb{R}^{k-1}) \cdot [\chi((\mathbb{R}^*)^n) - \chi(\mathbb{R}Y)]$$

and thus

$$\chi(\mathbb{R}X) = (-1)^{k-1} \left[(-2)^n - \chi(\mathbb{R}Y)\right].$$

By additivity, we have

$$\chi(\mathbb{R}X) = \sum_{I \subset \{1,\dots,n\}} \chi(\mathbb{R}X_I)$$

and the result follows. □

8. Statement of the main result

We use the notations of Section 6. Consider real polynomials $f_1, \dots, f_k \in \mathbb{K}[z_1, \dots, z_n]$ with Newton polytopes $\Delta_1, \dots, \Delta_k$. Denote by Y^{trop} the corresponding tropical intersection in (the tropical torus) $\mathbb{R}^n$. Let $\mathbb{R}Y^{\text{trop}}$ denote the real tropical intersection with respect to either the map W or the map $V_{\mathbb{C}}$: $\mathbb{R}Y^{\text{trop}} = \mathbb{R}Y^{\text{trop}}_W$ or $\mathbb{R}Y^{\text{trop}} = \mathbb{R}Y^{\text{trop}}_{V_{\mathbb{C}}}$ ($\mathbb{R}Y^{\text{trop}}_W$ and $\mathbb{R}Y^{\text{trop}}_{V_{\mathbb{C}}}$ are homeomorphic). Recall that Y^{trop} is a a non degenerate tropical complete intersection if and only if the corresponding convex polyhedral subdivision of the Cayley polytope $C(\Delta_1, \dots, \Delta_k)$ is a primitive triangulation.

Theorem 8.1. *Assume that Y^{trop} is a non degenerate tropical complete intersection.*

1. *The Euler characteristic of $\mathbb{R}Y^{\text{trop}}$ depends only on the polytopes $\Delta_1, \dots, \Delta_k$ and is equal to the mixed signature of a generic intersection in the complex torus $(\mathbb{C}^*)^n$ of algebraic hypersurfaces with Newton polytopes $\Delta_1, \dots, \Delta_k$, respectively. In other words, if Y^{alg} denotes such a generic intersection in $(\mathbb{C}^*)^n$, then its mixed signature $\tilde{\sigma}(Y_{alg})$ depends only on $\Delta_1, \dots, \Delta_k$ and we have*

$$\chi(\mathbb{R}Y^{\text{trop}}) = \tilde{\sigma}(Y_{alg}).$$

2. *The Euler characteristic of $\overline{\mathbb{R}Y}^{\text{trop}}$ depends only on the polytopes $\Delta_1, \dots, \Delta_k$ and is equal to the mixed signature of a generic intersection in the projective toric variety $X(\Delta)$ of algebraic hypersurfaces with Newton polytopes $\Delta_1, \dots, \Delta_k$, respectively. In other*

words, if $\overline{Y}^{alg}$ *denotes such a generic intersection in* $X(\Delta)$*, then* $\tilde{\sigma}(\overline{Y}_{alg})$ *depends only on* $\Delta_1, \ldots, \Delta_k$ *and we have*

$$\chi(\overline{\mathbb{R}Y}^{\mathrm{trop}}) = \tilde{\sigma}(\overline{Y}_{alg}).$$

Proof of Theorem 8.1. Part (2) follows from part (1) using the stratification by torus orbits and the additivity of the Euler characteristic and that of the mixed signature. Now, by Proposition 7.3 and Proposition 7.5, to prove part (1) it suffices to prove the case $k = 1$, that is, the toric hypersurface case. This is the content of the rest of the paper (see Theorem 11.1). □

9. Mixed signature of a complex toric hypersurface

Let f be a non degenerate Laurent polynomial with Newton polytope $\Delta \subset \mathbb{R}^n$ and assume that Δ has non empty interior. Denote by $Z \subset (\mathbb{C}^*)^n$ the nonsingular hypersurface defined by f.

Let $C \subset \mathbb{R}^{n+1}$ be the cone with vertex 0 over $\Delta \times \{1\} \subset \mathbb{R}^n \times \mathbb{R}$. The set of faces of C with the order given by the inclusion and the rank function ρ given by the dimension form an *Eulerian poset* that we denote by P ([1], Example 2.3). Hereafter, we refer to [1] for detailed definitions. Taking the dual cones of elements in P, we get the dual poset P^* which is an Eulerian poset with rank function $\rho^*(z^*) = n + 1 - \rho(z)$ and rank $n + 1$. If $x \in P$ is any face of C, then we denote by $[x, \hat{1}]$ the sub-poset of P formed by all the faces of C having x as a face. This is an Eulerian poset with rank function $z \mapsto \rho(z) - \rho(x)$ and rank $n + 1 - \rho(x) = \rho(C) - \rho(x)$. The dual poset $[x, \hat{1}]^*$ is an eulerian poset of rank $n + 1 - \rho(x)$ and with rank function $\rho^*(z^*) = n + 1 - \rho(z)$.

Let M denote the lattice $\mathbb{Z}^{n+1}$ in which the cone C has its vertices and which contains the vertices of $\Delta \times \{1\}$. If $m \in C \cap M$, define $x(m) \in P$ to be the minimal face of C containing m and $deg\, m$ to be the last coordinate of m (hence, $m = (m_0, deg\, m)$, where $m_0 \in deg\, m \cdot \Delta$). The following result gives a closed formula for $E(Z; u, v)$ in terms of so-called *B-polynomials* of the sub-posets of P^*, which are defined by induction on the rank (see [1], Definition 2.7).

Theorem 9.1 ([1], Theorem 3.24)**.**

$$E(Z; u, v) = \frac{(uv - 1)^n}{uv} + \frac{(-1)^{n+1}}{uv} \sum_{m \in C \cap M} (v - u)^{\rho(x(m))} B([x(m), \hat{1}]^*; u, v) \left(\frac{u}{v}\right)^{deg\, m}.$$

Define two functions (see [1], Definition 3.5)

$$S(C, t) := (1 - t)^{n+1} \sum_{m \in C \cap M} t^{deg\, m}$$

and

$$T(C, t) := (1 - t)^{n+1} \sum_{m \in Int(C) \cap M} t^{deg\, m},$$

where $Int(C)$ is the interior of C. They satisfy the duality relation ([1], Proposition 3.6)

$$S(C, t) = t^{n+1} T(C, t^{-1}) \tag{9.1}$$

In fact, $S(C, t)$ is a polynomial of degree n (see, for example, [8] or Lemma 9.2 below). The sum

$$\sum_{m \in Int(C) \cap M} t^{deg\, m}$$

can be written as

$$\sum_{\lambda=0}^{+\infty} Ehr_{\Delta}(\lambda)t^{\lambda},$$

where $Ehr_{\Delta}(\lambda)$ is the number of integer points in $\lambda \cdot \Delta$. The number $Ehr_{\Delta}(\lambda)$ can be expressed as a polynomial of degree $n = \dim(\Delta)$ in λ called the *Ehrhart polynomial* of Δ. Let a_l^{Δ} be the coefficient of λ^l in this polynomial:

$$Ehr_{\Delta}(\lambda) = \sum_{l=0}^{n} a_l^{\Delta}\lambda^l.$$

Let ψ_i be the coefficient of t^i in $S(C,t)$:

$$S(C,t) = \sum_{i=0}^{\infty} \psi_i t^i.$$

The following lemma can be found in Section 4.1 of [10] (see also [11] p. 233) or [3] Lemma 2.2 p.3.

Lemma 9.2. *One has*

$$\psi_i = \sum_{l=0}^{n}\left(\sum_{p=0}^{i}(-1)^{i-p}\mathrm{C}_{n+1}^{i-p}p^l\right)a_l^{\Delta}$$

and $\psi_i = 0$ *for* $i \geq n+1$.

We are now able to state our main formula for the mixed signature of toric hypersurfaces.

Proposition 9.3. *One has*

$$\tilde{\sigma}(Z) = -(-2)^n + \sum_{l=0}^{n} a_l^{\Delta}(\sum_{i=0}^{n}\sum_{p=0}^{i}(-1)^{n+p}\mathrm{C}_{n+1}^{i-p}p^l). \tag{9.2}$$

Proof. Recall that $\tilde{\sigma}(Z) = \varphi(-1)$, and that $\varphi(u) = [E(Z;1,u) + E(Z;u,1)]/2$. Writing I_m for $[x(m),\hat{1}]^*$ in Theorem 9.1 yields

$$\begin{aligned}\varphi(u) \quad = \quad & \frac{(u-1)^n}{u} + \\ & \frac{(-1)^{n+1}}{2u}\sum_{m\in C\cap M}(u-1)^{\rho(x(m))}\left[B(I_m;1,u)u^{-deg\, m} + (-1)^{\rho(x(m))}B(I_m;u,1)u^{deg\, m}\right]\end{aligned}$$

¿From [1], Definition 2.7 and Proposition 2.10, we have that $B(I_m;1,u) = 1$ if $m \in Int(C)$ and $B(I_m;1,u) = 0$ otherwise, and that $B(I_m;u,1) = (1-u)^{n+1-\rho(x(m))}$. It follows that

$$\varphi(u) \quad = \quad \frac{(u-1)^n}{u} + \frac{(u-1)^{n+1}}{2u}\sum_{m\in C\cap M}u^{deg\, m} + \frac{(1-u)^{n+1}}{2u}\sum_{m\in Int(C)\cap M}u^{-deg\, m}$$

The second and third terms in this sum are easily shown to be equal to $\frac{(-1)^{n+1}}{2u}S(C,u)$ and $\frac{(-u)^{n+1}}{2u}T(C,u^{-1})$, respectively. Now, in view of the duality (9.1), this gives

$$\varphi(u) \quad = \quad \frac{(u-1)^n}{u} + \frac{(-1)^{n+1}}{u}S(C,u).$$

The result follows then using Lemma 9.2 and putting $u = -1$. $\square$

10. Euler Characteristic of a real nonsingular tropical toric hypersurface

Let X be a real nonsingular tropical hypersurface with Newton polytope $\Delta \subset \mathbb{R}^n$ where Δ is assumed to have non empty interior. Hence, the dual polyhedral subdivision $\mathcal{S}$ of Δ is a primitive triangulation.

Lemma 10.1 ([17])**.** *Consider a k-simplex of $\mathcal{S}$ which is contained in the interior of Δ. Its number of non empty symmetric copies is $2^n - 2^{n-k}$.*

Denote by nb_k^Δ the number of k-simplices of $\mathcal{S}$ which are contained in the interior of Δ. We will see that these numbers are in fact independent of the chosen primitive triangulation of Δ.

Let S_2 be the Stirling number of the second kind defined by

$$S_2(i,j) = \frac{1}{j!} \sum_{t=0}^{j} (-1)^{j-t} \mathrm{C}_j^t t^i.$$

Proposition 10.2 (see [3])**.** *We have*

$$nb_k^\Delta = \sum_{l=k}^{n} k! S_2(l+1, k+1)(-1)^{n-l} a_l^\Delta$$

Proof. See [3] Proposition 2.5 p.4. $\square$

Proposition 10.3. *The Euler characteristic of X verifies*

$$\chi(X) = (-1)^n \sum_{k=1}^{n} \frac{2^n - 2^{n-k}}{k+1} \sum_{l=k}^{n} \sum_{t=0}^{k+1} (-1)^{t+l} \mathrm{C}_{k+1}^t t^{l+1} a_l^\Delta$$

Proof. The $(k-1)$-cells in the cellular decomposition of the T-hypersurface corresponding to X are given by the non-empty symmetric copies of the k-simplices of $\mathcal{S}$ contained in the interior of Δ. Hence, this number of $(k-1)$-cells is equal to $nb_k^\Delta(2^n - 2^{n-k})$ by Lemma 10.1. This gives

$$\chi(X) = \sum_{k=1}^{n} (-1)^{k-1} nb_k^\Delta (2^n - 2^{n-k}).$$

Using Proposition 10.2, we obtain

$$\chi(X) = \sum_{k=1}^{n} (-1)^{k-1} (2^n - 2^{n-k}) \sum_{l=k}^{n} (-1)^{n-l} a_l^\Delta \sum_{t=0}^{k+1} (-1)^{k+1-t} \mathrm{C}_{k+1}^t t^{l+1},$$

and the result follows. $\square$

11. Main result for a toric hypersurface

Let X be any real nonsingular tropical hypersurface with Newton polytope $\Delta \subset \mathbb{R}^n$ where Δ is assumed to have non empty interior. Let $Z \subset (\mathbb{C}^*)^n$ be any nonsingular hypersurface defined by a polynomial with Newton polytope Δ.

Theorem 11.1. *We have*

$$\chi(X) = \tilde{\sigma}(Z).$$

Before giving the proof, we need intermediate results. From Proposition 9.3, we have

$$\tilde{\sigma}(Z) = -(-2)^n + \sum_{l=0}^{n} a_l^{\Delta} (\sum_{i=0}^{n} \sum_{p=0}^{i} (-1)^{n+p} \mathrm{C}_{n+1}^{i-p} p^l). \tag{11.1}$$

On the other hand, Proposition 10.3 tell us that

$$\chi(X) = (-1)^{n+1} \sum_{k=1}^{n} \frac{2^n - 2^{n-k}}{k+1} \sum_{l=k+1}^{n+1} \sum_{t=0}^{k+1} (-1)^{t+l} \mathrm{C}_{k+1}^{t} t^l a_{l-1}^{\Delta} \tag{11.2}$$

Note that the sum on l can be taken from 1 (and in fact from 0) according to lemma 12.1. Write

$$\tilde{\sigma}(Z) = -(-2)^n + \sum_{l=0}^{n} S_{l,n} \cdot a_l^{\Delta},$$

with

$$S_{l,n} = (-1)^n \sum_{i=0}^{n} \sum_{p=0}^{i} (-1)^p \mathrm{C}_{n+1}^{i-p} p^l, \tag{11.3}$$

and

$$\chi(X) = \sum_{l=1}^{n} C_{l,n} \cdot a_l^{\Delta}$$

with

$$C_{l,n} = (-1)^{n-l} \sum_{k=1}^{n} \frac{2^n - 2^{n-k}}{k+1} \sum_{t=0}^{k+1} (-1)^t \mathrm{C}_{k+1}^{t} t^{l+1}. \tag{11.4}$$

Lemma 11.2. *We have*

$$\begin{array}{rcl} S_{l,n+1} & = & -2 S_{l,n} \quad \textit{if} \quad l \neq 0 \\ C_{l,n+1} & = & -2 C_{l,n}, \end{array}$$

and $S_{0,n} = (-2)^n$.

Proof. We have

$$\begin{aligned}
S_{0,n} &= (-1)^n \sum_{i=0}^{n} \sum_{p=0}^{i} (-1)^p \mathrm{C}_{n+1}^{i-p} \\
&= (-1)^n \sum_{i=0}^{n} \sum_{b=0}^{i} (-1)^{i-b} \mathrm{C}_{n+1}^{b} \\
&= (-1)^n \sum_{b=0}^{n} (-1)^b \mathrm{C}_{n+1}^{b} \sum_{t=b}^{n} (-1)^t \\
&= (-1)^n \sum_{b=0,\, b=n \bmod 2}^{n} \mathrm{C}_{n+1}^{b}.
\end{aligned}$$

The sum and difference

$$\sum_{b=0,\, b=n \bmod 2}^{n} \mathrm{C}_{n+1}^{b} \pm \sum_{b=0,\, b=n+1 \bmod 2}^{n+1} \mathrm{C}_{n+1}^{b}$$

are equal to 2^{n+1} and 0, respectively. This yields $S_{0,n} = (-2)^n$.

We have

$$S_{l,n+1} = (-1)^n \sum_{i=0}^{n+1} \sum_{p=0}^{i} (-1)^{p+1} \mathrm{C}_{n+2}^{i-p} p^l.$$

Just use that $\mathrm{C}_{n+2}^{i-p} = \mathrm{C}_{n+1}^{i-p-1} + \mathrm{C}_{n+1}^{i-p}$ to split the inner sum. Then we get

$$(-1)^n S_{l,n+1} = \sum_{i=0}^{n+1} \sum_{p=0}^{i} (-1)^{p+1} \mathrm{C}_{n+1}^{i-p-1} p^l + \sum_{i=0}^{n+1} \sum_{p=0}^{i} (-1)^{p+1} \mathrm{C}_{n+1}^{i-p} p^l.$$

By Lemma 12.1, $\sum_{p=0}^{n+1} (-1)^{p+1} \mathrm{C}_{n+1}^{n+1-p} (p)^l = 0$ since $l \leq n$. The term $\sum_{p=0}^{0} (-1)^{p+1} \mathrm{C}_{n+1}^{i-p-1} p^l$ is zero if $l \neq 0$, and we also have $\mathrm{C}_{n}^{i-i-1} = 0$. Hence for $l \neq 0$, we have

$$\begin{aligned}
(-1)^n S_{l,n+1} &= \sum_{i=1}^{n+1} \sum_{p=0}^{i-1} (-1)^{p+1} \mathrm{C}_{n+1}^{i-p-1} p^l + \sum_{i=0}^{n} \sum_{p=0}^{i} (-1)^{p+1} \mathrm{C}_{n+1}^{i-p} p^l \\
&= \sum_{j=0}^{n} \sum_{p=0}^{j} (-1)^{p+1} \mathrm{C}_{n+1}^{j-p} p^l + \sum_{i=0}^{n} \sum_{p=0}^{i} (-1)^{p+1} \mathrm{C}_{n+1}^{i-p} p^l,
\end{aligned}$$

with the change of index $j = i - 1$. This gives the equality $S_{l,n+1} = -2S_{l,n}$ for $l \neq 0$.

Finally, Let us show that $C_{l,n+1} = -2C_{l,n}$. We have

$$
\begin{aligned}
C_{l,n+1}(-1)^{n-l+1} &= \sum_{k=1}^{n+1} \frac{2^{n+1}-2^{n+1-k}}{k+1} \sum_{t=0}^{k+1} (-1)^t \mathrm{C}_{k+1}^t t^{l+1} \\
&= \sum_{k=1}^{n} \frac{2^{n+1}-2^{n+1-k}}{k+1} \sum_{t=0}^{k+1} (-1)^t \mathrm{C}_{k+1}^t t^{l+1} + \frac{2^{n+1}-1}{n+2} \sum_{t=0}^{n+2} (-1)^t \mathrm{C}_{n+2}^t t^{l+1} \\
&= 2\sum_{k=1}^{n} \frac{2^{n}-2^{n-k}}{k+1} \sum_{t=0}^{k+1} (-1)^t \mathrm{C}_{k+1}^t t^{l+1}
\end{aligned}
$$

since $\sum_{t=0}^{n+2}(-1)^t \mathrm{C}_{n+2}^t t^{l+1}$ by Lemma 12.1 $\square$

Lemma 11.3. *We have $S_{n,n} = C_{n,n}$*

Proof.

$$
\begin{aligned}
C_{n,n} &= \sum_{k=1}^{n} \frac{2^{n}-2^{n-k}}{k+1} \sum_{t=0}^{k+1} (-1)^t \mathrm{C}_{k+1}^t t^{n+1} \\
&= \sum_{k=0}^{n} \frac{2^{n}-2^{n-k}}{k+1} \sum_{t=1}^{k+1} (-1)^t \mathrm{C}_{k+1}^t t^{n+1}.
\end{aligned}
$$

Just notice that the two changes of range do not affect the sum. Then use that $\frac{1}{k+1}\mathrm{C}_{k+1}^t t^{n+1} = \mathrm{C}_k^{t-1} t^n$ to get

$$
\begin{aligned}
C_{n,n} &= \sum_{k=0}^{n} (2^n - 2^{n-k}) \sum_{t=1}^{k+1} (-1)^t \mathrm{C}_k^{t-1} t^n && (11.5)\\
&= \sum_{k=0}^{n} 2^n \sum_{t=1}^{k+1} (-1)^t \mathrm{C}_k^{t-1} t^n - \sum_{k=0}^{n} 2^{n-k} \sum_{t=1}^{k+1} (-1)^t \mathrm{C}_k^{t-1} t^n && (11.6)\\
&= 2^n \sum_{t=1}^{n+1} (-1)^t t^n \sum_{k=t-1}^{n} \mathrm{C}_k^{t-1} - \sum_{t=1}^{n+1} (-1)^t t^n \sum_{k=t-1}^{n} 2^{n-k} \mathrm{C}_k^{t-1} && (11.7)\\
&= 2^n \sum_{t=1}^{n+1} (-1)^t t^n \mathrm{C}_{n+1}^t - \sum_{t=1}^{n+1} (-1)^t t^n \sum_{m=t}^{n+1} \mathrm{C}_{n+1}^m && (11.8)
\end{aligned}
$$

by Lemma 12.2 and the fact that $\sum_{k=t-1}^{n} \mathrm{C}_k^{t-1} = \mathrm{C}_{n+1}^t$. Then, the first term is 0 by Lemma 12.1 and we get

$$\begin{aligned} -C_{n,n} &= \sum_{t=1}^{n+1}(-1)^t t^n \sum_{m=t}^{n+1} \mathrm{C}_{n+1}^m \\ &= \sum_{m=1}^{n+1}\sum_{t=1}^{m}(-1)^t t^n \mathrm{C}_{n+1}^m \\ &= \sum_{k=0}^{n}\sum_{t=0}^{k}(-1)^t t^n \mathrm{C}_{n+1}^{k-t} \end{aligned}$$

with the change of indices $k = (t-m)+n$ and using the fact that $\sum_{t=0}^{n+1}(-1)^t t^{n-1}\mathrm{C}_{n+1}^{k-t} = 0$ by Lemma 12.2. On the other hand, we have

$$S_{n,n} = (-1)^n \sum_{i=0}^{n}\sum_{p=0}^{i}(-1)^p \mathrm{C}_{n+1}^{i-p} p^n.$$

Therefore, we get $C_{n,n} = (-1)^{n+1}S_{n,n}$ which is the desired equality for n odd. Suppose now that n is even. Taking $l = n$ in (11.3), and noting that the first sum can be taken until $i = n+1$ due to Lemma 12.1, we get

$$\begin{aligned} (-1)^n S_{n,n} &= \sum_{i=0}^{n+1}\sum_{p=0}^{i}(-1)^p \mathrm{C}_{n+1}^{i-p} p^n && (11.9)\\ &= \sum_{i=0}^{n+1}\sum_{m=0}^{i}(-1)^{i-m}\mathrm{C}_{n+1}^{m}(i-m)^n && (11.10)\\ &= \sum_{k=0}^{n+1}\sum_{m=0}^{n+1-k}(-1)^{n+1-k-m}\mathrm{C}_{n+1}^{m}(n+1-k-m)^n && (11.11)\\ &= \sum_{k=0}^{n+1}\sum_{t=k}^{n+1}(-1)^{t-k}\mathrm{C}_{n+1}^{n+1-t}(t-k)^n && (11.12)\\ &= \sum_{k=0}^{n+1}\sum_{t=k+1}^{n+1}(-1)^{k-t}\mathrm{C}_{n+1}^{t}(k-t)^n. && (11.13) \end{aligned}$$

with the successive changes of indices $m = i-p$, $k = n+1-i$ and $t = n+1-m$ and using that n is even. Suming up (1.6) and (1.9) yields

$$2(-1)^n S_{n,n} = \sum_{k=0}^{n+1}\sum_{t=0}^{n+1}(-1)^{k-t}\mathrm{C}_{n+1}^{t}(k-t)^n$$

which is zero by Lemma 12.1. □

Proof of Theorem 11.1 We have $S_{0,n} = (-2)^n$ by Lemma 11.2 and $a_0^\Delta = 1$ by definition of the Ehrhart polynomial. Hence, Formula (11.3) can be written as $\tilde{\sigma}(Z) = \sum_{l=1}^{n} S_{l,n}\cdot a_l^\Delta$. Comparing with Formula (11.4), it remains to prove that $S_{l,n} = C_{l,n}$ for $l = 1,\dots,n$. But this clearly follows from Lemma 11.2 and Lemma 11.3. □

12. Appendix

We give technical results that are repeately used.

Lemma 12.1. *[See [3] Lemma 6 p.16] Let l and i be nonnegative integers. Then, for $l+1 \le i$, one has (See [20] p. 71)*

$$\sum_{q=0}^{i}(-1)^q \mathrm{C}_i^q q^l \;=\; \sum_{q=0}^{i}(-1)^q \mathrm{C}_i^q (i-q)^l = 0 \tag{12.1}$$

and, as a consequence, for any integer p,

$$\sum_{q=0}^{i}(-1)^q \mathrm{C}_i^q (p-q)^l = 0. \tag{12.2}$$

Lemma 12.2 (See [3] Lemma 7 p.17)**.** *One has $\sum_{t=0}^{p} 2^{p-t}\mathrm{C}_t^k = \sum_{l=k+1}^{p+1} \mathrm{C}_{p+1}^l$.*

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