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Benoît BERTRAND

Institut de Recherche Mathématique de Rennes

École Doctorale MATISSE

U.F.R. de Mathématiques

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Hypersurfaces et intersections complètes maximales dans les variétés toriques

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COMPOSITION DU JURY :

M.	D. Cerveau	
M.	M. Coste	
M.	I. Itenberg	Directeur de Thèse
M.	V. Kharlamov	rapporteur
M.	E. Shustin	rapporteur
M.	J.-J. Risler	rapporteur
M.	O. Viro	

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Chapitre 1

Introduction

Les travaux fondateurs de la topologie des variétés algébriques réelles sont dus à A. Harnack (voir [Har76]) et F. Klein. Une variété algébrique réelle (X, c) est une variété algébrique complexe X dotée d'une involution antiholomorphe c. On note $\mathbb{R}X$ et on appelle *partie réelle* de (X, c) l'ensemble des points fixes de c. Lorsque l'involution c ne prête pas à confusion, on notera tout simplement X la variété algébrique réelle (X, c).

Un polynôme homogène à $d+1$ variables à coefficients réels définit une hypersurface algébrique réelle $(Z, conj)$ de $\mathbb{C}P^d$, où $conj$ est la restriction sur Z de la conjugaison complexe dans $\mathbb{C}P^d$. Soit A une courbe algébrique réelle de degré m dans $\mathbb{C}P^2$ et l le nombre de composantes connexes de $\mathbb{R}A$. Harnack a montré que $l \leq \frac{(m-1)(m-2)}{2} + 1$. L'inégalité de Harnack peut aussi être écrite dans la forme $l \leq g+1$, où g est le genre de A. C'est sous cette forme que Klein a montré l'inégalité qui s'applique à toutes les courbes algébriques réelles non singulières compactes. Les courbes pour lesquelles l'inégalité de Harnack est une égalité sont dites *maximales*. On les appelle aussi *M-courbes*. Harnack a montré qu'il existait des courbes algébriques réelles maximales de tout degré dans le plan projectif.

D. Hilbert, dans son seizième problème, a posé la question de la classification à homéomorphisme près des paires $(\mathbb{R}P^2, \mathbb{R}A)$, où A est une courbe algébrique réelle non singulière de degré donné. Il a considéré aussi le problème de la classification à homéomorphisme près des paires $(\mathbb{R}P^3, \mathbb{R}S)$, où S est une surface algébrique réelle non singulière de degré donné. Plus généralement, ce problème peut être compris comme celui de la classification à homéomorphisme près des paires $(\mathbb{R}P^d, \mathbb{R}Z)$, où Z est une hypersurface algébrique réelle non singulière de degré donné.

L'inégalité de Smith-Thom généralise pour les variétés algébriques réelles l'inégalité de Harnack. Soit $b_*(X, \mathbb{Z}_2)$ la somme des dimensions des groupes d'homologie de X à coefficients dans $\mathbb{Z}_2$. Soit X une variété algébrique réelle. L'inégalité de Smith-Thom (voir, par exemple, [DK00] et [Wil78]) affirme que

$$b_*(\mathbb{R}X, \mathbb{Z}_2) \leq b_*(X, \mathbb{Z}_2).$$

Dans le cas des courbes, cela correspond bien à l'inégalité de Harnack. On dit que X est *maximale* (ou que X est une *M-variété*), si $b_*(\mathbb{R}X, \mathbb{Z}_2) = b_*(X, \mathbb{Z}_2)$.

Dans les années 1970, les travaux de V. Arnold, V. Rokhlin et V. Kharlamov donnent un nouvel essort à la topologie des variétés algébriques réelles. En 1979, O. Viro a inventé une méthode de construction d'hypersurfaces dans les variétés toriques projectives qui lui a permis d'obtenir la classification des courbes algébriques réelles non singulières de degré 7 dans $\mathbb{R}P^2$ (voir [Vir84]). Une version combinatoire de cette méthode, le *patchwork combinatoire* (ou *T-construction*) a été utilisée par I. Itenberg et O. Viro pour démontrer l'existence d'hypersurfaces algébriques réelles maximales dans les espaces projectifs en tout degré (voir [IV02]).

Dans son article [Stu94b], B. Sturmfels a proposé une version de la T-construction adaptée pour la construction d'intersections complètes dans des variétés toriques projectives. Itenberg et Viro l'ont utilisé pour démontrer l'existence d'intersections complètes maximales de k hypersurfaces de degré $m_1, \cdots, m_k$ dans $\mathbb{R}P^d$ pour tous les entiers strictement positifs d, $k \leq d$ et $m_1, \ldots, m_k$.

1.1 Résultats

Un polynôme P de polytope de Newton Δ définit une hypersurface Z_P dans la variété torique X_Δ associée à Δ. On dit que Δ est le polytope de Newton de Z_P.

On montre que, à partir de la dimension 3, il y a des polytopes de Newton Δ pour lesquels il n'existe pas d'hypersurfaces Z de X_Δ qui soient maximales et aient Δ pour polytope de Newton. Cela signifie qu'il n'est pas possible de généraliser directement aux variétés toriques le résultat d'Itenberg et Viro sur l'existence d'hypersurfaces maximales dans les espaces projectifs.

Cependant, pour tout polytope à sommets entiers dans $(\mathbb{R}^+)^n$, où $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, il existe des familles d'hypersurfaces dans X_Δ qui sont asymptotiquement maximales. Soit $\{Z_\lambda\}_{\lambda \in \mathbb{N}^*}$ une famille d'hypersurfaces avec des polytopes de Newton Δ_λ. On suppose que tous les polytopes Δ_λ correspondent à la même variété torique $X = X_{\Delta_\lambda}$ et que le volume de Δ_λ tend vers l'infini quand $\lambda \to +\infty$. On dit que la famille $\{Z_\lambda\}_{\lambda \in \mathbb{N}^*}$ est *asymptotiquement maximale* si $b_*(\mathbb{R}Z_\lambda; \mathbb{Z}_2)$ est équivalent à $b_*(Z_\lambda; \mathbb{Z}_2)$ quand $\lambda \to +\infty$.

Théorème 1.1
Soit Δ un polytope à sommets entiers dans $(\mathbb{R}^+)^d$, et $\{\lambda \cdot \Delta\}_{\lambda \in \mathbb{N}^*}$ la famille des multiples de Δ. Alors, il existe une famille asymptotiquement maximale d'hypersurfaces Z_λ de X_Δ telle que $\lambda \cdot \Delta$ est le polytope de Newton de Z_λ.

On prouve l'existence d'hypersurfaces maximales dans plusieurs variétés toriques de dimension 3. Les *polytopes de Nakajima* sont définis récursivement. Un polytope de Nakajima de dimension 0 est un point. Un polytope Δ de dimension d est un polytope de Nakajima s'il existe une fonction affine $l : \mathbb{R}^{d-1} \to \mathbb{R}$ et un polytope de Nakajima $\Delta' \subset \mathbb{R}^{d-1}$ de dimension $d-1$ tels que $l(\mathbb{Z}^{d-1}) \subset \mathbb{Z}$, l est positive sur Δ' et $\Delta = \{(x, x_d) \in \Delta' \times \mathbb{R}^+, x_d \leq l(x)\}$.

Théorème 1.2
Soit Δ un polytope de Nakajima de dimension 3 correspondant à une variété torique non singulière X_Δ. Alors, il existe une surface réelle maximale dans $\mathbb{R}X_\Delta$ ayant Δ comme polytope de Newton.

Soient α un entier positif ou nul et n, m et l des entiers strictement positifs. Notons $\delta_a^{m,n}$ l'enveloppe convexe des points $(0,0)$, $(n+ma, 0)$, $(0,m)$ et (n,m) dans $\mathbb{R}^2$. Comme corollaire du théorème 1.2, on obtient que le polytope $\delta_a^{m,n} \times [0,l]$ dans $\mathbb{R}^3$, est le polytope de Newton d'une surface maximale dans $\Sigma_a \times \mathbb{C}P^1$, où Σ_a est une surface rationnelle réglée associée à $\delta_a^{m,n}$.

La démonstration du théorème 1.2 est basée sur une relation entre la signature de certaines hypersurfaces obtenues par la T-construction et la caractéristique d'Euler de leur partie réelle. On dit qu'une triangulation à sommets entiers d'un polytope de dimension d est *primitive* si tous les d-simplexes de la triangulation sont de volume $\frac{1}{d!}$. Les T-hypersurfaces obtenues à partir d'une triangulation primitive sont appelées T-hypersurfaces primitives. Si Z est une hypersurface non singulière, on pose $\sigma(Z) = \sum_{p+q=0[2]} (-1)^p h^{p,q}(Z)$ où l'on somme pour tous les couples (p,q) tels que $p+q = 0 \mod 2$. Si la dimension de Z est paire, alors $\sigma(Z)$ est la signature de Z. On note $\chi(\mathbb{R}Z)$ la caractéristique d'Euler de $\mathbb{R}Z$.

Théorème 1.3
Soit Δ un polytope correspondant à une variété torique non singulière, et soit Z une T-hypersurface primitive de X_Δ. Alors

$$\sigma(Z) = \chi(\mathbb{R}Z).$$

Dans le cas de l'espace projectif de dimension 3, le théorème 1.3 est dû a Itenberg (voir [Ite97]).

Pour tout entier $d \geq 3$, on donne des exemples de polytopes $\Delta_d \subset (\mathbb{R}^+)^d$ à sommets entiers tels que les hypersurfaces définissant une intersection complète maximale dans X_{Δ_d} ne peuvent pas toutes avoir le polytope de Newton Δ_d.

En utilisant la construction de Sturmfels, pour toute variété torique projective X, on prouve l'existence de familles d'intersections complètes dans X qui sont asymptotiquement maximales. Soit $\Delta \subset (\mathbb{R}^+)^d$ un polytope à sommets entiers de dimension d, et soit $\{(Z_{1,m}, \cdots, Z_{k,m})\}_{m\in\mathbb{N}^*}$ une famille de k-uplets d'hypersurfaces algébriques réelles dans X_Δ telles que le polytope de Newton de $Z_{i,m}$ (pour tout $i = 1, \ldots, k$ et tout entier strictement positif m) est un multiple de Δ. Notons $\lambda_{i,m}$ l'entier tel que $\lambda_{i,m} \cdot \Delta$ est le polytope de Newton de $Z_{i,m}$. On suppose que,

- pour tout m, la variété $Y_m = Z_{1,m} \cap \cdots \cap Z_{k,m}$ est une intersection complète,
- pour tout $i = 1, \ldots, k$, la suite $\{\lambda_{i,m}\}$ tend vers l'infini quand $m \to +\infty$.

On dit que la famille $\{Y_m\}_{m\in\mathbb{N}^*}$ est asymptotiquement maximale si $b_*(\mathbb{R}Y_m; \mathbb{Z}_2)$ est équivalent à $b_*(Y_m; \mathbb{Z}_2)$ quand m tend vers l'infini.

Théorème 1.4
Soit Δ un polytope à sommets entiers de dimension d, et X_Δ la variété torique associée à Δ. Soient k un entier strictement positif inférieur ou égal à d et $(\lambda_{1,m}, \cdots, \lambda_{k,m})$ une suite de k-uplets telle que chaque $\lambda_{i,m}$ est un entier strictement positif et, pour tout $i = 1, \ldots, k$, la suite $\{\lambda_{i,m}\}$ tend vers l'infini quand $m \to +\infty$. Alors, il existe une suite de k-uplets $(Z_{1,m}, \cdots, Z_{k,m})$ d'hypersurfaces algébriques réelles dans X_Δ telle que

- *$\lambda_{i,m} \cdot \Delta$ est le polytope de Newton de $Z_{i,m}$,*
- *pour tout m, la variété $Y_m = Z_{1,m} \cap \cdots \cap Z_{k,m}$ est est une intersection complète, et la famille $\{Y_m\}_{m\in\mathbb{N}^*}$ est asymptotiquement maximale.*

On construit par la méthode de Sturmfels les intersections complètes décrites dans le théorème ci-dessous. Pour des entiers positifs ou nuls α, m tels que $(\alpha, m) \neq (0,0)$ et pour tout entier strictement positif n, considérons le polygone $\delta_\alpha^{n,m}$ dans $\mathbb{R}^2$ aux sommets $(0,0)$, $(m+n\alpha, 0)$, $(0,n)$ et (m,n). Remarquons que tout polygone de Nakajima de dimension 2 est égal à un des polygones $\delta_\alpha^{n,m}$ (à permutation des coordonnées près). Si m est non nul, la variété torique associée à $\delta_\alpha^{n,m}$ est une surface rationnelle réglée Σ_α. Pour $i = 1, 2$, notons $\Delta_i^{\alpha,m_i,n_i,l_i}$ l'enveloppe convexe des points $(0,0,0)$, $(m_i + \alpha n_i, 0, 0)$, $(0, n_i, 0)$, $(m_i, n_i, 0)$ et $(0,0,l_i)$ dans $\mathbb{R}^3$, où α, m_1, m_2 sont des entiers positifs et l_1, n_1,n_2, l_2 sont des entiers strictement positifs. La pyramide $\Delta_i^{m_i,n_i,l_i}$ est un cône de sommet $(0,0,l_i)$ sur $\delta_\alpha^{n_i,m_i} \times \{0\}$. On va supposer que $m_i = \lambda l_i$ et $n_i = \mu l_i$, où μ est un entier strictement positif et $(\lambda, \alpha) \neq (0,0)$.

Théorème 1.5
Soient α, m_1, m_2 des entiers positifs ou nuls et n_1, l_1, n_2 et l_2 des entiers strictement positifs qui vérifient les égalités $m_i = \lambda l_i$ et $n_i = \mu l_i$ pour certains entiers λ et μ tels que $(\alpha, \lambda) \neq (0,0)$. Alors, il existe une intersection complète maximale dans $X_{\Delta_1^{\alpha,m_1,n_1,l_1}} = X_{\Delta_2^{\alpha,m_1,n_1,l_1}}$ de deux surfaces dont les polytopes de Newton sont $\Delta_1^{\alpha,m_1,n_1,l_1}$ et $\Delta_2^{\alpha,m_2,n_2,l_2}$, respectivement.

Ce résultat généralise celui de L. Brusotti [Bru28] qui a construit, pour tous entiers strictement positifs d_1 et d_2, une intersection complète maximale dans $\mathbb{R}P^3$ de deux surfaces de degrés d_1 et d_2, respectivement.

Le texte est organisé de la façon suivante. Le chapitre 3 est consacré à la démonstration de la relation $\sigma(Z) = \chi(\mathbb{R}Z)$ sous les hypothèses du théorème 1.3. Les constructions de variétés maximales dans des variétés toriques de dimension 3 font l'objet du chapitre 4 pour les hypersurfaces et 8 pour les intersections complètes. Au chapitre 5 (respectivement, 7), on prouve l'existence de familles d'hypersurfaces (respectivement, d'intersections complètes) asymptotiquement maximales dans les variétés toriques projectives. Les exemples de polytopes $\Delta_d \subset (\mathbb{R}^+)^d$ tels que les hypersurfaces définissant une intersection complète maximale dans X_{Δ_d} ne peuvent pas toutes avoir le polytope de Newton Δ_d sont présentés dans les chapitres 4 et 6.

Chapter 2

Preliminaries

2.1 Toric varieties

We fix here some conventions and notations, and recall briefly the construction of toric varieties following [Ful93].

2.1.1 Cones and affine toric varieties

Fix an orthonormal basis of $\mathbb{R}^d$ and thus an inclusion $\mathbb{Z}^d \to \mathbb{R}^d$. This inclusion defines a lattice N in $\mathbb{R}^d$.

Definition 2.1.1 *A* ***cone*** σ *in* $\mathbb{R}^d$ *is the set of positive combinations of a set of points* $\{A_i\}$ *in* $\mathbb{R}^d$*, i.e.* $\sigma = \{\lambda_i A_i, \lambda_i \in \mathbb{R}^+\}$*. We say that* $\{A_i\}$ ***generates*** σ*. A cone* σ *is called* ***polyhedral*** *if it can be generated by a finite number of points. A cone is called* ***rational*** *if it can be generated by integer points. A cone is* ***strongly convex*** *if it contains no linear subspace.*

In what follows we use only polyhedral rational cones, and we will call them cones for simplicity. Denote by M the dual lattice $\mathrm{Hom}(N, \mathbb{Z})$ of N.

Definition 2.1.2 *Let* σ *be a (polyhedral rational) cone. The* ***dual cone*** *of* σ *is the cone* $\sigma^\vee = \{u \in M \otimes \mathbb{R} \ : \ \forall x \in \sigma \text{ one has } < u, x > \geq 0\}$.

Proposition 2.1 *Let* σ *be a polyhedral rational cone. Then* $\sigma^\vee$ *is also a polyhedral rational cone.*

Denote by S_σ the semigroup $(\sigma^\vee \cap M, +)$. Associate to S_σ its semigroup $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$.

Proposition 2.2 (Gordon Lemma) *Let σ be a strongly convex polyhedral rational cone. Then S_σ is a finitely generated semigroup and $\mathbb{C}[S_\sigma]$ is a finitely generated commutative $\mathbb{C}$-algebra.*

Associate to σ the affine algebraic variety $U_\sigma = Spec(\mathbb{C}[S_\sigma])$. The variety U_σ is called an ***affine toric variety***.

2.1.2 Fans and toric varieties

Definition 2.1.3 *A fan $\mathfrak{E}$ in $N \otimes \mathbb{R}$ is a set of strongly convex polyhedral rational cones in $N \otimes \mathbb{R}$ such that*

1. *any face of a cone in $\mathfrak{E}$ is a cone in $\mathfrak{E}$,*

2. *the intersection of two cones in $\mathfrak{E}$ is a face of each.*

For two cones σ and τ in $\mathfrak{E}$ the affine toric variety $U_{\sigma\cap\tau}$ is a principal open subset of both U_σ and U_τ.

To a fan $\mathfrak{E}$ we associate an algebraic variety $X(\mathfrak{E})$. First take the disjoint union of the affine toric varieties associated to the cones of $\mathfrak{E}$. Then glue each pair (U_σ, U_τ) along $U_{\sigma\cap\tau}$. The variety $X(\mathfrak{E})$ is called the ***toric variety*** associated to the fan $\mathfrak{E}$.

Definition 2.1.4 *A fan in $N \otimes \mathbb{R}$ is called* ***complete*** *if the union of its cones is $N \otimes \mathbb{R}$.*

Proposition 2.3 *If $\mathfrak{E}$ is a complete fan, then $X(\mathfrak{E})$ is complete variety.*

2.1.3 Polytopes and toric varieties

Here we consider polytopes in $M \otimes \mathbb{R}$. Unless explicitly specified, ***polytope*** will mean convex polytope whose vertices are integer (i.e. belong to the lattice M). Let Δ be a polytope in $M \otimes \mathbb{R}$.

Definition 2.1.5 *Let Γ be a facet of Δ. The minimal inner normal vector of Γ is the smallest vector v in N such that*

1. *v is nonzero,*

2. *v is orthogonal to Γ,*

3. for any x in Δ, $< x, v >$ is non-positive.

Let p be a vertex of Δ and let $\Gamma_1, \cdots, \Gamma_k$ be the facets of Δ containing p. To p we associate the cone σ_p generated by the minimal inner normal vectors of $\Gamma_1, \cdots, \Gamma_k$.

Definition 2.1.6 *The inner normal fan $\mathfrak{E}_\Delta$ is the fan whose d-dimensional cones are the cones σ_p for all vertices p of Δ.*

Definition 2.1.7 *The toric variety X_Δ associated to Δ is the toric variety $X(\mathfrak{E})$.*

Proposition 2.4 *Let Δ be a d-dimensional polytope, and X_Δ be its associated toric variety. Then the torus $(\mathbb{C}^*)^d$ acts on X_Δ and has an open dense orbit.*

Let a be a vertex of Δ and let $A_1, \cdots, A_k$ be the edges of Δ containing a. Let b_i be the nearest integer point to a on A_i and denote v_i the vector ab_i.

Definition 2.1.8 *We say that Δ is very simple in a if $(v_i)_{i\in\{1,\cdots,k\}}$ is a basis of M. The polytope δ is very simple if it is very simple in all its vertices.*

Definition 2.1.9 *A d-dimensional polytope Δ is simple if for each vertex a of Δ, the number of edges of Δ containing a is d.*

Proposition 2.5 *The toric variety X_Δ is nonsingular if and only if the polytope Δ is very simple.*

2.2 Combinatorial patchworking

By a ***subdivision*** of a polytope we mean a subdivision in convex polytopes whose vertices have integer coordinates.

Definition 2.2.1 *A subdivision τ of a d-dimensional polytope Δ is called **convex** if there exists a convex piecewise-linear function $\Phi : \Delta \to \mathbb{R}$ whose domains of linearity coincide with the d-dimensional polytopes of τ.*

A piecewise-linear function ν on a polytope Δ defines a subdivision δ of Δ. The graph of ν has a natural decomposition δ_0 into linear pieces. The decomposition δ is induced by the natural projection of δ_0 on Δ. The procedure described in the following remark is often used in the sequel to construct convex piecewise-linear functions.

Remark 1 *Fix a real function ν_0 defined on a set A of integer points of a polytope Δ. We associate to ν_0 the convex piecewise-linear function ν : $\Delta \to \mathbb{R}$ whose graph is the lower part of the convex hull of the graph of ν_0. Note that the vertices of the decomposition δ defined by ν are points in A. However, the set of vertices of δ can be a proper subset of A.*

Let us describe the *combinatorial patchworking*, also called *T-construction*, which is a particular case of the Viro method. Given a triple (Δ, τ, D), where Δ is a polytope, τ a convex triangulation of Δ, and D a distribution of signs at the vertices of τ, the combinatorial patchworking, produces an algebraic hypersurface Z in X_Δ.

Let Δ be a d-dimensional polytope with integer vertices which belongs to the positive orthant $(\mathbb{R}^+)^d = \{(x_1, \dots x_d) \in \mathbb{R}^d \mid x_1 \geq 0, \dots, x_d \geq 0\}$, and τ be a convex triangulation of Δ. Denote by s_{x_i} the reflection with respect to the coordinate hyperplane $x_i = 0$ in $\mathbb{R}^d$. Consider the union Δ^* of all copies of Δ under the compositions of reflections s_{x_i} and extend τ to a triangulation τ^* of Δ^* by means of these reflections. Let $D(\tau)$ be a sign distribution at the vertices of the triangulation τ (i.e. each vertex is labelled with $+$ or $-$). We extend $D(\tau)$ to a distribution of signs at the vertices of τ^* using the following rule : for a vertex a of τ^*, one has $sign(s_{x_i}(a)) = sign(a)$ if the i-th coordinate of a is even, and $sign(s_{x_i}(a)) = -sign(a)$, otherwise.

Let σ be a d-dimensional simplex of τ^* with vertices of different sign, and E be the hyperplane piece which is the convex hull of the middle points of the edges of σ with endpoints of opposite signs. We separate vertices of σ labelled with $+$ from vertices labelled with $-$ by E. The union of all these hyperplane pieces forms a piecewise-linear hypersurface H^*.

Remark 2 *Note that we get a natural cell decomposition of H^* whose d-dimensional pieces are the hyperplane pieces described above.*

Definition 2.2.2 *Let x be an integer point (resp., a vector) in $\mathbb{R}^d$. We call the* ***parity*** *of x and denote by $\bar{x}$ the coordinatewise reduction modulo 2 of x.*

For any facet Γ of Δ^*, let $N^\Gamma = (\nu_1^\Gamma, \nu_2^\Gamma, \dots, \nu_d^\Gamma)$ be a minimal integer vector normal to Γ. Identify Γ with ${s_{x_1}}^{\nu_1^\Gamma} \circ {s_{x_2}}^{\nu_2^\Gamma} \cdots \circ {s_{x_d}}^{\nu_d^\Gamma}(\Gamma)$ via the map ${s_{x_1}}^{\nu_1^\Gamma} \circ {s_{x_2}}^{\nu_2^\Gamma} \cdots \circ {s_{x_d}}^{\nu_d^\Gamma}$. Denote by $\widetilde{\Delta}$ the result of the identifications.

Proposition 2.6 *(see, for example, [GKZ94]) The variety* $\widetilde{\Delta}$ *is homeomorphic to the real part* $\mathbb{R}X_\Delta$ *of* X_Δ.

Denote by $\widetilde{H}$ the image of H^* in $\widetilde{\Delta}$. Let Q be a polynomial with Newton polytope Δ. It defines a hypersurface Z_0 in the torus $(\mathbb{C}^*)^d$ contained in X_Δ. The closure of Z_0 in X_Δ is the hypersurface defined by Q in X_Δ.

Definition 2.2.3 *Let* Q *be a polynomial with Newton polytope* Δ *and* Z *be the hypersurface defined by* Q *in* X_Δ. *We call* Δ *the* ***Newton polytope*** *of* Z.

Theorem 2.7 (T-construction, O. Viro)
Under the hypotheses made above, there exists a hypersurface Z *of* X_Δ *with Newton polytope* Δ *and a homeomorphism* $h : \mathbb{R}X_\Delta \to \widetilde{\Delta}$ *such that* $h(\mathbb{R}Z) = \widetilde{H}$. *If* Δ *is very simple, then* Z *is nonsingular.*

Definition 2.2.4 *The hypersurface* Z *in the above theorem is called a* ***real algebraic*** T***-hypersurface***.

Definition 2.2.5 *A d-dimensional simplex with integer vertices is called* ***primitive*** *if its volume is equal to* $\frac{1}{d!}$. *A triangulation* τ *of a d-dimensional polytope is* ***primitive*** *if every d-simplex of the triangulation is primitive. A real algebraic* T*-hypersurface is called* ***primitive*** *if the triangulation* τ *used in its construction is primitive.*

We define here some useful notions related to the combinatorial patchworking.

Definition 2.2.6 *With the above notations a simplex of* τ^* *is called* ***non-empty*** *if it has vertices of different signs.*

Definition 2.2.7 *Let* Δ *be a d-dimensional polytope. We call* ***lattice volume*** *of* Δ *and denote by* $\mathrm{Vol}(\Delta)$ *the volume normalized so that a primitive d-simplex has volume* 1. *The usual volume is denoted by* $\mathrm{vol}(\delta)$.

Remark 3 *If* Δ *is a d-dimensional polytope, then* $\mathrm{Vol}(\Delta) = d!\,\mathrm{vol}(\Delta)$.

Definition 2.2.8 *Let* τ *be a triangulation with a distribution of signs at its vertices. A vertex* v *of* τ *is called* ***isolated****, if all the vertices in the star of* v *except* v *itself have the sign opposite to the sign of* v.

Let p be an isolated vertex of the triangulation τ^* of Δ^*. Assume that p is in the interior of Δ^*. Then the star of p contains a connected component of H^* homeomorphic to a sphere S^d. So, p corresponds to a d-dimensional sphere of the real part of the hypersurface Z obtained by Viro's theorem.

Definition 2.2.9 *A k-dimensional simplex in $\mathbb{R}^d$ is called **elementary** if the reductions modulo 2 of its vertices generate a k-dimensional affine space.*

The following proposition is due to Itenberg (see [Ite97] Proposition 3.1).

Proposition 2.8 *Let s be an elementary k-simplex of a triangulation τ of a d-dimensional polytope. Assume that s is contained in j coordinate hyperplanes. Then the union of the symmetric copies of s contains exactly $(2^d - 2^{d-k})/2^j$ cells of dimension $k-1$ of the cell decomposition of H^*.*

2.3 Sturmfels theorem for complete intersections

In [Stu94b] B. Sturmfels gave a combinatorial way to construct complete intersections. We quote here this theorem in the particular case we need. For the general statement and the proof we refer to [Stu94b].

Let Δ_0 be a d-dimensional polytope. Let $\lambda_1, \cdots, \lambda_k$ be positive integers, where $k \le d$. Denote by Δ_i the polytope $\lambda_i \cdot \Delta_0$ and by Δ the Minkowski sum $\Delta_1 + \cdots + \Delta_k$. Let ν_i be a piecewise-linear convex function on Δ_i defining a triangulation τ_i. For each Δ_i, choose a distribution of signs D_i at the vertices of τ_i.

The initial data of the procedure of construction of a complete intersection using Sturmfels' theorem are the polytopes Δ_i, the functions ν_i and the sign distributions D_i. Apply the T-construction for each triple (Δ_i, τ_i, D_i) to construct the hypersurfaces S_i. Let D_i^* be the sign distribution at the vertices of τ_i^*.

The functions $\nu_1, \cdots, \nu_k$ define a convex decomposition of Δ in the following way. Let x be an integer point of Δ. Denote by E_x the set of k-tuples $(x_1, \cdots, x_k)$ such that x_i is a vertex of τ_i and $x_1 + \cdots + x_k = x$. Set $\nu_0(x) = \min_{(x_1,\cdots,x_k)\in E_x}(\nu_1(x_1) + \cdots + \nu_k(x_k))$ and extend the function ν_0 to the piecewise-linear convex function ν whose graph is the lower part of the convex hull of the graph of ν_0 (cf. remark 1). The function $\nu : \Delta \to \mathbb{R}$ defines a convex subdivision δ of Δ. Sturmfels' theorem requires the following genericity condition on the functions ν_i.

Definition 2.3.1 *With the notation as above, the function ν is called **sufficiently generic**, if each polytope Γ in δ can be uniquely written as the sum of simplices Γ_i for Γ_i in τ_i, and one has $dim\Gamma = dim\Gamma_1 + \cdots + \dim \Gamma_k$.*

If δ satisfies the above genericity condition, it is called a ***mixed subdivision*** of Δ.

Let δ be a mixed subdivision of Δ. To each vertex v of δ we assign a sign vector $\epsilon(v) = (\epsilon_1, \cdots, \epsilon_k)$, where $\epsilon_i \in \{+, -\}$ is the sign of the vertex of τ_i corresponding to v.

Extend δ to a subdivision δ^* of Δ^* by means of the reflections with respect to coordinate hyperplanes. The extension of the sign distribution to δ^* is as follows. Let v be a vertex of δ^*, and let $v_1, \cdots, v_k$ be the vertices of $\tau_1, \cdots, \tau_k$ corresponding to v. Then

$$\epsilon_j(s_{x_i}(v)) = sign(s_{x_i}(v_j)).$$

For $j \in \{1 \cdots k\}$ construct the hypersurface $\widetilde{S}_j$ in the following way. Notice that any polytope in δ^* can be uniquely written as the sum of simplices Γ_i for Γ_i in τ_i^*. Put $S_j^* \cap \Gamma = \Gamma_1 + \cdots + S_j \cap \Gamma_j + \cdots + \Gamma_k$. Let $\widetilde{S}_j$ be the image of S_j^* in $\widetilde{\Delta}$.

Theorem 2.9 (B. Sturmfels)
With the above notation, there exist hypersurfaces Z_i with Newton polytopes Δ_i, respectively, and a homeomorphism $f : \mathbb{R}X_\Delta \to \widetilde{\Delta}$ such that the hypersurfaces Z_i define a complete intersection Y in X_Δ, and f sends $\mathbb{R}Z_i$ (resp, $\mathbb{R}Y$) onto $\widetilde{S}_i$. (resp., $\cap_{j=1\cdots k}\widetilde{S}_j$).

2.3.1 A very simple example

As an example we construct a maximal complete intersection of two curves of given degrees in $\mathbb{C}P^2$ (see Figures 2.1 and 2.2).

Let Δ_0 be the triangle in $\mathbb{R}^2$ with vertices $(0,0), (0,1)$and$(1,0)$. Let λ_1 and λ_2 be two positive integers. Denote by Δ_i the triangle $\lambda_i \cdot \Delta_0$ $(i = 1, 2)$. The triangle $\Delta = (\lambda_1 + \lambda_2)\Delta_0$ is the Minkowski sum of Δ_1 and Δ_2. Take any convex triangulations of Δ_1 and Δ_2 such that any integer point of the

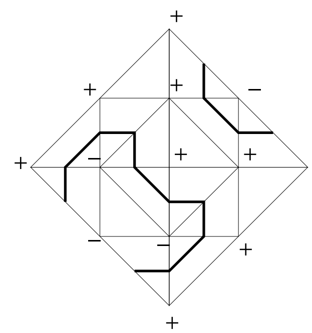

Figure 2.1: T-curve S_1 of degree 2

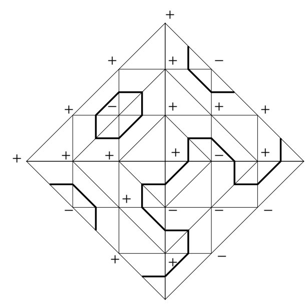

Figure 2.2: T-curve S_2 of degree 3

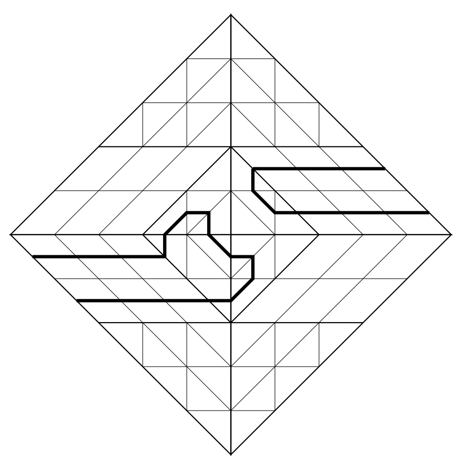

Figure 2.3: $\widetilde{S}_1$

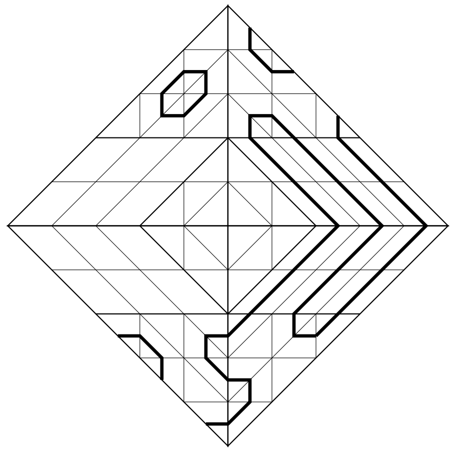

Figure 2.4: $\widetilde{S}_2$

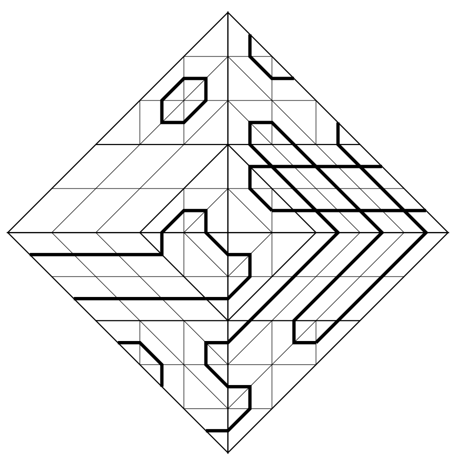

Figure 2.5: $\widetilde{S}_1$ and $\widetilde{S}_2$

edge $[(\lambda_1, 0), (0, \lambda_1)]$ (resp., $[(0,0), (\lambda_2, 0)]$) of Δ_1 (resp., Δ_2) is a vertex of the chosen triangulation of Δ_1 (resp., Δ_2). Let ν_1 and ν_2 be convex functions certifying the convexity of these triangulations. Add to ν_2 a linear function $\nu_3((x,y)) = C_1 y + C_2 x$ with $C_1 > C_2$. For C_2 large enough the obtained mixed subdivision of Δ is as follows. The triangle $((0,0), (\lambda_1, 0), (0, \lambda_1))$ is triangulated as Δ_1. The triangle $((0, \lambda_1), (0, \lambda_1 + \lambda_2), (\lambda_1, \lambda_2))$ is the image of Δ_2 under the translation t_{λ_1} by the vector $(0, \lambda_1)$. Take the image of the triangulation of Δ_2 by t_{λ_1}. The rest of Δ is a parallelogram Γ decomposed into parallelograms of area 1 having two edges parallel to $((1,0),(0,1))$ and the two other edges parallel to the x-axis. Notice that any parallelogram of area 1 has exactly one symmetric copy, where $\widetilde{S}_1$ and $\widetilde{S}_2$ intersect (see Figure 2.5). So, the number of intersection points of the real parts of our curves is the area of the parallelogram Γ, i.e. $\lambda_1 \lambda_2$. By Sturmfels' theorem 2.9 we associate to $\widetilde{S}_1$ and $\widetilde{S}_2$ (see Figures 2.3 and 2.4) two curves of degrees λ_1 and λ_2, respectively, defining a maximal complete intersection in the projective plane.

2.3.2 Cayley trick

Instead of constructing the complete intersection in the Minkowski sum of Newton polytopes, it is convenient to use so-called Cayley trick (see, for example, [Stu94a]).

Let Δ_1, ..., Δ_k be convex polytopes with integer vertices in $\mathbb{R}^d$ $(k \le d)$. For any $i = 1, \dots, k$ put

$$\hat{\Delta}_i = \{(x_1, \dots, x_{k+d}) \in \mathbb{R}^{k+d} \mid x_i = 1; x_j = 0 \text{ if } j \le k \text{ and } j \ne i; \\ (x_{k+1}, \dots, x_{k+d}) \in \Delta_i\}.$$

The convex hull of $\hat{\Delta}_1$, ..., $\hat{\Delta}_k$ in $\mathbb{R}^{k+d}$ is called *Cayley polytope* and is denoted by $C(\Delta_1, \dots, \Delta_k)$. The intersection of $C(\Delta_1, \dots, \Delta_k)$ with the subspace $B \subset \mathbb{R}^{k+d}$ defined by $x_1 = \ldots = x_k = 1/k$ is naturally identified with the Minkowski sum Δ of Δ_1, ..., Δ_k multiplied by $1/k$. Thus, any triangulation of the Cayley polytope $C(\Delta_1, \dots, \Delta_k)$ induces a subdivision of the Minkowski sum of Δ_1, ..., Δ_k.

The following lemma can be found, for example, in [Stu94a].

Lemma 1 *The correspondence described above establishes a bijection between the set of convex triangulations with integer vertices of $C(\Delta_1, \dots, \Delta_k)$ and the set of mixed subdivisions of the Minkowski sum of Δ_1, ..., Δ_k.* $\square$

Denote by C^* the union of the symmetric copies of $C(\Delta_1, \ldots, \Delta_k)$ under the reflections $s_{(i)}$, $i = k+1, \ldots, k+n$, where $s_{(i)}$ is the reflection of $\mathbb{R}^{k+d}$ with respect to the hyperplane $\{x_i = 0\}$, and compositions of these reflections.

Choose a convex triangulation τ of $C(\Delta_1, \ldots, \Delta_k)$ having integer vertices and a distribution of signs at the vertices of τ. Extend the triangulation τ to a symmetric triangulation τ^* of C^* and the distribution of signs at the vertices of τ to a distribution at the vertices of the extended triangulation by the same rule as in Section 2.2: passing from a vertex to its mirror image with respect to a coordinate hyperplane we preserve its sign if the distance from the vertex to the plane is even, and change the sign if the distance is odd.

For any $(k + d - 1)$-dimensional simplex γ of τ^* and any $j = 1, \ldots, k$ denote by γ_j the maximal face of γ which belongs to a symmetric copy of $\hat{\Delta}_j$. Let $K_j(\gamma)$ be the convex hull of the middle points of the edges of γ_j having endpoints of opposite signs, and let $H(\gamma)$ be the intersection of the join $K_1(\gamma) * \ldots * K_k(\gamma)$ with B. Denote by H^* the union of the intersections $H(\gamma)$, where γ runs over all the $(k + d - 1)$-dimensional simplices of τ^*, and denote by $\widetilde{H}$ the image of H^* in $\widetilde{(\frac{1}{k}\Delta)}$.

The following statement is an immediate corollary of Theorem 2.9.

Proposition 2.10 *Assume that all the polytopes Δ_1, ..., Δ_k are multiples of the same polytope Π with integer vertices. Then, there exist nonsingular real hypersurfaces Z_1, ..., Z_k in X_Π with Newton polytopes Δ_1, ..., Δ_k, respectively, and a homeomorphism $f : \mathbb{R}X_\Pi \to \widetilde{(\frac{1}{k}\Delta)}$ such that the hypersurfaces Z_1, ..., Z_k define a complete intersection Y in X_Π and f maps the set of real points $\mathbb{R}Y$ of Y onto $\widetilde{H}$.* □

Definition 2.3.2 *A $(k + d - 1)$-dimensional simplex γ of τ^* is said to be* ***nonempty*** *if $H(\gamma) \neq \emptyset$.*

Remark 4 *A $(k+d-1)$-dimensional simplex γ of τ^* is nonempty if and only if the simplices γ_j are all nonempty (i.e. have vertices of different signs).*

Let us reformulate the construction described in the example presented in the begining of this section (Subsection 2.3.1) using the Cayley polytope. Let Δ_1 be the convex hull of the points $(0,0), (0,\lambda_1), (\lambda_1, 0)$, and Δ_2 be the convex hull of the points $(0,0), (0,\lambda_2), (\lambda_2, 0)$. Consider a convex primitive triangulation τ_1 of Δ_1 and a convex primitive triangulation τ_2 of Δ_2. In the Cayley polytope $C(\Delta_1, \Delta_2)$ take the cone C_1 (resp., C_2) with the vertex $(1, 0, 0, \lambda_1)$ (resp., $(0, 1, 0, 0)$) over $\hat{\Delta}_2$ (resp., $\hat{\Delta}_1$). Let J be the join of the edges $[(0, 1, 0, 0), (0, 1, \lambda_2, 0)]$ and $[(1, 0, \lambda_1, 0), (1, 0, 0, \lambda_1)]$. The triangulations τ_1 and τ_2 induce primitive triangulations of C_1, C_2 and J which patch into a convex triangulation τ of $C(\Delta_1, \Delta_2)$. Choose any sign distribution at the vertices of τ apply the construction. Then, any tetrahedron in the triangulation of J has exactly one symmetric copy containing a point of H^*. Since J has lattice volume $\lambda_1\lambda_2$, the complete intersection constructed is maximal.

2.4 Some facts about triangulations of lattice polytopes

2.4.1 Ehrhart Polynomial

As we saw in section 2.2, the combinatorial patchworking deals with triangulations of lattice polytopes. We sum up here some results that will be used latter on. First, recall the results of E. Ehrhart on the number of integer points in a lattice polytope. Ehrhart showed that the number of integer points in the multiple $\lambda \cdot \Delta$ of a not necessarily convex polytope Δ, where λ is a positive integer, is a polynomial in λ (see [Ehr94]). We denote by $l(\Delta)$ and $l^*(\Delta)$ the numbers of integer points in Δ and in the interior of Δ, respectively.

Theorem 2.11 (E. Ehrhart)
Let Δ be a polytope with integer vertices. Then, the numbers $l(\lambda \cdot \Delta)$ and $l^*(\lambda \cdot \Delta)$ are polynomials in λ of degree $\dim \Delta$. Denote them respectively by $Ehr_\Delta(\lambda)$ and $Ehr^*_\Delta(\lambda)$. They satisfy the reciprocity law

$$(-1)^{dim \Delta} Ehr_\Delta(-\lambda) = Ehr^*_\Delta(\lambda).$$

One often considers the Ehrhart series

$$SE_p(t) = \sum_{\lambda=0}^{\infty} Ehr_\Delta(\lambda) t^\lambda.$$

Put $Q_\Delta(t) = (1-t)^{d+1} SE_p(t)$, where d is the dimension of Δ. In fact, $Q_\Delta(t)$ is a polynomial of degree d (see [Bri94] or lemma 2), and we define the numbers Ψ_j to be the coefficients of $Q_\Delta(t)$:

$$Q_\Delta(t) = \sum_{j=0}^{\infty} \Psi_j t^j.$$

Let a_i^Δ be the coefficient of λ^i in $Ehr_\Delta(\lambda)$.

Lemma 2 *One has*

$$\Psi_j = \sum_{i=0}^{d} (\sum_{n=0}^{j} (-1)^{j-n} \mathrm{C}_{d+1}^{j-n} n^i) a_i,$$

and $\Psi_j = 0$ *for* $j \geq d+1$.

We insert here technical lemmas that will be used quite often, mainly in chapter 3.

Lemma 3 *(cf. [VLW92] p. 71) Let α be a nonnegative integer such that $\alpha + 1 \leq i$ .Then one has $\sum_{a=0}^{i} (-1)^a \mathrm{C}_i^a (i-a)^\alpha = 0$, and equivalently $\sum_{a=0}^{i} (-1)^a \mathrm{C}_i^a a^\alpha = 0$.*

Lemma 4 *Let α be a nonnegative integer such that $\alpha + 1 \leq i$ .Then*

$$\sum_{a=0}^{i} (-1)^a \mathrm{C}_i^a (p-a)^\alpha = 0.$$

Proof . -
Write $p - a = (i-a) + (p-i)$ and then,

$$(p-a)^\alpha \quad = \quad \sum_{m=0}^{\alpha} \mathrm{C}_\alpha^m (i-a)^{\alpha-m} (p-i)^m.$$

So,

$$\begin{aligned} \sum_{a=0}^{i} (-1)^a \mathrm{C}_i^a (p-a)^\alpha &= \sum_{a=0}^{i} (-1)^a \mathrm{C}_i^a \sum_{m=0}^{\alpha} \mathrm{C}_\alpha^m (i-a)^{\alpha-m} (p-i)^m \\ &= \sum_{m=0}^{\alpha} \mathrm{C}_\alpha^m (p-i)^m \left(\sum_{a=0}^{i} (-1)^a \mathrm{C}_i^a (i-a)^{\alpha-m} \right) \end{aligned}$$

The last term is zero by lemma 3. □

Proof of lemma 2.
From the definitions we see that

$$Q_\Delta(t) = (\sum_{k=0}^{d+1} \mathrm{C}_{d+1}^k t^k (-1)^k) \cdot \sum_{n=0}^{\infty} (\sum_{i=0}^{d} a_i n^i) t^n,$$

and thus, $\Psi_j = \sum_{i=0}^d (\sum_{n=0}^j (-1)^{j-n} \mathrm{C}_{d+1}^{j-n} n^i) a_i^\Delta$.
Let $A_j = \sum_{n=0}^j (-1)^{j-n} \mathrm{C}_{d+1}^{j-n} n^i$. If $j \geq d+1$, then

$$A_j = \sum_{k=0}^{d+1} (-1)^k \mathrm{C}_{d+1}^k (j-k)^i,$$

which is zero according to lemma 4. □

2.4.2 Formula for the number of simplices of a primitive triangulation

In order to compute the Euler characteristic of a T-hypersurface $\mathbb{R}Z$ we often need to know the number of simplices of any dimension in the triangulation τ of Δ. In the case of a primitive triangulation, these numbers happen not to depend on the primitive triangulation chosen. This statement can be found in [Dai00]. Here we prove the statement for primitive convex triangulations.

Let nbs_r^Δ be the number of r-dimensional simplices in a primitive triangulation of Δ which are contained in the interior of Δ, and let i be the dimension of Δ. Let S_2 be the Stirling number of the second kind defined by $S_2(i,j) = 1/(j)! \sum_{k=0}^j (-1)^{j-k} \mathrm{C}_j^k k^i$.

Proposition 2.12 *Under the above hypotheses we have,*

$$nbs_r^\Delta = \sum_{l=r+1}^{i+1} stir(r+1, l)(-1)^{i-l+1}.a_{l-1}^\Delta,$$

where $stir(i,j) = (i-1)! S_2(j,i)$.

In order to prove this formula we first check that it is true for all primitive simplices.

Formula for a primitive simplex

Let s_i be a primitive simplex of dimension i. Then

$$Ehr_{s_i}(\lambda) = \frac{1}{i!}(\lambda+1)\dots(\lambda+i) = \frac{1}{i!}\sum_{j=1}^{i+1}(-1)^{i+1-j}S_1(i+1,j)\lambda^{j-1},$$

where S_1 is the first Stirling number defined by the formula $\sum_{m=0}^{n} S_1(n,m)x^m = x(x-1)\dots(x-n+1)$ (see [BV97]). Thus

$$a_{j-1}^{s_i} = \frac{(-1)^{i+1-j}}{i!}S_1(i+1,j).$$

It remains to show that

$$nbs_r^{s_i} = \frac{r!}{i!}\sum_{l=r+1}^{i+1} S_2(l,r+1).S_1(i+1,l).$$

We have $\sum_{l=r+1}^{i+1} S_2(l,r+1).S_1(i+1,l) = \delta_{r,i}$ (see [VLW92] p. 107), and thus, $\frac{r!}{i!}\sum_{l=r+1}^{i+1} S_2(l,r+1).S_1(i+1,l) = \delta_{r,i}$ which is exactly the number of r-dimensional simplices contained in the interior of s_i.

Proof for a general polytope

Definition 2.4.1 *A triangulation τ of a polytope of dimension d is called* ***shellable*** *if there exists a numbering of its d-simplices $s_1, s_2, \dots, s_k$ such that for $i \in \{2,\dots,k\}$*

$$s_i \cap \cup_{j=1}^{i-1} s_j$$

is a nonempty union of $(d-1)$-simplices of τ homeomorphic to a $(d-1)$-ball. This numbering is called a shelling of τ.

Theorem 2.13 (Ziegler ([Zie95], p. 243))
Every convex triangulation of a polytope is shellable.

The formula holds for a point (with the convention that the point is in its interior). We now assume that the formula is true in all dimensions less than d. Let Δ be a d-dimensional polytope, and τ be a primitive triangulation of Δ. Fix $s_1, s_2, \dots, s_t$ a shelling of τ, and put $U_j = \cup_{i=1}^{j} s_j$. Note that U_j need not to be convex. Assume that the formula is true for U_j. Note that the formula also holds for $U_j \cap s_{j+1}$ which is $(d-1)$-dimensional, eventhough it is not a polytope. We know that it is also true for s_{j+1}.

Lemma 5 *The numbers nbs_r satisfy the relation*

$$nbs_r^{U_j} + nbs_r^{s_{j+1}} + nbs_r^{U_j \cap s_{j+1}} = nbs_r^{U_{j+1}}.$$

Proof . -

This follows immediately from the fact that $\overset{\circ}{U_j} \sqcup \overset{\circ}{s}_{j+1} \sqcup (U_j \overset{\circ}{\cap} s_{j+1}) = \overset{\circ}{U}_{j+1}$ where $\overset{\circ}{U}$ stands for the interior of U. $\square$

By induction hypothesis,

$$nbs_r^{U_j \cap s_{j+1}} \quad = \quad \sum_{l=r+1}^{d} stir(r+1,l)(-1)^{d-l} a_{l-1}^{U_j \cap s_{j+1}},$$

and since $a_d^{U_j \cap s_{j+1}} = 0$, we can also write

$$nbs_r^{U_j \cap s_{j+1}} \quad = \quad \sum_{l=r+1}^{d+1} stir(r+1,l)(-1)^{d-l} a_{l-1}^{U_j \cap s_{j+1}}.$$

Then $nbs_r^{U_{j+1}} = \sum_{l=r+1}^{d+1} stir(r+1,l)(-1)^{d-l+1}.(a_{l-1}^{U_j} + a_{l-1}^{s_{j+1}} - a_{l-1}^{U_j \cap s_{j+1}})$, and $a_{l-1}^{U_j} + a_{l-1}^{s_{j+1}} - a_{l-1}^{U_j \cap s_{j+1}}$ is precisely the coefficient $a_{l-1}^{U_{j+1}}$ in the Ehrhart polynomial of U_{j+1}. This completes the proof of the formula.

2.5 Danilov and Khovanskii formulae

V. Danilov and A. Khovanskii [DK87] computed the Hodge numbers of a smooth hypersurface in a toric variety X_Δ in terms of the polytope Δ. For a face F of Δ, denote by a_i^F the coefficient of the term of degree i of the Ehrhart polytope of F. Let $\mathcal{F}_i(\Delta)$ be the set of i-dimensional faces of Δ, and f_i be the cardinality of $\mathcal{F}_i(\Delta)$.

Theorem 2.14 (V. Danilov, A. Khovanskii)
Let Δ be a simple polytope of dimension d, and Z be a smooth or quasi-

smooth algebraic hypersurface in X_Δ. Then, for $p \neq \frac{d-1}{2}$

$$\begin{aligned}
h^{p,p}(Z) &= (-1)^{p+1}\sum_{i=p+1}^{d}(-1)^i \mathrm{C}_i^{p+1} f_i(\Delta)\\
h^{\frac{d-1}{2},\frac{d-1}{2}}(Z) &= (-1)^{\frac{d+1}{2}}\sum_{i=\frac{d+1}{2}}^{d}(-1)^i \mathrm{C}_i^{\frac{d+1}{2}} f_i(\Delta) - \sum_{i=\frac{d+1}{2}}^{d}\sum_{F\in\mathcal{F}_i(\Delta)}(-1)^i\Psi_{\frac{d+1}{2}}(F)\\
h^{p,d-1-p}(Z) &= (-1)^d\sum_{i=p+1}^{d}\sum_{F\in\mathcal{F}_i(\Delta)}(-1)^i\Psi_{p+1}(F)\\
h^{p,q} &= 0 \text{ if } q\neq p \text{ or } p\neq d-1-p,
\end{aligned}$$

where $\Psi_{p+1}(F) = \sum_{\alpha=1}^{i+1}\sum_{a=0}^{p+1}(-1)^a \mathrm{C}_{i+1}^a (p+1-a)^{\alpha-1} a_{\alpha-1}^F$.

Let X be a topological space, and K a field. We denote by $b_i(X;K)$ the i-th Betti number of X with coefficients in K, i.e. the dimension of $H_i(X;K)$. Put $b_*(X;K) = \sum_{i=0}^{\infty} b_i(X;K)$. Sometimes, we write $b_i(X)$ and $b_*(X)$ for $b_i(X;\mathbb{Z}_2)$ and $b_*(X;\mathbb{Z}_2)$, respectively.

The following lemma is an immediate corollary of theorem 2.14.

Lemma 6 *Let Δ be a 3-dimensional simple polytope, and Z be an algebraic hypersurface of X_Δ with Newton polytope Δ. Then $b_*(Z;\mathbb{C}) = l^*(2\Delta) - 2l^*(\Delta) - \sum_{\Gamma\in\mathcal{F}_2(\Delta)}(l^*(\Gamma)-1) - 1$.*

The following two propositions can be derived from Khovanskii's results (see [Hov77] and [Hov78]) or can be found in [Mik].

Proposition 2.15 *Let Δ be a polytope, and $\{Z_\lambda\}_{\lambda\in\mathbb{N}}$ be a family of algebraic hypersurfaces in X_Δ with Newton polytopes $\lambda\cdot\Delta$. Then $b_*(Z_\lambda;\mathbb{Z}_2)$ is equivalent to $\mathrm{Vol}(\lambda\cdot\Delta)$ when λ tends to infinity.*

Denote by $\mathrm{Vol}(\Delta_1,\cdots,\Delta_k)$ the mixed volume of the polytopes $\Delta_1,\cdots,\Delta_k$. We choose a normalization of the mixed volume in such a way that for a primitive simplex σ we have $\mathrm{Vol}(\sigma,\cdots,\sigma)=1$.

Proposition 2.16 *Let Δ be a d-dimensional polytope, and k be a positive integer satisfying $k\leq d$. Assume that for any collection $\lambda_1,\cdots,\lambda_k$ of positive integers we have a collection of k hypersurfaces $Z_{\lambda_1},\cdots,Z_{\lambda_k}$ in X_Δ with Newton polytopes $\lambda_1\cdot\Delta,\ldots,\lambda_k\cdot\Delta$, respectively, such that $Z_{\lambda_1},\cdots,Z_{\lambda_k}$ define* *a complete intersection* $Y_{\lambda_1,\dots,\lambda_k}$ *in* X_Δ. *Then* $b_*(Y_{\lambda_1,\dots,\lambda_k};\mathbb{Z}_2)$ *is equivalent to* $\mathrm{Vol}(\lambda_1\cdot\Delta,\cdots,\lambda_k\cdot\Delta)$ *when* λ_i *tends to infinity for all* i.

We also use the following result of Khovanskii on the Euler characteristic of a complete intersection in the torus $(\mathbb{C}^*)^d$ (see [Hov78]).

Theorem 2.17 (A. Khovanskii)
Let Y *be a complete intersection in* $(\mathbb{C}^*)^d$ *defined by polynomials* $P_1,\cdots,P_k$ *with Newton polytopes* $\Delta_1,\cdots,\Delta_k$, *respectively. Then, the Euler characteristic of* Y *is the homogeneous term of degree* d *of*

$$\Delta_1(1+\Delta_1)^{-1}\cdot\ \cdots\ \cdot\Delta_k(1+\Delta_k)^{-1},$$

where the product of d *polytopes stands for their mixed volume and* $(1+\Delta_i)^{-1}$ *stands for the series* $\sum_{j=0}^{\infty}(-1)^j(\Delta_i)^j$.

In the case of two 3-dimensional polytopes we use the following direct consequence of Theorem 2.17.

Corollary 2.18
Let Δ *be a simple 3-dimensional polytope and* λ_1 *and* λ_2 *be positive integers. For* $i=1,2$ *put* $\Delta_i=\lambda_i\cdot\Delta$. *Let* Y *be a complete intersection in* X_Δ *defined by polynomials* P_1 *and* P_2 *with Newton polytopes* Δ_1 *and* Δ_2, *respectively. Then,* $b_*(Y;\mathbb{C})=(\lambda_1^2\lambda_2+\lambda_2^2\lambda_1)\,\mathrm{Vol}(\Delta)-\sum_{\Gamma\in\mathcal{F}_2(\Delta)}\lambda_1\lambda_2\,\mathrm{Vol}(\Gamma)+4$.

Proof . - By theorem 2.17, the Euler characteristic $\chi(Y)$ of Y is given by $\chi(Y)=-(\lambda_1^2\lambda_2+\lambda_2^2\lambda_1)\,\mathrm{Vol}(\Delta)+\sum_{\Gamma\in\mathcal{F}_2(\Delta)}\lambda_1\lambda_2\,\mathrm{Vol}(\Gamma)$. Since $b_*(Y;\mathbb{C})=-\chi(Y)+4$, we have the desired result. $\square$

Chapter 3

Primitive T-hypersurfaces

3.1 Statement

Let Δ be a d-dimensional polytope such that the toric variety X_Δ associated with P is nonsingular. Let Z be a primitive T-hypersurface with Newton polytope Δ, and τ be the primitive triangulation used in the construction of Z. Let $\iota : H_{d-1}(Z;\mathbb{Z})/tors \times H_{d-1}(Z;\mathbb{Z})/tors \to \mathbb{Z}$ be the intersection form of Z. If d is odd, then ι is a symmetric bilinear form, and its signature is called the signature of Z and is denoted $\sigma(Z)$. One can express $\sigma(Z)$ in terms of Hodge numbers of Z (see, for example, [GH78]):

$$\sigma(Z) = \sum_{p+q=0\ [2]} (-1)^p h^{p,q}(Z).$$

If d is even, denote by $\sigma(Z)$ the number defined by the same formula. We have the following statement.

Theorem 3.1
If Z is a primitive real algebraic T-hypersurface in a nonsingular toric variety X_Δ, then

$$\chi(\mathbb{R}Z) = \sigma(Z),$$

where $\chi(\mathbb{R}Z)$ is the Euler characteristic of the real part $\mathbb{R}Z$ of Z.

Recall that $l^*(\Delta)$ is the number of integer points in the interior of Δ.

Corollary 3.2
Assume that Δ is a 3-dimensional polytope, X_Δ is nonsingular, and Z is a primitive T-surface in X_Δ with Newton polytope Δ. If the number $b_0(\mathbb{R}Z)$ of connected components of $\mathbb{R}Z$ is at least $l^*(\Delta)+1$, then $b_0(\mathbb{R}Z) = l^*(\Delta)+1$ and Z is maximal.

Proof . - First, note that $l^*(\Delta) = h^{2,0}(Z)$. Then $\chi(\mathbb{R}Z) \geq 2h^{2,0}(Z) + 2 - b_1(\mathbb{R}Z)$. Now using the equalities $\chi(\mathbb{R}Z) = \sigma(Z) = 2h^{2,0}(Z) + 2 - h^{1,1}(Z)$ one gets $b_1(\mathbb{R}Z) \geq h^{1,1}(Z)$. Furthermore $h^{1,0}(Z) = h^{0,1}(Z) = 0$ (see Theorem 2.14), and thus $b_*(Z;\mathbb{C}) = 2h^{2,0}(Z) + 2 + h^{1,1}(Z)$. Hence $b_*(\mathbb{R}Z;\mathbb{Z}_2) \geq b_*(Z;\mathbb{C}) = b_*(Z;\mathbb{Z}_2)$. The Smith-Thom inequality implies that $b_*(\mathbb{R}Z;\mathbb{Z}_2) = b_*(Z;\mathbb{Z}_2)$, and thus, that $b_1(\mathbb{R}Z) = h^{1,1}(Z)$ and $b_0(\mathbb{R}Z) = h^{2,0}(Z) + 1$. □

3.2 Proof of Theorem 3.1

If d is even, the proof of 3.1 is straightforward. Indeed, in this case Z is a (smooth) odd dimensional hypersurface, so $\chi(\mathbb{R}Z) = 0$. On the other hand, we have the equality $h^{p,q}(Z) = h^{d-1-p,d-1-q}(Z)$ for any p and q, and $d-1$ is odd. Thus, $\sigma(Z) = \sum_{p+q=0\,[2]} (-1)^p h^{p,q}(Z) = 0$.

Assume now that d is odd and denote by $\mathcal{F}_i(\Delta)$ the set of i-dimensional faces of Δ. The number of i-dimensional faces of Δ is denoted by $f_i(\Delta)$.

Lemma 7 *We have*

$$\chi(\mathbb{R}Z) = \sum_{i=1}^{d} \sum_{F\in\mathcal{F}_i(\Delta)} \sum_{l=2}^{i+1} \chi_{l,i+1} a^F_{l-1} \quad \textit{and} \quad \sigma(Z) = \sum_{i=1}^{d} \sum_{F\in\mathcal{F}_i(\Delta)} \sum_{l=2}^{i+1} \sigma_{l,i+1} a^F_{l-1},$$

where a^F_l is the coefficient of the degree l term of the Ehrhart polynomial of F (see Section 2.4.1) and

$$\begin{aligned} \chi_{l,i+1} &= (-1)^{i-l+1} \sum_{j=0}^{i-1} \frac{(2^i - 2^j)}{i-j+1} \sum_{k=0}^{i-j+1} (-1)^k \mathrm{C}^k_{i-j+1} k^l, \\ \sigma_{l,i+1} &= \sum_{p=0}^{d-1} (-1)^i (-1)^{p+1} \sum_{q=0}^{p+1} (-1)^q \mathrm{C}^q_{i+1} (p+1-q)^{l-1}. \end{aligned}$$

Proof . - The triangulation τ^* induces a cell decomposition $\mathcal{D}$ of $\widetilde{H}$. Let $\widetilde{I_F}$ be the image in $\widetilde{\Delta}$ of the union of the symmetric copies of the interior of a face F and $\mathcal{D}(\widetilde{I_F})$ be the set of cells of $\mathcal{D}$ contained in $\widetilde{I_F}$. Put $\chi_F = \sum_{\delta\in\mathcal{D}(\widetilde{I_F})}(-1)^{dim(\delta)}$. Then $\chi(\mathbb{R}Z) = \sum_{i=1}^{d}\sum_{F\in\mathcal{F}_i(\Delta)}\chi_F$. According to Proposition 2.8, if Q is a k-simplex of τ contained in j coordinate hyperplanes then the union of the symmetric copies of Q contains exactly $(2^d-2^{d-k})/2^j$ cells of dimension $k-1$. An i-face F of Δ^* contained in j coordinate hyperplanes is identified with $2^{d-i-j}-1$ other copies of F when passing from Δ^* to $\widetilde{\Delta}$. Thus, the number of $(k-1)$-cells in $\mathcal{D}(\widetilde{I_F})$ is equal to $\frac{(2^d-2^{d-k})}{2^{d-i}}nbs_k^F$, where nbs_k^F is the number of k-simplices in the interior of F. According to Proposition 2.12 of section 2.4.2

$$nbs_k^F = \sum_{l=k+1}^{i+1} k!S_2(l,k+1)(-1)^{i-l+1}a_{l-1}^F,$$

where S_2 is the Stirling number of the second kind defined by $S_2(i,j) = 1/j!\sum_{k=0}^{j}(-1)^{j-k}\mathrm{C}_j^k k^i$. It finishes the proof of the formula for $\chi(\mathbb{R}Z)$.

To compute $\sigma(Z)$, we use the Danilov and Khovanskii formulae (see Theorem 2.14). One obtains the following expression for $\sigma(Z)$:

$$\sigma(Z) = \sum_{p=0}^{d-1}\sum_{i=p+1}^{d}(-1)^i\sum_{F\in\mathcal{F}_i(\Delta)}\left(-\mathrm{C}_i^{p+1}+(-1)^{p+1}\sum_{l=1}^{i+1}\sum_{q=0}^{p+1}(-1)^q\mathrm{C}_{i+1}^q(p+1-q)^{l-1}a_{l-1}^F\right).$$

Consider $\sigma(Z)$ as a polynomial in the variables a_l^F. Denote by σ^{cst} the constant term of $\sigma(Z)$, and by σ^1 the sum of monomials in variables a_0^F. We have

$$\sigma^{cst} = -\sum_{p=0}^{d-1}\sum_{i=p+1}^{d}(-1)^i\mathrm{C}_i^{p+1}f_i(P)$$

and

$$\sigma^1 = \sum_{p=0}^{d-1}\sum_{i=p+1}^{d}(-1)^i(-1)^{p+1}\sum_{F\in\mathcal{F}_i(P)}\sum_{a=0}^{p+1}(-1)^a\mathrm{C}_{i+1}^a a_0^F.$$

Lemma 8 *We have* $\sigma^{cst}+\sigma^1 = 0$.

Proof . - Since $a_F^0 = 1$, we obtain

$$\begin{aligned}\sigma^1 &= \sum_{p=0}^{d-1}\sum_{i=p+1}^{d}(-1)^i(-1)^{p+1}\sum_{F\in\mathcal{F}_i(P)}\sum_{a=0}^{p+1}(-1)^a \mathrm{C}_{i+1}^a\\ &= \sum_{p=0}^{d-1}\sum_{i=p+1}^{d}(-1)^i(-1)^{p+1}f_i(P)\sum_{a=0}^{p+1}(-1)^a \mathrm{C}_{i+1}^a.\end{aligned}$$

Furthermore, $\sum_{a=0}^{p+1}(-1)^a\mathrm{C}_{i+1}^a = (-1)^{p+1}\mathrm{C}_i^{p+1}$, and thus,

$$\sigma^1 = \sum_{p=0}^{d-1}\sum_{i=p+1}^{d}(-1)^i\mathrm{C}_i^{p+1}f_i(P).$$

□

Lemma 9 *Let l and i be nonnegative integers. Then, for $l+1 \le i$, one has $\sum_{q=0}^{i}(-1)^q\mathrm{C}_i^q(i-q)^l = 0$ (cf. [VLW92] p. 71) and, as a consequence,*

$$\sum_{q=0}^{i}(-1)^q\mathrm{C}_i^q(p-q)^l = 0.$$

Using Lemma 9 one shows that

$$\sigma_{l,i+1} = \sum_{p=0}^{i-1}(-1)^i(-1)^{p+1}\sum_{q=0}^{p+1}(-1)^q\mathrm{C}_{i+1}^q(p+1-q)^{l-1}.$$

Then we can prove the following lemma.

Lemma 10 *For $2 \le l \le i+1$, one has the following equalities*

$$\begin{aligned}\sigma_{l,i+2} &= -2\sigma_{l,i+1},\\ \chi_{l,i+2} &= -2\chi_{l,i+1}.\end{aligned}$$

Proof . - To prove the first equality, write $\mathrm{C}_{i+2}^q = \mathrm{C}_{i+1}^{q-1} + \mathrm{C}_{i+1}^q$ to get

$$\sigma_{l,i+2} = (-1)^{i+1}\sum_{p=0}^{i}(-1)^{p+1}\sum_{q=1}^{p+1}(-1)^q\mathrm{C}_{i+1}^{q-1}(p+1-q)^{l-1}$$

$$+(-1)^{i+1}\sum_{p=0}^{i}(-1)^{p+1}\sum_{q=0}^{p+1}(-1)^q\mathrm{C}_{i+1}^q(p+1-q)^{l-1}.$$

Notice that $\sum_{q=1}^{1}(-1)^q \mathrm{C}_{i+1}^{q-1}(p+1-q)^{l-1}$ is 0. Similarly, in the second term, $\sum_{q=0}^{i+1}(-1)^q \mathrm{C}_{i+1}^{q}(i+1-q)^{l-1}$ makes no contribution, because

$$\sum_{q=0}^{i+1}(-1)^q \mathrm{C}_{i+1}^{q}(i+1-q)^{l-1} = 0$$

for $l \leq i+1$ by Lemma 9.

Then,

$$\sigma_{l,i+2} = (-1)^{i+1}\sum_{p=1}^{i}(-1)^{p+1}\sum_{q=1}^{p+1}(-1)^q \mathrm{C}_{i+1}^{q-1}(p+1-q)^{l-1}$$
$$+(-1)^{i+1}\sum_{p=0}^{i-1}(-1)^{p+1}\sum_{q=0}^{p+1}(-1)^q \mathrm{C}_{i+1}^{q}(p+1-q)^{l-1}.$$

So, with the changes of indices $c = q-1$ and $r = p-1$ one gets

$$\begin{aligned}
\sigma_{l,i+2} &= (-1)^{i+1}\sum_{r=0}^{i-1}(-1)^{r}\sum_{c=0}^{r+1}(-1)^{c+1} \mathrm{C}_{i+1}^{c}(r+1-c)^{l-1} \\
&\quad +(-1)^{i+1}\sum_{p=0}^{i-1}(-1)^{p+1}\sum_{q=0}^{p+1}(-1)^q \mathrm{C}_{i+1}^{q}(p+1-q)^{l-1} \\
&= -2\sigma_{l,i+1}.
\end{aligned}$$

To prove the second equality stated in the lemma, use Lemma 9 to get rid of the term with $j=0$ in the sum $\sum_{j=0}^{i-1}\frac{(2^i-2^j)}{i-j+1}\sum_{k=0}^{i-j+1}(-1)^k \mathrm{C}_{i-j+1}^{k} k^l$:

$$\begin{aligned}
\chi_{l,i+2} &= (-1)^{i-l+2}\sum_{j=0}^{i}\frac{(2^{i+1}-2^j)}{i-j+2}\sum_{k=0}^{i-j+2}(-1)^k \mathrm{C}_{i-j+2}^{k} k^l \\
&= (-1)^{i-l}\frac{(2^{i+1}-1)}{i+2}\sum_{k=0}^{i+2}(-1)^k \mathrm{C}_{i+2}^{k} k^l \\
&\quad +(-1)^{i-l}\sum_{j=1}^{i}\frac{(2^{i+1}-2^j)}{i-j+2}\sum_{k=0}^{i-j+2}(-1)^k \mathrm{C}_{i-j+2}^{k} k^l \\
&= (-1)^{i-l}\sum_{j=1}^{i}\frac{(2^{i+1}-2^j)}{i-j+2}\sum_{k=0}^{i-j+2}(-1)^k \mathrm{C}_{i-j+2}^{k} k^l.
\end{aligned}$$

With the change of index $m = j - 1$ we get

$$
\begin{aligned}
\chi_{l,i+2} &= (-1)^{i-l} \sum_{m=0}^{i-1} \frac{(2^{i+1} - 2^{m+1})}{i - m + 1} \sum_{k=0}^{i-m+1} (-1)^k \mathrm{C}_{i-m+1}^k k^l \\
&= -2\chi_{l,i+1}.
\end{aligned}
$$

□

Remark 5 *One has* $\sum_{n=0}^{p} 2^{p-n} \mathrm{C}_n^k = \sum_{l=k+1}^{p+1} \mathrm{C}_{p+1}^l$.

Proof . - We use the fact that $\sum_{n=k}^{p} \mathrm{C}_n^k = \mathrm{C}_{p+1}^{k+1}$ and write

$$
\begin{aligned}
\sum_{n=0}^{p} 2^{p-n} \mathrm{C}_n^k &= \sum_{n=k}^{p} 2^{p-n} \mathrm{C}_n^k \\
&= \sum_{n=k}^{p} \mathrm{C}_n^k + \sum_{i=0}^{p} (2^i) \sum_{n=k}^{p-i} \mathrm{C}_n^k \\
&= \mathrm{C}_{p+1}^{k+1} + \sum_{i=0}^{p-k} (2^i) \sum_{i=0}^{p-k} \mathrm{C}_{p+1-i}^{k+1} \\
&= \mathrm{C}_{p+1}^{k+1} + \sum_{n=k+1}^{p+1} 2^{p+1-n} \mathrm{C}_n^{k+1},
\end{aligned}
$$

and the result follows by induction. □

Lemma 11 *For* $2 \leq l \leq d + 1$*, we have* $\chi_{l,l} = \sigma_{l,l}$.

Proof . - By Lemma 10 it is enough to prove the equality $\chi_{l,l} = \sigma_{l,l}$. Using the fact that $\sum_{n=0}^{p} 2^{p-n} \mathrm{C}_n^k = \sum_{l=k+1}^{p+1} \mathrm{C}_{p+1}^l$, write

$$
\chi_{l,l} = 2^{l-1} \sum_{k=1}^{l} (-1)^k \mathrm{C}_l^k k^{l-1} - \sum_{k=1}^{l} (-1)^k \sum_{i=k}^{l} \mathrm{C}_l^i k^{l-1}
$$

and

$$
\sigma_{l,l} = (-1)^{l-1} \sum_{b=1}^{l} \sum_{j=b}^{l} (-1)^b \mathrm{C}_l^j b^{l-1}.
$$

By Lemma 9, $\chi_{l,l} = -\sum_{k=1}^{l}(-1)^k \sum_{i=k}^{l} \mathrm{C}_l^i k^{l-1}$ and $\chi_{l,l} = (-1)^l \sigma_{l,l}$. This is the desired equality for l even. For an odd $l > 2$ we use the symmetry of the expression of $\sigma_{l,l}$ and write

$$\sigma_{l,l} = \sum_{p=0}^{\frac{l-3}{2}}((-1)^{p+1}\sum_{q=0}^{p+1}(-1)^q \mathrm{C}_l^q (p+1-q)^{l-1}$$
$$+(-1)^{l-1-p}\sum_{q=0}^{l-1-p}(-1)^q \mathrm{C}_l^q (l-1-p-q)^{l-1}).$$

Thus, $\sigma_{l,l} = \sum_{p=0}^{\frac{l-3}{2}}(-1)^{p+1}\sum_{q=0}^{l}(-1)^q \mathrm{C}_l^q (p+1-q)^{l-1}$. The right hand side of the last equality is zero by Lemma 9, and thus, $\sigma_{l,l} = \chi_{l,l} = 0$. $\square$

According to Lemma 11 the coefficients of $\sigma(Z)$ and $\chi(\mathbb{R}Z)$ in the expressions of Lemma 7 are equal, and thus, $\sigma(Z) = \chi(\mathbb{R}Z)$. $\square$

Chapter 4

Nakajima polytopes and M-surfaces

In this chapter we give examples of families of M-surfaces obtained by T-construction as hypersurfaces of 3-dimensional toric varieties. In section 4.3 we show that, in dimension $d \geq 3$, there exist polytopes Δ such that no hypersurface in X_Δ with Newton polytope Δ is maximal.

4.1 Nakajima polytopes

Definition 4.1.1 *A polytope P in $\mathbb{R}^d$ is a* ***Nakajima polytope*** *if either P is 0-dimensional or there exists a Nakajima polytope $\bar{P}$ in $\mathbb{R}^{d-1}$ and a linear function $f : \mathbb{R}^{d-1} \to \mathbb{R}$, nonnegative on $\bar{P}$, such that $f(\mathbb{Z}^{d-1}) \subset \mathbb{Z}$ and $P = \{(x, x_d) \in \bar{P} \times \mathbb{R} \mid 0 \leq x_d \leq f(x)\}$.*

Definition 4.1.2 *A polytope P in $\mathbb{R}^d$ is a* ***nondegenerate Nakajima polytope*** *if either P is 0-dimensional or there exists a nondegenerate Nakajima polytope $\bar{P}$ in $\mathbb{R}^{d-1}$ and a linear function $f : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $f(\mathbb{Z}^{d-1}) \subset \mathbb{Z}$, f is positive on $\bar{P}$, and $P = \{(x, x_d) \in \bar{P} \times \mathbb{R} \mid 0 \leq x_d \leq f(x)\}$.*

4.2 Construction of M-surfaces

Theorem 4.1
Let P be a 3-dimensional Nakajima polytope corresponding to a nonsingular toric variety X_P. Then, there exists a maximal surface in X_P with the

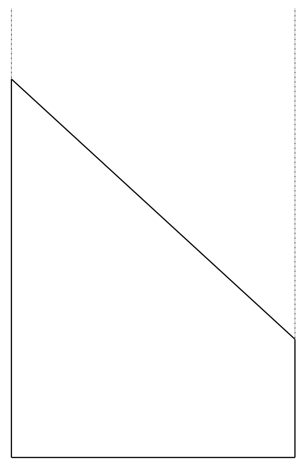

Figure 4.1: A 2-dimensional Nakajima polytope.

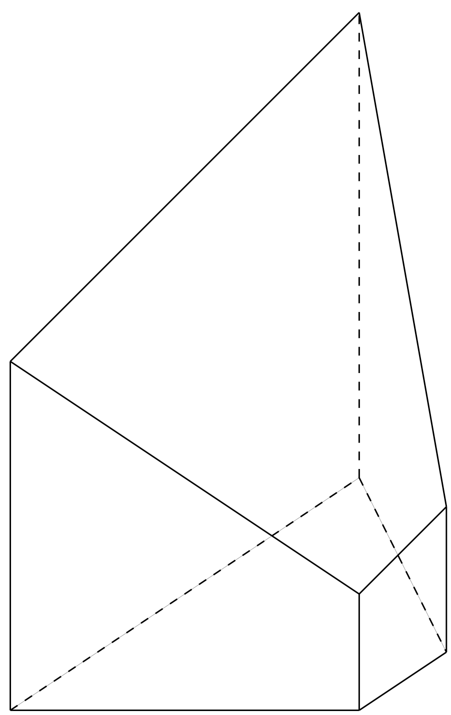

Figure 4.2: A 3-dimensional Nakajima polytope.

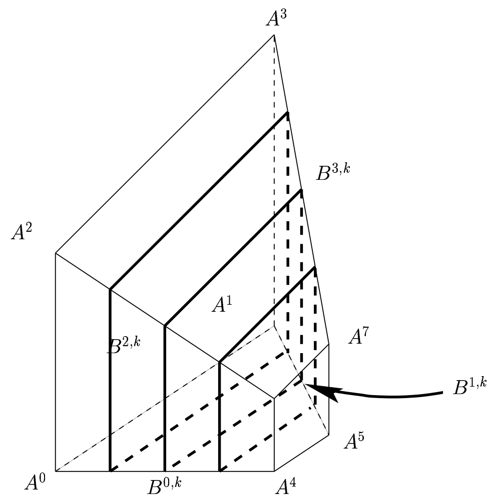

Figure 4.3: Slicing a nondegenerate Nakajima polytope.

Newton polytope P.

Proof . - Let us first assume that P is nondegenerate.

Decompose P into slices $T_k = P \cap \{(x,y,z) \in (\mathbb{R}^+)^3, k-1 \leq y \leq k\}$. Let s_k be the section $P \cap \{(x,y,z) \in (\mathbb{R}^+)^3, y = k\}$. Denote by $B^{0,k}$, $B^{1,k}$, $B^{2,k}$, and $B^{3,k}$ the vertices of s_k (see Figure 4.3). We subdivide the slices T_k in two cones and two joins. Take the cones of apex $B^{0,2k}$ and $B^{0,2k+2}$ on s_{2k+1} and the cones of apex $B^{3,2k-1}$ and $B^{3,2k+1}$ on s_{2k}. Take primitive triangulations of the sections s_k. They induce a primitive triangulation of the cones. The joins $[B^{0,2k}, B^{1,2k}] * [B^{1,2k+1}, B^{3,2k+1}]$, $[B^{0,2k}, B^{2,2k}] * [B^{2,2k+1}, B^{3,2k+1}]$, $[B^{0,2k+2}, B^{1,2k+2}] * [B^{1,2k+1}, B^{3,2k+1}]$, $[B^{0,2k+2}, B^{2,2k+2}] * [B^{2,2k+1}, B^{3,2k+1}]$ are also naturally primitively triangulated. Take the following distribution of signs at the integer points of P:

any integer point (i,j,k) gets "-" if j and k are both odd, and it gets "+", otherwise.

The polytope P is now equipped with a convex primitive triangulation and a sign distribution, so we can apply the T-construction as in chapter 2 section 2.2.

Let p be an integer interior point of P and $Star(p)$ be its star. Assume that p belongs to a section s_l. Then, $Star(p)$ has two vertices c_1 and c_2 outside s_l. They are the apices of the two cones over s_l. Since they have the same parity, with the chosen distribution of signs, in each octant their symmetric copies carry the same sign. Consider an octant where the symmetric copy q_0 of p is isolated in s_l (i.e. all vertices of $Star(q_0) \cap s_l$ except q_0 carry the sign opposite to the sign of q_0. Then, either in this octant the symmetric copies of c_1 and c_2 carry the sign opposite to the sign of q_0 (and, hence, q_0 is isolated), or $r(q_0)$ is isolated, where r is the reflection with respect to the coordinate plane $x = 0$. Thus, for each integer interior point p of P, there exists a symmetric copy q of p such that q is surrounded by a sphere $S^2(p) = Star(q) \cap H^*$. Moreover, one can check that at least one component of H^* intersects the coordinate planes. Thus, the T-surface constructed has at least $l^*(P) + 1$ connected components, and Corollary 3.2 shows that this surface is maximal.

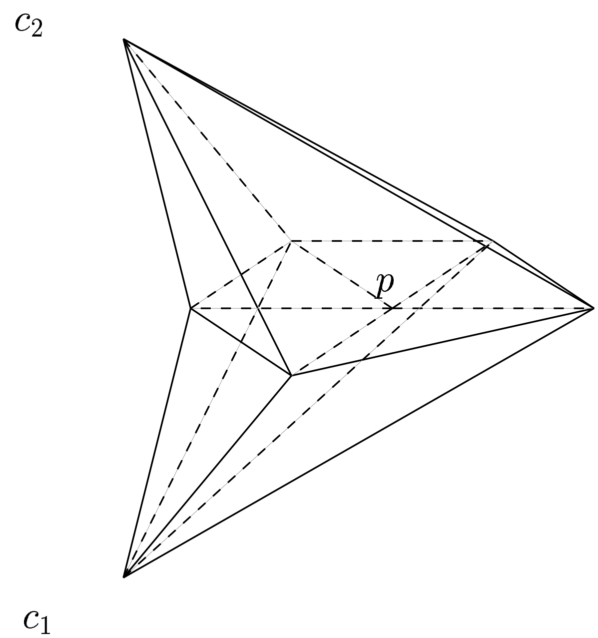

Figure 4.4: The star of p.

The degenerate case splits into two subcases. Either the Nakajima polytope is a truncated cylinder over a triangle corresponding to the projective plane, or it is a tetrahedron corresponding to the projective 3-space. The existence of M-surfaces in the latter case is contained in Itenberg-Viro's theorem. In the former case the Nakajima polytope P is the convex hull of the triangles $((0,0,0),(m,0,0),(0,m,0))$ and $((0,0,l),(m,0,l+me),(0,m,l+mf))$ for some integers m, l, e and f (see Figure 4.5).

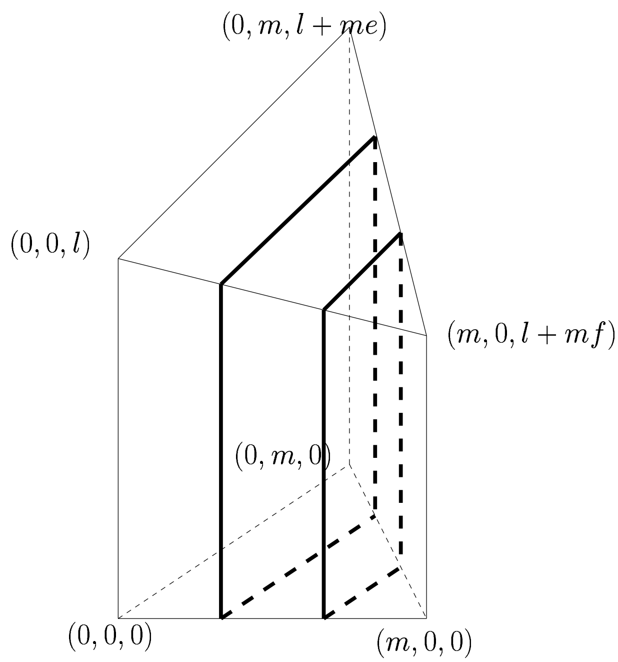

Figure 4.5: A degenerate Nakajima polytope.

Decompose P into slices $T_k = P \cap \{(x,y,z) \in (\mathbb{R}^+)^3, k-1 \leq y \leq k\}$. Let s_k be the section $P \cap \{(x,y,z) \in (\mathbb{R}^+)^3, y = k\}$. We triangulate $P \cap \{(x,y,z) \in (\mathbb{R}^+)^3, y \leq m-1\}$ using the same triangulation that in the nondegenerate case. Take the cone over s_{m-1} of apex $(m,0,0)$ if m is even or $(m,0,l+mf)$, otherwise. Take in the cone the triangulation induced by the triangulation of s_{m-1}. Subdivide the only one remaining non-primitive tetrahedron into primitive ones in the unique possible way. Take the same distribution of signs that in the nondegenerate case.

Then, as in the nondegenerate case, any interior point has an isolated symmetric copy. There is also a component intersecting the coordinate planes. Thus, the surface is maximal. $\square$

Note that Theorem 4.1 produces, in particular, M-surfaces in $\Sigma_\alpha \times \mathbb{C}P^1$.

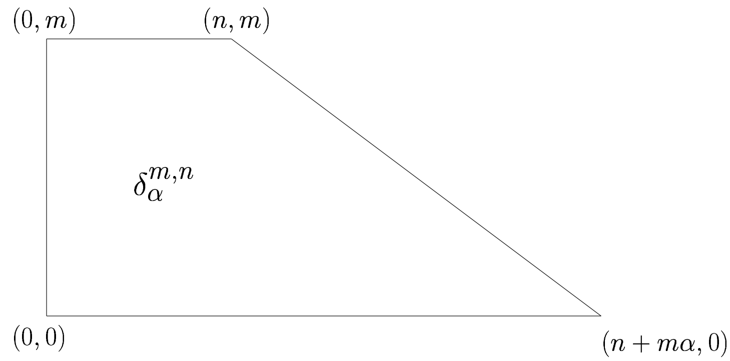

Figure 4.6: $\delta_\alpha^{m,n}$.

For a non-negative integer α and positive integers m and n denote by $\delta_\alpha^{m,n}$ the polygon having the vertices $(0, 0)$, $(n + m\alpha, 0)$, $(0, m)$, and (n, m) in $\mathbb{R}^2$. The toric variety associated with $\delta_\alpha^{m,n}$ is a rational ruled surface Σ_α. Consider now the truncated cylinder $P_\alpha^{l,m,n}$ of base $\delta_\alpha^{m,n}$ whose vertices are

$$(0, 0, 0), (n + m\alpha, 0, 0), (0, m, 0), (n, m, 0),$$
$$(0, 0, l), (n + m\alpha, 0, l), (0, m, l), \text{ and } (n, m, l),$$

where l is a positive integer. The toric variety $X_{P_\alpha^{l,m,n}}$ associated with $P_\alpha^{l,m,n}$ is $\Sigma_\alpha \times \mathbb{C}P^1$. Since $P_\alpha^{l,m,n}$ is a Nakajima polytope, the following statement is a corollary of Theorem 4.1.

Corollary 4.2
For any non-negative integer α and any positive integers m, n, and l, there exists a maximal surface in $\Sigma_\alpha \times \mathbb{C}P^1$ with the Newton polytope $P_\alpha^{l,m,n}$. $\square$

4.3 Newton polytopes without maximal hypersurfaces

In this section we show that Itenberg-Viro's theorem of existence of M-hypersurfaces of any degree in the projective spaces of any dimension cannot be generalized straightforwardly to all projective toric varieties. More precisely, we show that there exist polytopes Δ such that no hypersurface in X_Δ

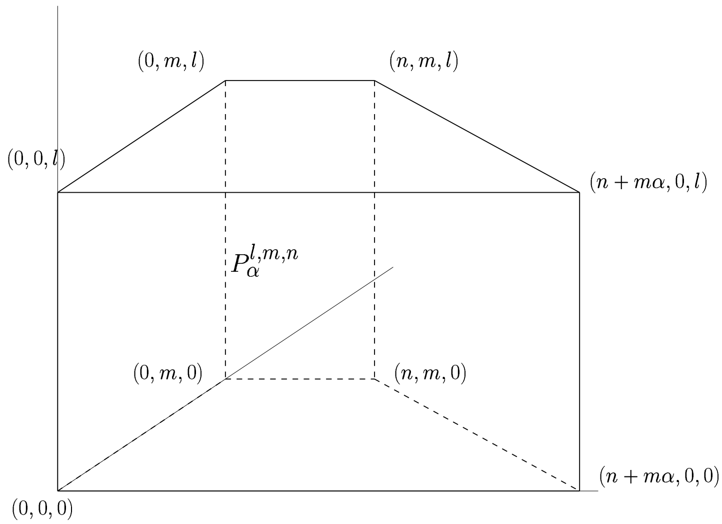

Figure 4.7: $P_\alpha^{l,m,n}$.

with Newton polytope Δ is maximal. Note that this does not mean that the toric variety X_Δ does not admit M-hypersurfaces.

Clearly, if Δ is an interval $[a, b]$ in $\mathbb{R}$, where a and b are nonnegative integers, then there exists a maximal 0-dimensional subvariety in $\mathbb{C}P^1 = X_\Delta$ with the Newton polygon Δ.

If Δ is a polygon in the first quadrant of $\mathbb{R}^2$, then again there exists a maximal curve in X_Δ with the Newton polygon Δ. Such a curve can be constructed by the combinatorial patchworking: it suffices to take as initial data a primitive convex triangulation of Δ equipped with the following distribution of signs: an integer point (i, j) of Δ gets the sign "-" if i and j are both even, and gets the sign "+", otherwise (cf., for example, [Ite95], [IV96], [Haa98]).

However, in dimension 3 there are polytopes Δ such that no hypersurface in X_Δ with the Newton polytope Δ is maximal.

Let k be a positive integer number, and Δ_k be the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0), (1,0,0), (0,1,0)$, and $(1,1,k)$. Note that the only integer points of Δ_k are its vertices.

Proposition 4.3 *For any odd $k \geq 3$ and any even $k \geq 8$, there is no maximal surface in X_{Δ_k} with the Newton polytope Δ_k.*

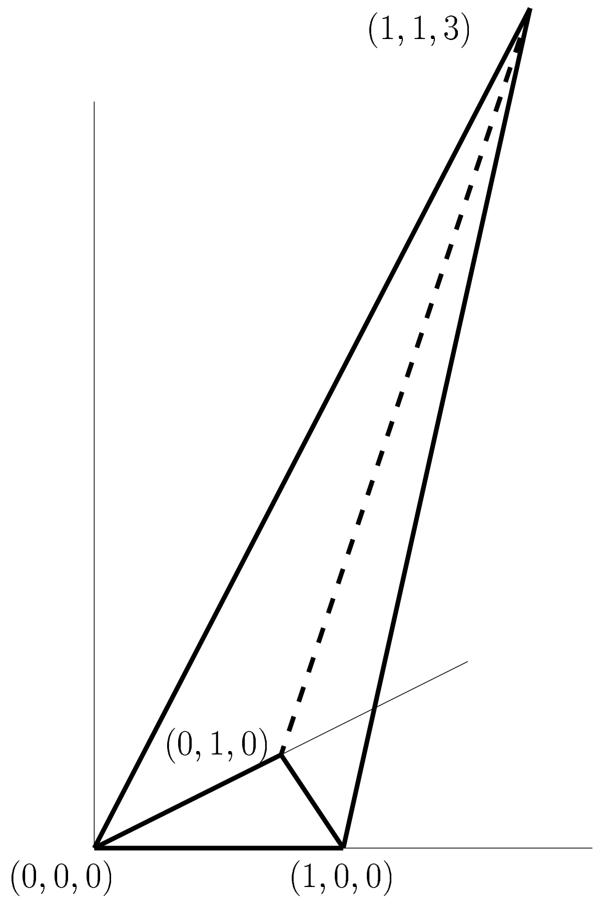

Figure 4.8: Tetrahedron Δ_3.

Proof . - The proof relies on the estimation of the Betti numbers of the complex and real parts of a real algebraic surface Z_k in X_{Δ_k} with Newton polytope Δ_k. The Betti numbers $b_*(Z_k;\mathbb{C})$ are given by lemma 6. We have $b_*(Z_k;\mathbb{C}) = l^*(2\Delta_k) - 2l^*(\Delta_k) - \sum_{\Gamma\in\mathcal{F}_2(\Delta_k)}(l^*(\Gamma)-1)-1$. Since $l^*(2\Delta_k) = k-1$ and $l^*(\Delta_k) = 0$, we get $b_*(Z_k;\mathbb{C}) = k+2$. Thus, $b_*(Z_k;\mathbb{Z}_2) \geq k+2$.

To estimate $b_*(\mathbb{R}Z_k;\mathbb{Z}_2)$ we consider two cases. If k is odd, Δ_k is an elementary tetrahedron, and $\mathbb{R}Z_k$ is homeomorphic to the projective plane. Thus, in this case, $b_*(\mathbb{R}Z_k;\mathbb{Z}_2) = 3$. If k is even, either Δ_k has 8 nonempty symmetric copies, or it has 6 nonempty symmetric copies. Furthermore, the boundary of any symmetric copy of Δ_k is identified with the boundary of another symmetric copy of Δ_k. Hence, we get either three or four connected components of $\mathbb{R}Z_k$ homeomorphic to a sphere, and $b_*(\mathbb{R}Z_k;\mathbb{Z}_2) \leq 8$.

Thus, for k greater or equal to 7, there is no maximal surface in X_{Δ_k} with the Newton polytope Δ_k. □

It is easy to generalize the above examples in dimension 3 to higher dimensions.

Proposition 4.4 *For any integer $d \geq 3$ there exist d-dimensional polytopes Δ such that no hypersurface in X_Δ with the Newton polytope Δ is maximal.*

Proof . - Fix an integer $d \geq 3$ and consider a family $\{\sigma_k\}_{k\in\mathbb{N}}$ of d-dimensional simplices in $\mathbb{R}^d$ such that their vertices are their only integer points and $\mathrm{Vol}(\sigma_k) = k$. For example, one can take for σ_k the simplex in $\mathbb{R}^d$ with vertices

$$(0,0,\dots,0,0),\ (1,0,\dots,0,0),\ (0,1,\dots,0,0),\ \dots,(0,0,\dots,1,0),$$
$$\text{and } (1,1,\dots,1,k).$$

Let Z_k be any hypersurface in X_{σ_k}. By Proposition 2.15 $b_*(Z_k;\mathbb{C})$ tends to infinity when k does, and so does $b_*(Z_k;\mathbb{Z}_2)$. Meanwhile, $b_*(\mathbb{R}Z_k;\mathbb{Z}_2)$ is bounded (for example, by the number of simplices in σ_k^*). So there exists a number k_0 such that for any integer $k > k_0$ and any hypersurface Z_k in X_{σ_k} one has $b_*(\mathbb{R}Z_k;\mathbb{Z}_2) < b_*(Z_k;\mathbb{Z}_2)$. □

Chapter 5

Asymptotically Maximal Families of Hypersurfaces

The question "does a given family of real algebraic varieties contain maximal elements?" is one of the problems in topology of real algebraic varieties. For the family of the hypersurfaces of a given degree in $\mathbb{R}P^d$ a positive answer is obtained in [IV02] using the T-construction. Since this question is, in general, a difficult problem, it is natural to tackle the following weaker question. Let Δ be a polytope d-dimensional polytope in $(\mathbb{R}^+)^d$, X_Δ the toric variety associated with Δ, and $\{\lambda \cdot \Delta\}_{\lambda \in \mathbb{N}}$ the family of the multiples of Δ. Suppose that there exists a collection of polynomials $\{P_\lambda\}_{\lambda \in \mathbb{N}}$ satisfying the following conditions :

1. the polytope $\lambda \cdot \Delta$ is the Newton polytope of P_λ,
2. let Z_λ be the hypersurface in X_Δ defined by P_λ; the total Betti numbers $b_*(\mathbb{R}Z_\lambda; \mathbb{Z}_2)$ and $b_*(Z_\lambda; \mathbb{Z}_2)$ are equivalent when λ tends to infinity.

In this case we say that the family $\{Z_\lambda\}_{\lambda \in \mathbb{N}}$ is ***asymptotically maximal***. Given a d-dimensional polytope Δ in $(\mathbb{R}^+)^d$, does there exist an asymptotically maximal family of hypersurfaces in X_Δ? In this chapter we give a positive answer to this question. From now on by polytope we mean a polytope in $(\mathbb{R}^+)^d$.

Theorem 5.1
For any polytope Δ there exists an asymptotically maximal family of hypersurfaces $\{Z_\lambda\}_{\lambda \in \mathbb{N}}$ in X_Δ such that for any λ the Newton polytope of Z_λ is $\lambda \cdot \Delta$.

5.1 Auxiliary statements

The proof of the existence of asymptotically maximal families is based on two important results.

In [IV02] I. Itenberg and O. Viro, using the T-construction, proved that there exist M-hypersurfaces of any degree in the projective space of any dimension.

Theorem 5.2 (I. Itenberg and O. Viro)
Let d and m be natural numbers, and T_1^d be a primitive d-dimensional simplex. Put $T_m^d = m \cdot T_1^d$. Then, there exists a primitive convex triangulation $\tau_{T_m^d}$ of T_m^d and a sign distribution $D(\tau_{T_m^d})$ at the vertices of $\tau_{T_m^d}$ such that the T-hypersurface Z_d^m obtained via the combinatorial patchworking from $\tau_{T_m^d}$ and $D(\tau_{T_m^d})$ is maximal.

In fact, we use only an asymptotical version of 5.2. The proof of this asymptotical version is much simpler than the proof of 5.2. It can be extracted from [IV02] and was communicated to us by the authors. The asymptotical version of 5.2 is reproduced in Section 5.2 below.

The second important result we use is due to F. Knudsen and D. Mumford [KKMSD67].

Theorem 5.3 (F. Knudsen and D. Mumford)
Let Δ be a polytope. There exists a positive integer l such that $l \cdot \Delta$ admits a convex primitive triangulation.

In the sequel, when there is no ambiguity on the triangulation of a polytope Δ and the sign distribution chosen, we denote by H_Δ the piecewise-linear hypersurface in Δ obtained by T-construction, H_Δ^* the piecewise-linear hypersurface in Δ_*, $\widetilde{H_\Delta}$ its image in $\widetilde{\Delta}$, and Z_Δ the corresponding hypersurface in X_Δ.

5.2 Itenberg-Viro asymptotical construction

Theorem 5.4
For any positive integers m and d such that $m \geq d+1$, there exists a hypersurface X of degree m in $\mathbb{R}P^d$ such that

$$b_*(\mathbb{R}X) \geq (m-2)(m-3)\dots(m-d-1).$$

5.2.1 Proof of Theorem 5.4

We describe a triangulation τ of the standard simplex $T = T_m^d$ and a distribution of signs at the integer points of T which provide via the combinatorial patchworking theorem a hypersurface with the properties formulated in Theorem 5.4.

To construct the triangulation τ, we use induction on d. If $d = 1$, the triangulation of $[0, m]$ is formed by m intervals $[0, 1]$, $\ldots$, $[m - 1, m]$ for any m. Assume that for all natural $k < d$ the triangulations of the standard k-dimensional simplices of all sizes are constructed and consider the d-dimensional one of size m.

Denote by $x_1, \ldots, x_d$ the coordinates in $\mathbb{R}^d$. Let $T_j^{d-1} = T \cap \{x_d = m - j\}$ and T_j be the image of T_j^{d-1} under the orthogonal projection to the coordinate hyperplane $\{x_d = 0\}$. Numerate the vertices of each simplex $T_1, \ldots, T_{m-1}, T_m = T_m^{d-1}$ as follows: assign 1 to the vertex at the origin and $i + 1$ to the vertex with nonzero coordinate at the i-th place. Assign to the vertices of $T_1^{d-1}, \ldots, T_{m-1}^{d-1}$ the numbers of their projections. A triangulation of each simplex $T_0, \ldots, T_{m-1}$ is constructed. Take the corresponding triangulations in the simplices T_j^{d-1}.

Let l be a nonnegative integer not greater than $d - 1$. If $m - j$ is even, denote by $T_j^{(l)}$ the l-face of T_j^{d-1} which is the convex hull of the vertices with numbers $1, \ldots, l + 1$. If $m - j$ is odd denote by $T_j^{(l)}$ the l-face of T_j^{d-1} which is the convex hull of the vertices with numbers $d - l, \ldots, d$.

Now for any integer $0 \le j \le m - 1$ and any integer $0 \le l \le d - 1$, take the join $T_{j+1}^{(l)} * T_j^{(d-1-l)}$. The triangulations of $T_{j+1}^{(l)}$ and $T_j^{(d-1-l)}$ define a triangulation of $T_{j+1}^{(l)} * T_j^{(d-1-l)}$. This gives rise to the desired triangulation τ of T. One can see that τ is convex.

The distribution of signs at the vertices of τ is given by the following rule. The vertex gets the sign "+" if the sum of its coordinates is even, and it gets the sign "−" otherwise.

Lemma 12 *For the hypersurface X of degree m in $\mathbb{R}P^d$ provided according to the combinatorial patchworking theorem by the triangulation τ and the distribution of signs defined above, one has*

$$b_*(\mathbb{R}X) \ge \begin{cases} (m-2)(m-3)\ldots(m-d-1), & \textit{if } m \ge d+1, \\ 0, & \textit{otherwise.} \end{cases}$$

To prove Lemma 12 we define a collection of cycles c_i, $i \in I$ of $\widetilde{H}$ (in fact, any c_i is also a cycle of the hypersurface $H \subset T_*$, and moreover, of the hypersurface $H \cap (\mathbb{R}^*)^d$). The cycles c_i are called *narrow*.

The collection of narrow cycles c_i is constructed together with a collection of *dual cycles* b_i. Any dual cycle b_i is a $(d-1-p)$-cycle in $\widetilde{T} \backslash \widetilde{H}$ (where p is the dimension of c_i) composed by simplices of τ_* and representing a homological class such that its linking number with any p-dimensional narrow cycle c_k is δ_{ik}.

Let us fix some notations. For any simplex $T_j^{(l)}$ (where $1 \leq j \leq m$ and $0 \leq l \leq d-1$), denote by $(T_j^{(l)})_*$ the union of the symmetric copies of $T_j^{(l)}$ under the reflections with respect to coordinate hyperplanes $\{x_i = 0\}$, where $i = 1, \ldots, l$, if $m-j$ is even, and $i = d-l, \ldots, d-1$, if $m-j$ is odd, and compositions of these reflections.

Any simplex $T_j^{(l)}$ is naturally identified with the standard simplex T_j^l in $\mathbb{R}^l$ with vertices $(0, \ldots, 0)$, $(j, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, j)$ via the linear map $\mathcal{L}_j^l : T_j^{(l)} \to T_j^l$ sending

1. the vertex with number i of $T_j^{(l)}$ to the vertex of T_j^l with the same number, if $m-j$ is even,

2. the vertex with number i of $T_j^{(l)}$ to the vertex of T_j^l with the number $i-d+l+1$, if $m-j$ is odd.

It is easy to see that $\mathcal{L}_j^l$ is simplicial with respect to the chosen triangulations of $T_j^{(l)}$ and T_j^l. The natural extension of $\mathcal{L}_j^l$ to $(T_j^{(l)})_*$ identifies $(T_j^{(l)})_*$ with $(T_j^l)_*$ and respects the chosen triangulations.

By *a symmetry* we mean a composition of reflections with respect to coordinate hyperplanes. Let $s_{(i)}$ be the reflection of $\mathbb{R}^d$ with respect to the hyperplane $\{x_i = 0\}$, $i = 1, \ldots, d$. Denote by s_j^l the symmetry of $(T_j^{l+1})_*$ which is identical if $m-j$ is even, and coincides with the restriction of $s_{(d-l-1)} \circ \ldots \circ s_{(d-1)}$ on $(T_j^{l+1})_*$ if $m-j$ is odd.

The narrow cycles and their dual cycles are defined below using induction on d. For $d = 1$ the narrow cycles are the pairs of points

$$(1/2, 3/2), \ldots, ((2m-5)/2, (2m-3)/2).$$

The dual cycles are pairs of vertices

$$(1, m-1), (2, m), (3, m+1), \ldots, (m-2, m),$$

if m is even, and pairs of vertices

$$(1, m), (2, m-1), (3, m), \ldots, (m-2, m),$$

if m is odd.

Assume that for all natural m and all natural $k < d$ the narrow cycles c_i in the hypersurface $\widetilde{H} \subset \widetilde{T}^k_m$ and the dual cycles b_i in $\widetilde{T}^k_m \setminus \widetilde{H}$ are constructed. The narrow cycles of the hypersurface in $\widetilde{T}^d_m$ are divided into 3 families.

Horizontal Cycles. The initial data for constructing a cycle of the first family consist of an integer j satisfying inequality $1 \leq j \leq m-1$ and a narrow cycle of the hypersurface in T^{d-1}_* constructed at the previous step. In the copy $(T^{d-1}_j)_*$ of T^{d-1}_*, take the copy c of this cycle and b of its dual cycle.

There exists exactly one symmetric copy of T^0_{j+1} incident to b. It is T^0_{j+1} itself, if $m-j$ is odd, and either T^0_{j+1}, or $s_{(d-1)}(T^0_{j+1})$, if $m-j$ is even. If the sign of the symmetric copy $s(T^0_{j+1})$ of T^0_{j+1} incident to b is opposite to the sign of c, we include c in the collection of narrow cycles of $\widetilde{H}$. Otherwise take $s_{(d)}(c)$ as a narrow cycle of $\widetilde{H}$. The dual cycle of c (resp., $s_{(d)}(c)$) is the suspension of b (resp., $s_{(d)}(b)$) with the vertex $s(T^0_{j+1})$ (resp., $s_{(d)}(s(T^0_{j+1}))$) and with the vertex $s(T^0_{j-1})$ (resp., $s_{(d)}(s(T^0_{j-1}))$).

Co-Horizontal Cycles. The initial data for constructing a cycle of the second family are the same as in the case of the horizontal cycles: the data consist of an integer j satisfying inequality $1 \leq j \leq m-1$ and a narrow cycle of the hypersurface in T^{d-1}_*.

In the copy $(T^{d-1}_j)_*$ of T^{d-1}_*, take the copy c of this cycle and b of its dual cycle. If the sign of the symmetric copy $s(T^0_{j+1})$ of T^0_{j+1} incident to b coincides with the sign of c, take b as dual cycle of a narrow cycle of $\widetilde{H}$. Otherwise take $s_{(d)}(b)$. The corresponding narrow cycle is a suspension of c (resp., $s_{(d)}(c)$).

Join Cycles. The initial data consist of integers j and l satisfying inequalities $1 \leq j \leq m-1$, $1 \leq l \leq d-2$, the copy $c_1 \subset (T^l_{j+1})_*$ of a narrow cycle of the hypersurface in $(T^l_{j+1})_*$, the copy $c_2 \subset (T^{d-1-l}_j)_*$ of a narrow cycle of the hypersurface in $(T^{d-1-l}_j)_*$ and the copies $b_1 \subset (T^l_{j+1})_*$ and $b_2 \subset (T^{d-1-l}_j)_*$ of the dual cycles of these narrow cycles.

One of the joins $b_1 * b_2$ and $s^l_{j+1}(b_1) * s^{d-1-l}_j(b_2)$, belongs to τ_*; denote it by J. If the signs of c_1 and c_2 coincide, take J as the dual cycle of a cycle of

$\widetilde{H}$. Otherwise take $s_{(d)}(J)$. The corresponding narrow cycle is either $c_1 * c_2$, or $s_{j+1}^l(c_1) * s_j^{d-1-l}(c_2)$, or $s_{(d)}(c_1 * c_2)$, or $s_{(d)}(s_{j+1}^l(c_1) * s_j^{d-1-l}(c_2))$.

Proof of Lemma 12. Both c_i and b_i with $i \in I$ are $\mathbb{Z}_2$-cycles homologous to zero in $\widetilde{T}$, which is homeomorphic to the projective space of dimension d. The sum of dimensions of c_i and b_i is $d-1$. Thus we can consider the linking number of c_i with $i \in I$ and b_k, $k \in I$ taking values in $\mathbb{Z}_2$. Each c_i bounds an obvious ball in $\widetilde{T}$. This ball meets b_i in a single point transversally and is disjoint with b_k for $k \neq i$ and $i, k \in I$. Hence the linking number of c_i and b_k is δ_{ik}.

Therefore the collections of homology classes realized in $\widetilde{T} \setminus \widetilde{H}$ and $\widetilde{H}$ by $b_i, i \in I$ and $c_i, i \in I$, respectively, generate subspaces of $H_*(\widetilde{T} \setminus \widetilde{H}; \mathbb{Z}_2)$ and $H_*(\widetilde{H}; \mathbb{Z}_2)$ and are dual bases of the subspaces with respect to the restriction of the Alexander duality. Hence c_i with $i \in I$ realize linearly independent Z_2-homology classes of $\widetilde{H}$.

It remains to show that the number of narrow cycles is at least

$$(m-2)(m-3)\dots(m-d-1),$$

if $m \geq d+1$. The statement can be proved by induction on d. The base $d=1$ is evident. To prove the induction step notice, first, that the statement is evidently true for $m = d+1$. Now, we use the induction on m and obtain the required statement from the inequality

$$\begin{aligned} &(m-3)(m-4)\dots(m-d-2) + 2(m-3)(m-4)\dots(m-d-1) \\ &+\sum_{k=1}^{d-2} [(m-2)(m-3)\dots(m-k-1)]\,[(m-3)(m-4)\dots(m-d+k-1)] \\ &\geq (m-2)(m-3)\dots(m-d-1). \end{aligned}$$

This finishes the proofs of Lemma 12 and Theorem 5.4. □

Remark 6 *The family of hypersurfaces in $\mathbb{R}P^d$ constructed in Theorem 5.4 is asymptotically maximal.*

Proof . - Indeed, the total Betti number of a nonsingular hypersurface of degree m in $\mathbb{C}P^d$ is equal to $\frac{(m-1)^{d+1}-(-1)^{d+1}}{m} + d + (-1)^{d+1}$. This number is equivalent to $(m-2)(m-3)\dots(m-d-1)$ when m tends to infinity. □

5.3 Proof of theorem 5.1

For a positive integer λ put $\Delta_\lambda = \lambda \cdot \Delta$. Let l be a positive integer such that Δ_l admits a primitive convex triangulation τ (see 5.3). Denote by ν a function certifying the convexity of τ. Let τ_λ be the triangulation of $\Delta_{\lambda l}$ obtained from τ by multiplication of its simplices by λ.

We can assume that $\lambda > d+1$. Let δ be a d-dimensional simplex of τ. The convex hull of the interior integer points of $\lambda \cdot \delta$ is a d-dimensional simplex $(\lambda - (d+1)) \cdot \delta$. Put $\delta_\lambda = \lambda \cdot \delta$ and $\delta'_\lambda = (\lambda - (d+1)) \cdot \delta$. For any d-dimensional simplex δ_λ of τ_λ, apply the construction of Lemma 12 to the convex hull δ'_λ of the interior integer points of δ_λ. Complete the triangulation of δ'_λ to a convex triangulation of δ_λ whose only extra vertices are the vertices of δ_λ in the following way. Let $\nu_{\lambda-(d+1)}$ be a convex piecewise-linear function certifying the convexity of the triangulation of δ'_λ. Define a convex function ν^δ_λ on δ_λ choosing the values of $\nu_{\lambda-(d+1)}$ at the integer points of δ'_λ and the value v at the vertices of δ_λ, where v is large enough (see Remark 1). Note that ν^δ_λ restricted to δ'_λ coincides with $\nu_{\lambda-(d+1)}$. If the decomposition defined by ν^δ_λ is not a triangulation, we slightly perturb $\nu_{\lambda-(d+1)}$ (without changing the triangulation of δ'_λ) to break the polytopes of the subdivision which are not simplices. Denote by τ^δ_λ the obtained triangulation of δ_λ.

The only vertices of τ^δ_λ in $\delta_\lambda \setminus \delta'_\lambda$ are the vertices of δ_λ. One can choose the same value v of the functions ν^δ_λ at the vertices of all the d-dimensional simplices δ of τ_λ. Hence, the functions ν^δ_λ can be glued together to form a piecewise-linear function ν_λ on $\Delta_{\lambda l}$ which is, by construction, convex on each d-dimensional simplex of τ_λ. Let ν' be a function certifying the convexity of τ_λ. Then, for sufficiently small $\epsilon > 0$ the function $\nu = \nu' + \epsilon\nu_\lambda$ certifies the convexity of the triangulation obtained by gluing the triangulations of the d-dimensional simplices of τ_λ. Thus, one gets a convex triangulation τ^l_λ of $\Delta_{\lambda l}$. Choose a sign distribution $D(\tau^l_\lambda)$ at the vertices of τ^l_λ in such a way that on each simplex δ'_λ the distribution coincides with the one Lemma 12. Let $Z_{\Delta_{\lambda l}}$ be the hypersurface obtained via the combinatorial patchworking from τ^l_λ and $D(\tau^l_\lambda)$.

Proposition 5.5 *The family of hypersurfaces $Z_{\Delta_{\lambda l}}$ of X_Δ constructed above is asymptotically maximal.*

Proof . - The total Betti number of $Z_{\Delta_{\lambda l}}$ is equivalent to $\mathrm{Vol}(\Delta_{\lambda l})$ when λ tends to infinity (see Proposition 2.15). For each d-dimensional simplex δ of τ_λ consider the narrow cycles of $H^*_{\Delta_{\lambda l}} \cap (\delta'_\lambda)_*$ which are constructed in the proof of Lemma 12. Since the narrow cycles are constructed with the dual cycles, the union of the obtained collections of narrow cycles consists of linearly independent cycles. Thus, $b_*(\mathbb{R}Z_{\Delta_{\lambda l}};\mathbb{Z}_2) \geq \mathrm{Vol}(\Delta_l)n_\lambda$, where n_λ is the number of narrow cycles in each δ'_λ. Since $n_\lambda \sim \mathrm{Vol}(\delta'_\lambda)$, we have $n_\lambda \sim \mathrm{Vol}(\delta_\lambda)$. So, $b_*(\mathbb{R}Z_{\Delta_{\lambda l}};\mathbb{Z}_2)$ is equivalent to $\mathrm{Vol}(\Delta_l)\,\mathrm{Vol}(\delta_\lambda)$. The latter number is equal to $\mathrm{Vol}(\Delta_{\lambda l})$. $\square$

Chapter 6

Newton polytopes without maximal complete intersection

In this chapter we show that for any integer d greater than 2 there exist polytopes $\Delta^d \subset (\mathbb{R}^+)^d$ of dimension d such that the hypersurfaces defining a maximal complete intersection in X_{Δ_d} cannot all have the Newton polytope Δ_d.

Let Δ_k be the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(1,1,k)$. Note that the only integer points of Δ_k are its vertices.

Proposition 6.1 *Let $k \geq 5$ be an integer, and Z_1 and Z_2 be real algebraic surfaces in X_{Δ_k} with Newton polytope Δ_k. Assume that Z_1 and Z_2 define a complete intersection Y_k in X_{Δ_k}. Then Y_k is not maximal.*

The proof relies on the estimation of the Betti numbers of the complex and real parts of the complete intersection Y_k of two surfaces whose Newton polytopes coincide with Δ_k.

Lemma 13 *Let Y_k be the complete intersection of two surfaces in X_{Δ_k} whose Newton polytopes coincide with Δ_k. Then $b_*(Y_k;\mathbb{C}) = 2k$.*

Proof . - By corollary 2.18, we have

$$b_*(Y_k;\mathbb{C}) = 2\,\mathrm{Vol}(\Delta_k) - \sum_{\Gamma \in \mathcal{F}_2(\Delta_k)} \mathrm{Vol}(\Gamma) + 4.$$

So, we get $b_*(Y_k;\mathbb{C}) = 2k$. $\square$

Proof of Proposition 6.1. According to Lemma 13, we have $b_*(Y_k;\mathbb{C}) = 2k$. Thus, $b_*(Y_k;\mathbb{Z}_2) \geq 2k$.

Let f_1 and f_2 be the polynomials defining the two surfaces. Then,

$$f_l(x,y,z) = a_l x + b_l y + c_l z^k + d_l \ (l = 1,2)$$

for some (a_l, b_l, c_l, d_l) in $\mathbb{R}^4$. The change of variables $\Lambda_k : x \mapsto x$, $\Lambda_k : y \mapsto y$, $\Lambda_k : z \mapsto z^{\frac{1}{k}}$ is a diffeomorphism of the first octant $(\mathbb{R}^*_+)^3$, where $\mathbb{R}^*_+ = \{x \in \mathbb{R} : x > 0\}$. Let Q_i be another octant, and ϕ_i be the diffeomorphism from Q_i to $(\mathbb{R}^*_+)^3$ defined by $\phi_i(x,y,z) = (|x|,|y|,|z|)$. Then $\psi_i = \phi_i^{-1} \circ \Lambda_k \circ \phi_i$ is a diffeomorphism from Q_i to itself. The diffeomorphism ψ_i maps the zeros of f_l to the zeroes of $\psi_{i*}(f_l)$ and $\psi_{i*}(f_l)(x,y,z) = a_l x + b_l y + c_l z + d_l$. Thus, in each octant, Y_k is diffeomorphic to the intersection of two plans. Hence, the number of connected components of Y_k is at most 8. So, Y_k is not maximal for $k \geq 5$. □

The example above is to compare with the following result in dimension 2.

Proposition 6.2 *Let Δ be a two-dimensional polygon. For any positive integers λ_1 and λ_2 there exist algebraic curves C_1 et C_2 in X_Δ such that*

- *the Newton polygons of C_1 et C_2 are $\lambda_1 \cdot \Delta$ and $\lambda_2 \cdot \Delta$, respectively,*
- *the curves C_1 et C_2 define a 0-dimensional maximal complete intersection in X_Δ.*

Proof . - We use here the Cayley trick. Take any primitive convex triangulation τ of Δ. By homothety, τ induces a triangulation τ_i on $\lambda_i \cdot \Delta$. Put $\Delta_i = \lambda_i \cdot \Delta$. Consider the following subdivision δ_0 of the Cayley polytope $C(\Delta_1, \Delta_2)$. In the faces of $C(\Delta_1, \Delta_2)$ corresponding to Δ_1 and Δ_2 take the triangulations τ_1 and τ_2, respectively. Each 3-dimensional polytope of the subdivision δ_0 is the convex hull of a triangle of τ_1 and a triangle of τ_2 which are the multiples of the same triangle of τ. Since τ is convex, δ_0 is also convex. Let ν_0 be a convex function certifying the convexity of δ_0, and let ν_1 be the convex function defined by $\nu_1(0,1,x,y) = C_1 y + C_2 x$ with $C_1 > C_2 > 0$ and $\nu_1(1,0,x,y) = 0$. Put $\nu_3 = \nu_1 + \nu_2$. If C_1 is sufficiently small, the function ν_3 induces the following refinement δ_1 of δ_0. Each 3-dimensional polytope of δ_0 is subdivided into two cones whose bases are triangles in $\hat{\Delta}_1$ and $\hat{\Delta}_2$, respectively, and a join J of two edges: one in $\hat{\Delta}_1$ and the other one in $\hat{\Delta}_2$. Take any convex primitive triangulations τ_1' and τ_2' refining τ_1 and τ_2, respectively. They define a convex primitive refinement δ_2 of δ_1. Choose a sign distribution at the vertices of δ_2 and apply the procedure of the combinatorial patchworking. Let J be a join of the decomposition δ_1 described above. It is triangulated into primitive tetrahedra t_i and has lattice volume $\lambda_1\lambda_2$. Each t_i has a symmetric copy containing a point of the T-complete intersection constructed. Thus, the number of intersection points obtained is $\lambda_1\lambda_2 \operatorname{Vol}(\Delta)$ and the complete intersection constructed is maximal. $\square$

Proposition 6.1 can be generalized in the following way.

Proposition 6.3 *For any positive integers d and $n \leq d$ there exists a d-dimensional polytope Δ_d such that n hypersurfaces defining a maximal complete intersection in X_{Δ_d} cannot all have the Newton polytope Δ_d.*

Proof . - Consider the simplex σ_k in $\mathbb{R}^d$ with the vertices

$$(0,0,\ldots,0,0),\ (1,0,\ldots,0,0),\ (0,1,\ldots,0,0),\ \ldots,(0,0,\ldots,1,0),$$
$$\text{and } (1,1,\ldots,1,k).$$

Let Y_k be a complete intersection of hypersurfaces in X_{σ_k} such that all these hypersurfaces have Newton polytope σ_k. Proposition 2.16 implies that $b_*(Y_k;\mathbb{Z}_2)$ tends to infinity when k tends to infinity.

Let $f_1,\ldots,f_n$ be the polynomials defining the hypersurfaces. Then,

$$f_l(x,y,z) = a_{l,0} + \sum_{i=1}^{d-1} a_{l,i}x_i + a_{l,d}{x_d}^k \ (l=1,\cdots,n)$$

for some $(a_{l,0},\cdots,a_{l,n})$ in $\mathbb{R}^{n+1}$. The change of variables $\Lambda_k : x_i \mapsto x_i$ for $i \neq n$, $\Lambda_k : x_n \mapsto {x_n}^{\frac{1}{k}}$ is a diffeomorphism of the first orthant $(\mathbb{R}_+^*)^d$, where $\mathbb{R}_+^* = \{x \in \mathbb{R} : x > 0\}$. Let Q_j be another orthant, and ϕ_j be the diffeomorphism from Q_j to $(\mathbb{R}_+^*)^d$ defined by $\phi_j(x_1,\cdots,x_d) = (|x_1|,\cdots,|x_d|)$. Then $\psi_j = \phi_j^{-1} \circ \Lambda_k \circ \phi_j$ is a diffeomorphism from Q_j to itself. The diffeomorphism ψ_j maps the zeros of f_l to the zeroes of ${\psi_j}_*(f_l)$ and ${\psi_j}_*(f_l)(x_1,\ldots,x_d) = a_{l,0}+\sum_{i=1}^d a_{l,i}x_i$. Thus, in each orthant, Y_k is diffeomorphic to the intersection of n hyperplanes. Hence, $b_*(\mathbb{R}Y_i;\mathbb{Z}_2)$ is bounded.

So, there exists a number k_0 such that for any $k \geq k_0$ and any complete intersection Y_k in X_{δ_k} one has $b_*(\mathbb{R}Y_k;\mathbb{Z}_2) < b_*(Y_k;\mathbb{Z}_2)$. $\square$

Chapter 7

Asymptotically maximal families of complete intersections

In this chapter we generalize the results of Chapter 5 to complete intersections.

7.1 Statement

Let Δ be a d-dimensional polytope in $\mathbb{R}^d$, and k be an integer such that $1 \leq k \leq d$. Knudsen-Mumford theorem 5.3 asserts that there exists a positive integer l such that $l \cdot \Delta$ admits a convex primitive triangulation. Let $\lambda_1, \cdots, \lambda_k$ be k positive integers. Denote by Δ_{λ_i} the polytope $\lambda_i l \cdot \Delta$. Let $\{(\lambda_{1,m}, \cdots, \lambda_{k,m})\}_{m\in\mathbb{N}}$ be a sequence of k-tuples of positive integers such that $\lambda_{i,m}$ tends to infinity for any $i = 1, \dots, k$. Let $\{(Z_{\lambda_{1,m}}, \cdots, Z_{\lambda_{k,m}})\}_m$ be a sequence of k-tuples of algebraic hypersurfaces in X_Δ such that $Z_{\lambda_{i,m}}$ has Newton polytope $\Delta_{\lambda_{i,m}}$. Assume that for any natural number m the variety $Y_m = Z_{1,m} \cap \cdots \cap Z_{k,m}$ is a complete intersection.

Definition 7.1.1 *Under the above hypotheses, the family $\{Y_m\}_{m\in\mathbb{N}}$ is called* ***asymptotically maximal****, if $b_*(\mathbb{R}Y_m; \mathbb{Z}_2)$ is equivalent to $b_*(Y_m; \mathbb{Z}_2)$ when m tends to infinity.*

Theorem 7.1
Let Δ be a d-dimensional polytope, and k be an integer number satisfying $1 \leq k \leq d$. Let $\{(\lambda_{1,m}, \cdots, \lambda_{k,m})\}_{m\in\mathbb{N}}$ be a sequence of k-tuples of natural numbers such that $\lambda_{i,m}$ tends to infinity for any $i = 1, \ldots, k$. Then, there exists a sequence of k-tuples $\{(Z_{\lambda_{1,m}}, \cdots, Z_{\lambda_{k,m}})\}_{m\in\mathbb{N}}$ of algebraic hypersurfaces in X_Δ such that

1. *$Z_{\lambda_{i,m}}$ has Newton polytope $\Delta_{\lambda_{i,m}}$*
2. *for any natural m, the variety $Y_m = Z_{1,m} \cap \cdots \cap Z_{k,m}$ is a complete intersection,*
3. *the family $\{Y_m\}_{m\in\mathbb{N}}$ is asymptotically maximal.*

The proof is based on the following result of Itenberg and Viro.

Theorem 7.2 (I. Itenberg and O. Viro)
Let Δ be a primitive d-dimensional simplex. For any k-tuple $\lambda_1, \cdots, \lambda_k$ of natural numbers, there exist piecewise-linear convex functions $\mu_1, \ldots, \mu_k$ on $\lambda_1 \cdot \Delta, \ldots, \lambda_k \cdot \Delta$, respectively, and sign distributions at the vertices of the corresponding triangulations of $\lambda_1 \cdot \Delta, \ldots, \lambda_k \cdot \Delta$ such that the real complete intersection in $X_\Delta = \mathbb{C}P^d$ obtained via Sturmfels' theorem 2.9 from these data is maximal.

In fact, as in Chapter 5, we use only an asymptotical version of 7.2. The proof of this asymptotical version is much simpler than the proof of 7.2. It can be extracted from [IV02] and was communicated to us by the authors. The asymptotical version of 7.2 is reproduced in Section 7.2 below.

7.2 Itenberg-Viro asymptotical statement

Theorem 7.3
For any positive integers k, $m_1, \ldots, m_k$ and d such that $k \leq d$ and $m_j \geq d+1$ $(j = 1, \ldots, k)$, there exists a complete intersection X of multi-degree $(m_1, \ldots, m_k)$ in $\mathbb{R}P^d$ such that

$$b_*(\mathbb{R}X) \geq \sum_{i_1+\ldots+i_k=d} \left(\prod_{j=1}^{k} (m_j-2)(m_j-3)\ldots(m_j-i_j-1) \right)$$

(the summation is over all possible decompositions $i_1 + \ldots + i_k = d$ of d in a sum of k positive integer numbers).

Proof of Theorem 7.3.

The notations used here are those of Section 5.2.1. Take the standard simplices $T^d_{m_1}, \dots, T^d_{m_k}$ and triangulate the Cayley polytope $C(T^d_{m_1}, \dots, T^d_{m_k})$ (see subsection 2.3.2) in the following way. Let i_1, ..., i_k be nonnegative integers such that $i_1 + \dots + i_k = d$, and put $i_0 = 0$. For any $j = 1, \dots, k$ consider the face of $T^d_{m_j}$ with the vertices having the numbers

$$i_1 + \dots + i_{j-1} + 1, \dots, i_1 + \dots + i_j + 1.$$

Denote by $J_{i_1, \dots i_k}$ the join of the corresponding faces of $C(T^d_{m_1}, \dots, T^d_{m_k})$. The simplices $J_{i_1, \dots i_k}$ (for all the possible choices of nonnegative integers such that $i_1 + \dots + i_k = d$) form a triangulation τ' of $C(T^d_{m_1}, \dots, T^d_{m_k})$.

Take for each simplex $T^d_{m_j}$ the triangulation and the distribution of signs described in subsection 5.2.1. For the simplices $\hat{T}^d_{m_1}, \dots, \hat{T}^d_{m_k}$ take the corresponding triangulations and distributions of signs. The triangulations of $\hat{T}^d_{m_1}, \dots, \hat{T}^d_{m_k}$ induce a refinement τ of τ'. Notice that τ is a primitive triangulation of $C(T^d_{m_1}, \dots, T^d_{m_k})$.

Lemma 14 *For the complete intersection X of multi-degree $m_1, \dots, m_k$ in $\mathbb{R}P^d$ provided according to Proposition 2.10 by the triangulation τ and the distribution of signs defined above, one has*

$$b_*(\mathbb{R}X) \geq \sum_{i_1 + \dots + i_k = d} \left(\prod_{j=1}^{k} (m_j - 2)(m_j - 3) \dots (m_j - i_j - 1) \right)$$

(the summation is over all the possible decompositions $i_1 + \dots + i_k = d$ of d in a sum of k positive integer numbers).

Proof. We define a collection of narrow cycles c_i, $i \in I$ of $\widetilde{H}$. The families of narrow cycles of $\widetilde{H}$ are indexed by the decompositions $i_1 + \dots + i_k = d$ of d in a sum of k positive integer numbers.

Fix a decomposition $\mathcal{I} : i_1 + \dots + i_k = d$ of d, where i_1, ..., i_k are positive integers. The initial data for constructing a narrow cycle of the corresponding family consist of narrow cycles $c_{(j)} \subset \widetilde{H}^{\mathcal{I}}_j$, $j = 1, \dots, k$, constructed in subsection 5.2.1 for the hypersurface $\widetilde{H}^{\mathcal{I}}_j$ in $\widetilde{T}^{i_j}_{m_j}$ produced via the combinatorial patchworking by the triangulation and distribution of signs described in subsection 5.2.1.

The i_j-dimensional face Δ^{i_j} of $T^d_{m_j}$ with the vertices having the numbers

$$i_1 + \dots + i_{j-1} + 1, \dots, i_1 + \dots + i_j + 1$$

are naturally identified with $T^{i_j}_{m_j}$ via the linear map $\mathcal{L}^{i_j} : \Delta^{i_j} \to T^{i_j}_{m_j}$ sending the vertex with number $i_1 + \ldots + i_{j-1} + r$ of Δ^{i_j} to the vertex with number r of $T^{i_j}_{m_j}$. The map $\mathcal{L}^{i_j}$ is simplicial with respect to the chosen triangulations of Δ^{i_j} and $T^{i_j}_{m_j}$. Denote by $\Delta^{i_j}_*$ the union of the symmetric copies of Δ^{i_j} under the reflections with respect to coordinate hyperplanes $\{x_i = 0\}$ in $\mathbb{R}^d$, where $i = i_1 + \ldots + i_{j-1} + 2, \ldots, i_1 + \ldots + i_j + 1$, and compositions of these reflections. The natural extension of $\mathcal{L}^{i_j}$ to $\Delta^{i_j}_*$ identifies $\Delta^{i_j}_*$ with $(T^{i_j}_{m_j})_*$ and respects the chosen triangulations. We also denote this extension by $\mathcal{L}^{i_j}$. Denote by $\hat{\Delta}^{i_j}_*$ the union of faces of $\hat{T}^d_{m_j}$ corresponding to $\Delta^{i_j}_*$, and by $\hat{\mathcal{L}}^{i_j}$ the corresponding map from $\hat{\Delta}^{i_j}_*$ to $(T^{i_j}_{m_j})_*$. Put $\hat{c}_{(j)} = (\hat{\mathcal{L}}^{i_j})^{-1}(c_{(j)})$.

Let $b_{(j)} \subset \widetilde{T}^{i_j}_{m_j} \setminus \widetilde{H}^{\mathcal{I}}_j$ be the dual cycle of $c_{(j)}$. Put $\hat{b}_{(j)} = (\hat{\mathcal{L}}^{i_j})^{-1}(b_{(j)})$. Consider the symmetric copies of $\hat{b}_{(1)}, \ldots, \hat{b}_{(k)}$ under the reflections with respect to coordinate hyperplanes $\{x_i = 0\}$ in $\mathbb{R}^{k+d}$ where $i = k+1, \ldots, k+d$, and compositions of these reflections. Among these symmetric copies there exist copies $\hat{b}'_{(1)}, \ldots, \hat{b}'_{(k)}$ of $\hat{b}_{(1)}, \ldots, \hat{b}_{(k)}$, respectively, such that

- the join $\hat{b}'_{(1)} * \ldots * \hat{b}'_{(k)}$ is the union of simplices of τ_*,
- all the vertices of $\hat{b}'_{(1)} * \ldots * \hat{b}'_{(k)}$ have the same sign.

Let $\hat{c}'_{(1)}, \ldots, \hat{c}'_{(k)}$ be the corresponding symmetric copies of $\hat{c}_{(1)}, \ldots, \hat{c}_{(k)}$, respectively. Then, take the intersection $B \cap (\hat{c}'_{(1)} * \ldots * \hat{c}'_{(k)})$ as a narrow cycle of $\widetilde{H}$.

The number of narrow cycles in the family indexed by $\mathcal{I}$ is at least

$$\prod_{j=1}^{k} (m_j - 2)(m_j - 3) \ldots (m_j - i_j - 1).$$

Thus, the total number of constructed narrow cycles in $\widetilde{H}$ is at least

$$\sum_{i_1 + \ldots + i_k = n} \left(\prod_{j=1}^{k} (m_j - 2)(m_j - 3) \ldots (m_j - i_j - 1) \right)$$

(the summation is over all the possible decompositions $i_1 + \ldots + i_k = d$ of d in a sum of k positive integer numbers). The linear independence of the narrow cycles of a hypersurface $H^l_{m_j} \subset T^l_{m_j}$ for any $1 \le l \le d$ and any $1 \le j \le k$ implies the linear independence of the narrow cycles constructed in $\widetilde{H}$. $\square$

Remark 7 *Denote by $Y^{\sigma}_{m_1,\ldots,m_k}$ the complete intersection constructed in Lemma 14. Then, the family $\{Y^{\sigma}_{m_1,\ldots,m_k}\}_{m_1,\ldots,m_k}$ is asymptotically maximal.*

Proof. - Note that $\sum_{i_1+\ldots+i_k=d}\left(\prod_{j=1}^{k}(m_j-2)(m_j-3)\ldots(m_j-i_j-1)\right)$ is equivalent to the mixed volume of $T^d_{m_1},\ldots,T^d_{m_k}$. Thus, by Proposition 2.16. $b_*(\mathbb{R}Y_{m_1,\ldots,m_k})$ is equivalent to $b_*(Y_{m_1,\ldots,m_k})$, when all m_i's tend to infinity.

7.3 Proof of Theorem 7.1

Let be a primitive convex triangulation of $l\cdot\Delta$, and $(\lambda_1,\cdots,\lambda_k)$ be a k-tuple of positive integers. Denote by Δ_{λ_i} the polytopes $\lambda_i l\cdot\Delta$. We can assume that λ_i is greater than $d+1$ for any i.

Let δ be a d-dimensional simplex of τ. Denote by $\hat{\delta}_1$, …, $\hat{\delta}_k$ the corresponding simplices in $\hat{\Delta}_{\lambda_1}$, …, $\hat{\Delta}_{\lambda_k}$, respectively. Subdivide the Cayley polytope $C(\Delta_{\lambda_1},\ldots,\Delta_{\lambda_k})$ into convex hulls of $\hat{\delta}_1$, …, $\hat{\delta}_k$, where δ runs over all d-dimensional simplices of τ. For a d-dimensional simplex δ of τ, put $\delta_i=\lambda_i\cdot\delta$ and $\delta'_i=(\lambda_i-(d+1))\cdot\delta$, where $i=1,\ldots,k$.

For any d-dimensional simplex δ of τ, take the triangulation of $C(\delta'_1,\ldots,\delta'_k)$ and the distribution of signs at the vertices of this triangulation described in the proof of Theorem 7.2. Extend the triangulations of the Cayley polytopes $C(\delta'_1,\ldots,\delta'_k)$ to a primitive convex triangulation $\hat{\tau}$ of $C(\Delta_{\lambda_1},\ldots,\Delta_{\lambda_k})$ in the same way as it was done in Chapter 5, Section 5.3. Extend also the distributions of signs at the integer points of polytopes $C(\delta'_1,\ldots,\delta'_k)$ to some distribution of signs $\hat{D}$ at the vertices of $\hat{\tau}$.

Let $Y_{\lambda_1,\ldots,\lambda_k}$ be the complete intersection in X_Δ obtained via Theorem 2.10 from $\hat{\tau}$ and $\hat{D}$.

Proposition 7.4 *The family of complete intersections $Y_{\lambda_1,\ldots,\lambda_k}$ constructed above is asymptotically maximal.*

Proof. - By the construction, we have $b_*(\mathbb{R}Y_{\lambda_1,\ldots,\lambda_k})\geq \mathrm{Vol}(\Delta_l)\cdot n_{\lambda_1,\ldots,\lambda_k}$, where $n_{\lambda_1,\ldots,\lambda_k}$ is the number of narrow cycles in each $C(\delta'_1,\ldots,\delta'_k)$. Note that $n_{\lambda_1,\ldots,\lambda_k}$ is equivalent to $b_*(\mathbb{R}Y^{\sigma}_{\lambda_1,\ldots,\lambda_k})$ when all numbers $\lambda_1,\ldots,\lambda_k$ tend to infinity. So, by Proposition 2.16 and Remark 7, we obtain that $b_*(\mathbb{R}Y_{\lambda_1,\ldots,\lambda_k})$ is equivalent to $b_*(Y_{\lambda_1,\ldots,\lambda_k})$ when the numbers $\lambda_1,\ldots,\lambda_k$ tend to infinity. □

Chapter 8

Maximal complete intersections in toric varieties corresponding to pyramids

In this section we describe a construction of maximal complete intersections of two hypersurfaces in X_Δ, where Δ is a pyramid whose basis is a 2-dimensional Nakajima polytope.

Let α and m be non-negative integers such that $(\alpha, m) \neq (0,0)$, and n be a natural number. Denote by $\delta_\alpha^{n,m}$ the polygon having vertices $(0,0), (m + n\alpha, 0), (0,n), (m,n)$ in $\mathbb{R}^2$. Notice that, up to the change of coordinate $\Lambda : (x_1, x_2) \mapsto (x_1, x_2)$, every 2-dimensional Nakajima polytope is equal to one of the polygons $\delta_\alpha^{n,m}$. If $m \neq 0$, the toric variety associated with $\delta_\alpha^{n,m}$ is a rational ruled surface Σ_α. For $i = 1, 2$ let $\Delta_i^{\alpha, m_i, n_i, l_i}$ be the convex hull of the points $(0,0,0)$, $(m_i + \alpha n_i, 0, 0)$, $(0, n_i, 0)$, $(m_i, n_i, 0)$ and $(0,0,l_i)$. The pyramid $\Delta_i^{m_i, n_i, l_i}$ is a cone of apex $(0,0,l_i)$ over $\delta_\alpha^{n,m} \times \{0\}$. Assume that $m_i = \lambda l_i$ for some nonnegative integer λ and $n_i = \mu l_i$ for some positive integer μ.

Theorem 8.1
Let α, m_2, m_1 be nonnegative integers, and n_1, n_2, l_1, l_2 be positive integers. Assume that there exist a non-negative integer λ and a positive integer μ such that $m_i = \lambda l_i$, $n_i = \mu l_i$, and $(\alpha, \lambda) \neq (0,0)$. Then, there exists a maximal complete intersection in $X_{\Delta_1^{\alpha,m_1,n_1,l_1}} = X_{\Delta_2^{\alpha,m_2,n_2,l_2}}$ of two surfaces with Newton polytopes $\Delta_1^{\alpha,m_1,n_1,l_1}$ and $\Delta_2^{\alpha,m_2,n_2,l_2}$, respectively.

The construction used in the proof of Theorem 8.1 is inspired by Itenberg-Viro's construction of maximal complete intersections in the projective spaces (see [IV02]).

8.1 Triangulation

Put $\Delta_i = \Delta_i^{\alpha,m_i,n_i,l_i}$. Let A_i^0, A_i^1, A_i^2, A_i^3 and A_i^4 be the vertices of the pyramid Δ_i (see Figure 8.1). We use here the Cayley trick (see Subsection 2.3.2) and describe a convex triangulation of $C(\Delta_1, \Delta_2)$. We construct the triangulation step by step to subdivide further and further the Cayley polytope $C(\Delta_1, \Delta_2)$. On each step the subdivision described is a refinement of the previous one. We describe the construction in the case when $\delta_\alpha^{n_1,m_1}$ and $\delta_\alpha^{n_2,m_2}$ are nondegenerate Nakajima polygons. If $\delta_\alpha^{n_1,m_1}$ and $\delta_\alpha^{n_2,m_2}$ are triangles, the description is the same except that the segments $[A_i^2, A_i^3]$ are reduced to a point, and the triangles (A_i^2, A_i^3, A_i^4) are reduced to a segment (see Figure 8.1). All figures are drawn in the case $\alpha = 0$.

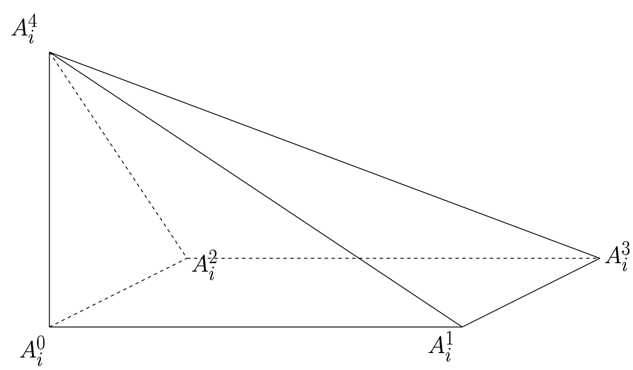

Figure 8.1: The pyramids Δ_i

To simplify the figures, we represent the pyramids as triangles and their convex hull as a 3-dimensional polytope. More precisely, we use a projection along $((0,0,0,0,0),(0,0,1,0,0))$ and draw the segments parallel to $((0,0,0,0,0),(0,0,1,0,0))$ as a point, the plan pieces containing a line parallel to $((0,0,0,0,0),(0,0,1,0,0))$ as segments, and so on. Figure 8.2 represents $C(\Delta_1, \Delta_2)$ with the above convention.

At the first step we set the value of ν_1 at A_1^4 to 1 and the values at other vertices of $C(\Delta_1, \Delta_2)$ to 0. We define ν_1 to be the function whose graph is the

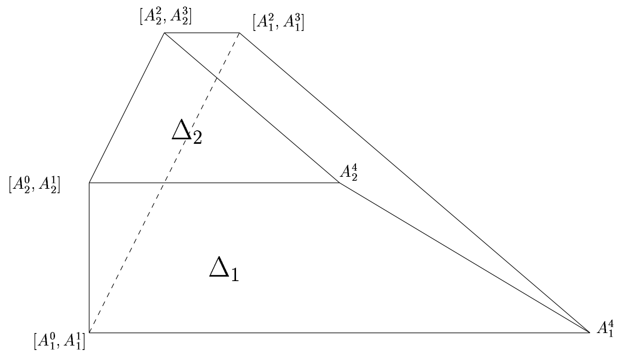

Figure 8.2: The Cayley polytope $C(\Delta_1, \Delta_2)$.

lower part of the convex hull of all the points $(A_i^j, \nu_1(A_i^j))$ (see Remark 1). The obtained subdivision δ_1 of $C(\Delta_1, \Delta_2)$ has two 4-dimensional polytopes. Denote by P_2 the 4-dimensional polytope which is the cone over $\hat{\Delta}_1$, and denote by P_1 the other 4-dimensional polytope of δ_1 (see Figure 8.3).

The second step consists in slicing the two pyramids orthogonally to the direction of (A_1^0, A_1^4) (see Figure 8.4). For that purpose we slightly perturb ν_1 by a piecewise-linear convex function μ_1 whose value at an integer point p of $C(\Delta_1, \Delta_2)$ with the fifth coordinate x_5 is equal to e^{x_5}. The lower part of the convex hull of the points $(p, \mu_1(x_5))$ is the graph of μ_1. We put $\nu_2 = \nu_1 + \epsilon_1 \cdot \mu_1$ for a positive sufficiently small ϵ_1. The function ν_2 defines a convex refinement δ_2 of δ_1.

Note that P_1 is now subdivided by the convex hulls of each slice of Δ_2 with the convex hull Q_1 of A_1^0, A_1^1, A_1^2 and A_1^3.

A part P^k of the decomposition of P_1 is the convex hull of three parallel quadrangles T_1, T_2^k, T_2^{k-1} for $k = l_2, \ldots, 2$. The only exception is the last part P^1 which is the convex hull of two quadrangles and the point A_2^4. We represent P^k on the figures as a prism.

We subdivide the decomposition δ_2 choosing slopes along the x_4-axis of the integer horizontal sections of the pyramids. Namely, choose positive numbers A, C, and E satisfying the inequalities $A > E > C$. Let μ_2 be the piecewise-linear function defined by the following values at integer points of $C(\Delta_1, \Delta_2)$: if an integer point $p \in C(\Delta_1, \Delta_2)$ has coordinates

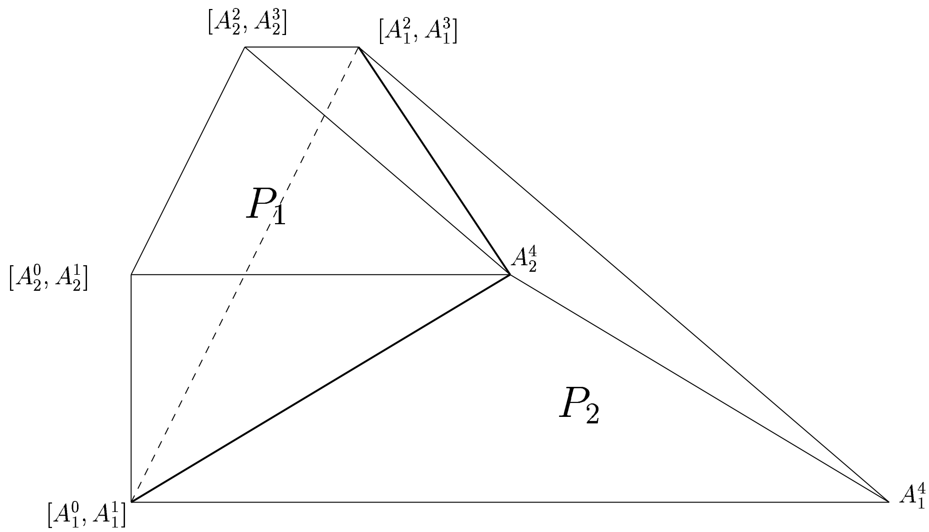

Figure 8.3: First decomposition of $C(\Delta_1, \Delta_2)$.

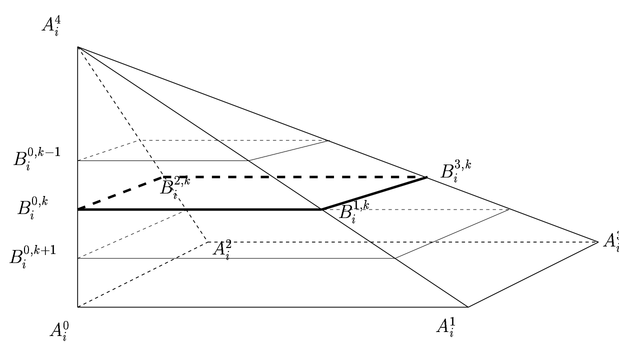

Figure 8.4: Slicing Δ_i

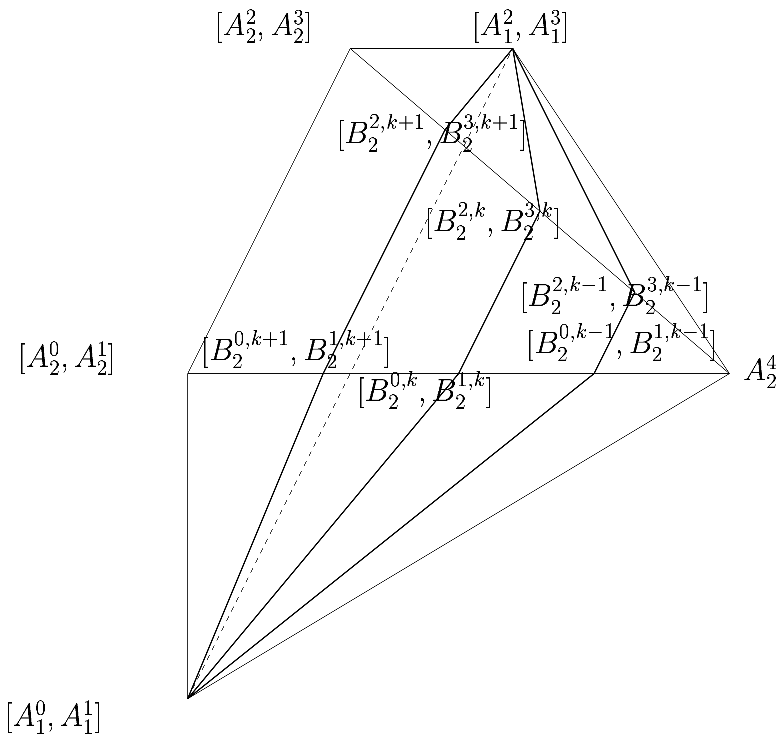

Figure 8.5: Decomposition of P_1

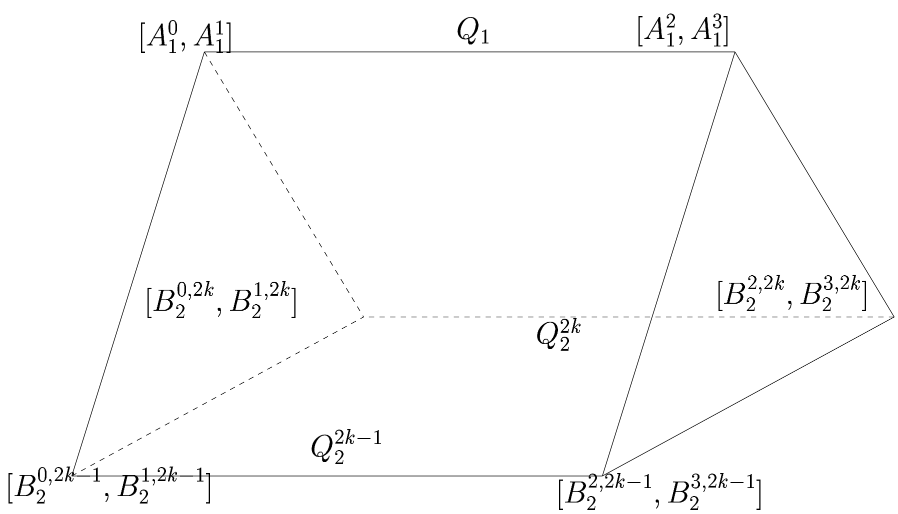

Figure 8.6: P^k

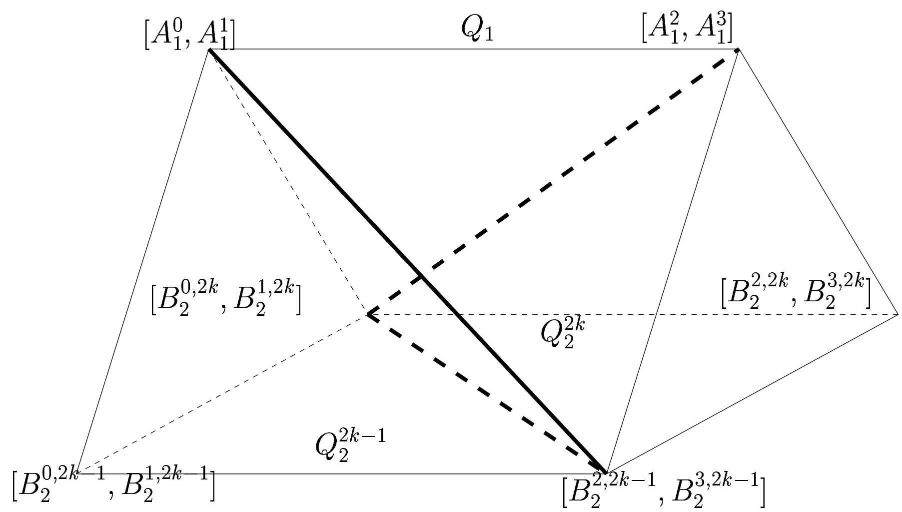

Figure 8.7: Subdivision of P^k

$(x_1, x_2, x_3, x_4, x_5)$, then

$$\begin{cases} \mu_2(p) &= A \cdot x_4 \text{ if } (x_1, x_2) = (0,1) \text{ and } x_5 = 0[2], \\ \mu_2(p) &= C \cdot x_4 \text{ if } (x_1, x_2) = (0,1) \text{ and } x_5 = 1[2], \\ \mu_2(p) &= E \cdot x_4 \text{ if } (x_1, x_2) = (1,0) \text{ and } x_5 = 0[2], \\ \mu_2(p) &= 0 \text{ if } (x_1, x_2) = (1,0) \text{ and } x_5 = 1[2] \end{cases}$$

We put $\nu_3 = \nu_2 + \epsilon_2 \cdot \mu_2$, for sufficiently small positive ϵ_2. Figure 8.8 shows the effect of the small perturbation $\epsilon \cdot \mu_2$ on the pyramids Δ_i.

We then subdivide the horizontal sections of the pyramids in a similar way. Of course, this induces a subdivision of the pyramids themselves and also of δ_3. More precisely, first, we slightly perturb ν_3 by a piecewise-linear convex function μ_3 whose value at an integer point p of $C(\Delta_1, \Delta_2)$ with the third coordinate x_3 is equal to e^{x_3}. The lower part of the convex hull of the points $(p, \mu_3(x_3))$ is the graph of μ_3. We put $\nu_4 = \nu_3 + \epsilon_3 \cdot \mu_3$ for a sufficiently small positive ϵ_3, and denote by δ_4 the corresponding refinement of δ_3.

Choose positive numbers a, b, c, d, e, and f satisfying the inequalities $c > e > a > d > f > b$. Let μ_4 be the piecewise-linear function defined by

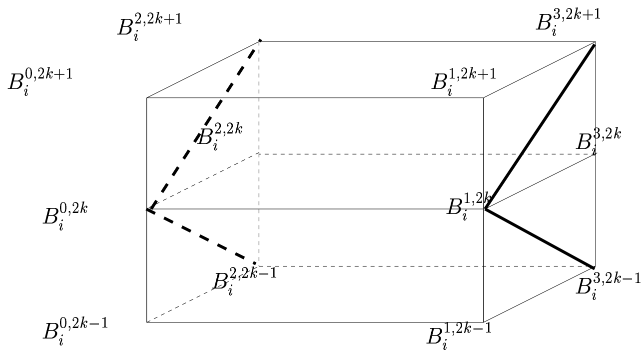

Figure 8.8: Subdivision of slices of pyramids Δ_i into prisms

the following values at integer points of $C(\Delta_1, \Delta_2)$:

$$\begin{cases} \mu_4(p) &= a \cdot x_3 \text{ if } (x_1, x_2) = (0,1),\ x_4 = 0[2] \text{ and } x_5 = 0[2], \\ \mu_4(p) &= b \cdot x_3 \text{ if } (x_1, x_2) = (0,1),\ x_4 = 1[2] \text{ and } x_5 = 0[2], \\ \mu_4(p) &= c \cdot x_3 \text{ if } (x_1, x_2) = (0,1),\ x_4 = n_2 - 1[2] \text{ and } x_5 = 1[2], \\ \mu_4(p) &= d \cdot x_3 \text{ if } (x_1, x_2) = (0,1),\ x_4 = n_2[2] \text{ and } x_5 = 1[2], \\ \mu_4(p) &= e \cdot x_3 \text{ if } (x_1, x_2) = (1,0),\ x_4 = 0[2] \text{ and } x_5 = 0[2], \\ \mu_4(p) &= f \cdot x_3 \text{ if } (x_1, x_2) = (1,0),\ x_4 = 1[2] \text{ and } x_5 = 0[2], \\ \mu_4(p) &= 0 \text{ if } (x_1, x_2) = (1,0) \text{ and } x_5 = 1[2] \end{cases}$$

Put $\nu_5 = \nu_4 + \epsilon_4 \cdot \mu_4$, for a sufficiently small positive ϵ_4, and denote by δ_5 the corresponding refinement of δ_4.

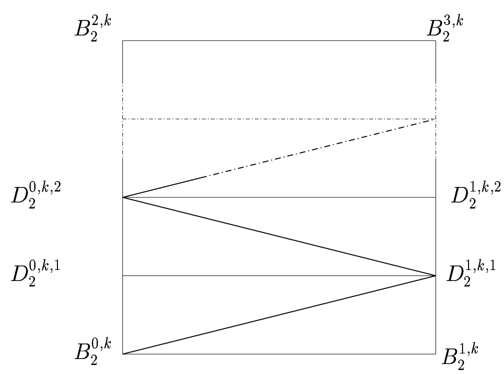

Figure 8.9: Subdivision of a quadrangle

	1	x_3	x_4	x_3x_4	x_5	x_5x_3	x_5x_4	$x_5x_3x_4$
10000	+	+	+	+	+	+	+	+
10010	+	+	−	−	+	+	−	−
10001	+	+	+	+	−	−	−	−
10011	+	+	−	−	−	−	+	+
10100	+	−	+	−	+	−	+	−
10110	+	−	−	+	+	−	−	+
10101	+	−	+	−	−	+	−	+
10111	−	+	+	−	+	−	−	+
01000	+	+	+	+	+	+	+	+
01010	+	+	−	−	+	+	−	−
01100	+	−	+	−	+	−	+	−
01110	+	−	−	+	+	−	−	+

Table 8.1: Sign distribution D.

The order chosen on a, b, c, d, e, f determines the triangulation of P_1 induced by ν_5 (see Figure 8.9). Finally, we refine the triangulation into a convex primitive one in the unique possible way.

We choose any convex triangulation of P_2 refining the subdivision described and patching with the chosen triangulation of P_1 to obtain a convex triangulation of $C(\Delta_1, \Delta_2)$. Take the following distribution D of signs at the integer points of $C(\Delta_1, \Delta_2)$:

any integer point p with coordinates $(x_1, x_2, x_3, x_4, x_5)$ gets "-" if the reduction of p modulo 2 is $(1, 0, 1, 1, 1)$ and it gets "+", otherwise.

Let Y be the complete intersection obtained applying Theorem 2.9 (more precisely, Proposition 2.10). Here and after the orthants will be named after the symmetry under which they are the image of the first one. For example, the orthant x_3 is the image of $(\mathbb{R}^+)^2 \times (\mathbb{R}^+)^3$ under the symmetry $s_{(3)}$ with respect to the hyperplane $x_3 = 0$, and so on. Table 8.1 gives in each orthant the sign of the integer points in P_1^* according to their parities (see Definition 2.2.2).

8.2 Counting Cycles

The nonempty 4-simplices in this construction are those that

- either have two vertices in $\hat{\Delta}_1^*$ and three vertices in $\hat{\Delta}_2^*$, or have three vertices in $\hat{\Delta}_1^*$ and two vertices in $\hat{\Delta}_2^*$,
- in each $\hat{\Delta}_i^*$ the vertices of the simplex do not all carry the same sign.

Thus, any nonempty 4-simplex is the convex hull of a triangle and a segment which do not belong to a symmetric copy of the same $\hat{\Delta}_i$, and both the triangle and the segment have vertices of different signs. We are now going to describe 1-cycles of the complete intersection constructed listing the unions of nonempty 4-simplices containing these 1-cycles. A nonempty 4-simplex will be represented by its intersections with $\hat{\Delta}_1^*$ and $\hat{\Delta}_2^*$. We describe the signs of the triangle and the segment corresponding to a nonempty 4-simplex up to a complete change of signs in $\hat{\Delta}_1^*$ and up to a complete change of signs in $\hat{\Delta}_2^*$. In the sequel of the text, we will often refer to a nonempty 4-simplex meaning the edge of the T-complete intersection contained in it.

Remark 8 *In the description we always assume that n_1 and n_2 are odd. The other cases are similar, and there is a one-to-one correspondence between the cycles we describe and those that appear in any other case.*

To simplify the notations in the description of 1-cycles, we identify integer points of $C(\Delta_1, \Delta_2)$ with their images under compositions of symmetries $s_{(i)}$ and precise explicitly in what orthant appear the cycles under description. We will often label on the figures the segments parallel to the x_3-axis by their slopes, and will refer to a segment of slope h as ***h-segment***.

8.2.1 Type I cycles

The cycles of type I are composed by 4-simplices having an edge on $\hat{\Delta}_1^*$ and a triangle on $\hat{\Delta}_2^*$. The edge on $\hat{\Delta}_1^*$ can be either on a symmetric copy of $[A_1^0, A_1^1]$, or on a symmetric copy of $[A_1^2, A_1^3]$. In the first case we say that the corresponding cycle is of type Ia, and in the second case we say that it is of type Ib.

Cycles of type Ia

A typical edge of a cycle of type Ia is represented on Figure 8.10.

This edge corresponds to a 4-simplex which is a symmetric copy of the convex hull of points $E_1^{0,l}$ and $E_1^{0,l+1}$ in $\hat{\Delta}_1$, and points $D_2^{0,2k+1,2j}$, $B_2^{1,2k}$, and

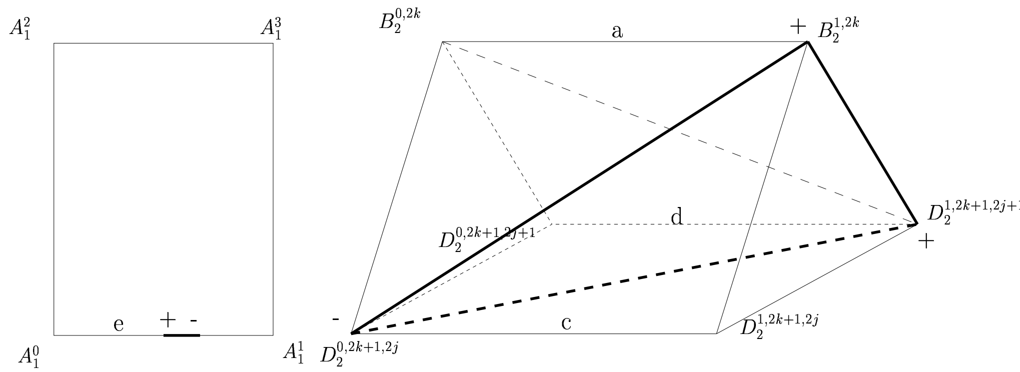

Figure 8.10: Typical edge of a cycle of type Ia

$D_2^{1,2k+1,2j+1}$ in $\hat{\Delta}_2$. Here, $[E_1^{0,l}, E_1^{0,l+1}]$ is a primitive segment in $[A_1^0, A_1^1]$. Let $t_{2k,j}$ be the triangle $(D_2^{0,2k+1,2j}, B_2^{1,2k}, D_2^{1,2k+1,2j+1})$. We describe how this edge is completed into a cycle. If $[D_2^{0,2k+1,2j}, D_2^{1,2k+1,2j}]$ is in the interior of $\hat{\Delta}_2$, then the edge $[D_2^{0,2k+1,2j-1}, D_2^{1,2k+1,2j-1}]$ exists. It is called the ***symmetric*** edge to $[D_2^{0,2k+1,2j+1}, D_2^{1,2k+1,2j+1}]$ with respect to $[D_2^{0,2k+1,2j}, D_2^{1,2k+1,2j}]$. Consider the prism

$$(D_2^{0,2k+1,2j}, D_2^{1,2k+1,2j}, D_2^{0,2k+1,2j+1}, D_2^{1,2k+1,2j+1}, A_1^0, A_1^1).$$

It contains the triangle $(D_2^{0,2k+1,2j}, B_2^{1,2k}, D_2^{1,2k+1,2j-1})$ which is called the ***symmetric*** triangle to $t_{2k,j}$ with respect to $[D_2^{0,2k+1,2j}, D_2^{1,2k+1,2j}]$. This triangle together with the edge $[E_1^{0,l}, E_1^{0,l+1}]$ form a nonempty 4-simplex (since we have a sign distribution depending only on the parity of coordinates). Notice that the common boundary of this 4-simplex and

$$(E_1^{0,l}, E_1^{0,l+1}, D_2^{0,2k+1,2j}, B_2^{1,2k}, D_2^{1,2k+1,2j+1})$$

is nonempty. Thus, the corresponding edges of the T-complete intersection patch together.

Replacing $B_2^{1,2k}$ by $B_2^{1,2k+2}$, when it is possible, we get two other triangles which together with $[E_1^{0,l}, E_1^{0,l+1}]$ form nonempty 4-simplices. We say that these triangles are ***symmetric*** to the previous ones with respect to the section $(B_2^{0,2k}, B_2^{1,2k}, B_2^{2,2k}, B_2^{3,2k})$. When there is no ambiguity we call this section the cd***-section*** and denote it by (c, d). The four edges described above patch together to form a 1-cycle.

We say that the 1-cycle described is the orbit of the edge $\mathfrak{E}$ described on Figure 8.10 under the action of the symmetries with respect to the c-segment and to the cd-section.

Remark 9 *The above cycles are in the orthant $x_5x_3x_4$.*

We can also have the sign distribution shown on Figure 8.11. We complete the edge described on Figure 8.11 by taking its orbit under the action of the symmetries with respect to the d-segment and to the cd-section (these symmetries are called ***d*-symmetry** and ***cd*-symmetry**, respectively).

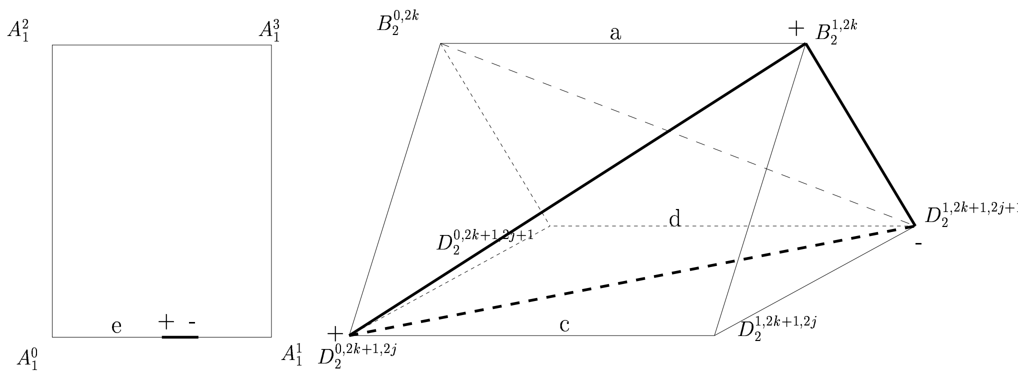

Figure 8.11: Typical edge of a cycle of type Ia

Remark 10 *The above cycles are in the orthant x_3.*

Cycles of type Ib

The cycles of type Ib are very similar to the cycles of type Ia. We could use exactly the same description replacing the segment $[A_1^0, A_1^1]$ by the segment $[A_1^2, A_1^3]$, and the prism whose segments parallel to the x-axis have slopes a, c, and d by the prism whose segments parallel to the x-axis have slopes a, b, and d.

Recall that n_1 is assumed to be odd, so $[A_1^2, A_1^3]$ has slope f. Consider the edge $\mathfrak{E}_1$ corresponding to the 4-simplex s shown on Figure 8.12. The 4-simplex s is the convex hull of a primitive segment of $[A_1^2, A_1^3]$ and the triangle $(D_2^{0,2k,2j}, B_2^{0,2k+1}, D_2^{1,2k+1,2j+1})$.

We complete the edge $\mathfrak{E}_1$ described on Figure 8.12 to a cycle by taking its orbit under the action of the symmetries with respect to the a-segment and to the ab-section.

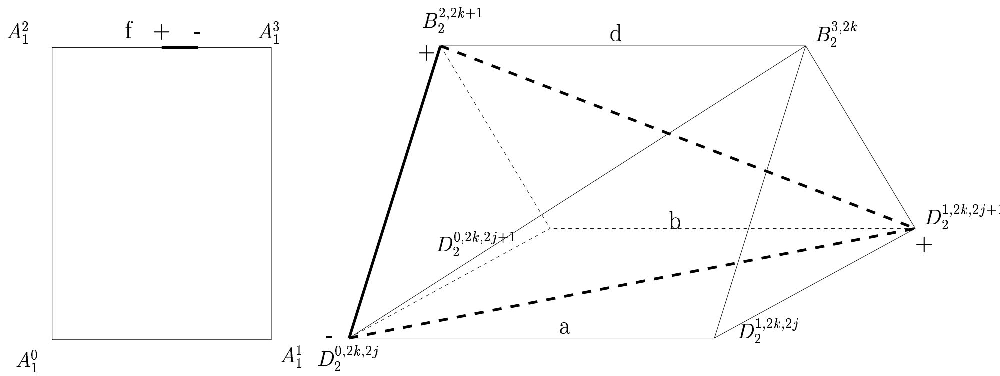

Figure 8.12: Typical edge of a cycle of type Ib

Remark 11 *The above cycles are in the orthant x_3x_4.*

Consider the edge $\mathfrak{E}_2$ described on Figure 8.13. It is completed by taking its orbit under the action of the symmetries with respect to the b-segment and to the ab-section.

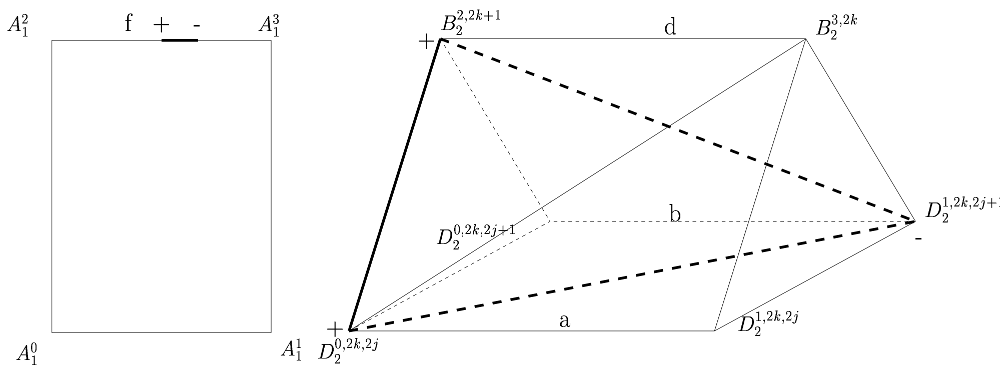

Figure 8.13: Typical edge of a cycle of type Ib

Remark 12 *The above cycles are in the orthant x_5x_4.*

Proposition 8.2 *Let e_1 be a primitive edge of $[A_1^0, A_1^1]$, and e_2 a segment of slope either c or d in the interior of $\hat{\Delta}_2$. Let e_1' be a primitive edge of $[A_1^2, A_1^3]$, and e_2' a segment of slope either a or b in the interior of $\hat{\Delta}_2$. Denote by E_1 the set of pairs (e_1, e_2) and (e_1', e_2'). Then, there is a one-to-one correspondence between the elements of E_1 and the cycles of type I. Thus, the number N_1 of* *cycles of type I is equal to* $\frac{1}{2}m_1(n_2-2)(l_2-1)+\alpha n_1(\mu\frac{1}{4}l_2^2-\frac{l_2}{2})$*, if* l_2 *is even, and* $\frac{1}{2}m_1(n_2-2)(l_2-1)+\alpha n_1(\mu\frac{1}{4}(l_2-1)(l_2+1)-\frac{l_2-1}{2})$*, if* l_2 *is odd.*

Proof . - The first part of the statement follows from the description of the cycles. The number of edges e_2 (resp., e_2') is the number of integer points with odd (resp., even) fifth coordinate x_5 in a triangular face of $\hat{\Delta}_2$ whose base has length n_2. □

8.2.2 Type II cycles

The cycles of type II are composed by 4-simplices having an edge on $\hat{\Delta}_2^*$ and a triangle on $(A_1^0, A_1^1, A_1^2, A_1^3)^*$. As cycles of type I they split into two subtypes.

Cycles of type IIa

First, describe cycles that we call cycles of type IIa. With the chosen distribution of signs these cycles are in the orthant $x_5x_3x_4$ and x_3. Consider the edge $\mathfrak{E}$ corresponding to the 4-simplex shown on Figure 8.14. The point p of the ef-section that carries the sign "-" is the vertex of four triangles in this section as shown on Figure 8.15. The point p is isolated in this section. The edges corresponding to the four triangles together with $[B_2^{1,2k}, B_2^{3,2k+1}]$ form the cycle containing $\mathfrak{E}$.

Cycles of type IIb

The cycles of type IIb are those containing an edge $\mathfrak{E}$ corresponding to the 4-simplex shown on Figure 8.16. With the chosen distribution of signs, they appear in the orthants x_3x_4 and x_5x_3. The description of cycles of type IIb is similar to the above one. Let $\mathfrak{E}$ be the edge shown on Figure 8.16. Let p be the point in the ef-section that carries the sign "-". The edge $\mathfrak{E}$ is completed into a cycle by considering the three other triangles in (e, f) containing p.

Proposition 8.3 *Let* e_1 *be a primitive segment of* $[A_2^0, A_2^4]$*, and* p *an integer point in the interior of the horizontal section* $(A_1^0, A_1^1, A_1^2, A_1^3)$*. There is a one-to-one correspondence between the pairs* (e_1, p) *and the cycles of type II. The number* N_2 *of cycles of type II is equal to* $l_2(m_1-1)(n_1-1)+\alpha\frac{l_2}{2}n_1(n_1-1)$*.*
□

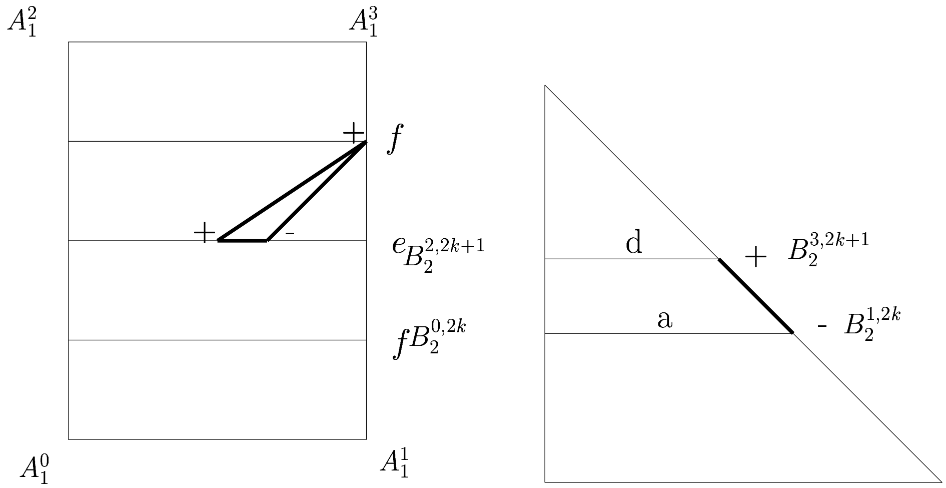

Figure 8.14: Typical edge of a cycle of type IIa

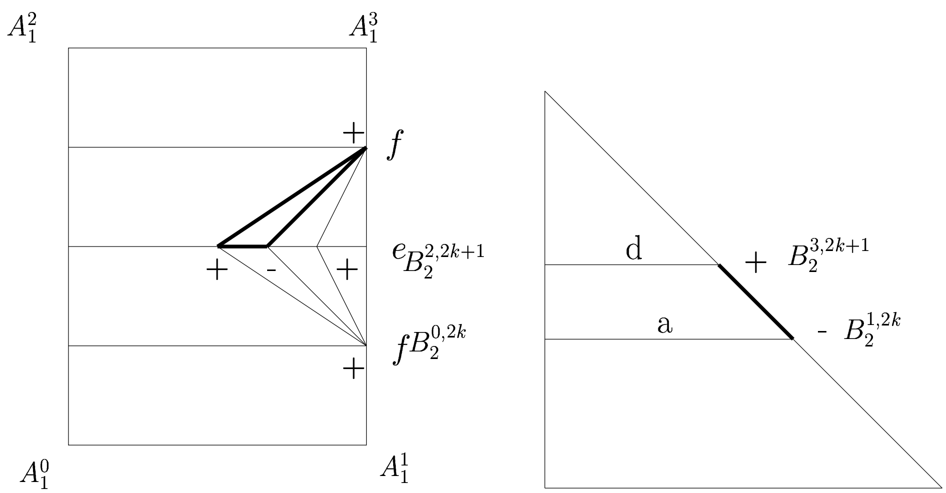

Figure 8.15: complete cycle IIa

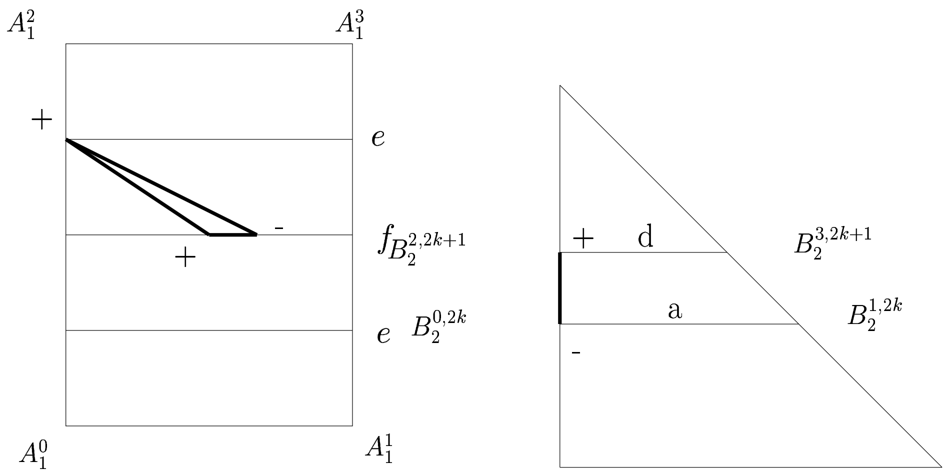

Figure 8.16: Typical edge of a cycle of type IIb

8.2.3 Type III cycles

The third type of cycles looks like the second one, except the fact that the roles of (e, f) and (a, d) are exchanged.

Cycles of type IIIa

With the chosen distribution of signs, the cycles of type IIIa below are in the orthants x_4 and x_3x_4. Let $\mathfrak{E}$ be the edge corresponding to the 4-simplex shown on Figure 8.17. Let p be the point in the part of $\hat{\Delta}_1$ represented on Figure 8.17 that carries the sign "-". The edge $\mathfrak{E}$ is completed to a cycle (as in the case of cycles of type II) by considering the three other triangles of the part of $\hat{\Delta}_1$ represented here that contain p.

Cycles of type IIIb

Let $\mathfrak{E}$ be the edge corresponding to the 4-simplex drawn on Figure 8.18. It can be completed to a cycle exactly as the cycles of type IIIa. These cycles appear in the orthants x_3 and x_3x_4.

Proposition 8.4 *Let e_1 be a primitive edge of $[A_1^0, A_1^2]$, and p an integer point in the interior of (A_2^0, A_2^1, A_2^4) (resp., (A_2^2, A_2^3, A_2^4)) with odd third coordinate (resp., even third coordinate). There is a one-to -one correspondence between the pairs (e_1, p) and the cycles of type III. The number N_3 of cycles*

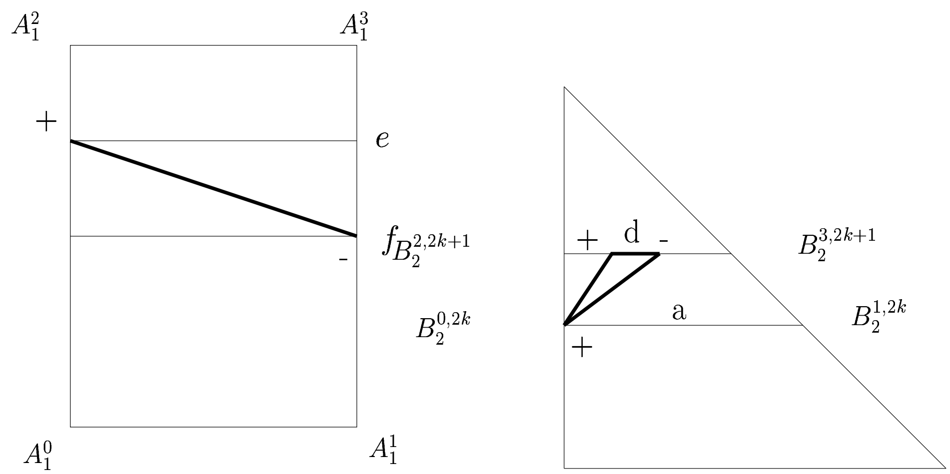

Figure 8.17: Typical edge of a cycle of type IIIa

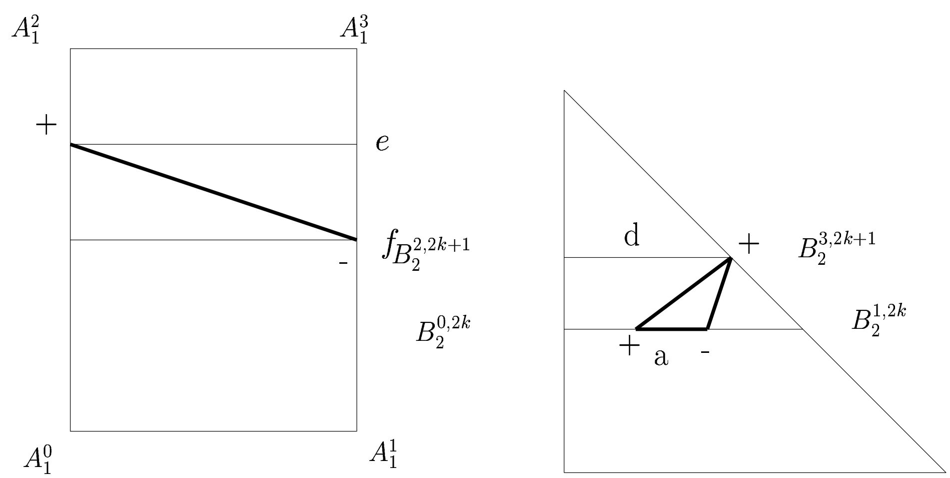

Figure 8.18: Typical edge of a cycle of type IIIb

of type III is equal to $(m_2-2)\frac{l_2-1}{2}n_1+\frac{1}{4}\alpha\mu n_1 l_2(l_2-2)$ *if* l_2 *is even, and* $(m_2-2)\frac{l_2-1}{2}n_1+\frac{1}{4}\alpha\mu n_1(l_2-1)^2$ *if* l_2 *is odd.* □

8.2.4 Mixed cycles

The cycles we describe here are called *mixed*, because each of them is composed by edges of two different types.

Type I-II cycles

Cycles of type I-II split into two subtypes.

Cycles of type I-IIa

We start with an edge $\mathfrak{E}_1$ of type IIa corresponding to a 4-simplex that has 2 vertices on $[A_1^0, A_1^1]$ (see Figure 8.19). The edge $\mathfrak{E}_1$ has a common endpoint with the edge $\mathfrak{E}_2$ of type Ia drawn on Figure 8.20. Notice that here the d-segment is on the boundary of $\hat{\Delta}_2$.

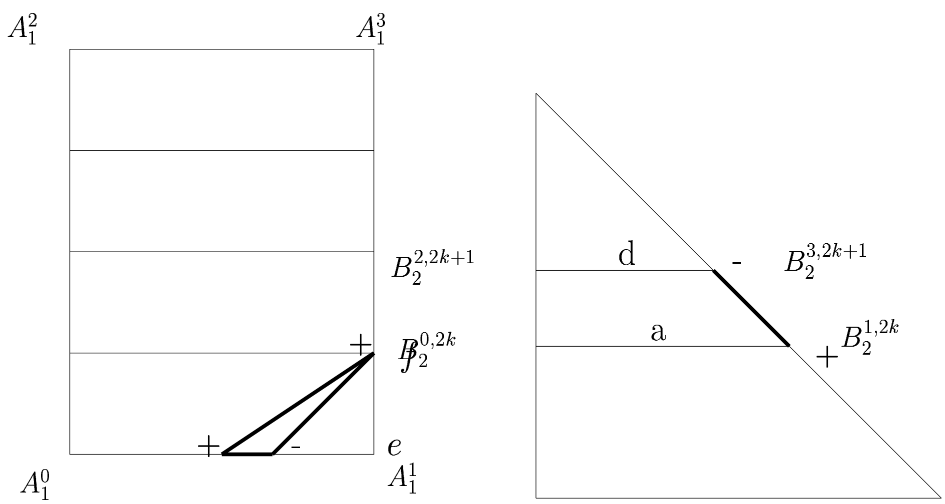

Figure 8.19: Type I-IIa : $\mathfrak{E}_1$

Consider two cases: either the point $D_2^{0,2k+1,2j}$ has the sign "+", or it has the sign "-". If $D_2^{0,2k+1,2j}$ has the sign "+", then the edge $\mathfrak{E}_2$ has a common endpoint with the edge $\mathfrak{E}_3$ represented on Figure 8.22. The edge $\mathfrak{E}_3$

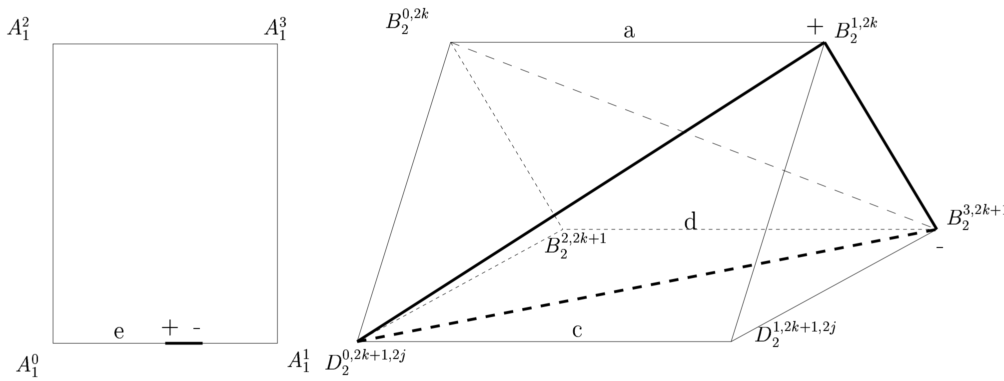

Figure 8.20: Type I-IIa : $\mathfrak{E}_2$

is obtained from $\mathfrak{E}_2$ by the symmetry with respect to the cd-section. Notice that on Figure 8.22, we represented the symmetric prism to the prism of Figure 8.20 by changing the segment of slope a.

In the case when $D_2^{0,2k+1,2j}$ carries the sign "-", the edge $\mathfrak{E}_2$ has a common endpoint with the edge $\mathfrak{E}_2{}'$ which is obtained from $\mathfrak{E}_2$ by the symmetry with respect to the c-segment. The segment $\mathfrak{E}_2{}'$ patches with its symmetric image with respect to the d-segment and so on until we reach the boundary of $\hat{\Delta}_2$. Denote by $\mathfrak{E}_2{}''$ the latter edge. It is contained in a 4-simplex having a facet on the boundary of $\hat{\Delta}_2$. Then using the c-symmetry we pass to another symmetric copy $s(\hat{\Delta}_2)$ of $\hat{\Delta}_2$ and get the edge $\mathfrak{E}_2{}'''$. Let p be the point in which the 4-simplex σ corresponding to $\mathfrak{E}_2{}'''$ intersects a d-segment (see Figure 8.21). Then $sign(p) = sign(B_2^{1,2k})$ and we apply the cd-symmetry to $\mathfrak{E}_2{}'''$ to get the edge $\mathfrak{E}_3{}'''$ which patches with it.

We describe the rest of the edges forming the cycle in the case when $D_2^{0,2k+1,2j}$ carries the sign "+" (the case when $D_2^{0,2k+1,2j}$ has the sign "-" is similar). The edge $\mathfrak{E}_3$ patches with the edge $\mathfrak{E}_4$ of type IIa shown on Figure 8.23. Switch now from the triangle on Figure 8.23 to the other triangle containing the point with sign "-". Then, we see that $\mathfrak{E}_4$ patches with the edge $\mathfrak{E}_5$ represented on Figure 8.24. Notice that $\mathfrak{E}_5$ has a common endpoint with a type Ia edge $\mathfrak{E}_6$ shown on Figure 8.25. Using the cd-symmetry we get the edge $\mathfrak{E}_7$ (see Figure 8.26). The edge $\mathfrak{E}_7$ patches with a type IIa edge $\mathfrak{E}_8$ of Figure 8.27 which has a common endpoint with $\mathfrak{E}_1$. This completes the cycle.

Remark 13 *The only condition for the existence of the above cycles is the*

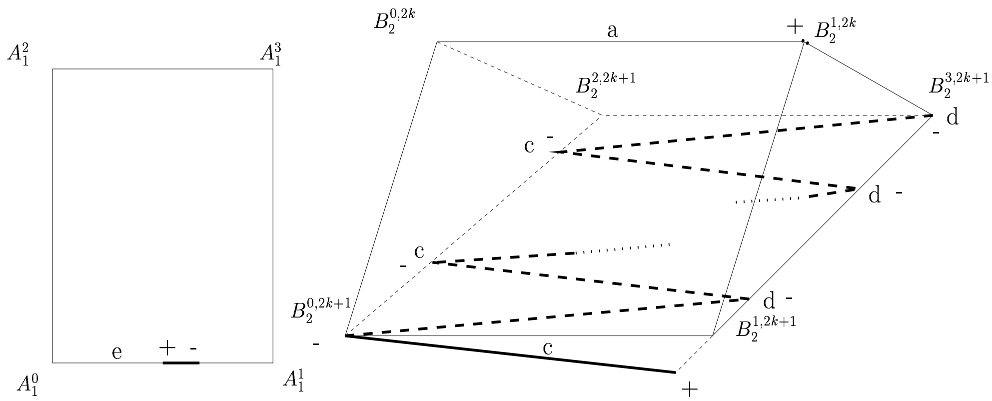

Figure 8.21: Type I-IIa : Crossing $\hat{\Delta}_2$.

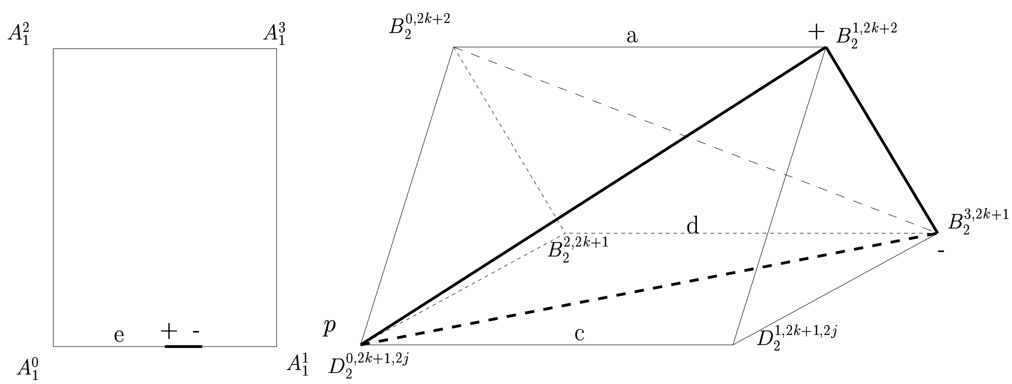

Figure 8.22: Type I-IIa : $\mathfrak{E}_3$

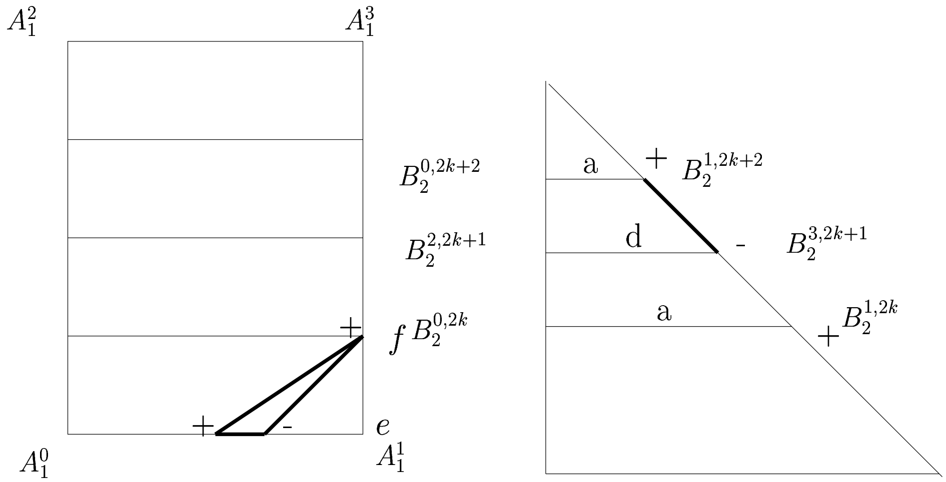

Figure 8.23: Type I-IIa : $\mathfrak{E}_4$

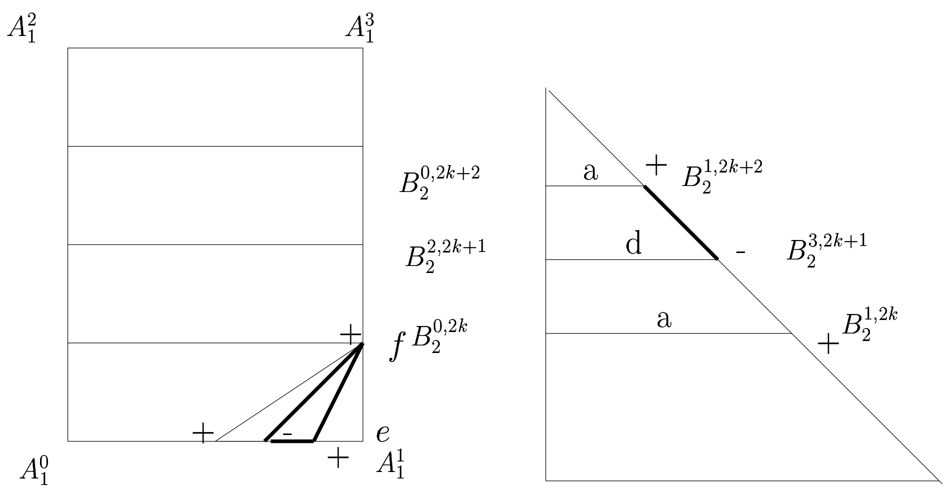

Figure 8.24: Type I-IIa : $\mathfrak{E}_5$

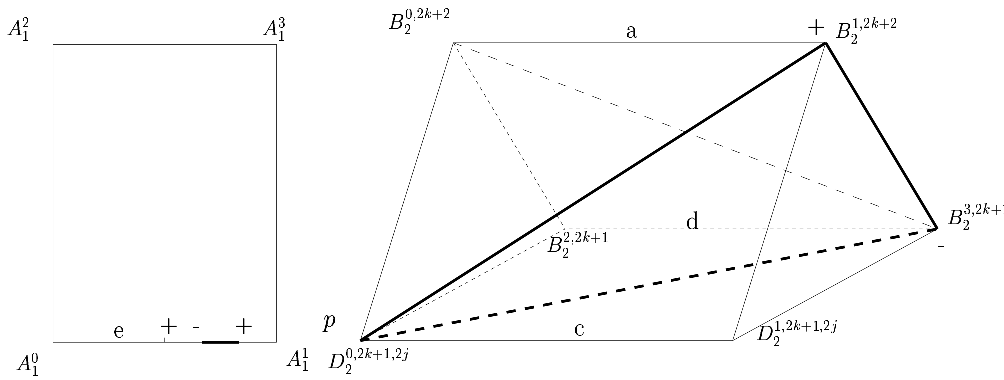

Figure 8.25: Type I-IIa : $\mathfrak{E}_6$

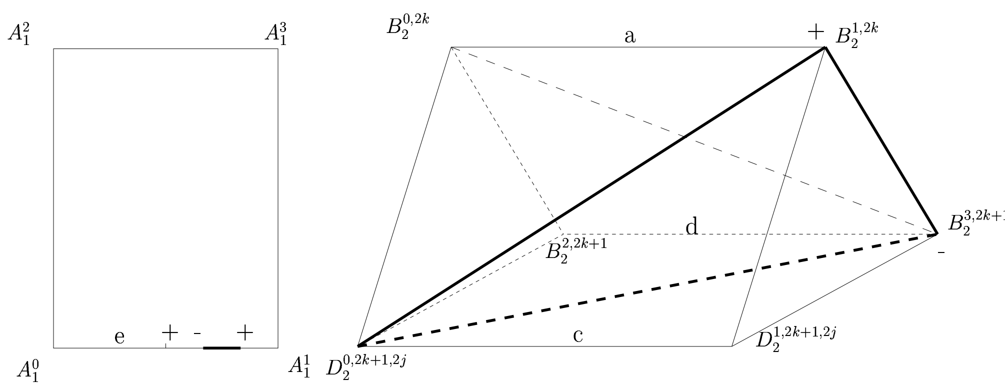

Figure 8.26: Type I-IIa : $\mathfrak{E}_7$

existence of an orthant in which the point p_1 of Proposition 8.5 is isolated in its horizontal section, and the points $B_2^{1,2k}$ and $B_2^{3,2k+1}$ are of opposite signs. This is achieved either in orthant x_3, or in orthant $x_5x_3x_4$.

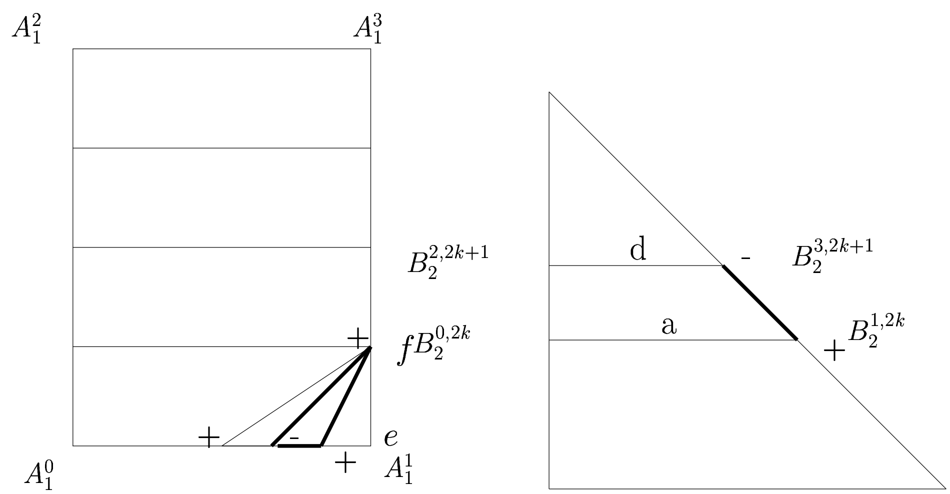

Figure 8.27: Type I-IIa : $\mathfrak{E}_8$

Cycles of type I-IIb

The same kind of description can be used in the case of cycles of type I-IIb. Namely, start with the type IIb edge $\mathfrak{E}_1$ described on Figure 8.28. In the orthant x_3x_4 the edge $\mathfrak{E}_1$ has a common endpoint with the edge $\mathfrak{E}_2$ shown on Figure 8.29. In the orthant x_5x_4 the edge $\mathfrak{E}_1$ has a common endpoint with the edge $\mathfrak{E}_{2'}$ having the same description as $\mathfrak{E}_2$ (except that one has to change the sign of $B_2^{0,2k}$ shown on Figure 8.29). The cycle can be completed in the same way as it was done for the I-IIa cycles.

Remark 14 *The only condition for the existence of the above cycles is the existence of the orthant in which the point p_1 of Proposition 8.5 is isolated in its horizontal section, and the points $B_2^{0,2k}$ and $B_2^{2,2k+1}$ are of opposite signs. This is achieved either in orthant x_3x_4, or in orthant x_3x_5.*

Proposition 8.5 *Let p_1 be an interior integer point of $[A_1^0, A_1^1]$ (resp., $[A_1^2, A_1^3]$), and p_2 an interior integer point of $[A_2^0, A_2^4]$ with odd (resp., even) third coordinate. The pairs (p_1, p_2) are in a one-to-one correspondence with the cycles of*

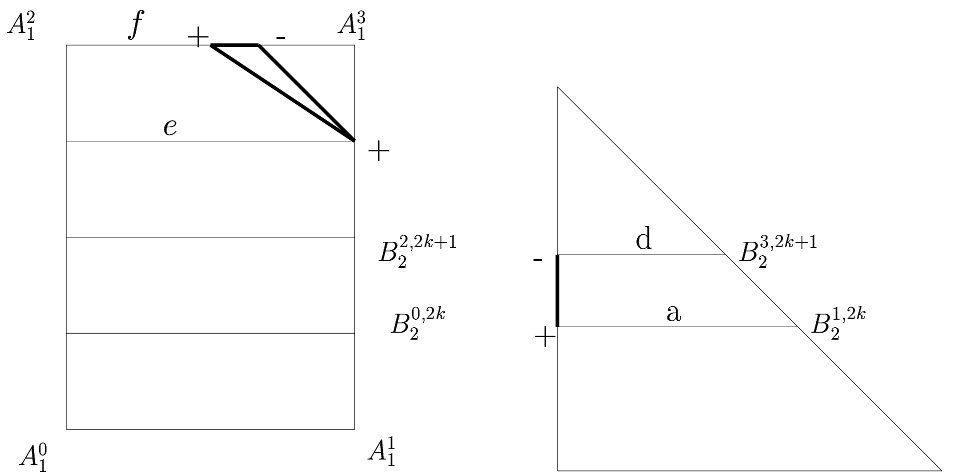

Figure 8.28: Type I-IIb : $\mathfrak{E}_1$

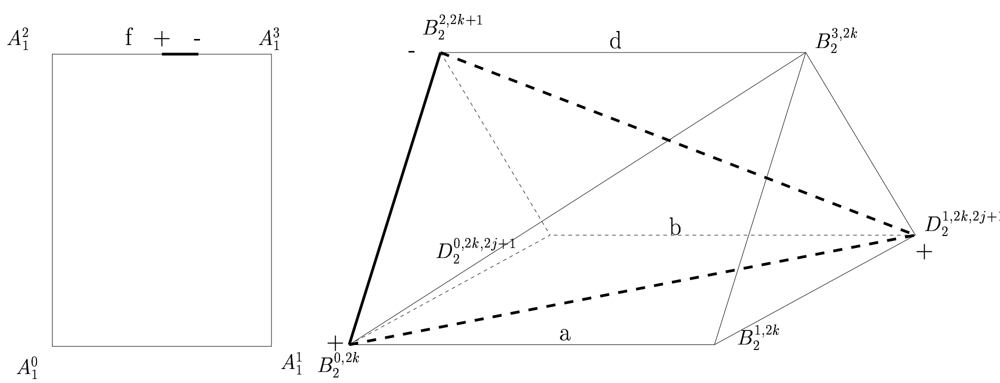

Figure 8.29: Type I-IIb : $\mathfrak{E}_2$

type I-II. The number N_4 of these cycles is equal to $(m_1-1)(l_2-1)+n_1\alpha\lfloor\frac{l_2}{2}\rfloor$.□

Type II-III cycles

There are four different sorts of cycles of type II-III. They differ by the symmetries that are used to describe them. For the first sort that we describe here, we use the d-symmetry and the f-symmetry. The sorts of cycles are labelled by the symmetries occurring in their description. For example, the first sort is called type II-III df. The three other sorts are type II-III ae, type II-III de, and type II-III af. The cycles of types II-III df, II-III ae, II-III de, and II-III af appear in the orthants x_4, x_3, x_5x_4, and x_5x_3, respectively.

Cycles of type II-III *df*

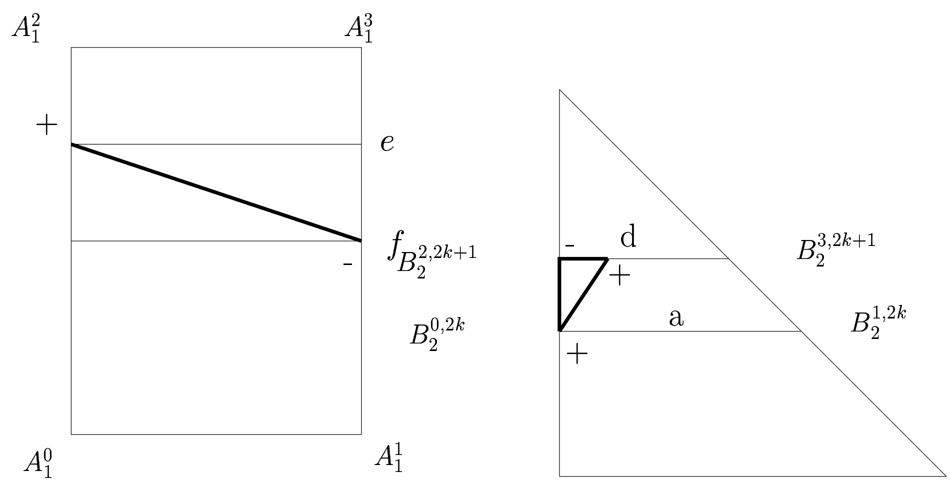

Figure 8.30: Type II-III df : $\mathfrak{E}_1$

We start with the edge $\mathfrak{E}_1$ of type IIIa shown on Figure 8.30. It has a common endpoint with the edge $\mathfrak{E}_2$ of type IIb described on Figure 8.31. All the points on the f-segment have the same sign.

The edge $\mathfrak{E}_2$ has a common endpoint with $\mathfrak{E}_2{}'$ which corresponds to the neighboring triangle in the ef-section (see Figure 8.32). Continuing to climb down this f-segment we get to an edge corresponding to a 4-simplex having a facet on the boundary of $\hat{\Delta}_2$. We then pass to another symmetric copy

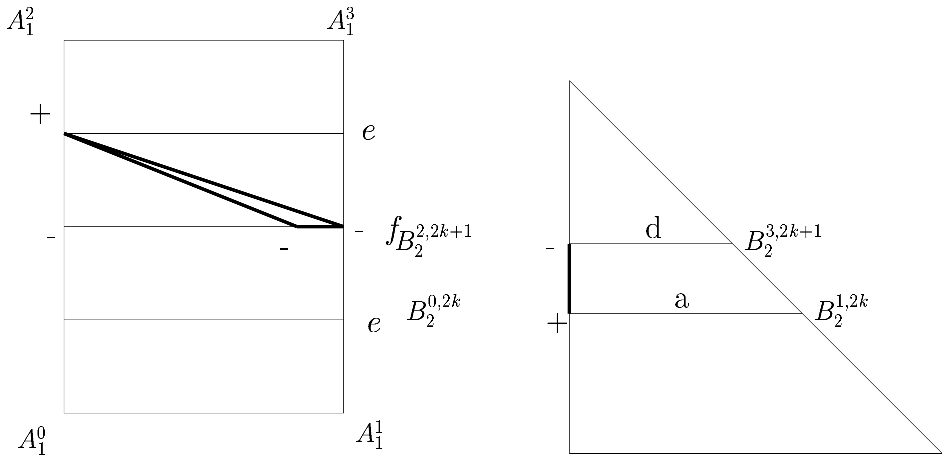

Figure 8.31: Type II-III df : $\mathfrak{E}_2$

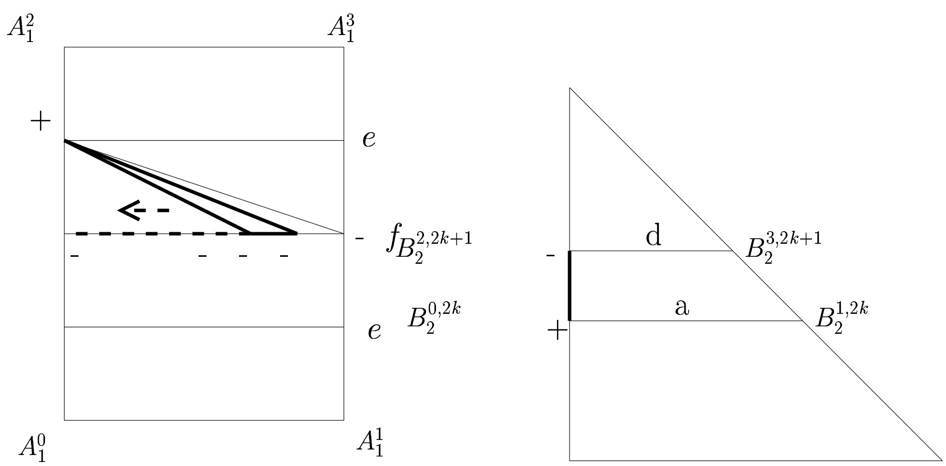

Figure 8.32: Type II-III df : Travelling along the f-segment.

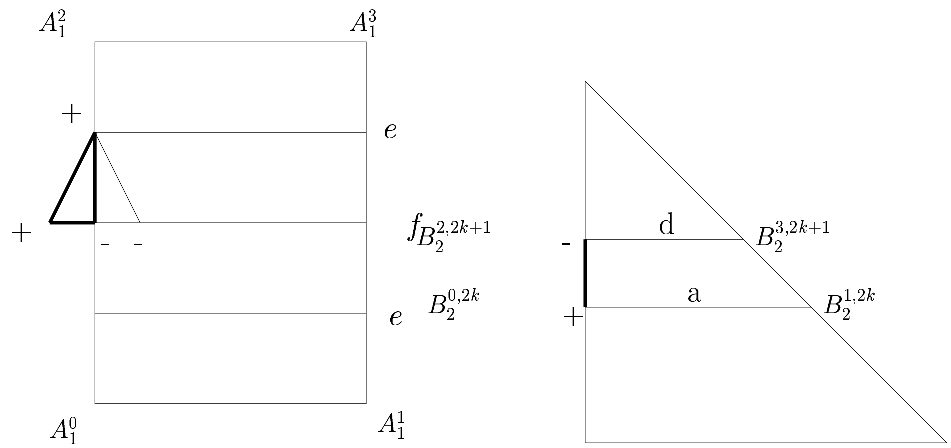

Figure 8.33: Type II-III df : Changing of symmetric copy

of $\hat{\Delta}_2$ (see Figure 8.33), where we apply the f-symmetry and climb up the f-segment to $\hat{\Delta}_2$ in a similar way. Finally, we get an edge which patches with the edge $\mathfrak{E}_3$ on Figure 8.34. To pass from $\mathfrak{E}_3$ to the edge $\mathfrak{E}_4$ shown on Figure 8.35 we use the d-symmetry. Then, we do the same trick backwards to complete the cycle.

The cycles of type II-III ae can be described in the same way replacing the d-symmetry by the a-symmetry and the f-symmetry by the e-symmetry. We start with the edge $\mathfrak{E}_1$ of type IIIb described on Figure 8.36. The edge $\mathfrak{E}_1$ patches with the edge $\mathfrak{E}_2$ of type IIb (see Figure 8.37). Then we continue as for the type II-III df.

We now describe cycles of type II-III de. Start from edge $\mathfrak{E}_1$ of Figure 8.38. It patches with edge $\mathfrak{E}_2$ shown on Figure 8.39. Then, following the cycle, we have edges $\mathfrak{E}_3$ (see Figure 8.40), $\mathfrak{E}_4$ (see Figure 8.41), $\mathfrak{E}_5$ (see Figure 8.42) and $\mathfrak{E}_6$ (see Figure 8.43). Then, we use the e-symmetry to $\mathfrak{E}_6$ to get the next edge and continue to follow the cycle to complete it.

The type II-III af cycles can be described in the same way exchanging d and a and also e and f. An edge $\mathfrak{E}_1$ to start is described on Figure 8.44.

Proposition 8.6 *Let p_1 be an interior integer point of $[A_1^0, A_1^2]$, and p_2 an interior integer point of $[A_2^0, A_2^4]$. The pairs (p_1, p_2) are in one-to-one correspondence with the cycles of type II-III. The number N_5 of these cycles is equal to $(n_1 - 1)(l_2 - 1)$.* $\square$

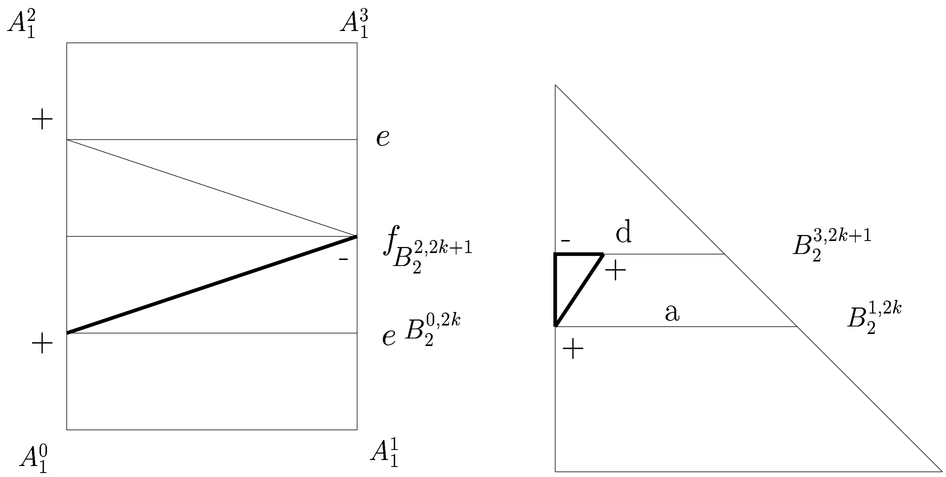

Figure 8.34: Type II-III df : $\mathfrak{E}_3$

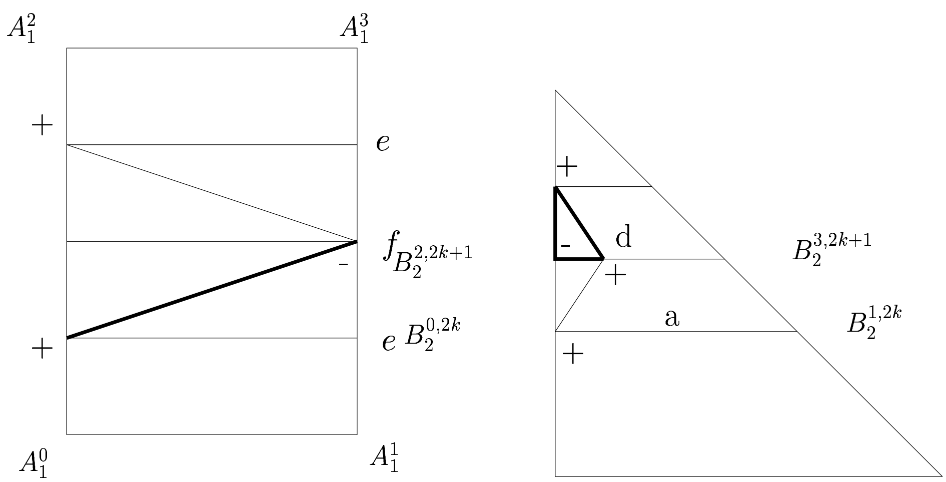

Figure 8.35: Type II-III df : $\mathfrak{E}_4$

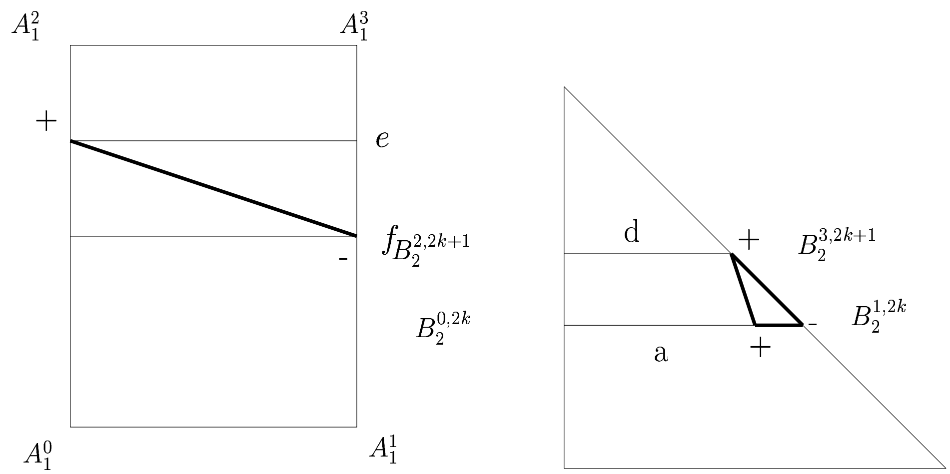

Figure 8.36: Type II-III ae : $\mathfrak{E}_1$

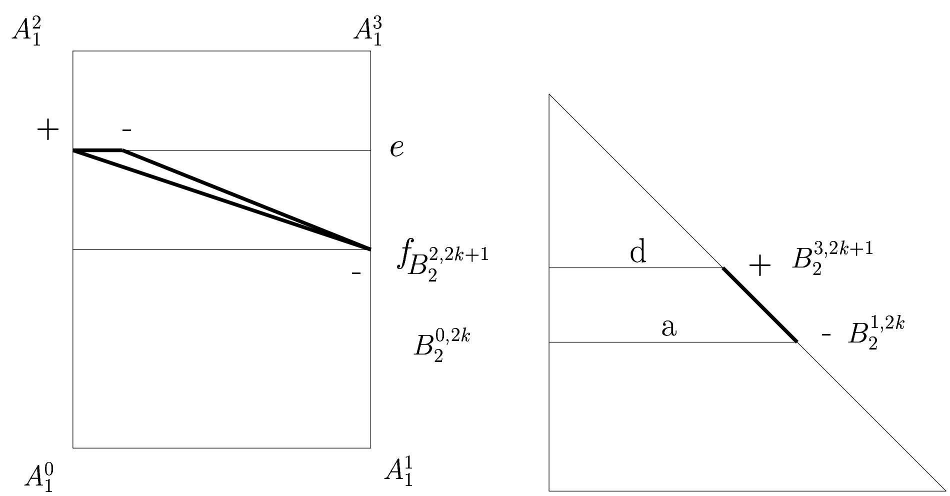

Figure 8.37: Type II-III ae : $\mathfrak{E}_2$

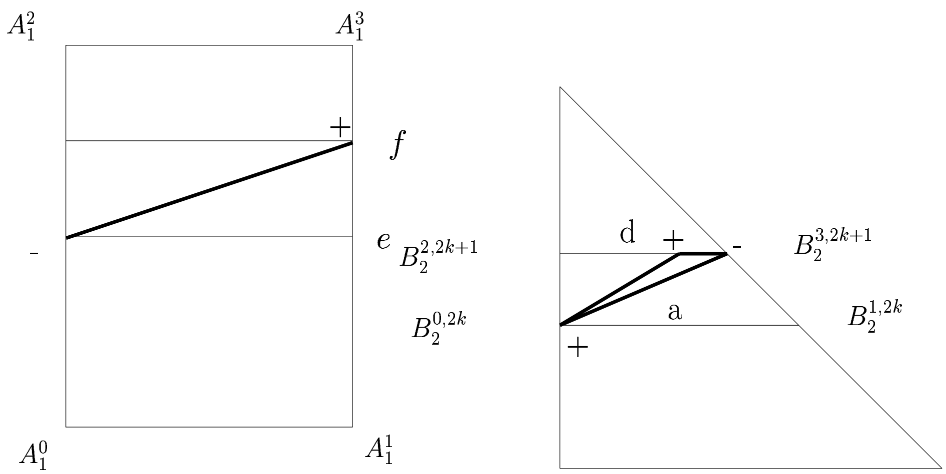

Figure 8.38: Type II-III de : $\mathfrak{E}_1$

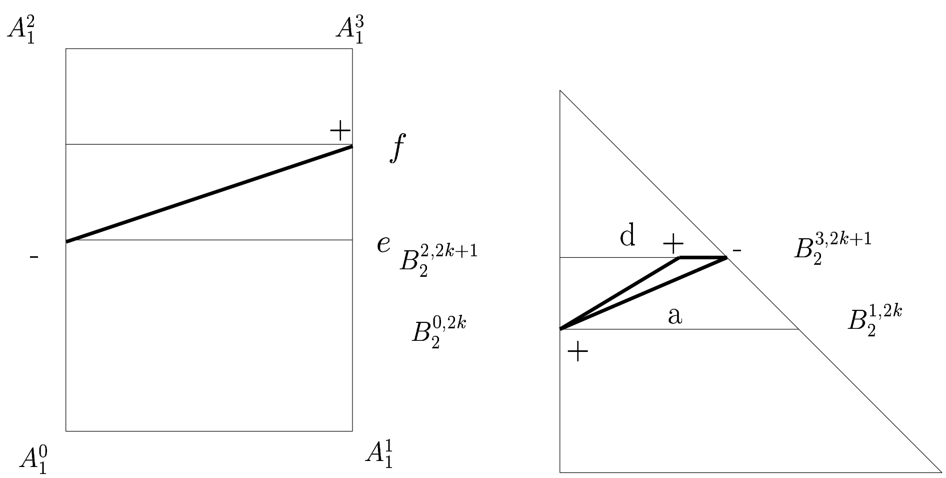

Figure 8.39: Type II-III de : $\mathfrak{E}_2$

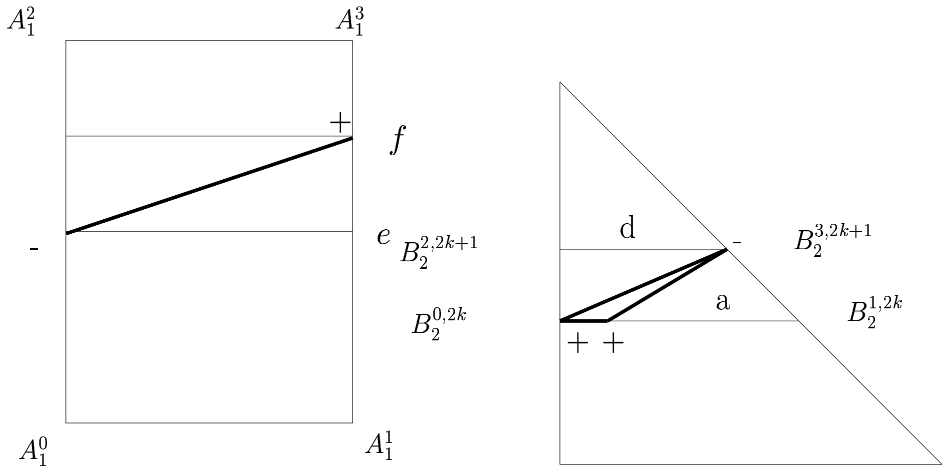

Figure 8.40: Type II-III de : $\mathfrak{E}_3$

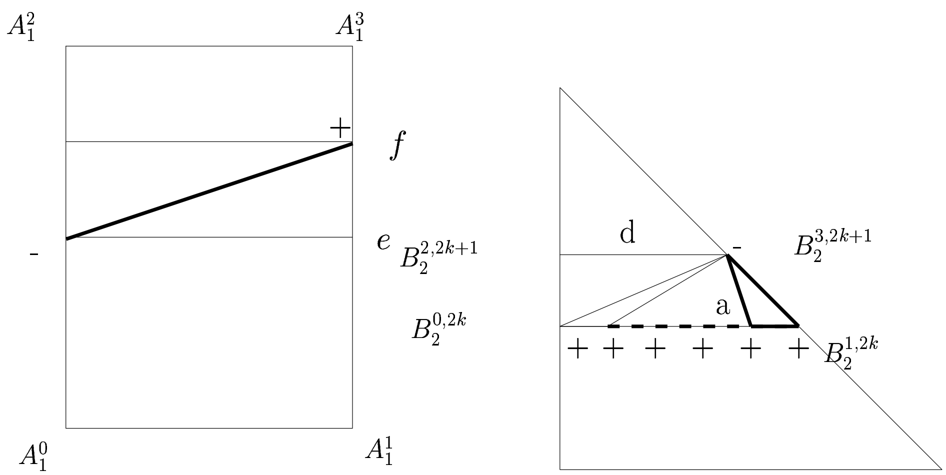

Figure 8.41: Type II-III de : $\mathfrak{E}_4$

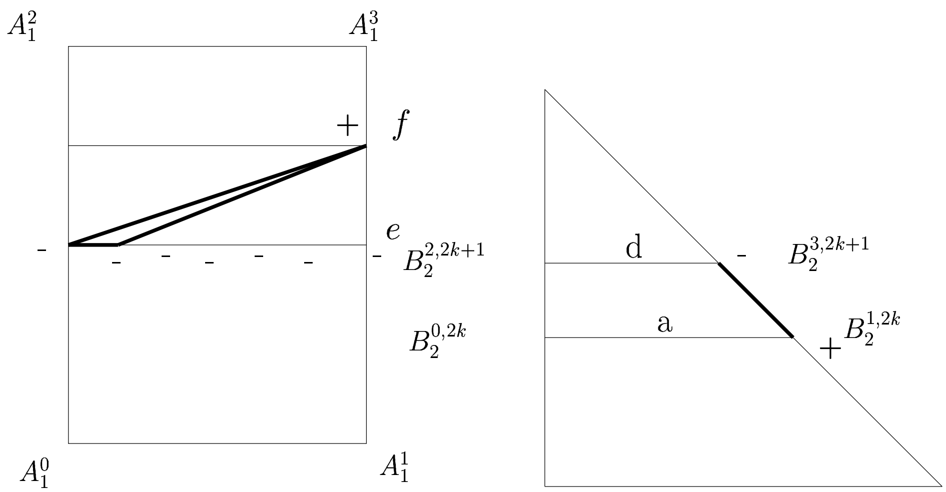

Figure 8.42: Type II-III de : $\mathfrak{E}_5$

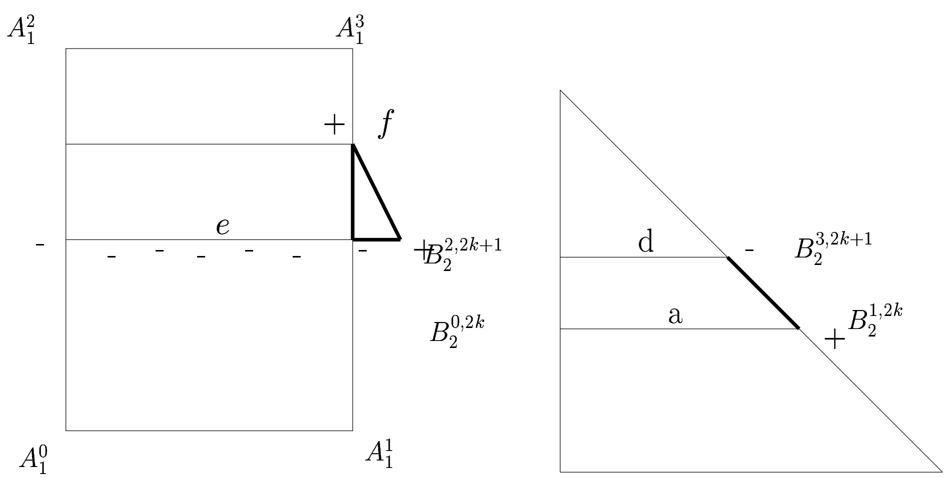

Figure 8.43: Type II-III de : $\mathfrak{E}_6$

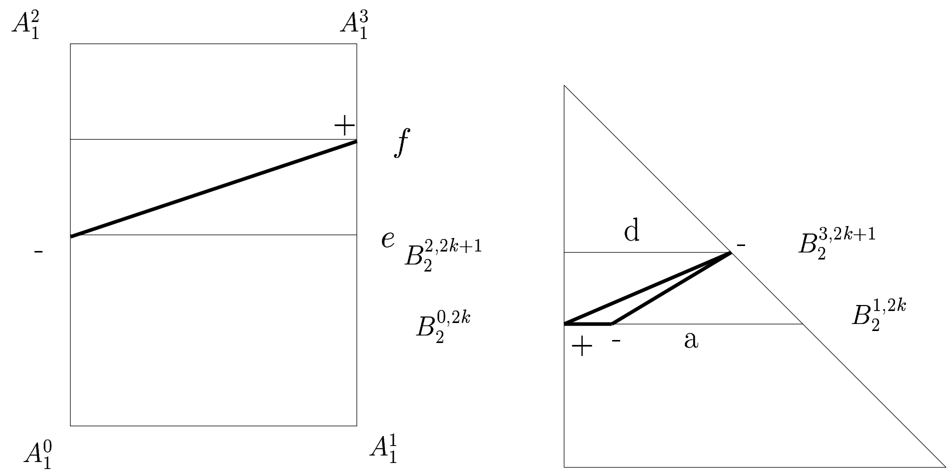

Figure 8.44: Type II-III af : $\mathfrak{E}_1$

8.2.5 Linear cycles

The cycles we describe here are composed by edges of three main types. We call these cycles *linear*. There are two sorts of linear cycles. The construction of the linear cycles of the first sort uses the cd-symmetry, and the construction of the linear cycles of the second sort uses the ab-symmetry. We describe here only the linear cycles of the first sort. With the chosen distribution of signs, these cycles appear in the orthant x_5x_4. The linear cycles of the second sort can be described in a completely similar way. They appear in the orthant x_5x_3.

Start from $\mathfrak{E}_1$ on Figure 8.45 and follow the cycle as described on Figures 8.46, 8.47, and 8.48. Notice that all the points of the e-segment on Figure 8.48 have the same sign. We climb up the e-segment and then switch to another symmetric copy of $\hat{\Delta}_1$. Apply the cd-symmetry to the edge $\mathfrak{E}_5$ and complete the cycle using the same trick backwards.

To describe the second sort kind of linear cycles, one can start with an edge described on Figure 8.50 and imitate the description above.

Proposition 8.7 *There is a one-to-one correspondence between the interior integer points of $[A_2^0, A_2^4]$ and the linear cycles. The number N_6 of linear cycles is equal to $l_2 - 1$.* $\square$

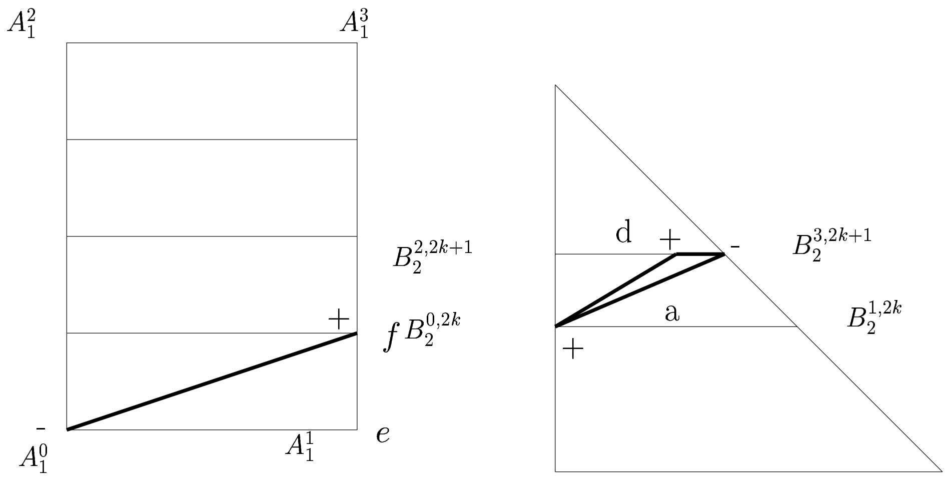

Figure 8.45: linear cycle : $\mathfrak{E}_1$

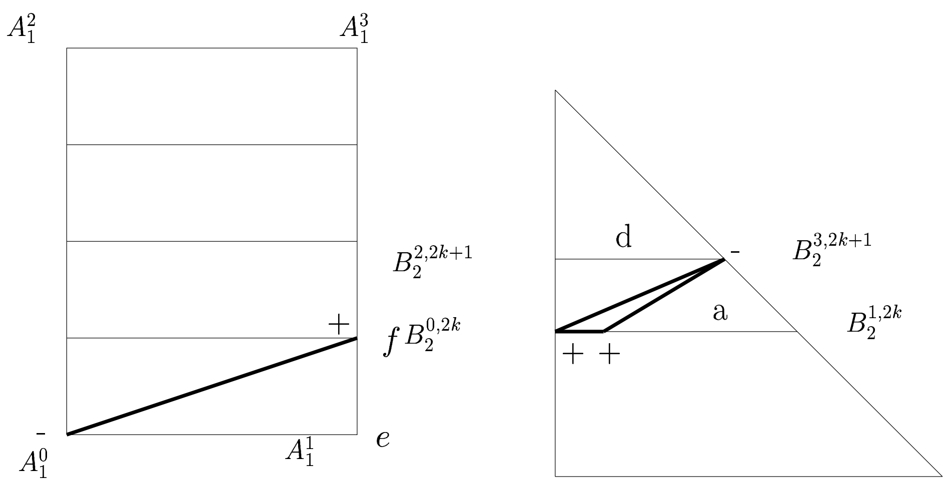

Figure 8.46: linear cycle : $\mathfrak{E}_2$

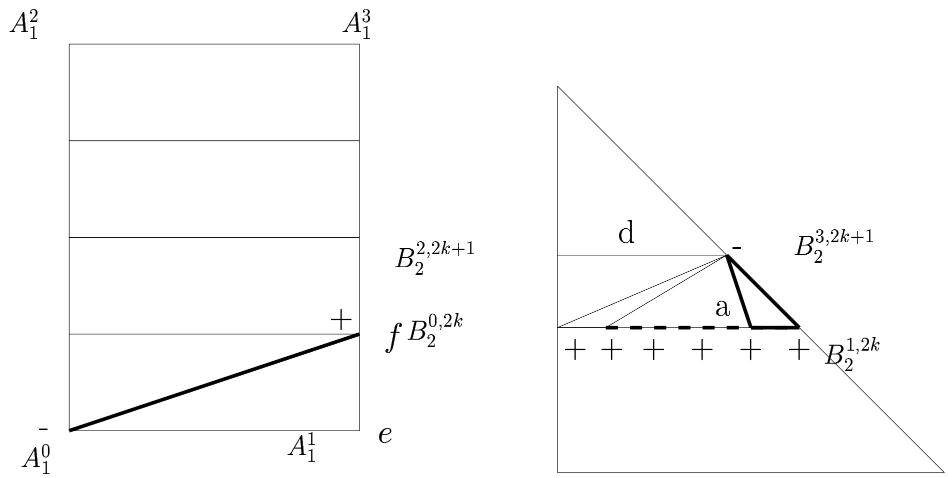

Figure 8.47: linear cycle : $\mathfrak{E}_3$

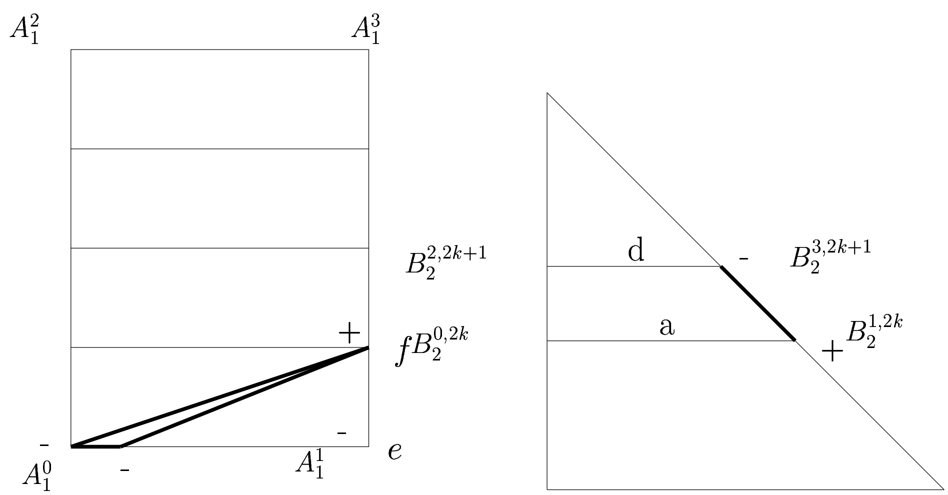

Figure 8.48: linear cycle : $\mathfrak{E}_4$

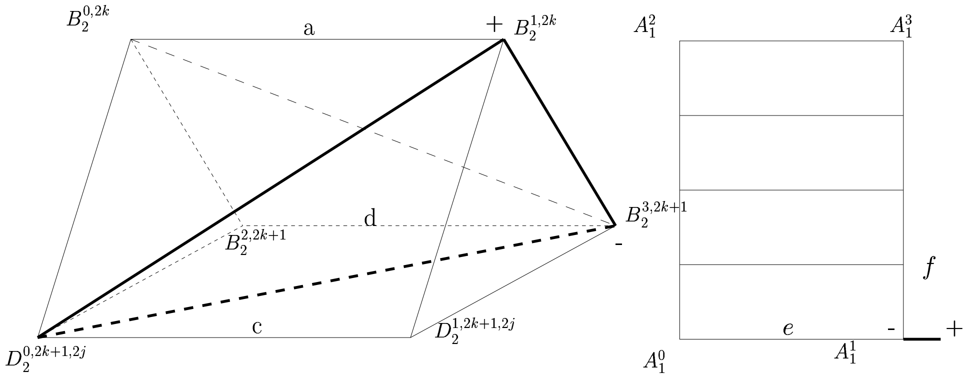

Figure 8.49: linear cycle : $\mathfrak{E}_5$

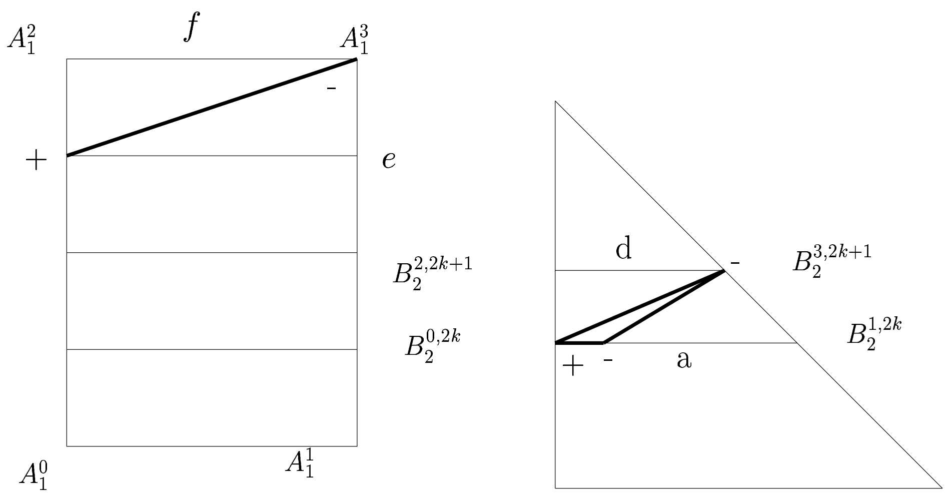

Figure 8.50: linear cycle of the second kind : $\mathfrak{E}_1$

8.2.6 Proof of Theorem 8.1

Lemma 15 *Under the hypotheses of Theorem 8.1, one has*

$$b_*(Y) = (2\lambda\mu + \alpha\mu^2)(l_1^2 l_2 + l_2^2 l_1) - (2\lambda\mu + \alpha\mu^2 + 2\lambda + 2\mu + \alpha\mu)l_1 l_2 + 4.$$

Proof . - The statement immediately follows from Theorem 2.18. □

Lemma 16 *Under the hypotheses of Theorem 8.1, the total number of cycles described is equal to* $(\lambda\mu+\frac{1}{2}\alpha\mu^2)(l_1^2 l_2+l_2^2 l_1)-(\lambda\mu+\frac{1}{2}\alpha\mu^2+\lambda+\mu+\frac{1}{2}\alpha\mu)l_1 l_2+1$.

Proof . - We evaluate the sum $\sum_{i=1}^{6} N_i$ substituting λl_i for m_i and μl_i for n_i, and get the result. □

There exists one cycle more that we did not describe. It contains, for example, the edges of type I intersecting the base of $\hat{\Delta}_2$. Thus, Y is maximal. □

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