Spring School "Tropical Geometry and Cluster Algebras"

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Cluster algebras: basic notions and key examples

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based on joint work with

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Main references

Cluster algebras I–IV:

J. Amer. Math. Soc. **15** (2002); (with A. Z.);
Invent. Math. **154** (2003); (with A. Z.);
Duke Math. J. **126** (2005) (with A. B. and A. Z.);
Compos. Math. **143** (2007) (with A. Z.).

Y-systems and generalized associahedra,
Ann. of Math. **158** (2003) (with A. Z.).

Cluster algebras and triangulated surfaces I–II:

Acta Math. **201** (2008) (with M. S. and D. T.);
preprint (2008–2012) (with D. T.).

Tensor diagrams and cluster algebras,
in preparation (2012) (with P. P.).

LECTURE 1

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The **Cluster Algebras Portal** at

⟨`http://www.math.lsa.umich.edu/~fomin/cluster.html`⟩

provides links to:

- $>$400 papers on the `arXiv`;
- a separate listing for lecture notes and surveys;
- conferences, seminars, courses, thematic programs, etc.

The **Cluster Algebras program at MSRI** in Fall 2012 will in particular include:

- introductory workshops: August 22 – September 7;
- research workshop: October 29 – November 2.

Plan

Total positivity.
First prototypical example: base affine space.
Quiver mutations. Basic notions of cluster algebras.

Basic structural results.
Second prototypical example: Grassmannians of planes.
Finite type classification. Cluster combinatorics.

Cluster algebras associated with surfaces.
Laminations and lambda lengths.
Finite mutation type classification.

Zamolodchikov periodicity.
Cluster structures in general Grassmannians.

Fundamentals of classical invariant theory.
Tensor diagrams.
Cluster structures in rings of SL_3-invariants.

Motivations and applications

Cluster algebras are a class of commutative rings equipped with a particular kind of combinatorial structure.

Motivation: algebraic/combinatorial study of *total positivity* and *dual canonical bases* in semisimple algebraic groups (G. Lusztig).

Cluster-algebraic structures have been identified and explored in several mathematical disciplines, including:

- Lie theory and quantum groups;
- quiver representations;
- Poisson geometry and Teichmüller theory;
- discrete integrable systems.

Total positivity

A real matrix is *totally positive* (resp., *totally nonnegative*) if all its minors are positive (resp., nonnegative).

Total positivity is a remarkably widespread phenomenon:

- 1930s-40s (F. Gantmacher–M. Krein, I. Schoenberg)
  classical mechanics, approximation theory

- 1950s-60s (S. Karlin, A. Edrei–E. Thoma)
  stochastic processes, asymptotic representation theory

- 1980s-90s (I. Gessel–X. Viennot, Y. Colin de Verdière)
  enumerative combinatorics, graph theory

- 1990s-2000s (G. Lusztig, S. F.-A. Zelevinsky)
  Lie theory, quantum groups, cluster algebras

Totally positive varieties
(informal concept)

Let X be a complex algebraic variety. Fix a collection $\boldsymbol{\Delta}$ of "important" regular functions on X. The corresponding *totally positive variety* $X_{>0}$ (resp., *totally nonnegative variety* $X_{\geq 0}$) is the set of points in X at which all functions in $\boldsymbol{\Delta}$ take positive (resp., nonnegative) real values.

Example: $X = \mathrm{GL}_n(\mathbb{C})$, $\boldsymbol{\Delta} = \{\text{all minors}\}$.

Example: the *totally positive/nonnegative Grassmannian*.

Generalizations to arbitrary semisimple Lie groups.

In these examples, the notion of positivity depends on a particular choice of coordinates.

Why study totally positive/nonnegative varieties?

(1) The structure of $X_{\geq 0}$ as a semialgebraic set can reveal important features of the complex variety X.

Example: unipotent upper-triangular matrices.

$$X = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$X_{\geq 0} = \left\{ \begin{array}{c} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ xz - y \geq 0 \end{array} \right\}$$

$$\begin{tikzcd}[column sep=small]
& X \arrow[dl, dash] \arrow[dr, dash] & \\
y=0 \arrow[d, dash] \arrow[drr, dash] & & xz-y=0 \arrow[d, dash] \arrow[dll, dash] \\
x=y=0 \arrow[dr, dash] & & y=z=0 \arrow[dl, dash] \\
& x=y=z=0 &
\end{tikzcd}$$

Why study totally positive/nonnegative varieties? (continued)

(2) Sometimes, totally positive varieties can be identified with important spaces.

Examples: decorated Teichmüller spaces (R. Penner, S.F.-D.T.); "higher Teichmüller theory" (V. Fock–A. Goncharov); Schubert positivity via Peterson's map (K. Rietsch, T. Lam).

(3) We would like to better understand the interplay between two objects associated with a complex algebraic variety X (endowed with some additional structure): the *tropicalization* of X and its positive part $X_{>0}$.

From positivity to cluster algebras

Which algebraic varieties X have a "natural" notion of positivity?

Which families $\mathbf{\Delta}$ of regular functions should one consider in defining this notion?

The concept of a cluster algebra can be viewed as an attempt to answer these questions.

Prototypical example of a cluster algebra

Consider the algebra

$$\mathcal{A} = \mathbb{C}[\mathsf{SL}_n]^N \subset \mathbb{C}[x_{11}, \ldots, x_{nn}]/\langle \mathsf{det}(x_{ij}) - 1 \rangle$$

of polynomials in the matrix entries of an $n \times n$ matrix $(x_{ij}) \in \mathsf{SL}_n$ which are invariant under the natural action of the subgroup

$$N = \left\{ \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \subset \mathsf{SL}_n(\mathbb{C})$$

by multiplication on the right.

$\mathcal{A}$ is the *base affine space* for $G = \mathsf{SL}_n(\mathbb{C})$.

Flag minors and positivity

Invariant theory: $\mathcal{A}=\mathbb{C}[\mathrm{SL}_n]^N$ is generated by the *flag minors*

$$\Delta_I : x \mapsto \det(x_{ij} | i \in I, j \leq |I|),$$

for $I \subsetneq \{1, \ldots, n\}$, $I \neq \emptyset$.

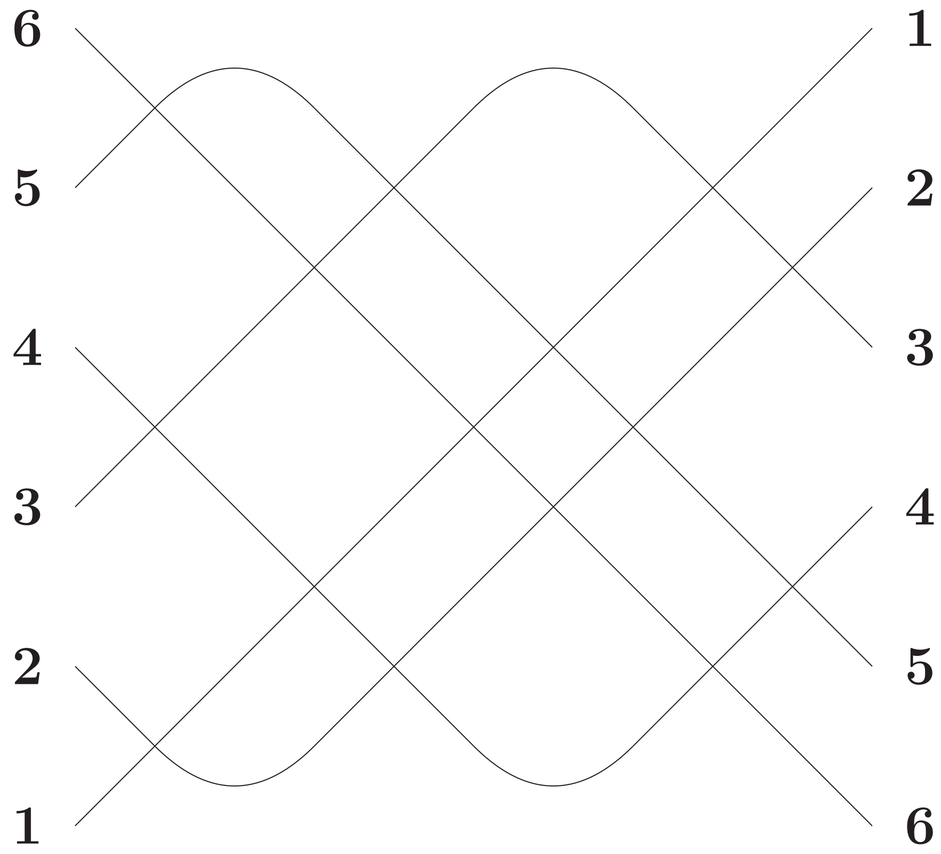
The flag minors Δ_I satisfy well-known homogeneous quadratic identities (*generalized Plücker relations*).

A point in G/N represented by a matrix $x \in G$ is *totally positive* (resp., *totally nonnegative*) if all flag minors Δ_I take positive (resp., nonnegative) values at x.

There are $2^n - 2$ flag minors. How many do we have to test in order to verify that a given point is totally positive?

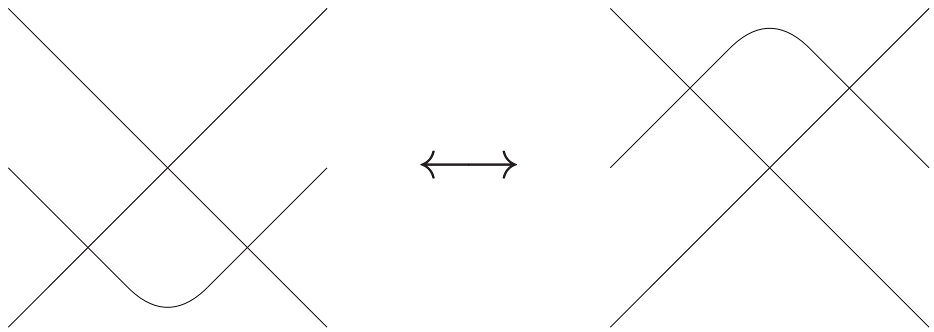
Answer: enough to check $\dim(G/N) = \frac{(n-1)(n+2)}{2}$ flag minors.

Pseudoline arrangements

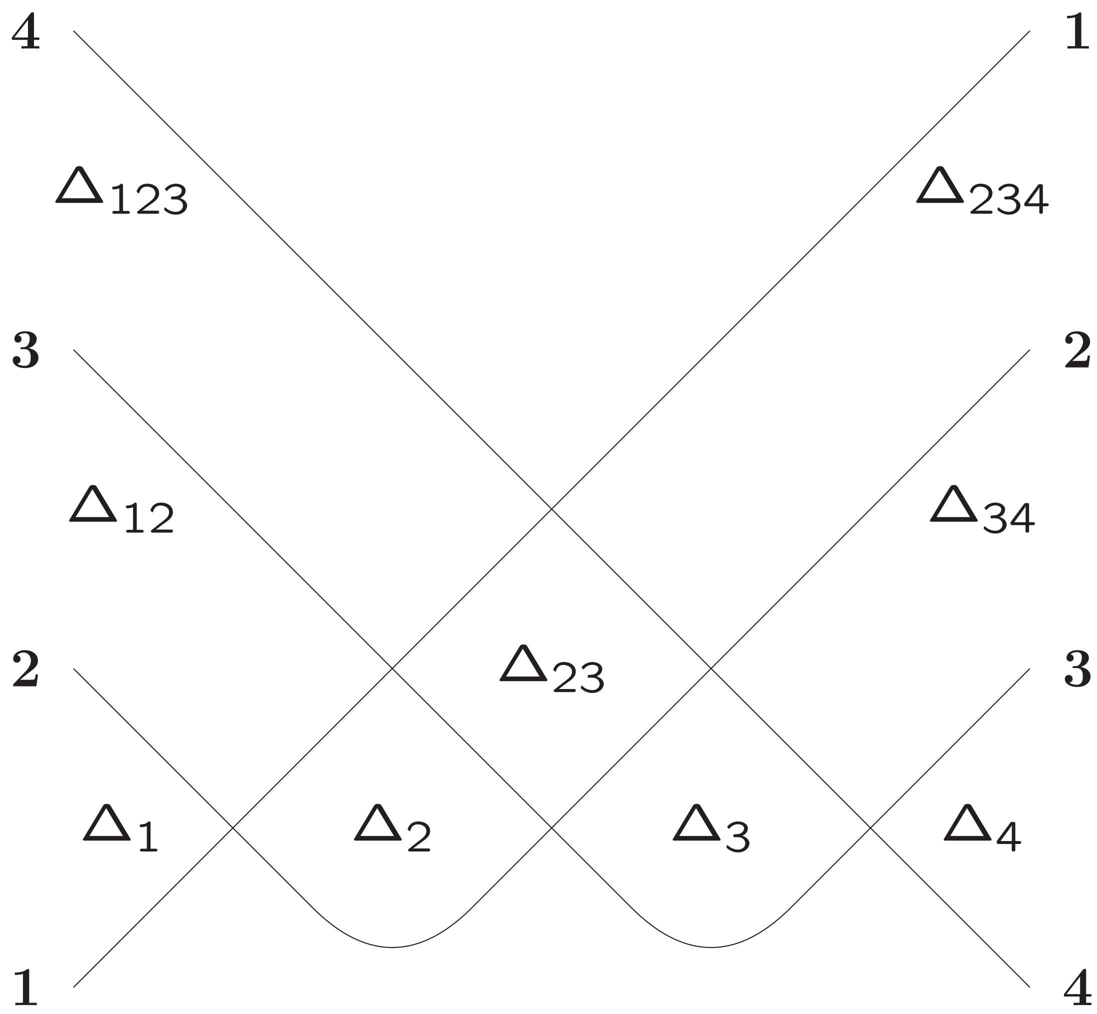


Braid moves

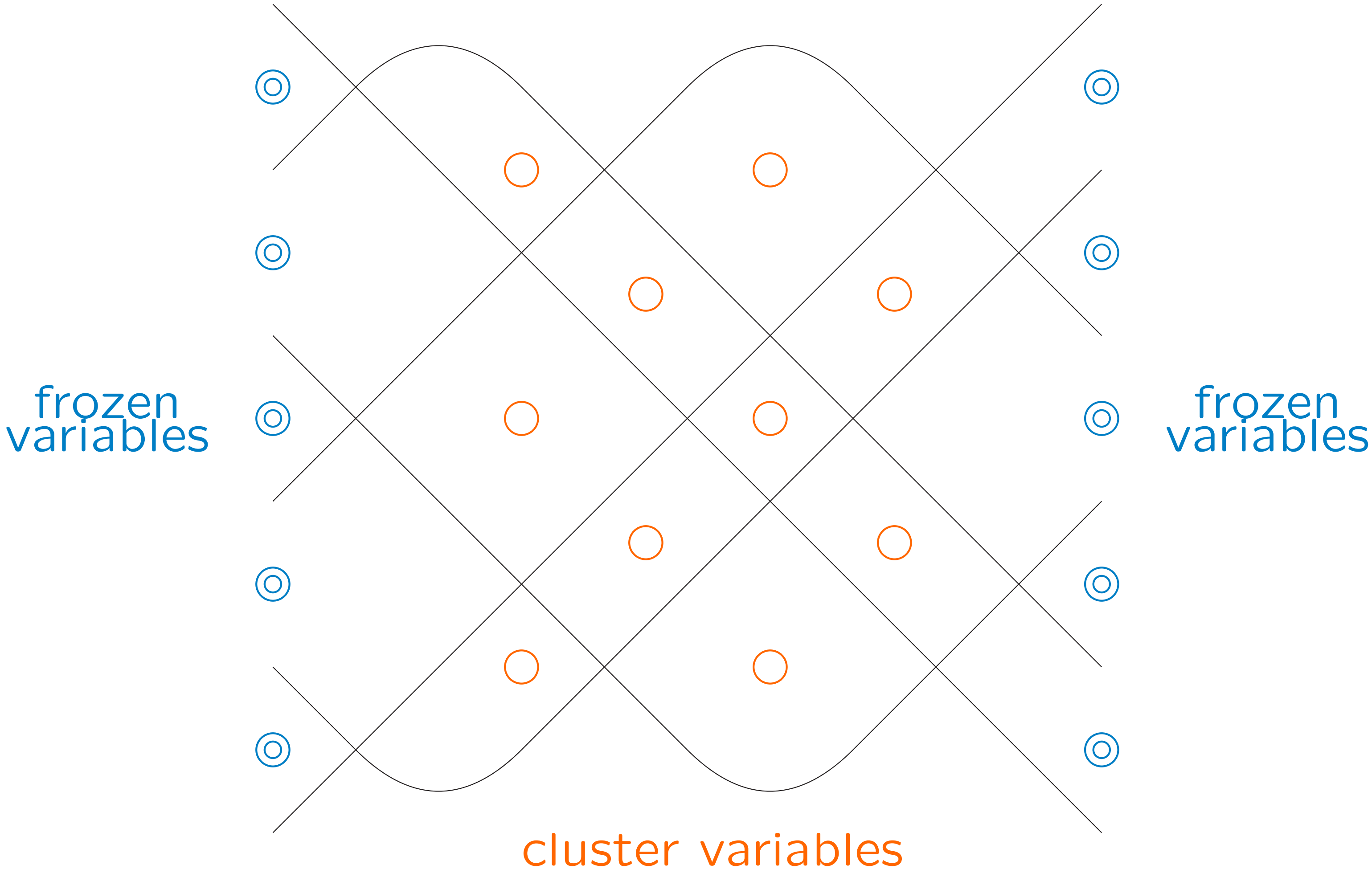
Any two pseudoline arrangements are related by *braid moves:*



Chamber minors



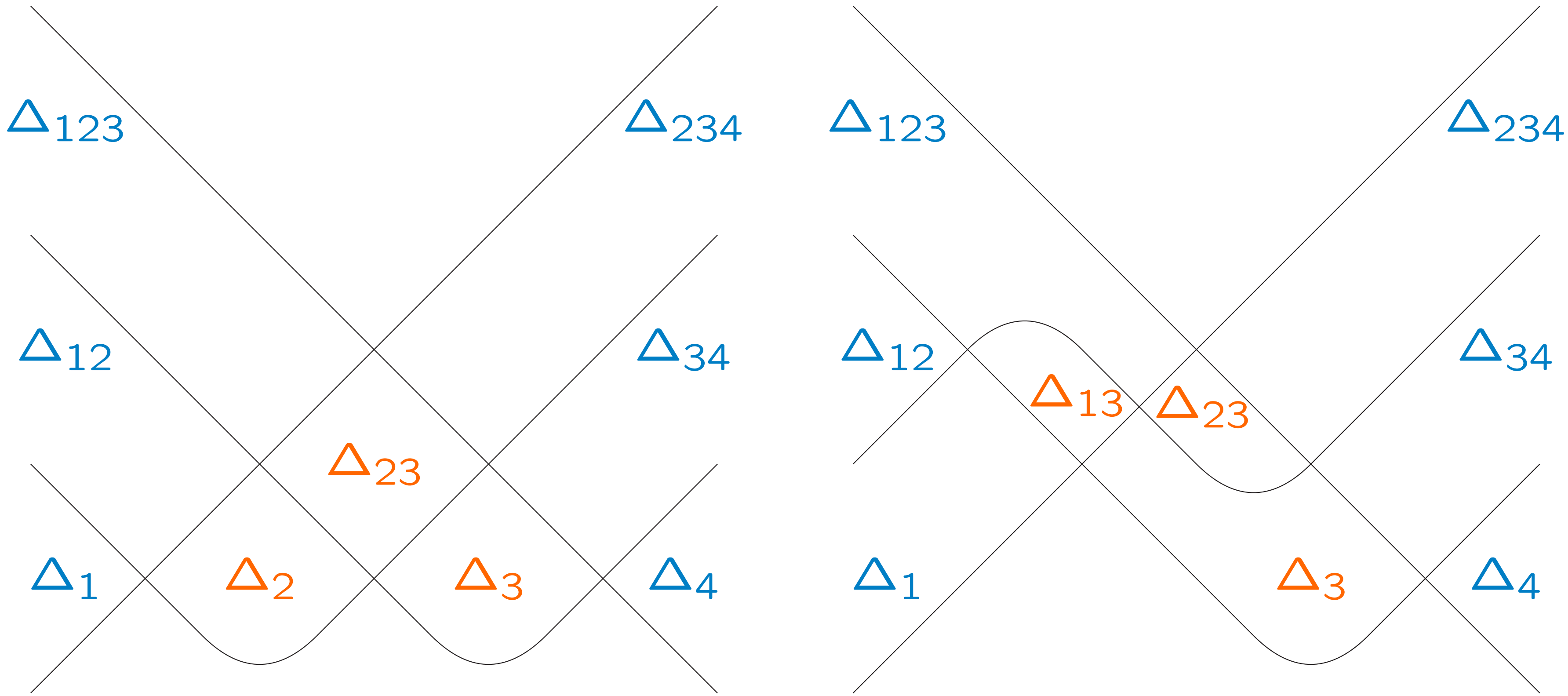
Cluster jargon

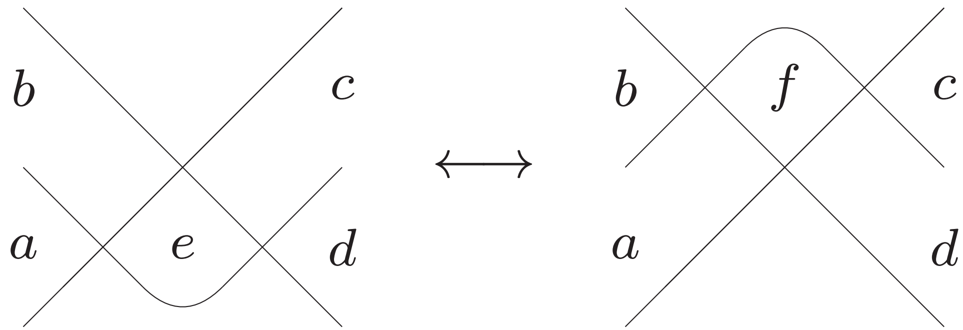


cluster = {cluster variables}

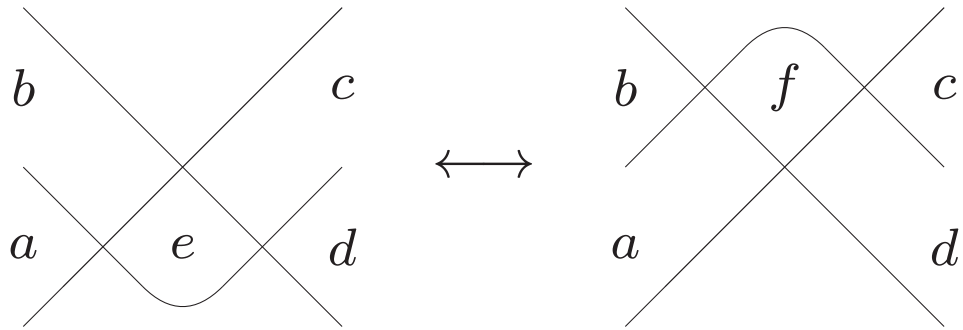
extended cluster = {frozen variables, cluster variables}

Braid moves and cluster exchanges





Exchange relations



The chamber minors a, b, c, d, e, f satisfy the *exchange relation*

$$ef = ac + bd.$$

(For example, $\Delta_2 \Delta_{13} = \Delta_{12}\,\Delta_3 + \Delta_1\,\Delta_{23}$.)

The rational expression $f = \frac{ac+bd}{e}$ is *subtraction-free*.

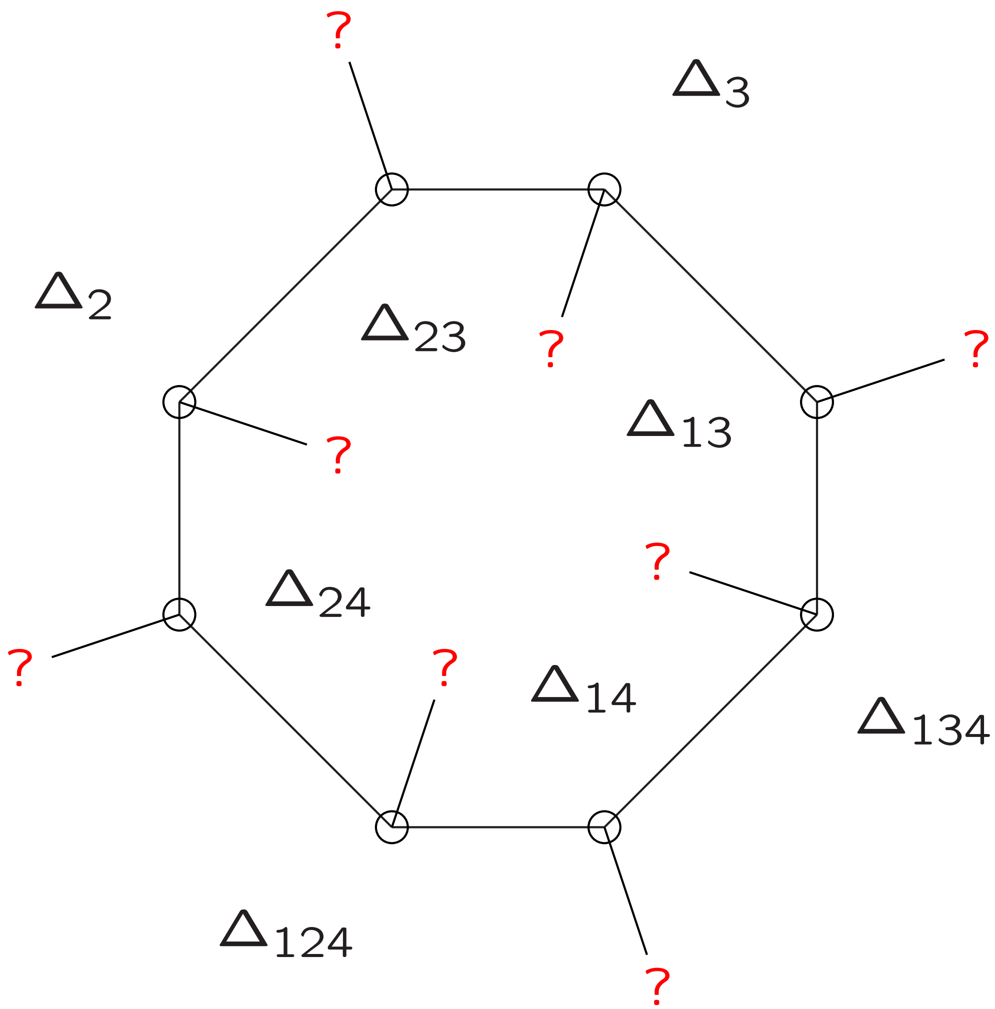
Theorem 1 *If the elements of a particular extended cluster evaluate positively at a given point, then so do all flag minors.*

Main features of a general cluster algebra setup

(illustrated by the prototypical example)

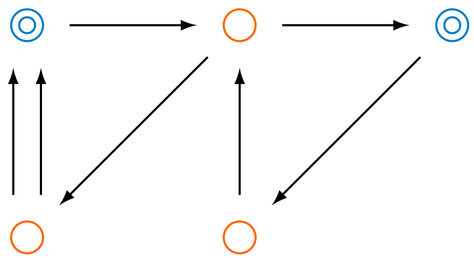
- a family of generators of the algebra (the flag minors);
- a finite subset of “frozen” generators;
- grouping of remaining generators into overlapping “clusters;”
- combinatorial data accompanying each cluster (a pseudoline arrangement);
- “exchange relations” that can be written using those data;
- a “mutation rule” for producing new combinatorial data from the given one (braid moves).

The missing mutations



Quivers

A *quiver* is a finite oriented graph. Multiple edges are allowed; oriented cycles of length 1 or 2 are not.



We view a quiver as combinatorial data accompanying a cluster. Accordingly, two types of vertices: "frozen" and "mutable."

Ignore edges connecting frozen vertices to each other.

Quiver mutations

Quiver analogues of braid moves.

Quiver mutation $\mu_z : Q \mapsto Q'$ is computed in three steps.

Step 1. For each instance of $x \to z \to y$, introduce an edge $x \to y$.

Step 2. Reverse the direction of all edges incident to z.

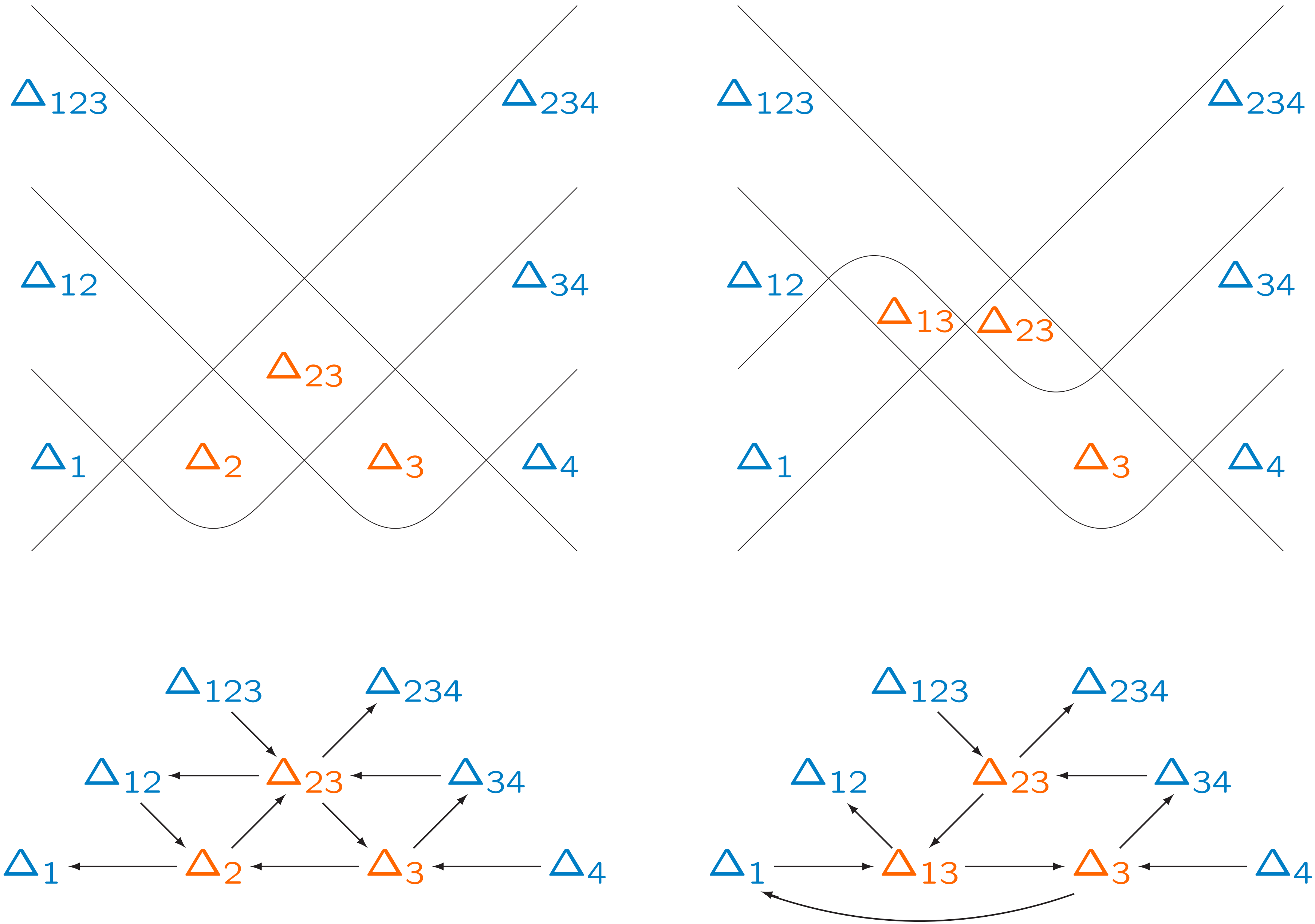
Step 3. Remove oriented 2-cycles.

$$\begin{tikzcd} \circ & z \arrow[l] \arrow[d] \\ \circ \arrow[u] \arrow[ur] & \circ \arrow[l] \end{tikzcd} \qquad \overset{\mu_z}{\longleftrightarrow} \qquad \begin{tikzcd} \circ \arrow[r] & z \arrow[dl] \\ \circ \arrow[u, shift left] \arrow[u, shift right] & \circ \arrow[u] \end{tikzcd}$$

$$Q \qquad\qquad\qquad\qquad Q'$$

Easy: mutation of Q' at z recovers Q.

Braid moves as quiver mutations



Cluster exchanges in the language of quivers

A *seed* $(Q, \mathbf{z})$ is a quiver Q labeled by the elements of an "extended cluster" $\mathbf{z}$, a collection of algebraically independent generators of some field of rational functions.

A *seed mutation* at a mutable vertex z replaces Q by $\mu_z(Q)$, and z by the new cluster variable z' defined by the exchange relation

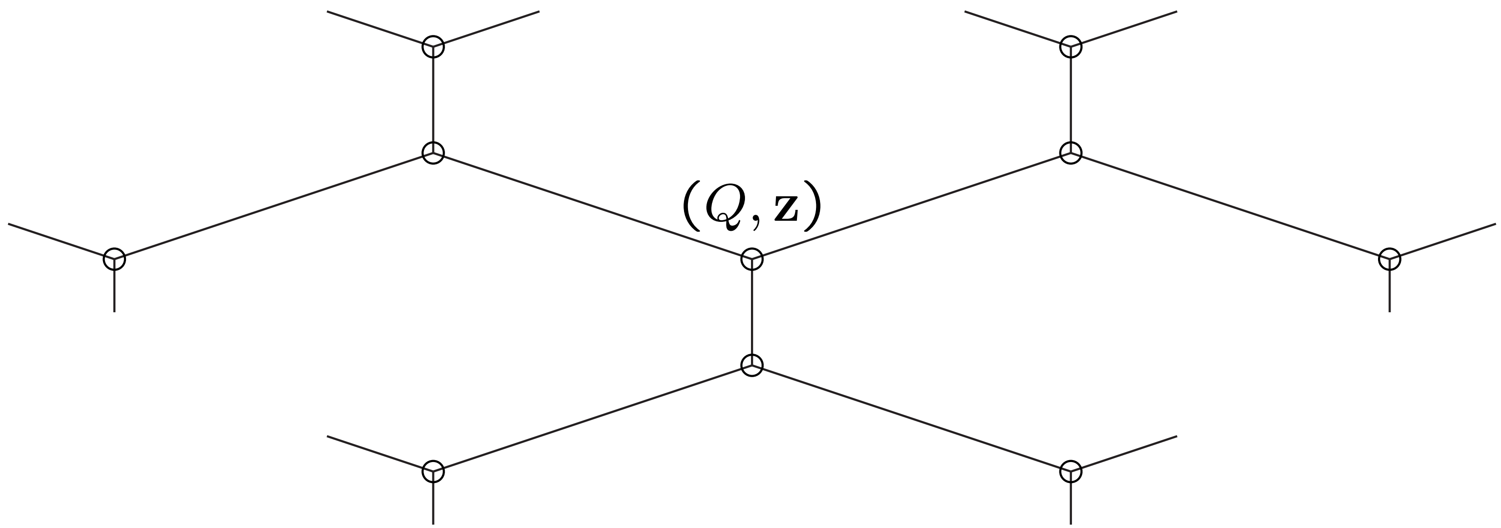
$$z\,z' = \prod_{z \leftarrow y} y + \prod_{z \to y} y\,.$$

The remaining elements of $\mathbf{z}$ stay put.

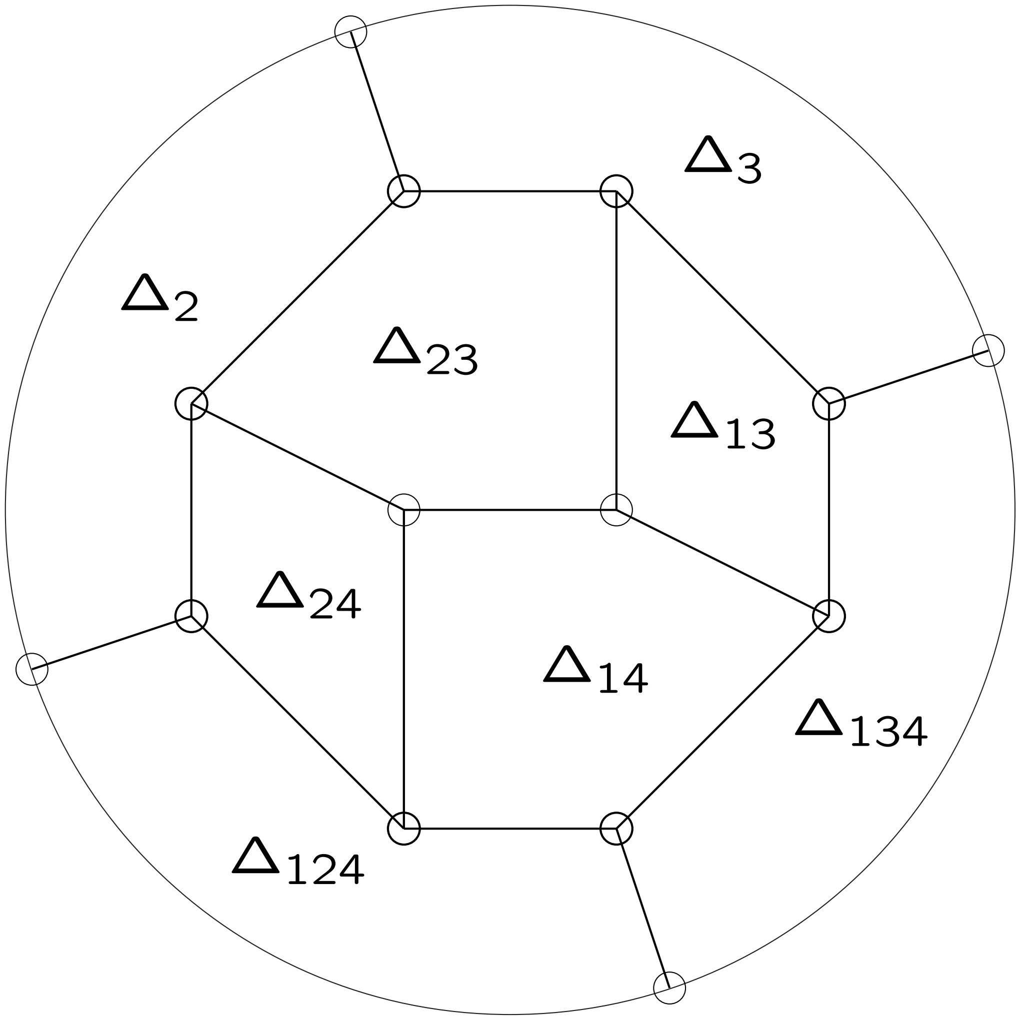
Definition of a cluster algebra $\mathcal{A}(Q)$

Assign a formal variable to each vertex of Q. These variables form the initial extended cluster $\mathbf{z}$.

The *cluster algebra* $\mathcal{A}(Q)$ is the subring generated inside the field of rational functions $\mathbb{C}(\mathbf{z})$ by all extended clusters obtained from the initial seed $(Q, \mathbf{z})$ by iterated mutations.



Cluster combinatorics for $\mathbb{C}[\mathrm{SL}_4]^N$



$$\Omega = -\Delta_1\Delta_{234} + \Delta_2\Delta_{134}$$

Examples of cluster algebras

Theorem 2 [J. Scott, *Proc. London Math. Soc.* **92** (2006)] *The homogeneous coordinate ring of any Grassmannian* $\mathrm{Gr}_{k,r}(\mathbb{C})$ *has a natural cluster algebra structure.*

Theorem 3 [C. Geiss, B. Leclerc, and J. Schröer, *Ann. Inst. Fourier* **58** (2008)] *The coordinate ring of any partial flag variety* $\mathrm{SL}_m(\mathbb{C})/P$ *has a natural cluster algebra structure.*

This can be used to build a cluster structure in each ring $\mathbb{C}[\mathrm{SL}_m]^N$.

Other examples include coordinate rings of G/P's, double Bruhat cells, Schubert varieties, etc.

What do we gain from a cluster structure?

1. A sensible notion of (total) positivity.

2. A “canonical basis,” or a part of it.

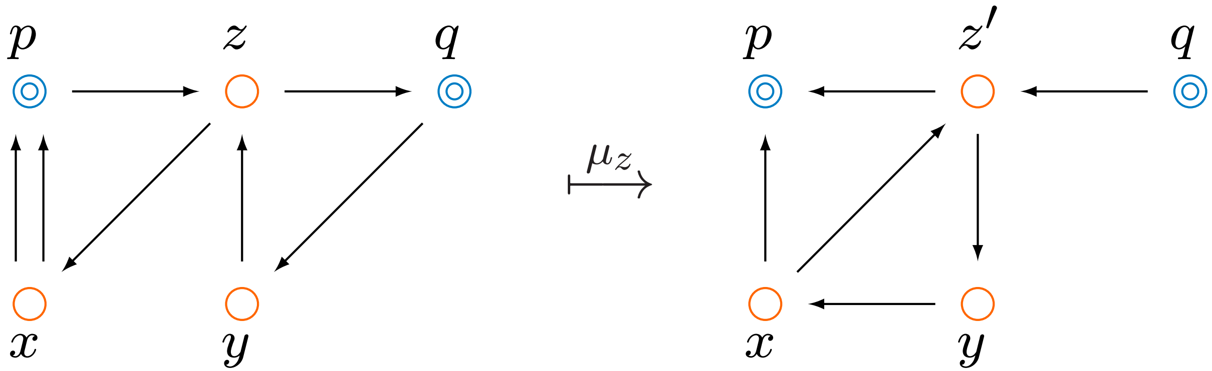
A *cluster monomial* is a product of (powers of) elements of the same extended cluster.

In $\mathcal{A} = \mathbb{C}[\mathrm{SL}_4/N]$, the cluster monomials form a linear basis. This is an example of Lusztig’s *dual canonical basis*.

3. A uniform perspective and general tools of cluster theory.

LECTURE 2

Review: quivers, clusters, seeds, and mutations



$$zz' = py + qx$$

The *cluster algebra* $\mathcal{A}(Q)$ is generated inside the ambient field of rational functions $\mathbb{C}(\mathbf{z})$ by all the elements of all extended clusters obtained from the initial seed $(Q, \mathbf{z})$ by iterated mutations.

Next: basic structural results of the theory of cluster algebras.

The Laurent phenomenon

Theorem 4 *All cluster variables are Laurent polynomials in the elements of the initial extended cluster.*

Exercise. Suppose that a sequence $x_0, x_1, x_2, \ldots$ satisfies the *Somos-5* recurrence relation

$$x_n x_{n+5} = x_{n+1} x_{n+4} + x_{n+2} x_{n+3} \,.$$

Interpret this recurrence as a special case of cluster mutation. Conclude that each x_n is expressed in terms of $x_0, \ldots, x_4$ as a Laurent polynomial.

Setting $x_0 = \cdots = x_4$ produces a sequence of integers

$$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, 22833, 165713, \ldots$$

Cluster monomials

A *cluster monomial* is a product of (powers of) elements of the same extended cluster.

In $\mathcal{A} = \mathbb{C}[\mathrm{SL}_4/N]$, the cluster monomials form a linear basis. This is an example of Lusztig's *dual canonical basis*.

In general, cluster monomials do not form a linear basis—unless the number of clusters is finite.

Theorem 5 [G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso, and P.-G. Plamondon, `arXiv:1203.1307`]. *In a cluster algebra defined by a quiver, the cluster monomials are linearly independent.*

Positivity conjectures

Conjecture 6 (Strong Positivity Conjecture) *Any cluster algebra has an additive basis which*

- *includes the cluster monomials and*
- *has nonnegative structure constants.*

Conjecture 7 (Laurent Positivity Conjecture) *When expressed in terms of an arbitrary extended cluster, each cluster variable is given by a Laurent polynomial with positive coefficients.*

It is not hard to see that Conjecture 6 implies Conjecture 7.

Cluster complex and exchange graph

Theorem 8 [M. Gekhtman, M. Shapiro, and A. Vainshtein, *Math. Res. Lett.* **15** (2008)] *Every seed is uniquely determined by its cluster. Two seeds are related by a mutation if and only of their clusters share all elements but one.*

The combinatorics of clusters and exchanges between them is encoded by the *cluster complex*, the simplicial complex whose vertices are the cluster variables and whose maximal simplices are the clusters.

By Theorem 8, the cluster complex is a *pseudomanifold*. Its dual graph is the *exchange graph* of the cluster algebra. The vertices of the exchange graph are labeled by the seeds/clusters while its edges correspond to mutations.

Cluster type

Conjecture 9 *The cluster complex does not depend on the frozen part of the initial quiver.*

Theorem 10 [G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso, and P.-G. Plamondon, `arXiv:1203.1307`]. *In a cluster algebra defined by an initial quiver Q, the exchange graph does not depend on the frozen part of Q.*

Cluster algebras which only differ in their frozen parts are said to have the same *(cluster) type*.

Example: cluster structures on SL_4/N and $\mathrm{Gr}_{2,6}$.

Example: Grassmannian $\mathrm{Gr}_{2,n+3}(\mathbb{C})$

Let $\mathcal{A}_n$ denote the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}_{2,n+3}(\mathbb{C})$ with respect to its Plücker embedding.

Let z be a $2 \times (n+3)$ matrix with generic entries:

$$z = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1,n+3} \\ z_{21} & z_{22} & \cdots & z_{2,n+3} \end{bmatrix}$$

One can view z as representing a subspace in $\mathrm{Gr}_{2,n+3}(\mathbb{C})$.

The *Plücker coordinates* P_{ij}, for $1 \le i < j \le n+3$, defined by

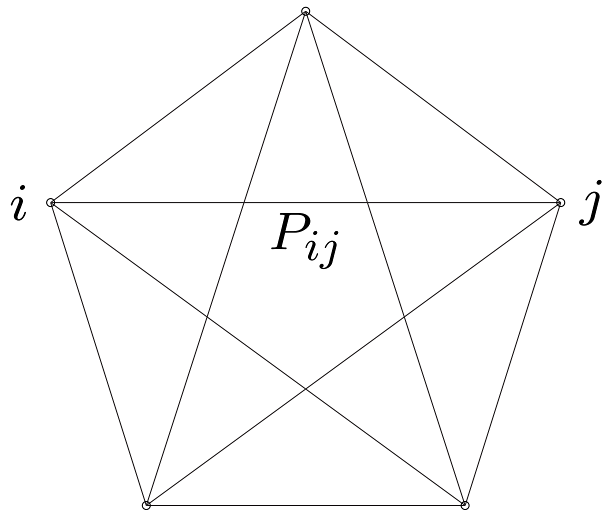
$$P_{ij} = \det \begin{bmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{bmatrix},$$

generate $\mathcal{A}_n$. They satisfy the *Grassmann-Plücker relations*

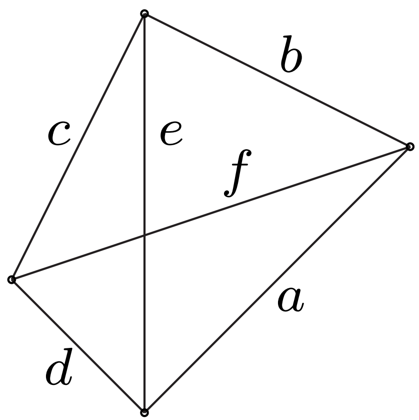
$$P_{ik}\,P_{jl} = P_{ij}\,P_{kl} + P_{il}\,P_{jk} \quad (i < j < k < l).$$

Ptolemy relations

The $\binom{n+3}{2}$ Plücker coordinates naturally correspond to the sides and diagonals of a convex $(n+3)$-gon:

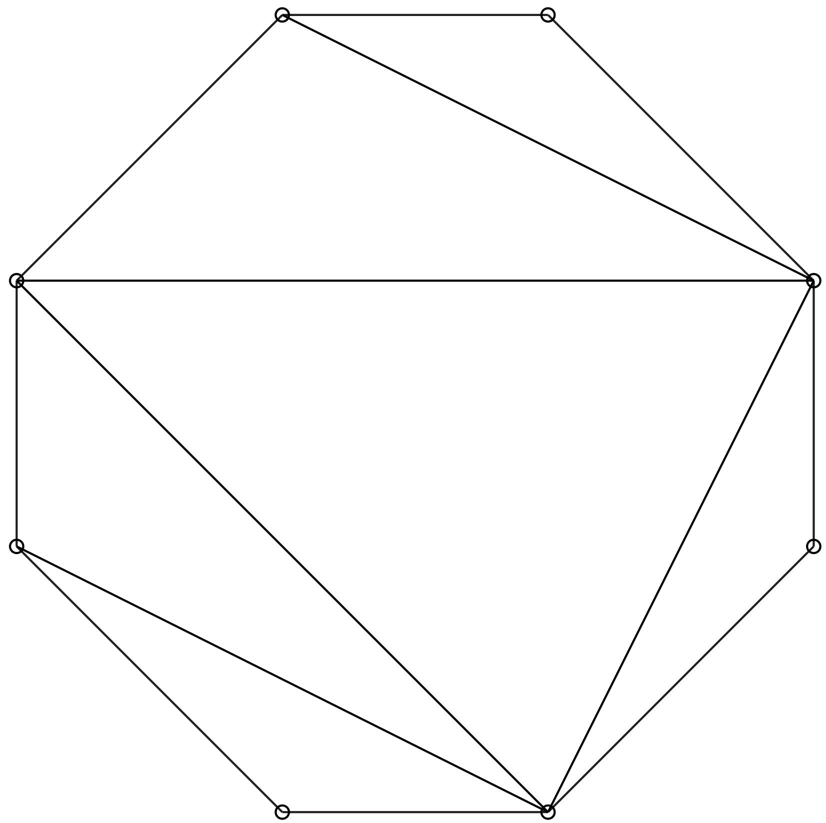


The Grassmann-Plücker relations become the *Ptolemy relations*:



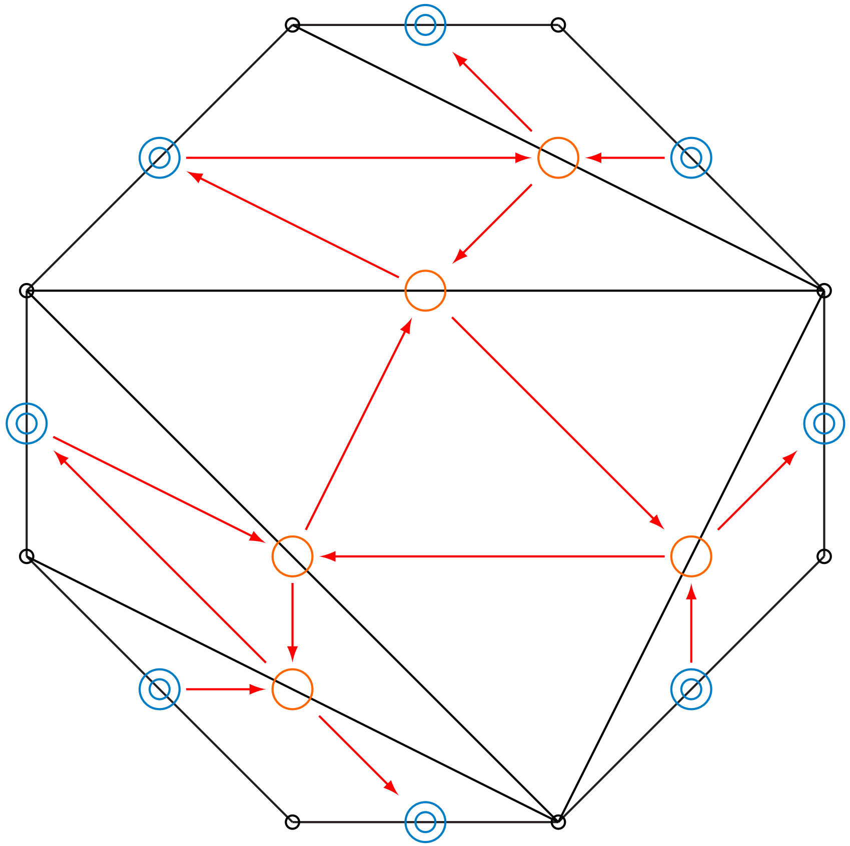
$$ef = ac + bd$$

Cluster structure on a Grassmannian $\mathrm{Gr}_{2,n+3}(\mathbb{C})$



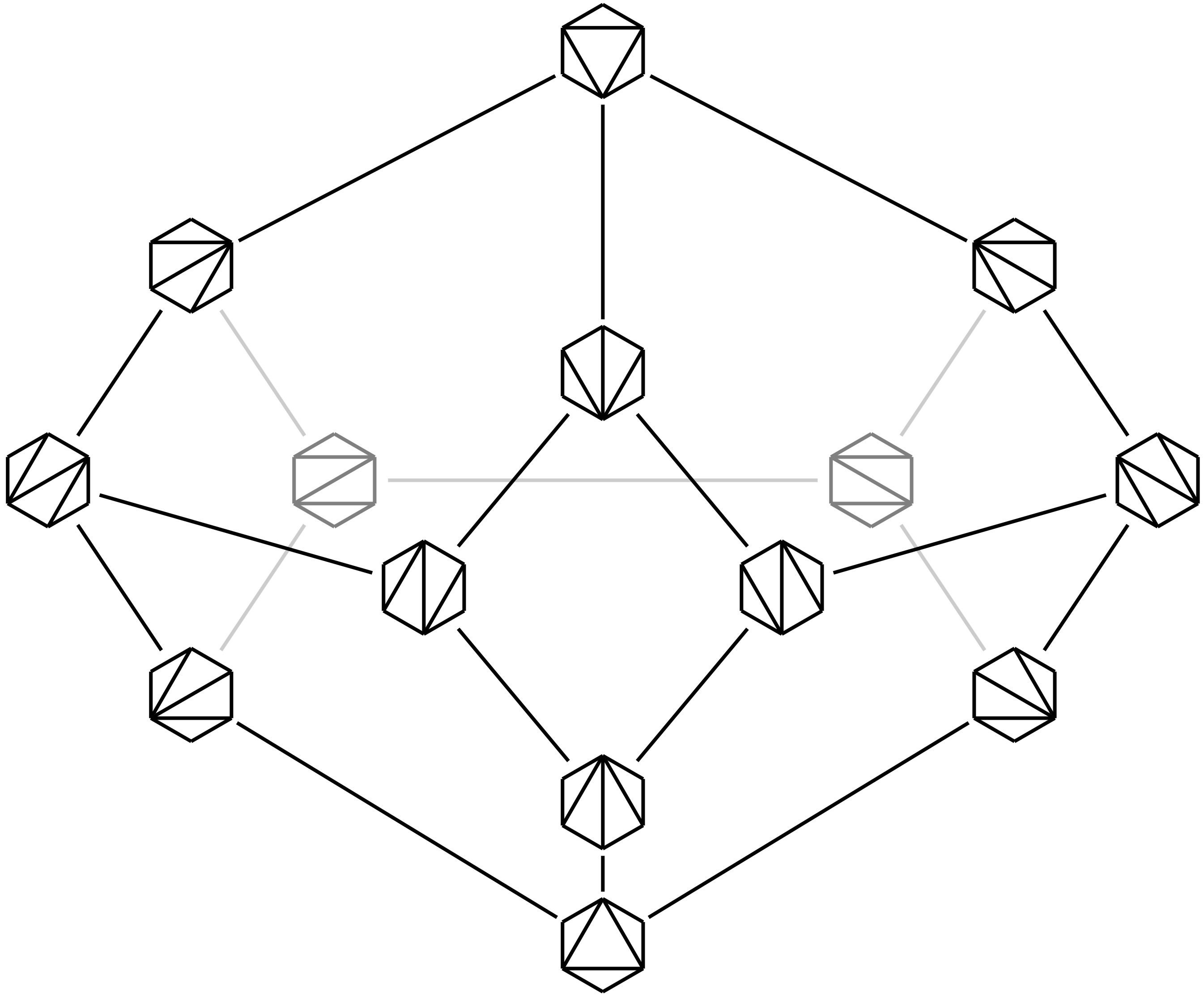
cluster variables	$\longleftrightarrow$	diagonals
frozen variables	$\longleftrightarrow$	sides
clusters/seeds	$\longleftrightarrow$	triangulations
mutations	$\longleftrightarrow$	flips
exchange relations	$\longleftrightarrow$	Grassmann-Plücker relations

Cluster structure on a Grassmannian $\mathrm{Gr}_{2,n+3}(\mathbb{C})$



cluster variables	$\longleftrightarrow$	diagonals
frozen variables	$\longleftrightarrow$	sides
clusters/seeds	$\longleftrightarrow$	triangulations
mutations	$\longleftrightarrow$	flips
exchange relations	$\longleftrightarrow$	Grassmann-Plücker relations

Exchange graph for $\mathbb{C}[\mathrm{Gr}_{2,n+3}]$: the associahedron



Cluster algebras of finite type

A cluster algebra is of *finite type* if it has finitely many seeds (equivalently, finitely many cluster variables).

The classification of cluster algebras of finite type turns out to be completely parallel to the classical Cartan-Killing classification of semisimple Lie algebras and finite root systems.

Theorem 11 *A cluster algebra is of finite type if and only if the mutable part of its quiver at some seed is an orientation of a (simply-laced) Dynkin diagram.*

The type of this Dynkin diagram in the Cartan-Killing nomenclature is uniquely determined by the cluster algebra.

Cluster types of some coordinate rings

The symmetry exhibited by the cluster type of a cluster algebra is usually not apparent at all from its geometric realization.

$\mathbb{C}[\mathrm{Gr}_{2,n+3}]$	A_n
$\mathbb{C}[\mathrm{Gr}_{3,6}]$	D_4
$\mathbb{C}[\mathrm{Gr}_{3,7}]$	E_6
$\mathbb{C}[\mathrm{Gr}_{3,8}]$	E_8
$\mathbb{C}[\mathrm{SL}_3]^N$	A_1
$\mathbb{C}[\mathrm{SL}_4]^N$	A_3
$\mathbb{C}[\mathrm{SL}_5]^N$	D_6

Cluster complexes in finite type

Theorem 12 [F. Chapoton, S.F., and A. Zelevinsky, *Canad. Math. Bull.* **45** (2002)] *The cluster complex of a cluster algebra of finite type is the dual simplicial complex of a simple convex polytope.*

These polytopes are called *generalized associahedra*.

In type A_n, we recover the ordinary associahedron (Stasheff's polytope).

Polyhedral realization of the cluster complex

Theorem 13 *The cluster variables in a cluster algebra of finite type are in bijection with the roots in the corresponding finite crystallographic root system which are either positive or negative simple.*

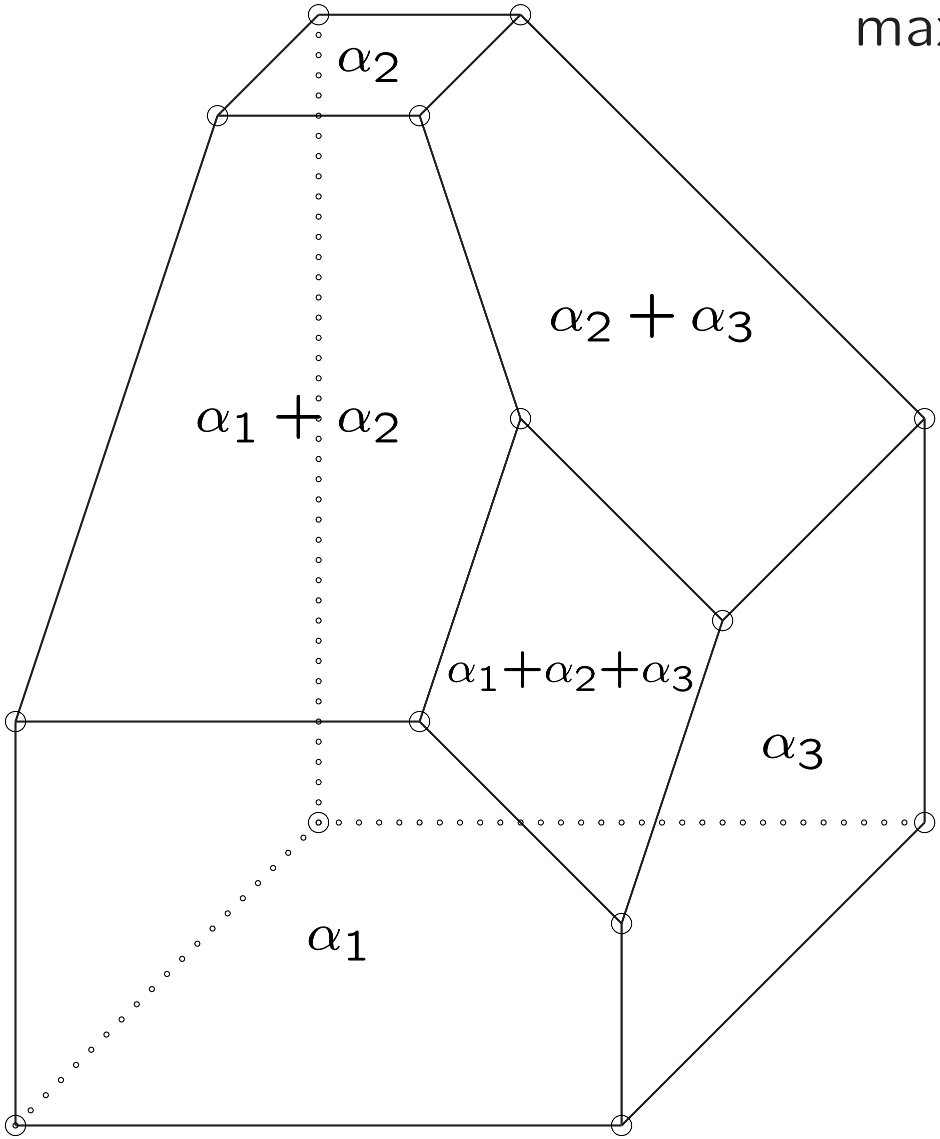
The "almost positive" root labeling a cluster variable is determined by the denominator of its Laurent expansion with respect to the distinguished cluster.

The cluster complex can be built on the ground set of almost positive roots. Its combinatorics, and the geometry of the associated simplicial fan (the normal fan of the generalized associahedron) can be explicitly described in root-theoretic terms.

Polyhedral realization of the associahedron of type A_3

$$\max(-z_1\,,\,-z_3\,,\,z_1\,,\,z_3\,,\,z_1+z_2\,,\,z_2+z_3)\ \le\ 3/2$$

$$\max(-z_2\,,\,z_2\,,\,z_1+z_2+z_3)\ \le\ 2$$



Enumerative results

Theorem 14 *The number of clusters in a cluster algebra of finite type is equal to*

$$N(\Phi)=\prod_{i=1}^{n}\frac{e_i+h+1}{e_i+1},$$

where $e_1,\ldots,e_n$ *are the exponents, and* h *is the Coxeter number.*

$N(\Phi)$ is the *Catalan number* associated with the root system Φ.

Φ	A_n	D_n	E_6	E_7	E_8
$N(\Phi)$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080

Catalan combinatorics of arbitrary type

Besides clusters, the numbers $N(\Phi)$ enumerate many other families of combinatorial objects related to the root system Φ:

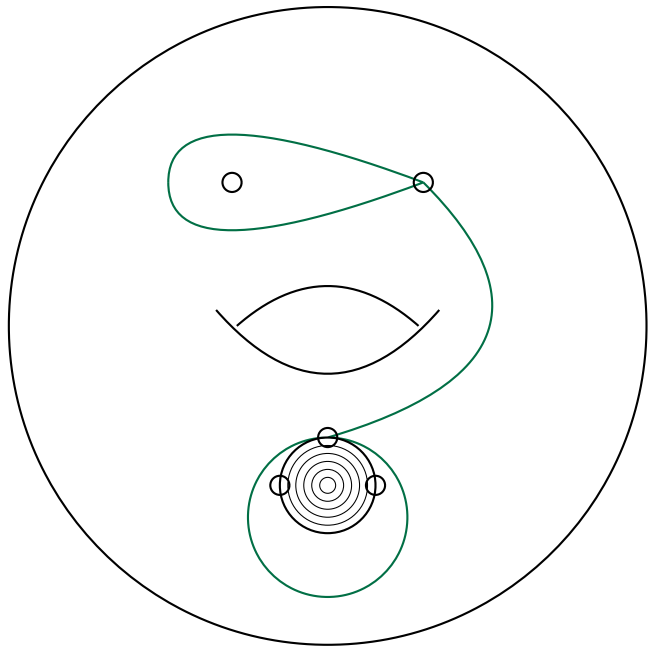
- ad-*nilpotent ideals* in a Borel subalgebra of a semisimple Lie algebra;
- antichains in the *root poset*;
- regions of the *Catalan arrangement* contained in the fundamental chamber;
- orbits of the Weyl group action on the quotient $Q/(h+1)Q$ of the root lattice;
- conjugacy classes of elements x of a semisimple Lie group which satisfy $x^{h+1} = 1$;
- *non-crossing partitions* of the appropriate type.

Cluster algebras associated with surfaces

Cluster-algebraic structures arising in the context of in Teichmüller theory were discovered and studied by V. Fock and A. Goncharov and independently by M. Gekhtman, M. Shapiro, and A. Vainshtein, making use of earlier foundational work by V. Fock, R. Penner, and W. Thurston.

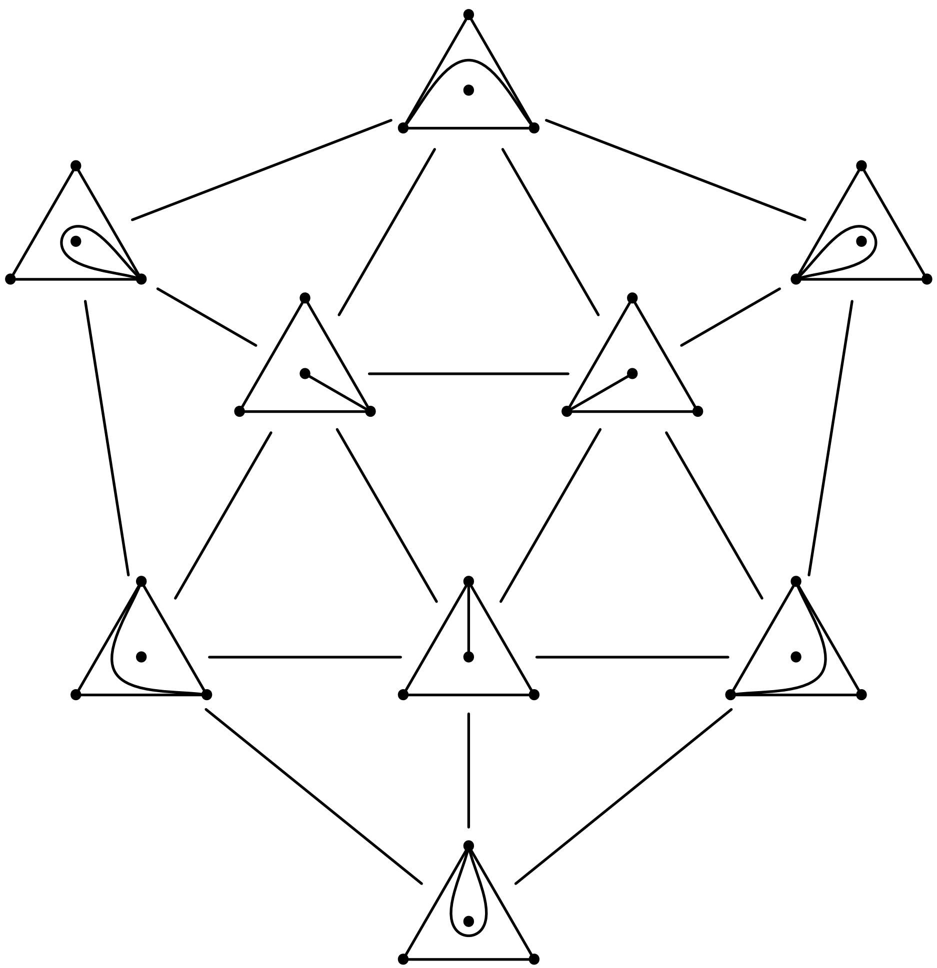
Arcs on a surface

Let $\mathbf{S}$ be a connected oriented surface with boundary. (Several small degenerate cases must be excluded.) Fix a finite non-empty set $\mathbf{M}$ of *marked points* in the closure of $\mathbf{S}$. An *arc* in $(\mathbf{S}, \mathbf{M})$ is a non-selfintersecting curve in $\mathbf{S}$, considered up to isotopy, which connects two points in $\mathbf{M}$, does not pass through $\mathbf{M}$, and does not cut out an unpunctured monogon or digon.



Arc complex

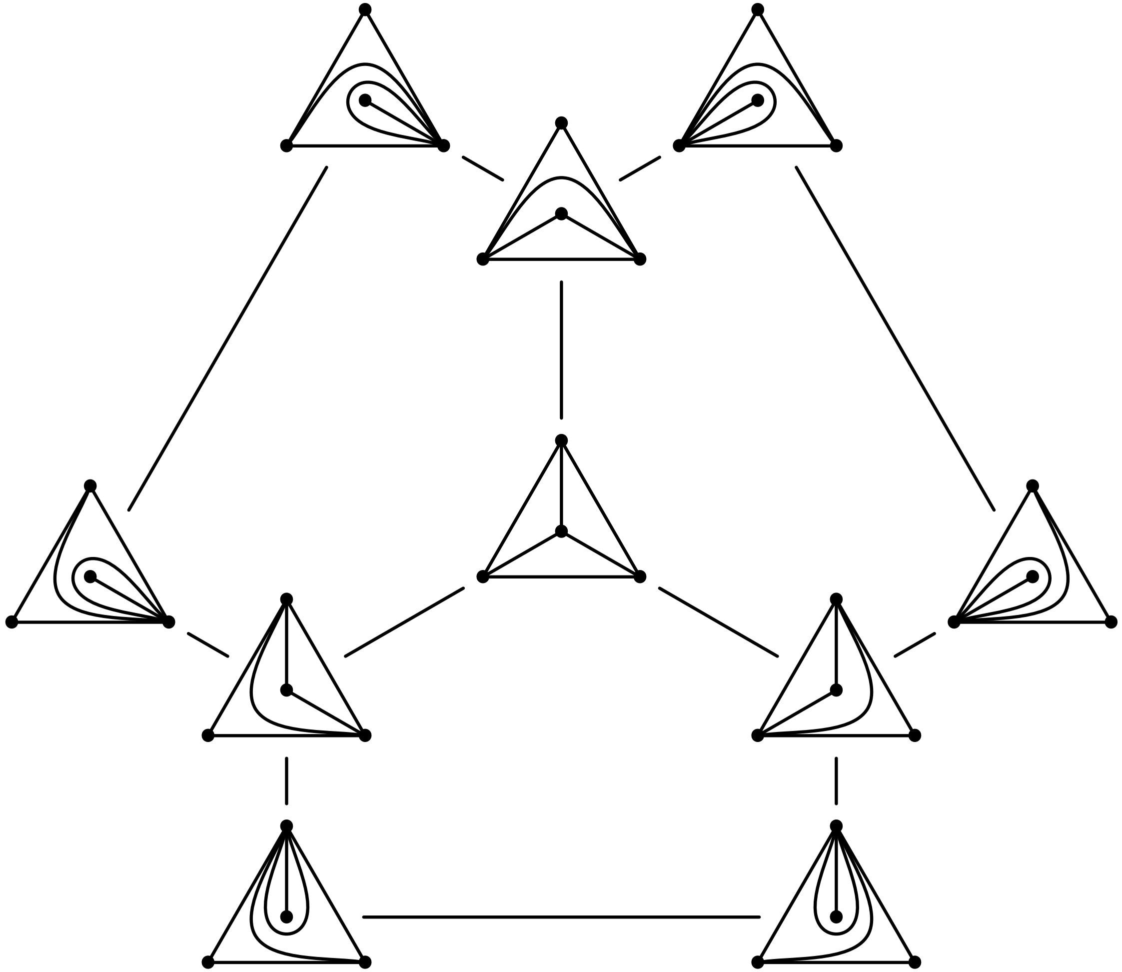
Arcs are *compatible* if they have non-intersecting realizations. Collections of pairwise compatible arcs are the simplices of the *arc complex*. Its facets correspond to *ideal triangulations*.



The arc complex is a *pseudomanifold with boundary*.

Flips on a surface

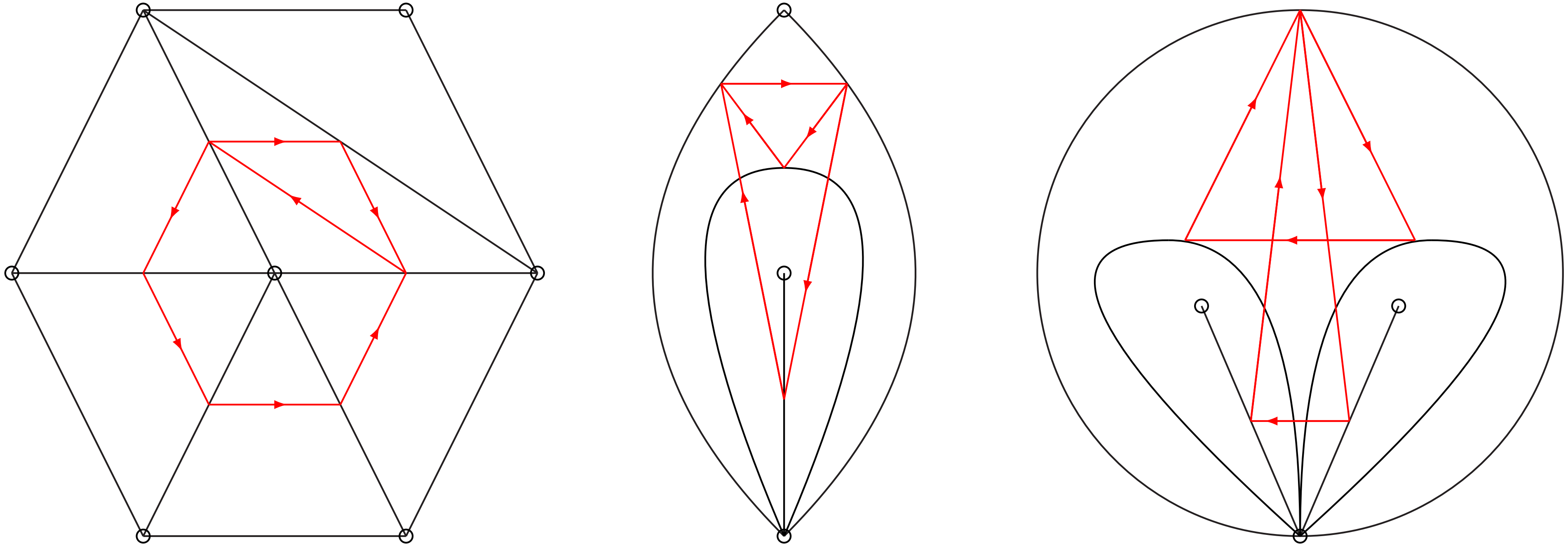
The edges of the dual graph of the arc complex correspond to *flips*.



An edge inside a *self-folded triangle* cannot be flipped.

Quiver associated with a triangulation

To a triangulation T we associate a quiver $Q(T)$.



Lemma 15 *Flips in ideal triangulations translate into mutations of the associated quivers.*

That is, if triangulations T' and T are related by a flip of the arc labeled k, then $Q(T') = \mu_k(Q(T))$.

Surface models for cluster algebras

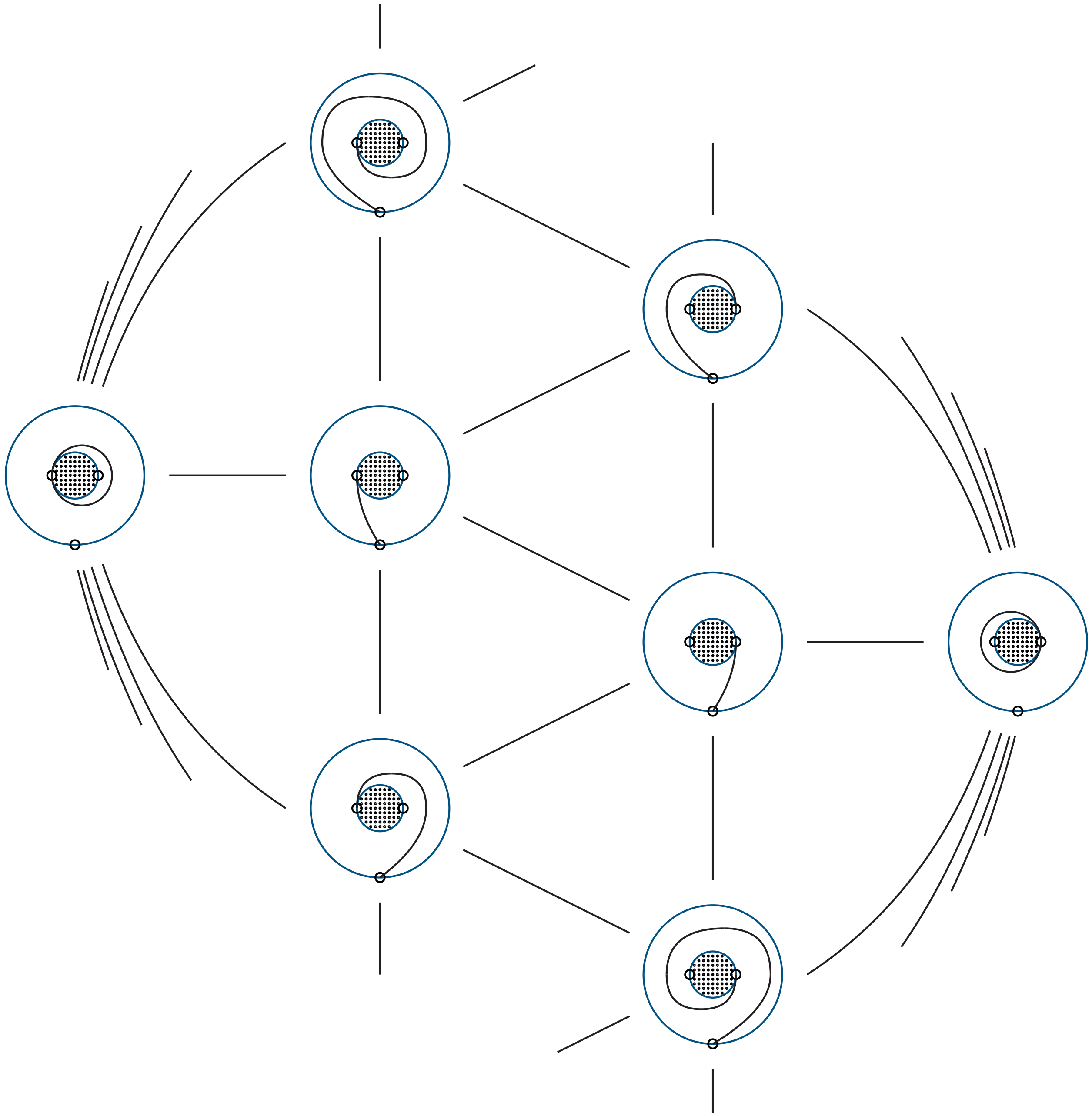
Lemma 15 suggests that if the mutable part of a quiver Q can be associated with a triangulated surface, then the underlying combinatorics of $\mathcal{A}(Q)$ can be modeled as follows:

- cluster variables correspond to arcs;
- clusters correspond to triangulations;
- exchanges correspond to flips.

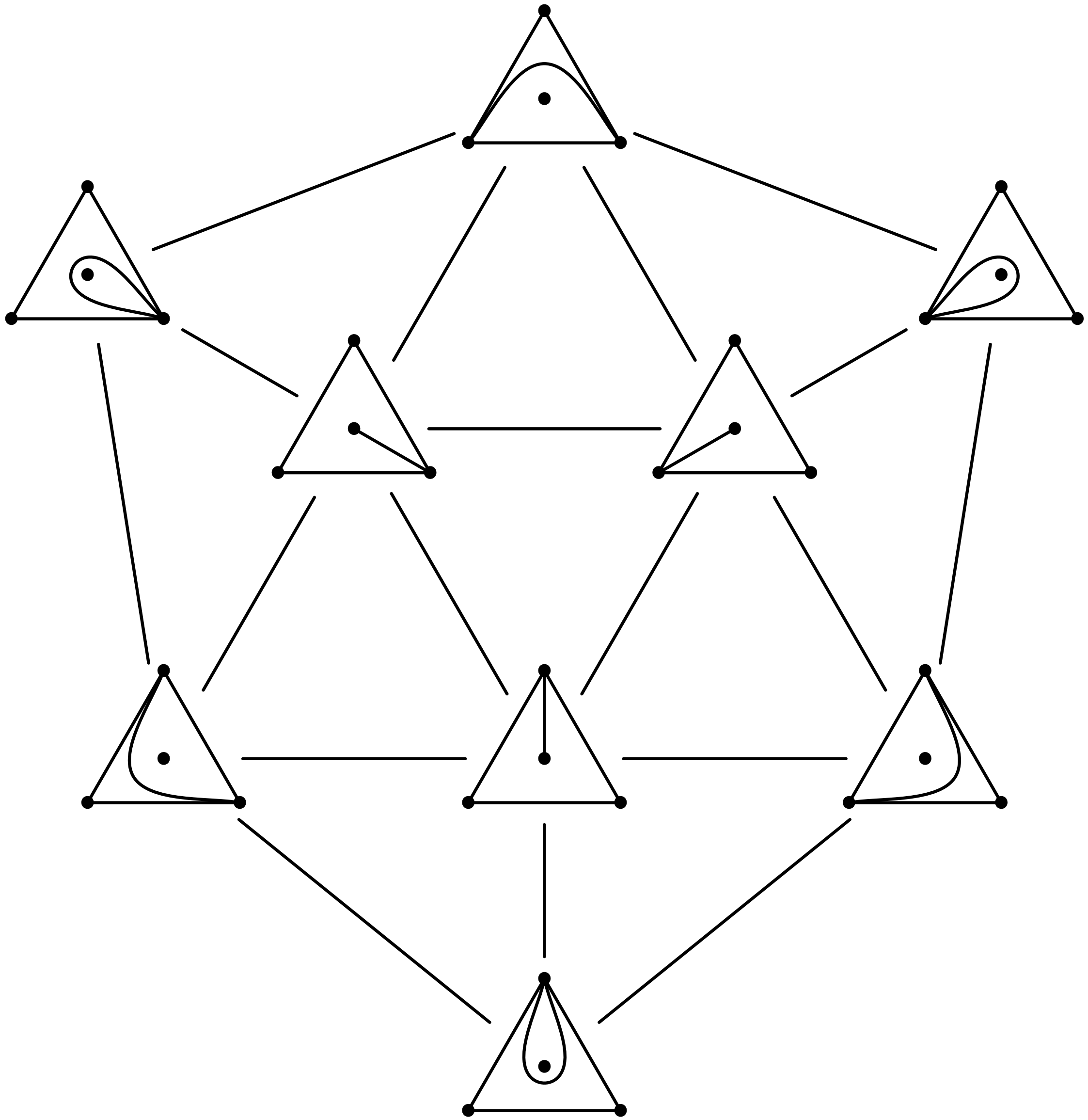
If there is at least one marked point in the interior, then we run into a problem: flips in some directions are not allowed. This problem can be solved by introducing *tagged arcs*.

LECTURE 3

Review: arc complex of a marked surface



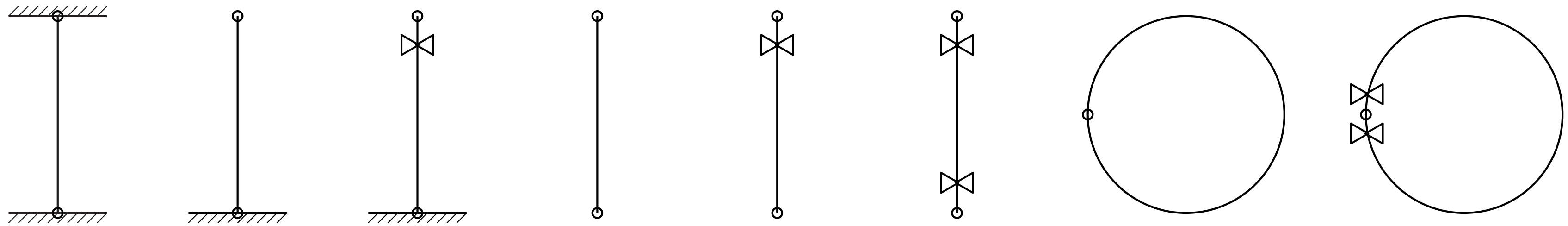
Review: boundary of the arc complex



Tagged arcs

A *tagged arc* is obtained by taking an arc that does not cut out a once-punctured monogon, and “tagging” each of its ends in one of two ways, *plain* or *notched*, so that

- an endpoint lying on the boundary is tagged plain;
- both ends of a loop are tagged in the same way.

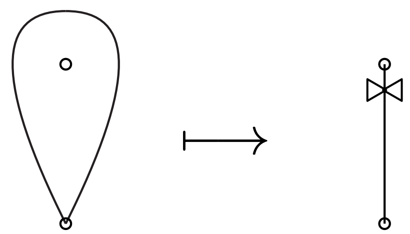


Compatibility of tagged arcs

Tagged arcs α and β are *compatible* if and only if

- their untagged versions α° and β° are compatible;
- if α and β share an endpoint, then the ends of α and β connecting to it must be tagged in the same way—unless $\alpha^\circ = \beta^\circ$, in which case at least one end of α must be tagged in the same way as the corresponding end of β.

Ordinary arcs can be viewed as a special case of tagged arcs:

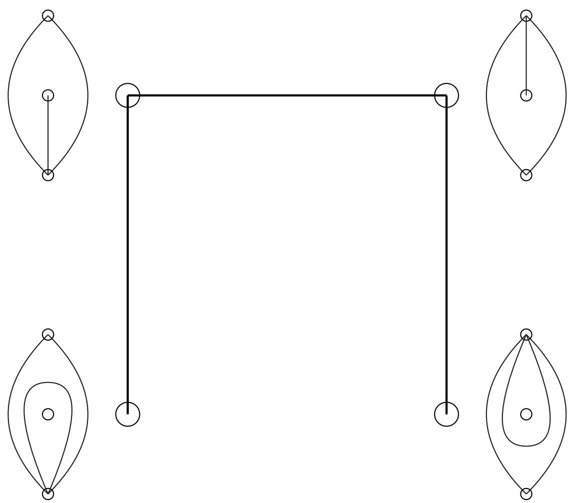


Under this identification, the notion of compatibility of tagged arcs extends the corresponding notion for ordinary arcs.

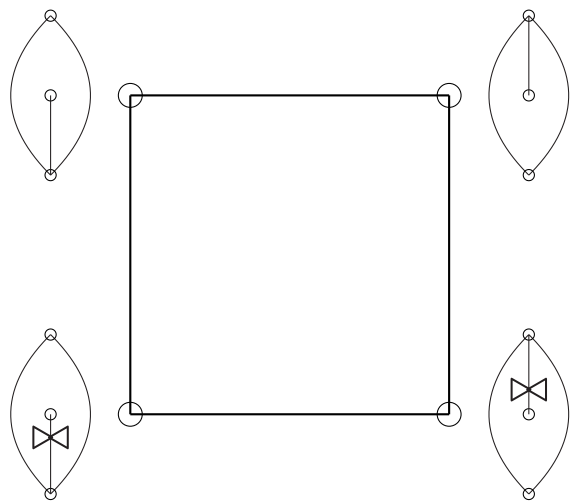
Tagged arc complex

The *tagged arc complex* is the simplicial complex whose vertices are tagged arcs and whose simplices are collections of pairwise compatible tagged arcs. The maximal simplices are called *tagged triangulations*.

The arc complex is a *subcomplex* of the tagged arc complex.

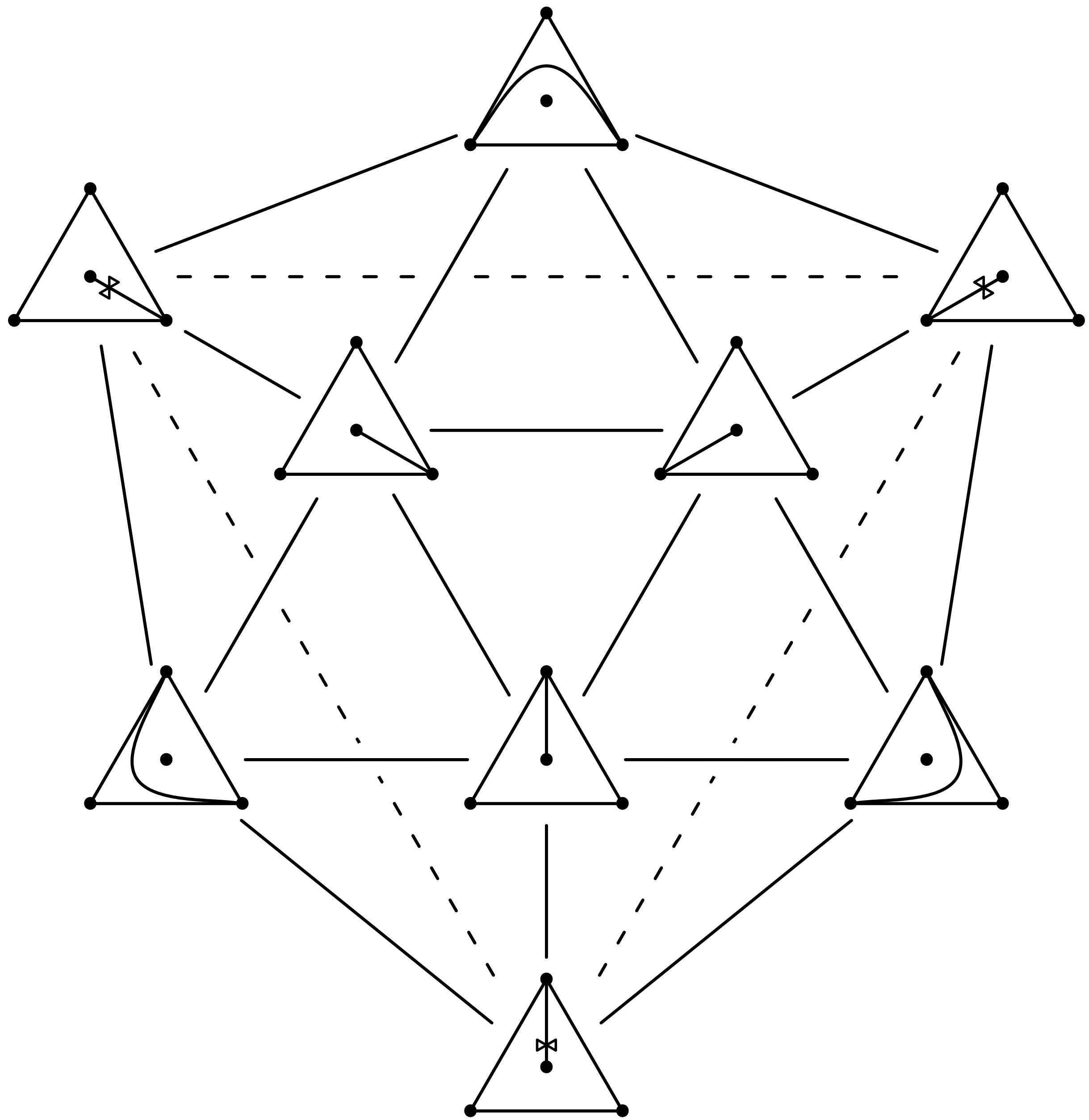


arc complex



tagged arc complex

Tagged arc complex of a once-punctured triangle



Properties of the tagged arc complex

Theorem 16 *The tagged arc complex is a pseudomanifold.*

Theorem 17 *The tagged arc complex is connected unless* $(\mathbf{S}, \mathbf{M})$ *is a closed surface with a single puncture, in which case it consists of two isomorphic connected components.*

Theorem 18 *The tagged arc complex is either contractible or homotopy equivalent to a sphere.*

The definition of the quiver $Q(T)$ can be extended to tagged triangulations T so that, as before, (tagged) flips correspond to quiver mutations.

Cluster complex associated with a marked surface

Theorem 19 *Let $\mathcal{A}$ be a cluster algebra whose quiver Q at some (equivalently, any) seed has mutable part that can be represented as $Q(T)$, for some triangulation T of some marked surface $(\mathbf{S}, \mathbf{M})$. Then the cluster complex of $\mathcal{A}$ is isomorphic to a connected component of the tagged arc complex of $(\mathbf{S}, \mathbf{M})$.*

Under this isomorphism,

- *cluster variables correspond to tagged arcs,*
- *clusters/seeds correspond to tagged triangulations,*
- *mutations correspond to flips, and*
- *the quiver at a seed associated with a tagged triangulation T is $Q(T)$.*

Classification of quivers of finite mutation type

A quiver has *finite mutation type* if its mutation equivalence class consists of finitely many quivers (up to isomorphism).

Theorem 20 [A. Felikson, P. Tumarkin, M. Shapiro, 2008]
Apart from a finite number of exceptions, a quiver has a finite mutation type if and only if it comes from a triangulated surface.

Which cluster algebras come from surfaces?

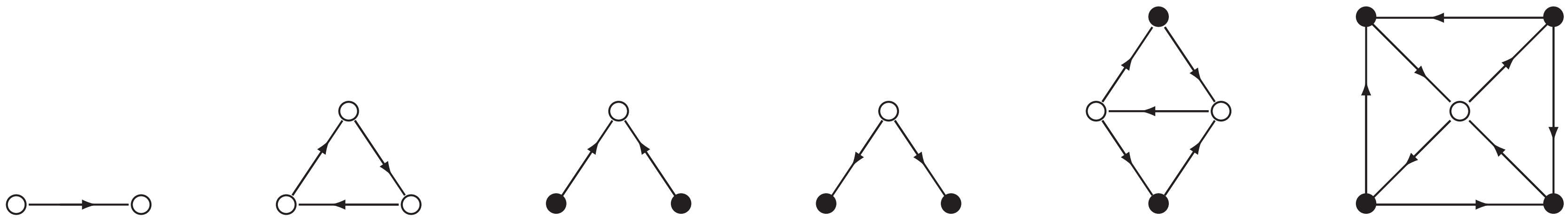
Examples of cluster algebras (say of rank n) which can be associated with triangulated surfaces include the following cluster types:

- finite type A_n (unpunctured $(n+3)$-gon);
- finite type D_n (once-punctured n-gon);
- affine type $\widetilde{A}(n_1, n_2)$, $n_1 + n_2 = n$ (unpunctured annulus);
- affine type $\widetilde{D}_{n-1}$ (twice-punctured $(n-3)$-gon).

Exceptional finite cluster types E_6, E_7, E_8 *cannot* be modeled by triangulated surfaces.

Block decompositions

Theorem 21 *A quiver corresponds to an ordinary (equivalently, tagged) triangulation of a bordered surface if and only if it can be obtained from a collection of blocks of the form shown below by gluing together some pairs of white vertices.*



Beyond the cluster complex

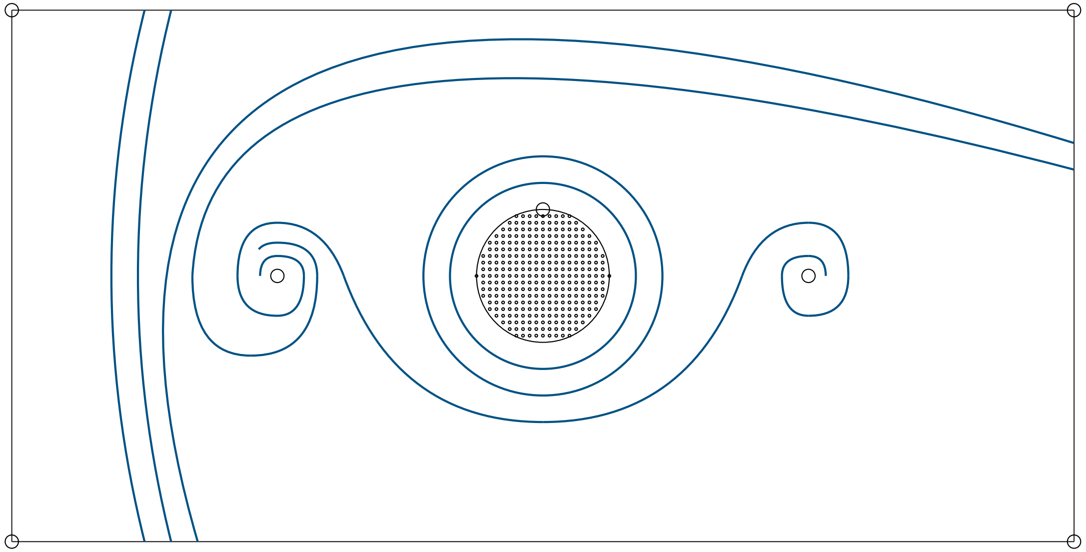
Once we understand the cluster complex (e.g., in finite type, or in the case of surfaces), the next goal is to obtain a direct (=non-recursive) description of the coefficients appearing in the exchange relations.

Case in point: the base affine space for $\mathrm{SL}_4(\mathbb{C})$ (cluster type A_3).

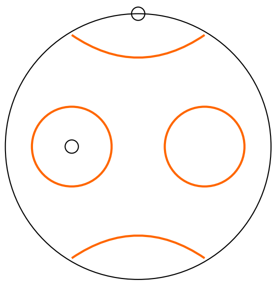
For cluster algebras associated with surfaces, this problem can be solved using W. Thurston's machinery of integral *laminations*.

Integral laminations

An *integral (unbounded measured) lamination* on $(\mathbf{S}, \mathbf{M})$ is a finite collection of non-selfintersecting and pairwise non-intersecting curves in $\mathbf{S}$, modulo isotopy:

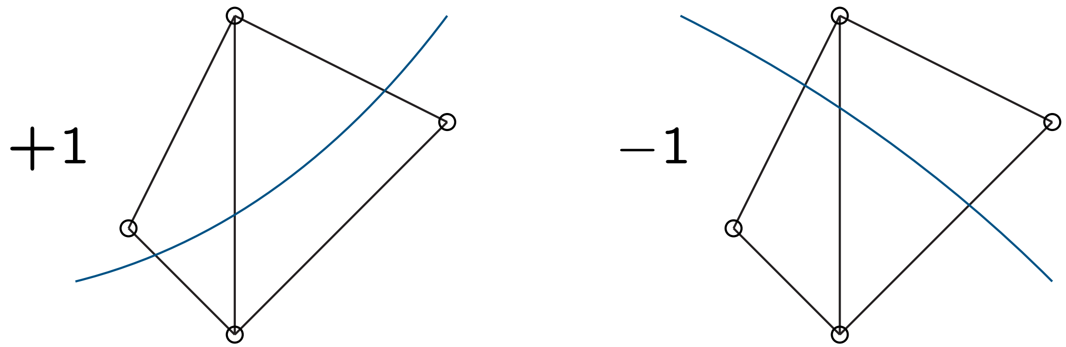


Curves that are not allowed in a lamination:



Shear coordinates

Let L be an integral lamination, and T a triangulation without self-folded triangles. For each arc γ in T, the *shear coordinate* $b_\gamma(T, L)$ is defined as a sum of contributions from all curves in L. Each such curve contributes $+1$ (resp., -1) to $b_\gamma(T, L)$ if it cuts through the quadrilateral surrounding γ as shown:

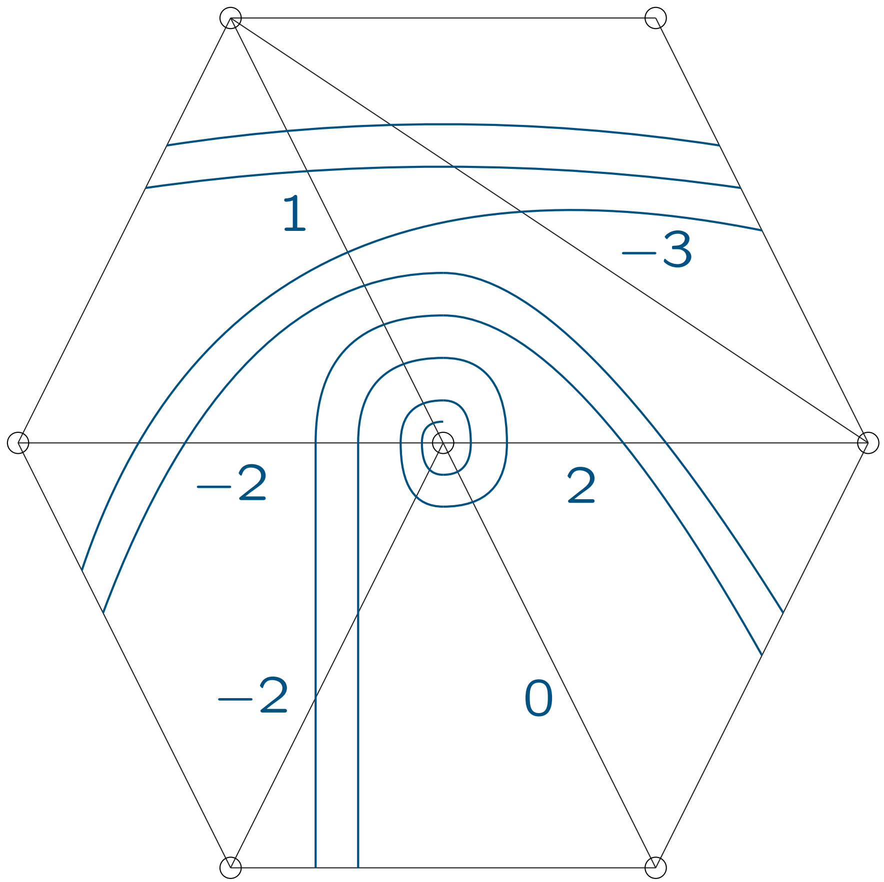


Thurston's coordinatization theorem

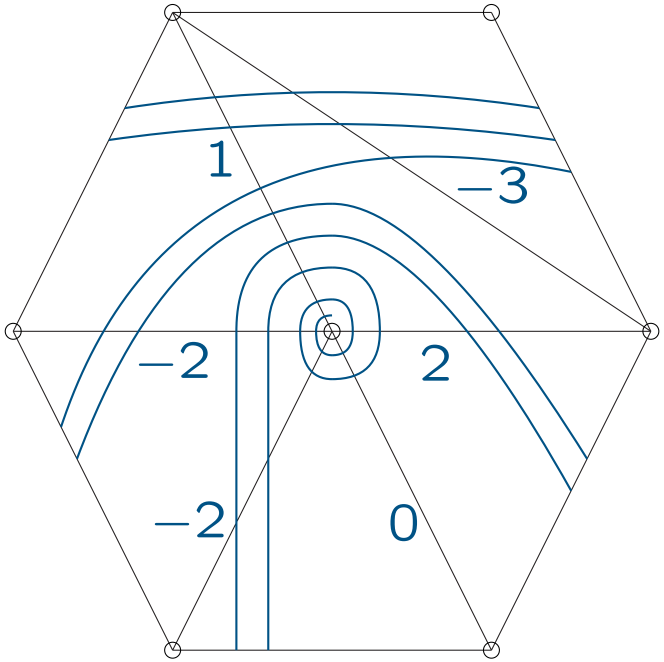
Theorem 22 [W.Thurston] *For a fixed triangulation* T*, the map*

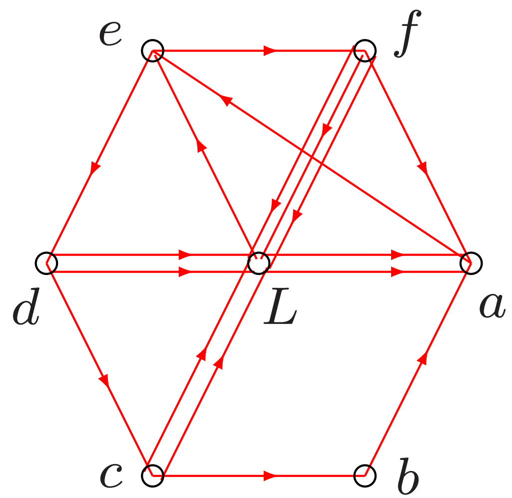
$$L \mapsto (b_\gamma(T, L))_{\gamma \in T}$$

is a bijection between integral laminations and $\mathbb{Z}^n$.



Extended quiver associated with a surface with laminations





$Q(T, \mathbf{L})$

Flips and mutations

The notion of shear coordinates of a lamination can be extended to the tagged case, leading to the definition of the quiver $Q(T, \mathbf{L})$ for any tagged triangulation T.

Theorem 23 *Tagged flips translate into mutations of associated extended quivers* $Q(T, \mathbf{L})$.

Cluster algebra associated with a multi-lamination

Theorem 24 *For any multi-lamination* $\mathbf{L}$ *on a bordered surface* $(\mathbf{S}, \mathbf{M})$*, there is a unique cluster algebra* $\mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ *in which*

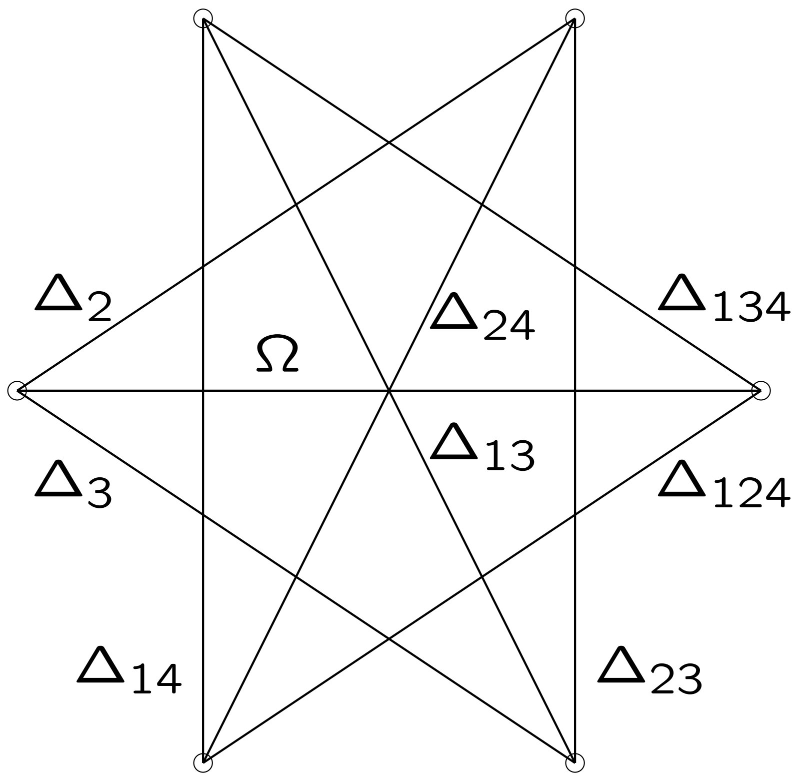
- *cluster variables are labeled by the tagged arcs in* $(\mathbf{S}, \mathbf{M})$*;*
- *coefficient variables are labeled by the laminations in* $\mathbf{L}$*;*
- *clusters correspond to the tagged triangulations in* $(\mathbf{S}, \mathbf{M})$*;*
- *exchange relations correspond to the tagged flips, and are encoded by the quivers* $Q(T, \mathbf{L})$*.*

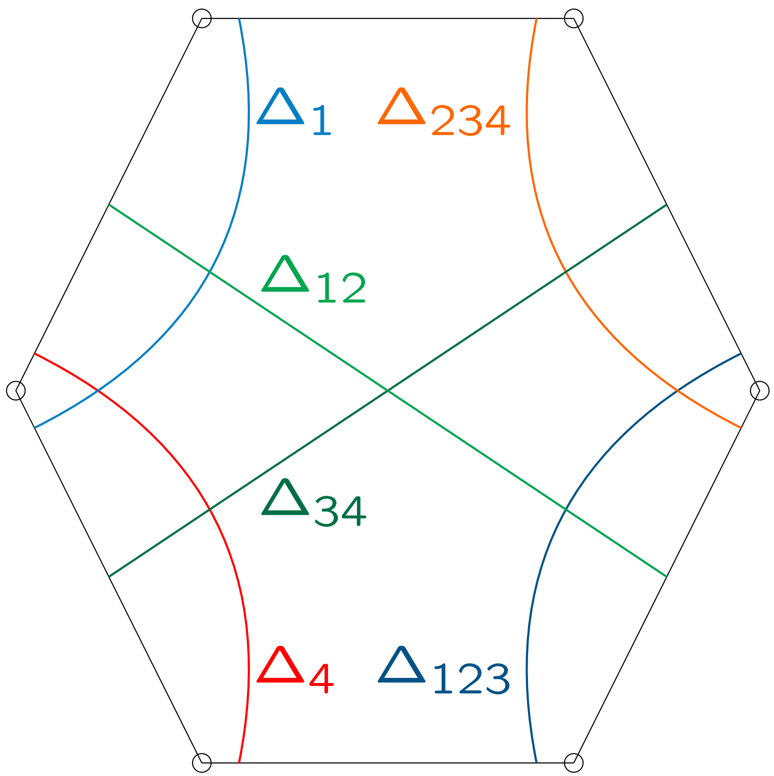
By Thurston's coordinatization theorem, *any* cluster algebra $\mathcal{A}(Q)$ such that the mutable part of Q can be associated with a triangulated surface has a topological realization of the above form, for some choice of multi-lamination $\mathbf{L}$.

Example: affine base space for SL_4

The cluster algebra $\mathbb{C}[SL_4]^N$ can be described by the multi-lamination shown below. Its generators correspond to

- the 9 diagonals of the hexagon (cluster variables) and
- the 6 laminations shown below (coefficient variables).





Cluster variables as lambda lengths

The cluster variables in a cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ can be given an intrinsic interpretation, as renormalized *lambda lengths* (or Penner coordinates) on a suitable *decorated Teichmüller space*.

In this geometric realization, the Teichmüller space plays the role of the *totally positive variety* associated with $\mathcal{A}$.

Decorated Teichmüller space

(after R. Penner)

Assume that all marked points in $\mathbf{M}$ lie on the boundary of $\mathbf{S}$. (This assumption greatly simplifies the general theory.)

The *Teichmüller space* $\mathcal{T}(\mathbf{S},\mathbf{M})$ consists of all complete finite-area hyperbolic structures with constant curvature -1 on $\mathbf{S} \setminus \mathbf{M}$, with geodesic boundary at $\partial\mathbf{S}\setminus\mathbf{M}$, and a cusp at each point of $\mathbf{M}$.

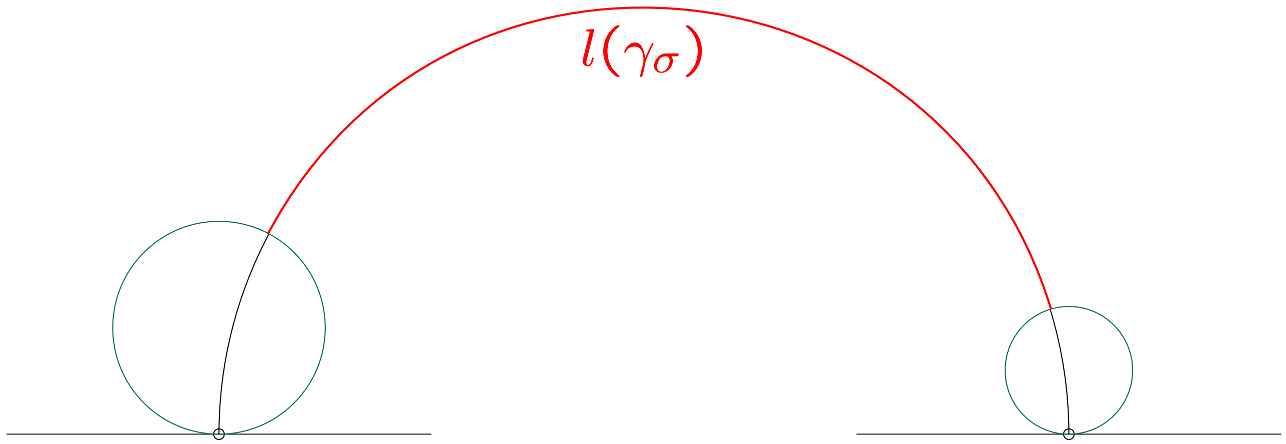
A point in the *decorated Teichmüller space* $\widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M})$ is a hyperbolic structure as above together with a collection of *horocycles*, one around each marked point.

Lambda lengths

Let γ be an arc or a boundary segment between adjacent marked points. For a decorated hyperbolic structure $\sigma \in \widetilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$, the *lambda length* $\lambda(\gamma) = \lambda_\sigma(\gamma)$ of γ is defined by

$$\lambda(\gamma) = \exp(l(\gamma_\sigma)/2),$$

where $l_\sigma(\gamma)$ denotes the signed distance between the horocycles at either end of γ along the unique geodesic γ_σ representing γ.



Penner's coordinatization

For a given γ, one can view the lambda length

$$\lambda(\gamma) : \sigma \mapsto \lambda_\sigma(\gamma)$$

as a function on the decorated Teichmüller space $\widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M})$. These functions can be used to coordinatize $\widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M})$:

Theorem 25 [R. Penner] *Let T be a triangulation of $(\mathbf{S},\mathbf{M})$ without self-folded triangles. Then the map*

$$\prod_\gamma \lambda(\gamma) : \widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M}) \to \mathbb{R}^m_{>0}$$

is a homeomorphism. Here γ runs over the arcs in T and the boundary segments of $(\mathbf{S},\mathbf{M})$, while m denotes the total number of such γ's.

Laminated lambda lengths

Fix a multi-lamination $\mathbf{L} = (L_i)_{i\in I}$. For an arc or boundary segment γ, the *laminated lambda length* of γ is defined by

$$\lambda_{\sigma,\mathbf{L}}(\gamma) = \lambda_\sigma(\gamma) \prod_{i\in I} q_i^{l_{L_i}(\gamma)/2},$$

where $q = (q_i)_{i\in I}$ is a collection of positive real parameters, and $l_{L_i}(\gamma)$ denotes the *transversal measure* of γ with respect to L_i.

A point (σ, q) in the *laminated Teichmüller space* $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ is a decorated hyperbolic structure $\sigma \in \widetilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ together with a vector $q \in \mathbb{R}^I_{>0}$ satisfying the boundary conditions $\lambda_{\sigma,\mathbf{L}}(\gamma) = 1$ for all boundary segments in $(\mathbf{S}, \mathbf{M})$.

Penner's theorem implies that $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ is coordinatized by the laminated lambda lengths of arcs in a fixed triangulation together with the parameters q_i:

$$\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L}) \cong \mathbb{R}^{n+|I|}_{>0}.$$

Geometric model, unpunctured case

Theorem 26 *Let* $\mathbf{L}$ *be a multi-lamination on a marked bordered surface* $(\mathbf{S}, \mathbf{M})$ *without interior marked points. Then the totally positive variety associated with the corresponding cluster algebra* $\mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ *is canonically isomorphic to the laminated Teichmüller space* $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$*. Under this isomorphism, the cluster variables are represented by the laminated lambda lengths of arcs, and the coefficient variables are represented by the parameters* q_i *associated with the laminations in* $\mathbf{L}$*.*

Thus, the lambda lengths $\lambda_{\sigma,\mathbf{L}}(\gamma)$ satisfy the exchange relations encoded by the extended signed adjacency matrices $\tilde{B}(T, \mathbf{L})$.

Geometric model, general case

If we allow marked points in the interior of $\mathbf{S}$, the construction becomes substantially more complicated, and requires us to:

- extend Thurston's theory of shear coordinates to tagged triangulations;

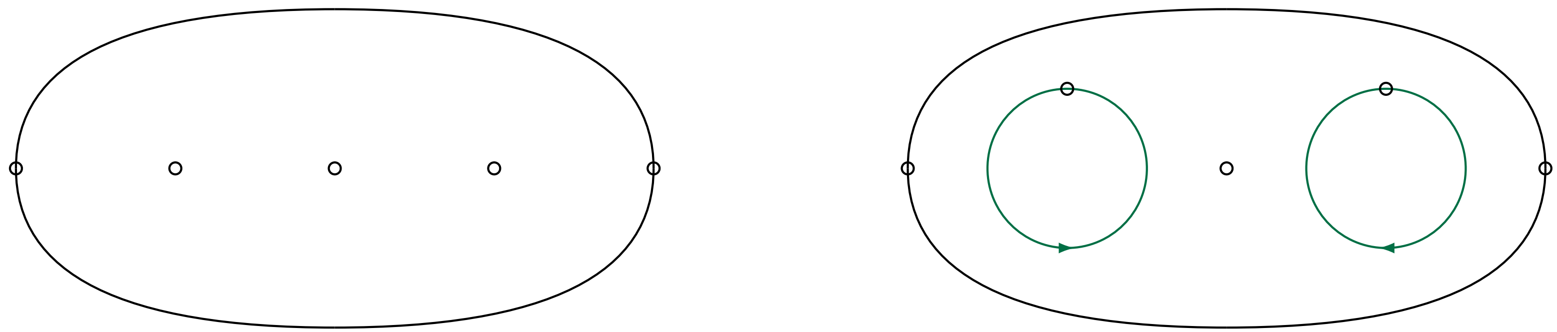
- lift laminations and arcs to the covering space to handle intersection numbers and lambda lengths in the presence of spiralling;

- set up the proper generalization of a decorated Teichmüller space;

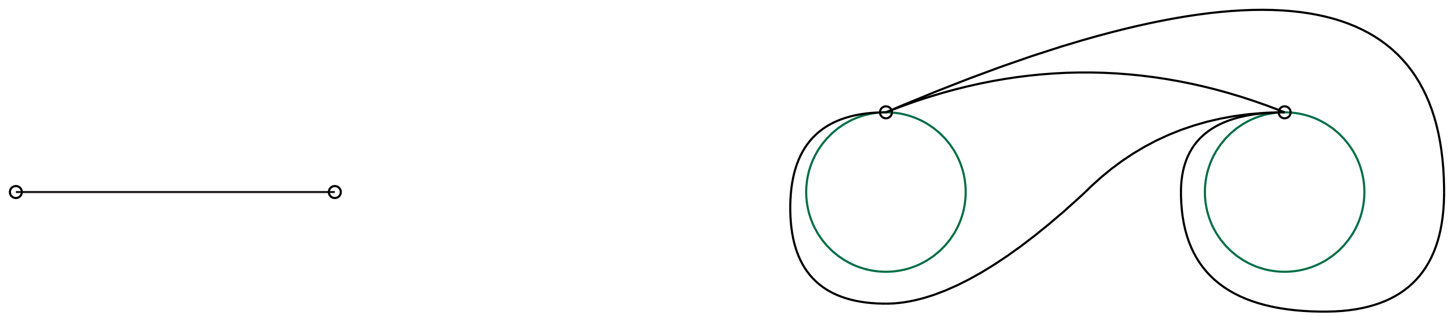
- realize cluster variables as well-defined functions on this space, independent on the choices of lifts.

Opened surface

We open each puncture in $\mathbf{M}$ to a circular boundary component; orient it in one of three ways: clockwise, counterclockwise, or degenerate (no opening); and pick a marked point on it.

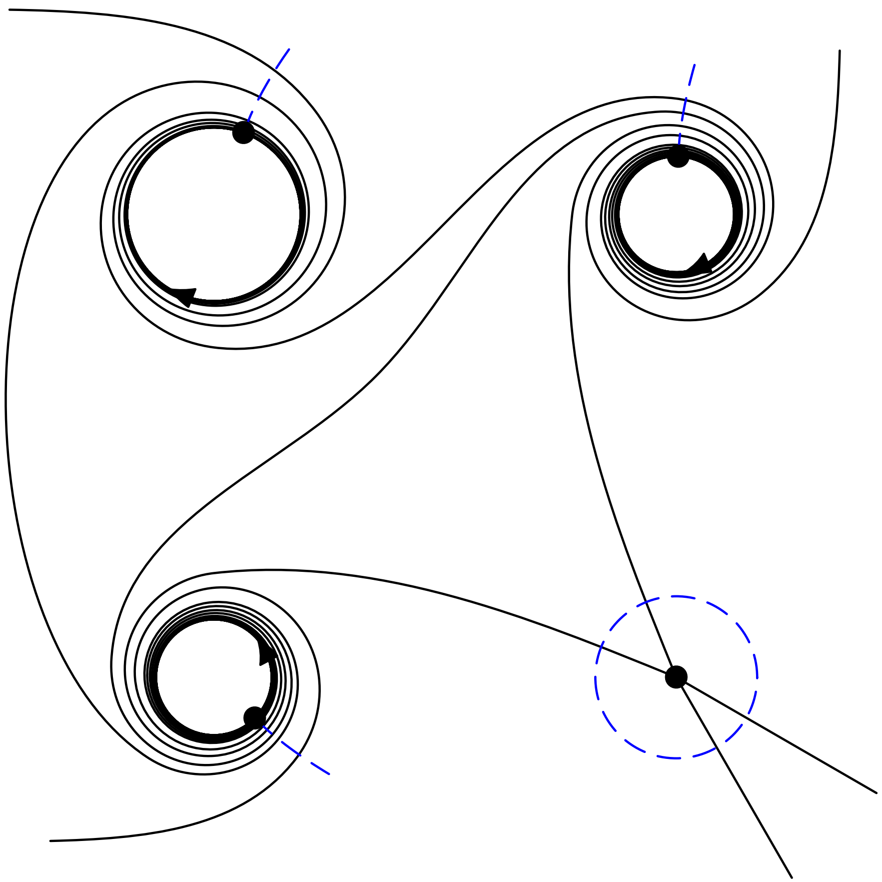


Each arc on $(\mathbf{S},\mathbf{M})$ incident to a puncture can be *lifted* to infinitely many different arcs on the opened surface.



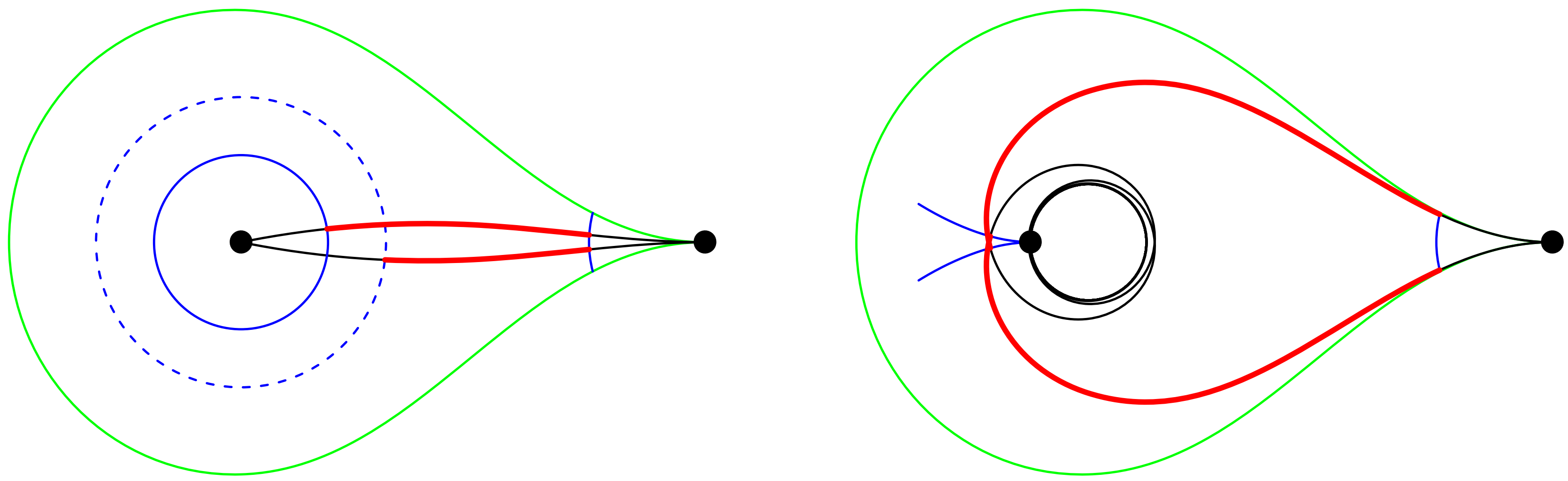
Teichmüller space $\overline{\mathcal{T}}(\mathrm{S},\mathrm{M})$ associated with opened surfaces

A point in $\overline{\mathcal{T}}(\mathbf{S},\mathbf{M})$ includes: an orientation of each nontrivial opening C; a decorated hyperbolic structure with geodesic boundary along C; and an horocycle at the marked point chosen on C. This horocycle should be perpendicular to C and to all geodesics that spiral into C in the chosen direction.



Lambda lengths of lifted tagged arcs

In order to extend the notion of a lambda length to tagged arcs on the opened surface, we use *conjugate* horocycles:



Combined with a suitable rescaling of lambda lengths at vertices (a *gauge transformation*), this definition ensures that the product of lambda lengths of two lifted tagged arcs as shown above equals the lambda length of the enclosing loop.

These lambda lengths can be used to coordinatize the space $\overline{\mathcal{T}}(\mathbf{S},\mathbf{M})$.

Laminated lambda lengths: general case

Fix a lift $\overline{L}$ of each lamination $L \in \mathbf{L}$ to the opened surface.

Choose a lift $\overline{\gamma}$ of each tagged arc γ to the opened surface.

Both the lambda length of $\overline{\gamma}$ and its transverse measure with respect to $\overline{L}$ will depend on the choice of the lift $\overline{\gamma}$.

It turns out however that if the parameters q_i and the lengths of the boundary segments and openings satisfy appropriate boundary conditions, then the same formula as before defines a "laminated lambda length" $\lambda_{\sigma,\mathbf{L}}(\overline{\gamma})$ which only depends on γ but not on the choice of its lift $\overline{\gamma}$.

These functions will satisfy the required exchange relations, provide a geometric realization of the corresponding cluster algebra, and identify the appropriate modification of the decorated Teichmüller space with the associated totally positive variety.