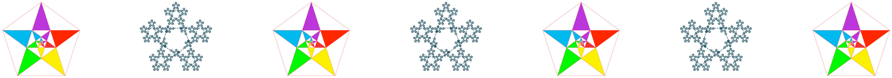
Poisson properties of cluster algebras: Pentagram map

Paris

Day 5, Afternoon, April 20, 2012

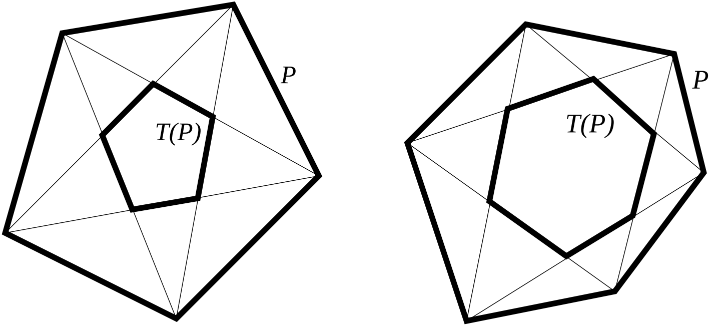
And the evening and the morning were the fifth day



R. Schwartz, V. Ovsienko, S. Tabachnikov, S. Morier-Genoud, M. Glick, F. Soloviev, G. Mari-Beffa, M. Gekhtman, M. Shapiro, A. Vainshtein, R. Kenyon, A. Goncharov, V. Fock, A. Marshakov

(almost) everything ArXived

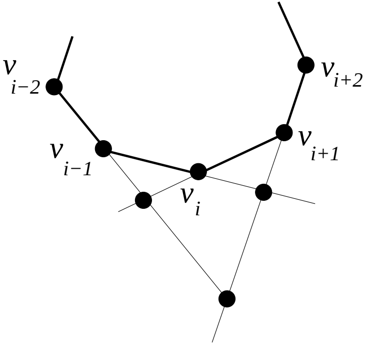
Pentagram Map T:



Acts on projective equivalence classes of closed and *twisted* n-gons with monodromy M. The latter constitute a $2n$-dimensional space, the former is $2n-8$-dimensional.

A good reference: `http://en.wikipedia.org/wiki/Pentagram_map`

Corner coordinates: left and right cross-ratios $X_1, Y_1, \ldots, X_n, Y_n$.



The map is as follows:

$$X_i^* = X_i \, \frac{1 - X_{i-1}\, Y_{i-1}}{1 - X_{i+1}\, Y_{i+1}}, \qquad Y_i^* = Y_{i+1} \, \frac{1 - X_{i+2}\, Y_{i+2}}{1 - X_i\, Y_i}.$$

Hidden *scaling symmetry*

$$(X_1, Y_1, \dots, X_n, Y_n) \mapsto (tX_1, t^{-1}Y_1, \dots, tX_n, t^{-1}Y_n)$$

commutes with the map.
"Easy" invariants:

$$O_n = \prod_{i=1}^{n} X_i, \quad E_n = \prod_{i=1}^{n} Y_i.$$

Monodromy invariants:

$$\frac{O_n^{2/3} E_n^{1/3} (\mathrm{Tr}\ M)}{(\det M)^{1/3}} = \sum_{k=1}^{[n/2]} O_k$$

are polynomials in (X_i, Y_i), decomposed into homogeneous components; likewise, for E_k with M^{-1} replacing M.

Theorem (OST 2010). The Pentagram Map is completely integrable on the space of twisted n-gons:

1). The monodromy invariants are independent integrals (there are $2[n/2]+2$ *of them).*

2). There is an invariant Poisson structure of corank 2 if n is odd, and corank 4 if n is even, such that these integrals Poisson commute.

Poisson bracket: $\{X_i, X_{i+1}\} = -X_i\, X_{i+1},\ \{Y_i, Y_{i+1}\} = Y_i\, Y_{i+1},$ and the rest $= 0$.

Complete integrability on the space of closed polygons has been proven as well:

F. Soloviev. *Integrability of the Pentagram Map*, arXiv:1106.3950;

V. Ovsienko, R. Schwartz, S. Tabachnikov. *Liouville-Arnold integrability of the pentagram map on closed polygons*, arXiv:1107.3633.

Cluster algebras connection:

M. Glick. *The pentagram map and Y-patterns*, Adv. Math., **227** (2011), 1019–1045.

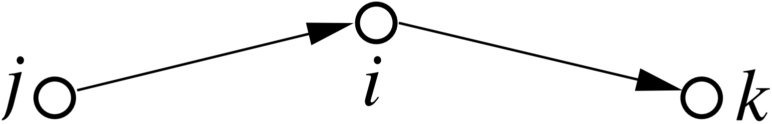
He considered the dynamics in the $2n-1$-dimensional quotient space by the scaling symmetry $(X, Y) \mapsto (tX, t^{-1}Y)$:

$$p_i = -X_{i+1}Y_{i+1}, \quad q_i = -\frac{1}{Y_i X_{i+1}},$$

and proved that it was a Y-type cluster algebra dynamics.

Cluster dynamics

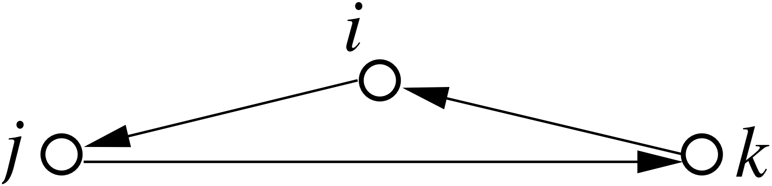
Given a *quiver* (an oriented graph with no loops or 2-cycles) whose vertices are labeled by variables τ_i (rational functions in some free variables), the mutation on vertex i is as follows:



$$\tau_i^* = \frac{1}{\tau_i}, \quad \tau_j^* = \frac{\tau_j \tau_i}{1+\tau_i}, \quad \tau_k^* = \tau_k(1+\tau_i);$$

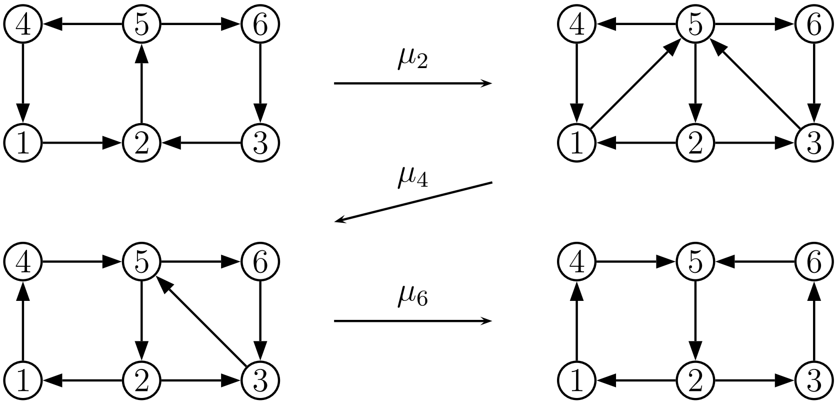
the rest of the variables are intact.

The quiver also mutates, in three steps:
(i) for every path $j \to i \to k$, add an edge $j \to k$;
(ii) reverse the orientation of the edges incident to the vertex i;
(iii) delete the resulting 2-cycles.

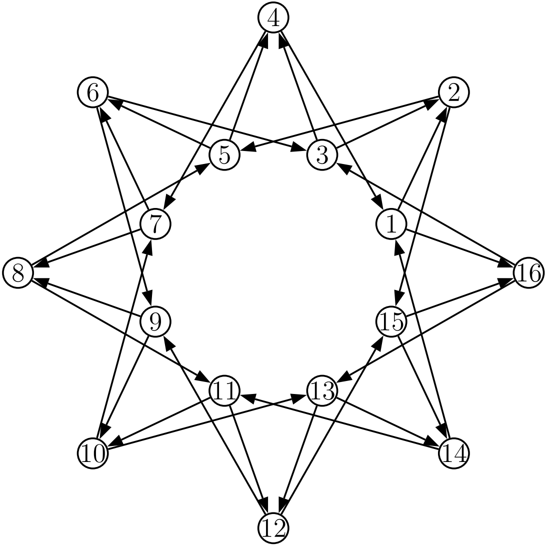


The mutation on a given vertex is an involution.

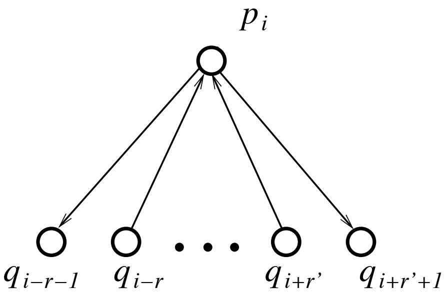
Example of mutations:



Glick's quiver ($n = 8$):



Joint work in progress with Michael Gekhtman, Sergey Tabachnikov, and Alek Vainshtein, ERA 19 (2012), 1–17.
Generalizing Glick's quiver (the case of $k = 3$), consider the homogeneous bipartite graph $\mathcal{Q}_{k,n}$ where $r = [k/2] - 1$, and $r' = r$ for k even and $r' = r + 1$ for k odd (each vertex is 4-valent):



Dynamics: mutations on all p-vertices, followed by swapping p and q; this is the map $\overline{T}_k$:

$$q_i^* = \frac{1}{p_i}, \quad p_i^* = q_i \frac{(1+p_{i-r-1})(1+p_{i+r+1})p_{i-r}p_{i+r}}{(1+p_{i-r})(1+p_{i+r})}, \quad k \text{ even},$$

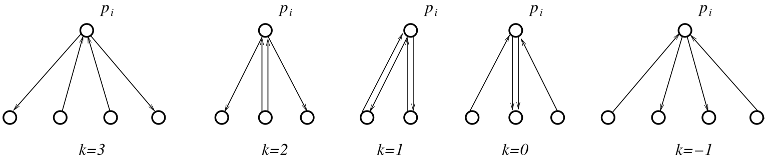
$$q_i^* = \frac{1}{p_{i-1}}, \quad p_i^* = q_i \frac{(1+p_{i-r-2})(1+p_{i+r+1})p_{i-r-1}p_{i+r}}{(1+p_{i-r-1})(1+p_{i+r})}, \quad k \text{ odd}.$$

The quiver is preserved. The function $\prod p_i q_i$ is invariant; we restrict to the subspace $\prod p_i q_i = 1$.

Invariant Poisson bracket: the variables Poisson commute, unless they are connected by an arrow: $\{p_i, q_j\} = \pm p_i q_j$ (depending on the direction).

(This bracket comes from the general theory: GSV, *Cluster algebras and Poisson geometry, AMS, 2010*).

The quivers, for small values of k, look like this (for $k = 1$, the arrows cancel out):



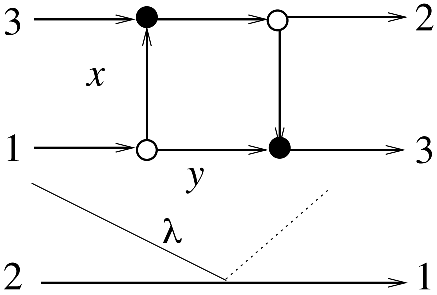
The map $\overline{T}_k$ is *reversible*: $\overline{D}_k \circ \overline{T}_k \circ \overline{D}_k = \overline{T}_k^{-1}$,
where

$$\overline{D}_k : p_i \mapsto \frac{1}{q_i},\ q_i \mapsto \frac{1}{p_i}, \quad k \text{ even},$$
$$\overline{D}_k : p_i \mapsto \frac{1}{q_{i+1}},\ q_i \mapsto \frac{1}{p_i}, \quad k \text{ odd}.$$

Goal: to reconstruct the x, y-dynamics and to interpret it geometrically.

Weighted directed networks on the cylinder and the torus (A. Postnikov math.CO/0609764, for networks in a disc; GSV book).

Example:



Two kind of vertices, white and black.

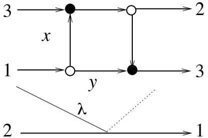
Convention: an edge weight is 1, if not specified.

The *cut* is used to introduce a *spectral parameter* λ.

Boundary measurements

:

the network



corresponds to the matrix

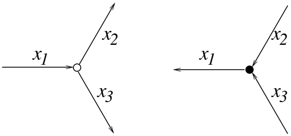
$$\begin{pmatrix} 0 & x & x+y \\ \lambda & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Concatenation of networks $\mapsto$ product of matrices.

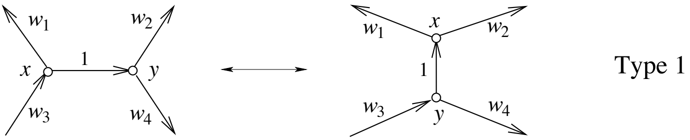
Gauge group: at a vertex, multiply the weights of the incoming edges and divide the weights of the outgoing ones by the same function. Leaves the boundary measurements intact.

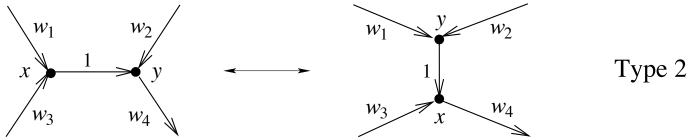
Face weights: the product of edge weights over the boundary (orientation taken into account). The boundary measurement map to matrix functions factorizes through the space of face weights. (They will be identified with the p, q-coordinates).

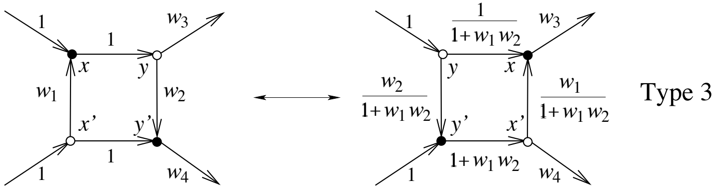
Poisson bracket (6-parameter): $\{x_i, x_j\} = c_{ij} x_i x_j,\ i \neq j \in \{1, 2, 3\}$



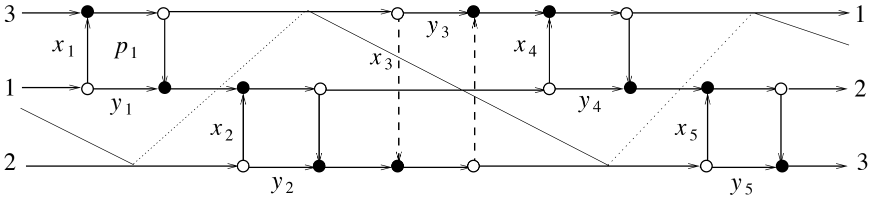
Postnikov moves (do not change the boundary measurements):





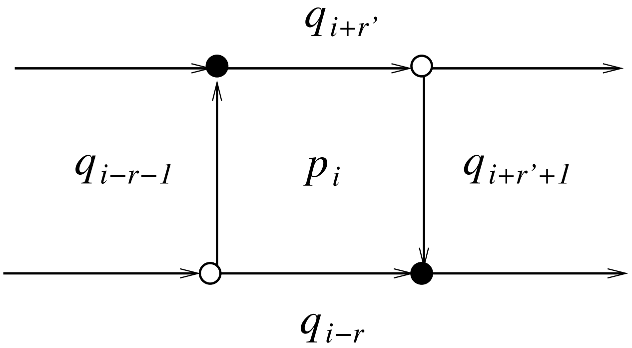


Consider a network whose dual graph is the quiver $\mathcal{Q}_{k,n}$.
It is drawn on the torus. Example, $k = 3, n = 5$:



Convention: white vertices of the graph are on the left of oriented edges of the dual graph.

The network is made of the blocks:

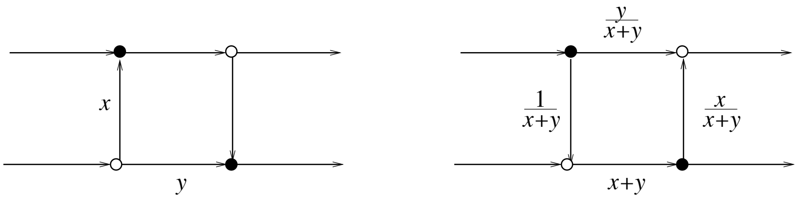


Face weights:

$$p_i = \frac{y_i}{x_i}, \quad q_i = \frac{x_{i+1+r}}{y_{i+r}}.$$

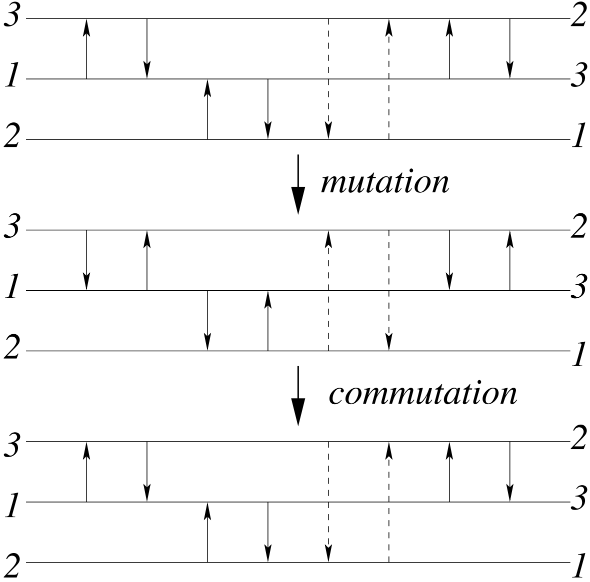
This is a projection $\pi : (x, y) \mapsto (p, q)$ with 1-dimensional fiber.

(x, y)-dynamics: mutation (Postinov type 3 move on each p-face),



followed by the Postnikov type 1 and 2 moves on the white-white and black-black edge (this interchanges p- and q-faces), including moving across the vertical cut, and finally, re-calibration to restore 1s on the appropriate edges. These moves preserve the conjugacy class of the boundary measurement matrix.

Schematically:



This results in the map T_k:

$$x_i^* = x_{i-r-1}\frac{x_{i+r}+y_{i+r}}{x_{i-r-1}+y_{i-r-1}}, \quad y_i^* = y_{i-r}\frac{x_{i+r+1}+y_{i+r+1}}{x_{i-r}+y_{i-r}}, \quad k \text{ even},$$
$$x_i^* = x_{i-r-2}\frac{x_{i+r}+y_{i+r}}{x_{i-r-2}+y_{i-r-2}}, \quad y_i^* = y_{i-r-1}\frac{x_{i+r+1}+y_{i+r+1}}{x_{i-r-1}+y_{i-r-1}}, \quad k \text{ odd}.$$

The map T_k is conjugated to the map $\overline{T}_k$: $\pi \circ T_k = \overline{T}_k \circ \pi$.
Relation with the pentagram map: the change of variables

$$x_i \mapsto Y_i, \quad y_i \mapsto -Y_i X_{i+1} Y_{i+1},$$

identifies T_3 with the pentagram map.

Complete integrability of the maps T_k

The ingredients are suggested by the combinatorics of the network.
Invariant Poisson bracket (in the "stable range" $n \geq 2k-1$):

$$\{x_i, x_{i+l}\} = -x_i x_{i+l}, 1 \leq l \leq k-2;\ \{y_i, y_{i+l}\} = -y_i y_{i+l}, 1 \leq l \leq k-1;$$
$$\{y_i, x_{i+l}\} = -y_i x_{i+l}, 1 \leq l \leq k-1;\ \{y_i, x_{i-l}\} = y_i x_{i-l}, 0 \leq l \leq k-2;$$

the indices are cyclic.
The functions $\prod x_i$ and $\prod y_i$ are Casimir. If n is even and k is odd, one has four Casimir functions:

$$\prod_{i \text{ even}} x_i, \quad \prod_{i \text{ odd}} x_i, \quad \prod_{i \text{ even}} y_i, \quad \prod_{i \text{ odd}} y_i.$$

Lax matrices, monodromy, integrals: for $k \geq 3$,

$$L_i = \begin{pmatrix} 0 & 0 & 0 & \dots & x_i & x_i + y_i \\ \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix},$$

and for $k = 2$,

$$L_i = \begin{pmatrix} \lambda x_i & x_i + y_i \\ \lambda & 1 \end{pmatrix}.$$

The boundary measurement matrix is $M(\lambda) = L_1 \cdots L_n$. The characteristic polynomial

$$\det(M(\lambda) - z) = \sum I_{ij}(x, y) z^i \lambda^j.$$

is T_k-invariant: the integrals I_{ij} are in involution.

Zero curvature (Lax) representation:

$$L_i^* = P_i L_{i+r-1} P_{i+1}^{-1}$$

where L_i are the Lax matrices and

$$P_i = \begin{pmatrix} 0 & \frac{x_i}{\lambda\sigma_i} & \frac{y_{i+1}}{\lambda\sigma_{i+1}} & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{x_{i+1}}{\sigma_{i+1}} & \frac{y_{i+2}}{\sigma_{i+2}} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{x_{i+k-4}}{\sigma_{i+k-4}} & \frac{y_{i+k-3}}{\sigma_{i+k-3}} & 0 \\ -\frac{1}{\sigma_{i+k-2}} & 0 & 0 & \dots & 0 & \frac{x_{i+k-3}}{\sigma_{i+k-3}} & 1 \\ \frac{1}{\sigma_{i+k-2}} & -\frac{1}{\lambda\sigma_{i+k-1}} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda\sigma_{i+k-1}} & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

with $\sigma_i = x_i + y_i$.

Geometric interpretations

Twisted corrugated polygons in $\mathbf{RP}^{k-1}$ and $k-1$-diagonal maps

Let $k \geq 3$. Let $\mathcal{P}_{k,n}$ be the space of projective equivalence classes of *generic* twisted n-gons in $\mathbf{RP}^{k-1}$; one has: dim $\mathcal{P}_{k,n} = n(k-1)$.
Let $\mathcal{P}^0_{k,n} \subset \mathcal{P}_{k,n}$ consist of the polygons with the following property: for every i, the vertices V_i, V_{i+1}, V_{i+k-1} and V_{i+k} span a projective plane. These are *corrugated* polygons. Projective duality preserves corrugated polygons.
The consecutive $k-1$-diagonals of a corrugated polygon intersect. The resulting polygon is again corrugated. One gets a pentagram-like $k-1$-diagonal map on $\mathcal{P}^0_{k,n}$. For $k = 3$, this is the pentagram map.

Coordinates: lift the vertices V_i of a corrugated polygon to vectors $\widetilde{V}_i$ in $\mathbf{R}^k$ so that the linear recurrence holds

$$\widetilde{V}_{i+k} = y_{i-1}\widetilde{V}_i + x_i\widetilde{V}_{i+1} + \widetilde{V}_{i+k-1},$$

where x_i and y_i are n-periodic sequences. These are coordinates in $\mathcal{P}^0_{k,n}$. In these coordinates, the map is identified with T_k.
The same functions x_i, y_i can be defined on polygons in the projective plane. One obtains integrals of the "deeper" diagonal maps on twisted polygons in $\mathbf{RP}^2$.

Case $k = 2$

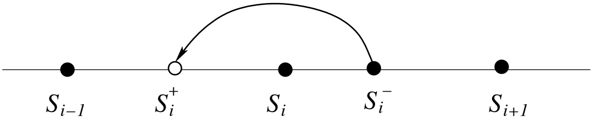
Consider the space $\mathcal{S}_n$ of pairs of twisted n-gons (S^-, S) in $\mathbf{RP}^1$ with the same monodromy. Consider the projectively invariant projection ϕ to the (x, y)-space (cross-ratios):

$$x_i = \frac{(S_{i+1} - S^-_{i+2})(S^-_i - S^-_{i+1})}{(S^-_i - S_{i+1})(S^-_{i+1} - S^-_{i+2})}$$

$$y_i = \frac{(S^-_{i+1} - S_{i+1})(S^-_{i+2} - S_{i+2})(S^-_i - S^-_{i+1})}{(S^-_{i+1} - S_{i+2})(S^-_i - S_{i+1})(S^-_{i+1} - S^-_{i+2})}.$$

Then x_i, y_i are coordinates in $\mathcal{S}_n / PGL(2, \mathbf{R})$.

Define a transformation $F_2(S^-, S) = (S, S^+)$, where S^+ is given by the following local leapfrog rule: given points $S_{i-1}, S_i^-, S_i, S_{i+1}$, the point S_i^+ is obtained by the reflection of S_i^- in S_i in the projective metric on the segment $[S_{i-1}, S_{i+1}]$:



The projection ϕ conjugates F_2 and T_2.

In formulas:

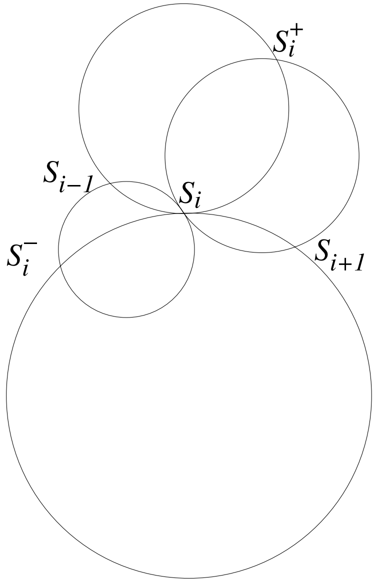
$$\frac{1}{S_i^+ - S_i} + \frac{1}{S_i^- - S_i} = \frac{1}{S_{i+1} - S_i} + \frac{1}{S_{i-1} - S_i},$$

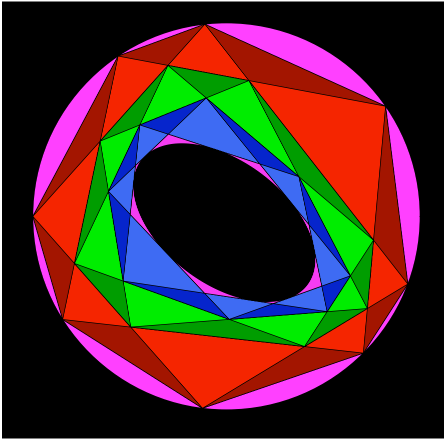
or, equivalently,

$$\frac{(S_i^+ - S_{i+1})(S_i - S_i^-)(S_i - S_{i-1})}{(S_i^+ - S_i)(S_{i+1} - S_i)(S_i^- - S_{i-1})} = -1,$$

(Toda-type equations).

In $\mathbf{CP}^1$, a circle pattern interpretation (generalized Schramm's pattern):





Thank you!