Poisson properties of cluster algebras

Paris

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τ-coordinates

Nondegenerate coordinate change:

$$\tau_i(t) = \begin{cases} \prod_{j\neq i} z_j(t)^{b_{ij}(t)} & \text{for } i \leqslant m, \\ \prod_{j\neq i} z_j(t)^{b_{ij}(t)}/z_i(t) & \text{for } m+1 \leqslant i \leqslant n. \end{cases}$$

Exchange in direction i:

$$\tau_i \mapsto \frac{1}{\tau_i}; \qquad \tau_j \mapsto \begin{cases} \tau_j(1+\tau_i)^{b_{ij}}, & \text{if } b_{ij} > 0, \\ \tau_j\left(\frac{\tau_i}{1+\tau_i}\right)^{-b_{ij}}, & \text{otherwise.} \end{cases}$$

Definition

We say that a skew-symmetrizable matrix A is *reducible* if there exists a permutation matrix P such that PAP^T is a block-diagonal matrix, and *irreducible* otherwise. The *reducibility* $\rho(A)$ is defined as the maximal number of diagonal blocks in PAP^T. The partition into blocks defines an obvious equivalence relation $\sim$ on the rows (or columns) of A.

Compatible Poisson structures

Theorem

For an $B \in \mathbb{Z}_{n,n+m}$ as above of rank n the set of Poisson brackets for which all extended clusters in $\mathfrak{A}(B)$ are log-canonical has dimension $\rho(B) + \binom{m}{2}$. Moreover, the coefficient matrices Ω^τ of these Poisson brackets in the basis τ are characterized by the equation $\Omega^\tau[m,n] = \Lambda B$ for some diagonal matrix $\Lambda =$ diagonal$(\lambda_1, \ldots, \lambda_n)$ where $\lambda_i = \lambda_j$ whenever $i \sim j$.

Degenerate exchange matrix

Example

Cluster algebra of rank 3 with trivial coefficients. Exchange matrix $B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. Compatible Poisson bracket must satisfy $\{x_1, x_2\} = \lambda x_1 x_2,\ \{x_1, x_3\} = \mu x_1 x_3,\ \{x_2, x_3\} = \nu x_2 x_3$

Exercise: Check that these conditions imply $\lambda = \mu = \nu = 0$.

Conclusion: Only trivial Poisson structure is compatible with the cluster algebra.

What to do?

We will use the dual language of 2-forms

Compatible 2-forms

Definition

2-form ω is compatible with a collecion of functions $\{f_i\}$ if $\omega = \sum_{i,j} \omega_{ij} \frac{df_i}{f_i} \wedge \frac{df_j}{f_j}$

Definition

2-form ω is compatible with a cluster algebra if it compatible with all clusters.

Exercise

Check that the form $\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} - \frac{dx_1}{x_1} \wedge \frac{dx_3}{x_3} + \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}$ is compatible with the example above.

Compatible 2-forms

Theorem

For an $B \in \mathbb{Z}_{n,n+m}$ the set of Poisson brackets for which all extended clusters in $\mathfrak{A}(B)$ are log-canonical has dimension $\rho(B) + \binom{m}{2}$. Moreover, the coefficient matrices $\Omega^{\mathbf{x}}$ of these 2-forms in initial cluster are characterized by the equation $\Omega^{\mathbf{x}}[m, n] = \Lambda B$, where $\Lambda = diagonal(\lambda_1, \ldots, \lambda_n)$, with $\lambda_i = \lambda_j \neq 0$ whenever $i \sim j$.

Recovering cluster algebra transformations

We recover cluster algebra exchange rules as unique involutive transformations of log-canonical bases satisfying certain additional restrictions.

Local data F is a family of rational functions in one variable ψ_w, $w = 0, \pm 1, \pm 2, \dots$, and an additional function in one variable φ.

For any Poisson bracket ω and any log-canonical (with respect to ω) basis $\tau = (\tau_1, \dots, \tau_m)$, the local data F gives rise to n transformations F_i^ω defined as follows:

(i) $F_i^\omega(\tau_i) = \varphi(\tau_i)$;

(ii) let $\Omega = (\omega_{ij})$ be the coefficient matrix of ω in the basis τ, then $F_i^\omega(\tau_j) = \tau_j \psi_{\omega_{ij}}(\tau_i)$ for $j \neq i$.

Cluster $\mathfrak{A}-$ and $\mathfrak{X}-$ manifolds

Coordinate ring	**Manifold**
$\mathcal{A}-$ cluster algebra with cluster coordinates x_i	$\mathfrak{A}-$ cluster variety with compatible 2-form
$\uparrow \pi^*$	$\pi:\tau_i=\prod_{j=1}^{n+m} x_j^{b_{ij}} \downarrow$
$\mathcal{X}-$ algebra generated by τ_i	$\mathfrak{X}-$ Poisson cluster variety

We say that local data F is canonical if for any Poisson bracket ω, any log-canonical (with respect to ω) basis τ, and any index i, the set $F_i^\omega(t)$ is a log-canonical basis of ω as well.

Local data is called involutive if any F_i^ω is an involution, and is called normalized if $\lim_{z\to 0}\psi_w(z) = \pm 1$ for any integer $w \geqslant 0$.

We say that a polynomial P of degree p is a-reciprocal if $P(0) \neq 0$ and there exists a constant c such that $\xi^p P(a/\xi) = cP(\xi)$ for any $\xi \neq 0$.

Description of normalized involutive canonical local data

Theorem

Any normalized involutive canonical local data has one of the following forms:

(i) $\varphi(\xi) = \xi$ *and* $\psi_w(\xi) = \pm 1$ *for any integer* w (*trivial local data*);

(ii) $\varphi(\xi) = -\xi$ *and* $\psi_w(\xi) = \pm \left(\frac{P(\xi)}{P(-\xi)}\right)^w$, *where* P *is a polynomial without symmetric roots;*

(iii) $\varphi(\xi) = \frac{a}{\xi}$, $\psi_w(\xi) = a_w \xi^{c_w} \psi_1^w(\xi)$, *and* $\psi_1(\xi) = \frac{P(\xi)}{Q(\xi)}$, *where* P *and* Q *are coprime* a*-reciprocal polynomials of degrees* p *and* q, *and the constants* a_w, c_w, p, q *satisfy relations* $a_{-1}^2 = a^{-c_{-1}}$, $p - q = c_{-1}$, *and*

$$a_w = \begin{cases} \pm 1 & \text{for } w \geqslant 0 \\ a_{-w}^{-1} a_{-1}^{-w} & \text{for } w < 0, \end{cases}$$

$$c_w = \begin{cases} 0 & \text{for } w \geqslant 0 \\ -wc_{-1} & \text{for } w < 0. \end{cases}$$

Finiteness

We say that local data F is finite if it has the following finiteness property: let $n = 2$, and let ω possess a log-canonical basis $\tau = (\tau_1, \tau_2)$ such that the corresponding coefficient matrix has the simplest form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; then the group generated by F_1^ω and F_2^ω has a finite order.

Theorem

Any nontrivial finite involutive canonical local data has one of the following forms:
(i) $\varphi(\xi) = a/\xi$ *with* $a \neq 0$, $\psi_w(\xi) = (\pm 1)^w a_w$, $a_w = \pm 1$ *and* $a_w = a_{-w}$;
(ii) $\varphi(\xi) = b^2/\xi$ *with* $b \neq 0$,

$$\psi_w(\xi) = \begin{cases} (\pm 1)^w a_w \left(\dfrac{\xi + b}{b} \right)^w & \text{for } w \geqslant 0 \\ (\pm 1)^w a_{-w} \left(\dfrac{\xi + b}{\xi} \right)^w & \text{for } w < 0, \end{cases}$$

where $a_w = \pm 1$ *and* $a_w = a_{-w}$.

Constructing cluster algebra for Grassmannian $G_k(n)$

Goal: construct one cluster containing only Plücker coordinates.

- Find $k \times (n-k)$ functions f_i such that $\{f_i, f_j\} = c_{ij} f_i f_j$.
  For $1 \le i \le k$, $1 \le j \le n-k$ denote by $s = \min(i, n-k-j+1)$. For an element $(\mathbf{1} \quad Y) \in G_k(n)$ denote by Y_{ij} its dense $s \times s$ minor whose left low corner is (i,j) and which is attached either to the first row or to the last column.
- Exchange matrix is computed as an "inverse" of coefficient matrix of Poisson bracket
- If the result of cluster mutations of Y_{ij} is regular function on $G_k(n)$ then we claim Y_{ij} to be cluster variable, otherwise we claim Y_{ij} to be frozen variable.
  Cluster mutations of this initial cluster correspond to matrix minor identities.

Cluster structure of $G_2(4)$

Example

$(\mathbf{1} \quad Y) \in G_2(4)$. $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$Y_{11} = a$; $Y_{12} = b$; $Y_{21} = det(Y) = ad - bc$; $Y_{22} = c$

Poisson bracket: Y_{21} is a Casimir function (Poisson commutes with all other functions)

$\{Y_{11}, Y_{12}\} = Y_{11}Y_{12}$; $\{Y_{11}, Y_{22}\} = 0$; $\{Y_{12}, Y_{22}\} = Y_{12}Y_{22}$;

Coefficient matrix $\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $rank\Omega = 2$

Example $G_2(4)$ continued...

Example

Top left part is invertible with inverse matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

By the degree homogenuity a candidate for exchange matrix must contain rows of type: $\begin{pmatrix} 0 & 1 & 2\ell+1 & -2(\ell+1) \\ -1 & 0 & -2s-1 & 2(s+1) \end{pmatrix}$.

Condition that the result of cluster mutation is regular function means that expression $ad^{2s+1} \pm (ad-bc)^{2(\ell+1)}$ is divisible by b which implies for $s=1$. No ℓ satisfies this condition.

Hence, the cluster algebra contains two clusters:

$$\{a, b, d, det(Y)\} \leftrightarrow \{c, b, d, \det(Y)\},$$

with two cluster variables a and c, and three frozen variables $b, d, \det(Y)$.

Cluster manifold

For an abstract cluster algebra of geometric type $\mathcal{A}$ of rank m we construct an algebraic variety $\mathfrak{A}$ (which we call cluster manifold)
Idea: $\mathfrak{A}$ is a "good" part of $Spec(\mathcal{A})$.
We will describe $\mathfrak{A}$ by means of charts and transition functions.
For each cluster t we define an open chart

$$\mathfrak{A}(t) = Spec(\mathbb{C}[\mathbf{x}(t), \mathbf{x}(t)^{-1}, \mathbf{y}]),$$

where $\mathbf{x}(t)^{-1}$ means $x_1(t)^{-1}, \ldots, x_m(t)^{-1}$.
Transitions between charts are defined by exchange relations

$$x_i(t')x_i(t) = \prod_{b_{ik}(t)>0} z_k(t)^{b_{ik}(t)} + \prod_{b_{ik}(t)<0} z_k(t)^{-b_{ik}(t)}$$

$$z_j(t') = z_j(t) \quad j \neq i,$$

Finally, $\mathfrak{A} = \cup_t \mathfrak{A}(t)$.

Nonsingularity of $\mathfrak{A}$

$\mathfrak{A}$ contains only such points $p \in Spec(\mathcal{A})$ that there is a cluster t whose cluster elements form a coordinate system in some neighborhood of p.

Observation The cluster manifold $\mathfrak{A}$ is nonsingular and possesses a Poisson bracket that is log-canonical w.r.t. any extended cluster.

Let ω be one of these Poisson brackets.

Casimir of ω is a function that is in involution with all the other functions on $\mathfrak{A}$. All rational casimirs form a subfield $\mathfrak{F}_C$ in the field of rational functions $\mathbb{C}(\mathfrak{A})$. The following proposition provides a complete description of $\mathfrak{F}_C$.

Lemma $\mathfrak{F}_C = \mathfrak{F}(\mathbf{m}_1, \dots, \mathbf{m}_s)$, where $\mathbf{m}_j = \prod y_i^{\alpha_{ji}}$ for some integral α_{ji}, and $s = \text{corank}\omega$.

Toric action

We define a local toric action on the extended cluster t as the $\mathbb{C}^*$-action given by the formula $z_i(t) \mapsto z_i(t) \cdot \xi^{w_i(t)}$, $\xi \in \mathbb{C}^*$ for some integral $w_i(t)$ (called weights of toric action).
Local toric actions are compatible if taken in all clusters they define a global action on $\mathcal{A}$. This toric action is said to be an extension of the above local actions.
$\mathfrak{A}^0$ is the regular locus for all compatible toric actions on $\mathfrak{A}$.
$\mathfrak{A}^0$ is given by inequalities $y_i \neq 0$.

Examples

- There exists a cluster algebra structure on SL_n compatible with Sklyanin Poisson bracket. $\mathfrak{A}^0$ is the maximal double Bruhat cell.
- There exists a cluster algebra structure on Grassmanian compatible with push-forward of Sklyanin Poisson bracket. $\mathfrak{A}^0$ determined by the inequalities $\{\text{solid Plücker coordinate} \neq 0\}$.
- (Decorated) Teichmüller space has a natural structure of cluster algebra. Weyl-Petersson symplectic form is the unique symplectic form "compatible" with the structure of cluster algebra.

Symplectic leaves

$\mathfrak{A}$ is foliated into a disjoint union of symplectic leaves of ω.
Given generators $q_1, \dots, q_s$ of the field of rational casimirs $\mathfrak{F}_C$ we have a map $Q : \mathfrak{A} \to \mathbb{C}^s$, $Q(x) = (q_1(x), \dots, q_s(x))$.
We say that a symplectic leaf $\mathbf{L}$ is generic if there exist s vector fields u_i on $\mathfrak{A}$ such that
a) at every point $x \in \mathbf{L}$, the vector $u_i(x)$ is transversal to the surface $Q^{-1}(Q(\mathbf{L}))$;
b) the translation along u_i for a sufficiently small time t gives a diffeomorphism between $\mathbf{L}$ and a close symplectic leaf $\mathbf{L}_t$.
Lemma $\mathfrak{A}^0$ is foliated into a disjoint union of generic symplectic leaves of the Poisson bracket ω.
Remark Generally speaking, $\mathfrak{A}^0$ does not coincide with the union of all "generic" symplectic leaves in $\mathfrak{A}$.

Connected components of $\mathfrak{A}^0$

Question: find the number $\#(\mathfrak{A}^0)$ of connected components of $\mathfrak{A}^0$.
Let $\mathbb{F}_2^n$ be an n-dimensional vector space over $\mathbb{F}_2$ with a fixed basis $\{e_i\}$.
Let B' be a $n \times n$- matrix with $\mathbb{Z}_2$ entries defined by the relation $B' \equiv B(t)\,(mod\ 2)$ for some cluster t, and let $\omega = \omega_t$ be a (skew-)symmetric bilinear form on $\mathbb{F}_2^n$, such that $\omega(e_i, e_j) = b'_{ij}$. Define a linear operator $\mathfrak{t}_i : \mathbb{F}_2^n \to \mathbb{F}_2^n$ by the formula $\mathfrak{t}_i(\theta) = \xi - \omega(\theta, e_i)e_i$, and let $\Gamma = \Gamma_t$ be the group generated by $\mathfrak{t}_i$, $1 \leqslant i \leqslant m$.
Theorem The number of connected components $\#(\mathfrak{A}^0)$ equals to the number of Γ_t-orbits in $\mathbb{F}_2^n$.
Application: we computed the number of connected components of $\mathfrak{A}^0$ for Grassmanians.

Cluster $\mathfrak{A}-$ and $\mathfrak{X}-$ manifolds

Example

Cluster algebra of rank 3.

Seed $\left((x_1, x_2, x_3); B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\right)$

The space (x_1, x_2, x_3) is equipped with the 2-form

$$\Omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} + \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} + \frac{dx_3}{x_3} \wedge \frac{dx_1}{x_1}.$$

$\pi : (x_1, x_2, x_3) \mapsto (\tau_1, \tau_2, \tau_3)$

$\tau_1 = x_2/x_3 \;\; \tau_2 = x_3/x_1 \;\; \tau_3 = x_1/x_2$

Relation $\tau_1\tau_2\tau_3 = 1$.

The space τ_1, τ_2, τ_3 is equipped with the Poisson structure

$\{\tau_1, \tau_2\} = \tau_1\tau_2$; $\{\tau_2, \tau_3\} = \tau_2\tau_3$; $\{\tau_3, \tau_1\} = \tau_3\tau_1$

$\tau_1\tau_2\tau_3$ is a Casimir function (it commutes with all functions).

Ω is a 2-form.
$Ker\Omega = \{$ vector $\xi : \Omega(\xi, \eta) = 0 \forall \eta\}$ provides a fibration of the vector space
$\{$ space of fibers of $Ker\Omega\} \to \operatorname{Im} \pi$ is a local diffeomorphism.
More generally, Ω is a 2-form on a cluster manifold $\mathfrak{A}$ of coefficient-free cluster algebra $\mathcal{A}$.
$Ker\Omega$ determines an integrable distribution in $T\mathfrak{A}$.
Generic fibers of $Ker\Omega$ form a smooth manifold $\tilde{\mathfrak{X}}$ whose dimension is rank(B).
$\pi : \mathfrak{A} \to \tilde{\mathfrak{X}}$ is a natural projection.
Then, $\widetilde{\Omega} = \pi_*(\Omega)$ is a symplectic form on $\tilde{\mathfrak{X}}$ dual to the Poisson structure.

Example: Teichmüller space $T_{g,s}$ of genus g surface with s punctures.

Let F be a topological surface of genus g with s punctures.
$\mathcal{M}$ is a space of all smooth Riemannian metrics on F.
M_{-1} is a subset of all metrics of curvature -1 (hyperbolic metrics).

$Diff_+$ orientation preserving diffeomorpism of F (acts on M).
$Diff_0 \subset Diff_+$ is the set of all diffeomorphisms isotopic to Id.
Teichmüller space $T_{g,s} = Diff_{-1}/Diff_0$.

Hyperbolic metric $\stackrel{1-1}{\longleftrightarrow}$ conform. structure ($\simeq$ compl. structure)

$F_\lambda \in T_{g,s}$ is equipped with metric $\lambda|dz|^2$.

Deformation of complex structure = infinitesimal Beltrami operator $\mu \frac{d\bar{z}}{dz}$.

Dual space = { quadratic differentials φdz^2}.
$\langle \mu \frac{d\bar{z}}{dz}, \varphi dz^2 \rangle = \int_{F_\lambda} \mu\varphi dz \wedge d\bar{z}$

Quadratic differentials on F_λ form a cotangent space $T^*_{F_\lambda} T_{g,s}$.

Hermitian structure on $T^*_{F_\lambda} T_{g,s}$ defined by

$$\langle \varphi dz^2, \psi dz^2 \rangle = \frac{i}{2} \int_{F_\lambda} \frac{\varphi\psi}{\lambda} dz \wedge d\bar{z}$$

determines Weil-Peterson Kähler co-metric on $T_{g,s}$.

The skew-symmetric part of Weil-Peterson metric is the Weil-Peterson symplectic form WP on $T_{g,s}$

$\mathcal{A}$ ═══ decorated Teichmüller space $\widetilde{T_{g,s}}$ (={curves with chosen horocycles about each puncture}) equipped with Penner coordinates $\dim \widetilde{T_{g,s}} = 6g - 6 + 3s$

$\pi^* \uparrow$ $\qquad$ $\pi \downarrow$ forgets horocycles

$\mathcal{X}$ ═══ Teichmüller space $T_{g,s}$ equipped with "shear coordinates" $\dim T_{g,s} = 6g - 6 + s$

Product of shear coordinates about each puncture =1

Proposition. $\widetilde{\Omega} = WP$

General construction

Cluster algebra $\mathcal{A}$ with coefficients.
Toric action on $\mathcal{A}$:
t- cluster,
$x_i(t)$ - cluster variable in cluster t,
local toric action in cluster t with weights $w_i(t)$:

$$x_i(t) \mapsto x_i(t) \cdot \xi^{w_i(t)}$$

Local actions are compatible if they define a global toric action in the union of all clusters.
Example: Rescaling of horocycles is a global toric on the cluster algebra of homogeneous coordinate ring of decorated Teichmüller space.

We assume $\mathrm{rank}(B) = n$.
$\mathfrak{A}$ is a cluster manifold
$\mathfrak{A}^0$ is a union of regular toric orbits

Proposition

$\mathfrak{A}^0 = \{p \in \mathfrak{A}$ such that frozen variables $x_{n+j} \neq 0 \;\; \forall j = 1, \ldots, m\}$

Example: $\mathcal{A}_1$ is the standard ("totally positive") cluster algebra on Mat_n.
Toric action: $Diag_n \times Diag_n : Mat_n \to Mat_n$, i.e., $(D_1, D_2) : X \to D_1 X D_2$
Then, $\mathfrak{A}^0$ is the maximal double Bruhat cell $B_+ w_0 B_+ \cap B_- w_0 B_-$.

Example 2: $\mathcal{A}_2$ is the standard ("totally positive") cluster algebra = coordinate ring of maximal Schubert cell in Grassmannian $G_k(k+l)$.

$$k\{\left(\overbrace{\mathbf{1}}^{k} \quad \overbrace{Q}^{l}\right)$$

Toric action: $Diag_k \times Diag_l : Q \to Q$,
i.e., $(D_1, D_2) : Q \to D_1 Q D_2$
Then, $\mathfrak{A}^0$ consists of Grassmann elements with nonzero cyclically dense minors.

Symplectic leafs of $\mathfrak{A}$

If $\mathrm{rank}(B) = n$ then $\mathfrak{A}$ is a Poisson manifold. It is foliated into symplectic leaves. Symplectic leaf L is generic if some its neighborhood is diffeomorphic to $L \times$ *open ball*. $\mathfrak{A}^0$ is a union of generic symplectic leafs.

Remark. For $\mathcal{A}_1$ and $\mathcal{A}_2$ the manifold $\mathfrak{A}^0$ is the union of all generic leafs (not true in general).

Question: Find the number $\#(\mathfrak{A}^0)$ of connected components of $\mathfrak{A}^0$.

Answer:

Let $\mathbb{F}_2^{k+l}$ be a $k+l$-dimensional vector space over $\mathbb{F}_2 = \mathbf{Z}/2\mathbf{Z}$.
$\{e_i\}$ is the standard basis.
$\hat{B}$ is any $(k+l) \times (k+l)$ skew-symmetrizable integer matrix such that $\hat{B}[k, k+l] = B$. $\hat{D}$ is a skew-symmetrizing $(k+l) \times (k+l)$ diagonal matrix.
Let $B' = DB \; mod \; 2$, $B' = (b'_{ij})$.
B' determines a skew-symmetric bilinear form η on $\mathbb{F}_2^{k+l}$: $\eta(e_i, e_j) = b'_{ij}$.

$\forall i \in [1, k]$ define a symplectic transvection $\mathfrak{t}_i : \mathbb{F}_2^{k+l} \to \mathbb{F}_2^{k+l}$ as
$\mathfrak{t}_i(\xi) = \xi - \eta(\xi, e_i)e_i$.pause
Γ is the group generated by $\mathfrak{t}_i$, $i \in [1, k]$.
Theorem. $\#(\mathfrak{A}^0) =$ the number of $\Gamma-$orbits in $\mathbb{F}_2^{k+l}$.

Corollary (i) The number of connected components of intersections of two Schubert cells $Bw_0 \cdot B \cap w_0 B_0 w_0 \cdot B$ is given in the following table

n	2	3	4	5	≥ 6
#	2	6	20	52	$3 \cdot 2^{n-1}$

(ii) The number of connected components of subset Grassmannian $G_k(n)$ with non-vanishing cyclically dense minors is

$$\begin{cases} 3 \cdot 2^{n-1}, & \text{if } k > 3,\ \ n > 7;\ \text{or } k = 3,\ \ n = 6; \\ (n-1) \cdot 2^{n-2}, & \text{if } k = 2, n > 3; \end{cases}$$

Cluster determines seed

Cluster algebra with coefficients. $\mathbb{P}$ - semifield (free multiplicative abelian group of a finite rank m with generators $g_1, \dots, g_m$ endowed with an additional operation $\oplus$ which is commutative, associative, and distributive w.r.t. multiplication.

Ambient field is the field $\mathfrak{F}$ of rational functions in n independent variables with coefficients in the field of fractions on the integer group ring $\mathbf{Z}\mathbb{P}$.

A seed is a triple $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ where $\mathbf{x} = (x_1, \dots, x_n)$ is a transcendence basis of $\mathfrak{F}$ over the fiels of fractions of $\mathbf{Z}\mathbb{P}$; $\mathbf{y} = (y_1, \dots, y_n)$ is an n-tuple of elements of $\mathbb{P}$ and B is a skew-symmetrizable integer $n \times n$ matrix.

A seed mutation in direction $k \in [1, n]$

$$\begin{array}{ccc} \Sigma & \xrightarrow{k} & \Sigma' \\ \Vert & & \Vert \\ (\mathbf{x}, \mathbf{y}, B) & \longrightarrow & (\mathbf{x}', \mathbf{y}', B') \end{array},$$

where ...

$\mathbf{x}' = (\mathbf{x} \setminus \{x_k\} \cup \{x'_k\})$, satisfying

$$x'_k x_k = \frac{y_k}{y_k \oplus 1} \prod_{b_{ki}>0} x_i^{b_{ki}} + \frac{1}{y_k \oplus 1} \prod_{b_{ki}<0} x_i^{-b_{ki}},$$

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j y_k^{b_{jk}} (y_k \oplus 1)^{-b_{jk}} & \text{if } j \neq k \text{ and } b_{jk} > 0; \\ y_j (y_k \oplus 1)^{-b_{jk}} & \text{if } j \neq k \text{ and } b_{jk} \leq 0; \end{cases}$$

and

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

$\mathbb{P}$ is a tropical semifield if $\oplus$ is defined by

$$\prod_{i=1}^{m} g_i^{\alpha_i} \oplus \prod_{i=1}^{m} g_i^{\beta_i} = \prod_{i=1}^{m} g_i^{\min(\alpha_i,\beta_i)}$$

$\mathcal{A}$ is of geometric type if $\mathbb{P}$ is a tropical semifield.
(Each $y_i = \prod_{j=1}^{m} g_j^{\alpha_{ij}}$).
Example Frozen variables $x_{k+1}, \ldots, x_{k+l}$ play the role of generators $g_1, \ldots, g_l$. Transformation rules can be rewritten as

$$x_k x_k' = \prod_{\substack{1 \leqslant i \leqslant k+l \\ b_{ki}>0}} x_i^{b_{ki}} + \prod_{\substack{1 \leqslant i \leqslant k+l \\ b_{ki}<0}} x_i^{-b_{ki}}$$

Two conjectures

Conjecture 1 The exchange graph of a cluster algebra depends only on the initial exchange matrix B.

Conjecture 2

1. Every seed is uniquely defined by its cluster; thus, the vertices of the exchange graph can be identified with the clusters up to a permutation of cluster variables;
2. Two clusters are adjacent in the exchange graph iff they have exactly $(n-1)$ common cluster variables.

Both conjectures are proven for cluster algebras of finite type ([FZ], 2003).

Conjecture 2 is proved for acyclic cluster algebras ([BMRT], 2006)

Existance of presymplectic form implies:

Theorem A

Conjecture 2 holds for a cluster algebra $\mathcal{A}$:

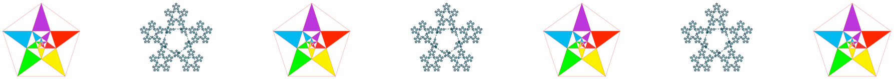
1. when $\mathcal{A}$ is of a geometric type;
2. when B is nondegenerate.

Theorem B

Let B be nondegenerate. Then the exchange graphs of all cluster algebras with the same initial exchange matrix B coincide.

Pentagram map

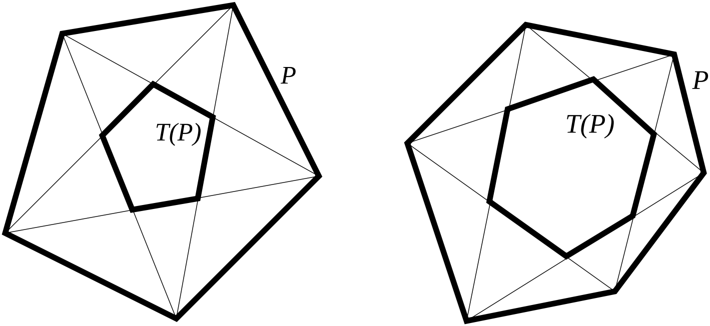
And the evening and the morning were the fifth day, April 19



R. Schwartz, V. Ovsienko, S. Tabachnikov, S. Morier-Genoud, M. Glick, F. Soloviev, G. Mari-Beffa, M. Gekhtman, M. Shapiro, A. Vainshtein, R. Kenyon, A. Goncharov, V. Fock, A. Marshakov

(almost) everything ArXived

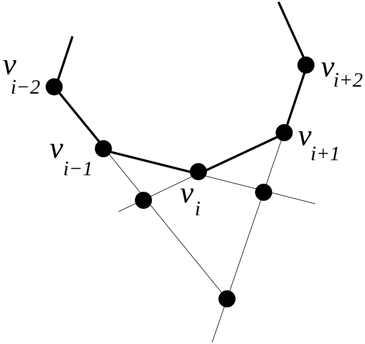
Pentagram Map T:



Acts on projective equivalence classes of closed and *twisted* n-gons with monodromy M. The latter constitute a $2n$-dimensional space, the former is $2n-8$-dimensional.

A good reference: `http://en.wikipedia.org/wiki/Pentagram_map`

Corner coordinates: left and right cross-ratios $X_1, Y_1, \ldots, X_n, Y_n$.



The map is as follows:

$$X_i^* = X_i \frac{1 - X_{i-1}\, Y_{i-1}}{1 - X_{i+1}\, Y_{i+1}}, \quad Y_i^* = Y_{i+1} \frac{1 - X_{i+2}\, Y_{i+2}}{1 - X_i\, Y_i}.$$

Hidden *scaling symmetry*

$$(X_1, Y_1, \ldots, X_n, Y_n) \mapsto (tX_1, t^{-1}Y_1, \ldots, tX_n, t^{-1}Y_n)$$

commutes with the map.
"Easy" invariants:

$$O_n = \prod_{i=1}^{n} X_i, \quad E_n = \prod_{i=1}^{n} Y_i.$$

Monodromy invariants:

$$\frac{O_n^{2/3} E_n^{1/3} (\mathrm{Tr}\ M)}{(\det M)^{1/3}} = \sum_{k=1}^{[n/2]} O_k$$

are polynomials in (X_i, Y_i), decomposed into homogeneous components; likewise, for E_k with M^{-1} replacing M.

Theorem (OST 2010). The Pentagram Map is completely integrable on the space of twisted n-gons:
1). The monodromy invariants are independent integrals (there are $2[n/2]+2$ *of them).*
2). There is an invariant Poisson structure of corank 2 if n is odd, and corank 4 if n is even, such that these integrals Poisson commute.
Poisson bracket: $\{X_i, X_{i+1}\} = -X_i X_{i+1}, \ \{Y_i, Y_{i+1}\} = Y_i Y_{i+1}$, and the rest $= 0$.
Complete integrability on the space of closed polygons has been proven as well:
F. Soloviev. *Integrability of the Pentagram Map*, arXiv:1106.3950;
V. Ovsienko, R. Schwartz, S. Tabachnikov. *Liouville-Arnold integrability of the pentagram map on closed polygons*, arXiv:1107.3633.

Cluster algebras connection:

M. Glick. *The pentagram map and Y-patterns*, Adv. Math., **227** (2011), 1019–1045.

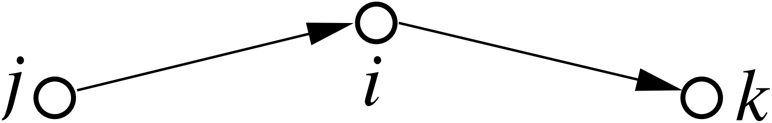
He considered the dynamics in the $2n-1$-dimensional quotient space by the scaling symmetry $(X, Y) \mapsto (tX, t^{-1}Y)$:

$$p_i = -X_{i+1}Y_{i+1}, \quad q_i = -\frac{1}{Y_iX_{i+1}},$$

and proved that it was a Y-type cluster algebra dynamics.

Cluster dynamics

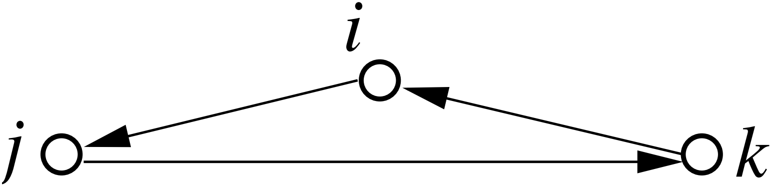
Given a *quiver* (an oriented graph with no loops or 2-cycles) whose vertices are labeled by variables τ_i (rational functions in some free variables), the mutation on vertex i is as follows:



$$\tau_i^* = \frac{1}{\tau_i}, \quad \tau_j^* = \frac{\tau_j \tau_i}{1+\tau_i}, \quad \tau_k^* = \tau_k(1+\tau_i);$$

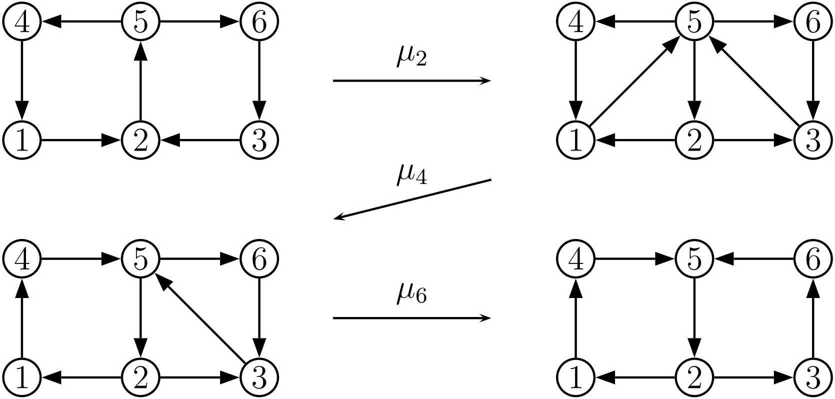
the rest of the variables are intact.

The quiver also mutates, in three steps:
(i) for every path $j \to i \to k$, add an edge $j \to k$;
(ii) reverse the orientation of the edges incident to the vertex i;
(iii) delete the resulting 2-cycles.

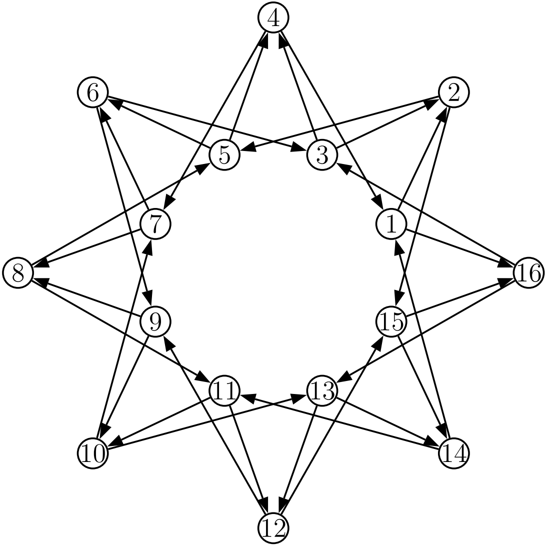


The mutation on a given vertex is an involution.

Example of mutations:

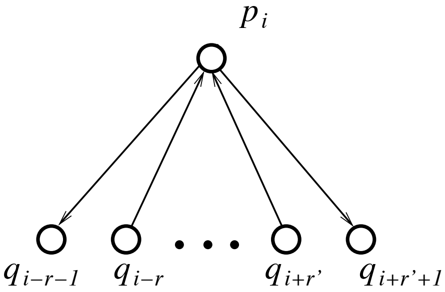


Glick's quiver ($n = 8$):



Joint work in progress with Michael Gekhtman, Sergey Tabachnikov, and Alek Vainshtein, ERA 19 (2012), 1–17.

Generalizing Glick's quiver (the case of $k=3$), consider the homogeneous bipartite graph $\mathcal{Q}_{k,n}$ where $r=[k/2]-1$, and $r'=r$ for k even and $r'=r+1$ for k odd (each vertex is 4-valent):



Dynamics: mutations on all p-vertices, followed by swapping p and q; this is the map $\overline{T}_k$:

$$q_i^* = \frac{1}{p_i}, \quad p_i^* = q_i \frac{(1+p_{i-r-1})(1+p_{i+r+1})p_{i-r}p_{i+r}}{(1+p_{i-r})(1+p_{i+r})}, \quad k \text{ even},$$

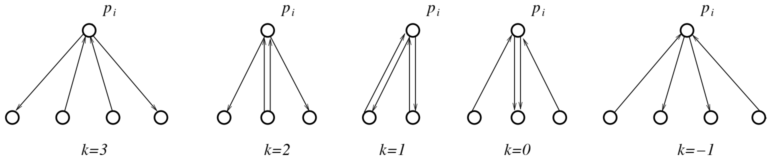
$$q_i^* = \frac{1}{p_{i-1}}, \quad p_i^* = q_i \frac{(1+p_{i-r-2})(1+p_{i+r+1})p_{i-r-1}p_{i+r}}{(1+p_{i-r-1})(1+p_{i+r})}, \quad k \text{ odd}.$$

The quiver is preserved. The function $\prod p_i q_i$ is invariant; we restrict to the subspace $\prod p_i q_i = 1$.

Invariant Poisson bracket: the variables Poisson commute, unless they are connected by an arrow: $\{p_i, q_j\} = \pm p_i q_j$ (depending on the direction).

(This bracket comes from the general theory: GSV, *Cluster algebras and Poisson geometry, AMS, 2010*).

The quivers, for small values of k, look like this (for $k = 1$, the arrows cancel out):



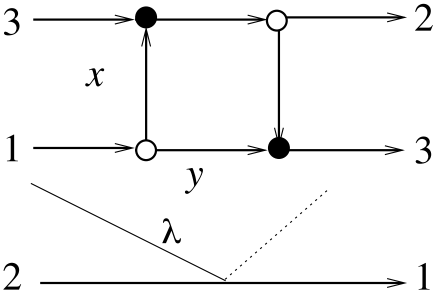
The map $\overline{T}_k$ is *reversible*: $\overline{D}_k \circ \overline{T}_k \circ \overline{D}_k = \overline{T}_k^{-1}$, where

$$\overline{D}_k : p_i \mapsto \frac{1}{q_i}, \; q_i \mapsto \frac{1}{p_i}, \quad k \text{ even},$$
$$\overline{D}_k : p_i \mapsto \frac{1}{q_{i+1}}, \; q_i \mapsto \frac{1}{p_i}, \quad k \text{ odd}.$$

Goal: to reconstruct the x, y-dynamics and to interpret it geometrically.

Weighted directed networks on the cylinder and the torus (A. Postnikov math.CO/0609764, for networks in a disc; GSV book).

Example:



Two kind of vertices, white and black.

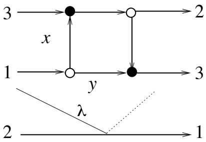
Convention: an edge weight is 1, if not specified.

The *cut* is used to introduce a *spectral parameter* λ.

Boundary measurements

:

the network



corresponds to the matrix

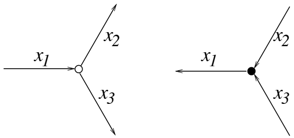
$$\begin{pmatrix} 0 & x & x+y \\ \lambda & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Concatenation of networks $\mapsto$ product of matrices.

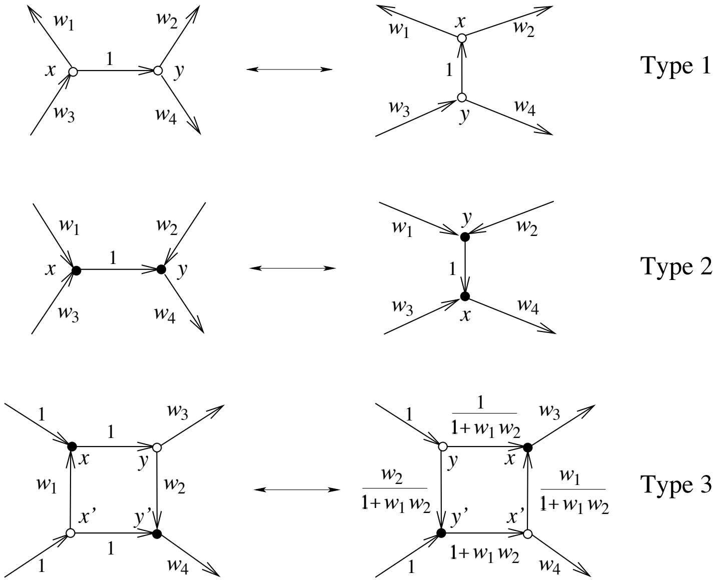
Gauge group: at a vertex, multiply the weights of the incoming edges and divide the weights of the outgoing ones by the same function. Leaves the boundary measurements intact.

Face weights: the product of edge weights over the boundary (orientation taken into account). The boundary measurement map to matrix functions factorizes through the space of face weights. (They will be identified with the p, q-coordinates).

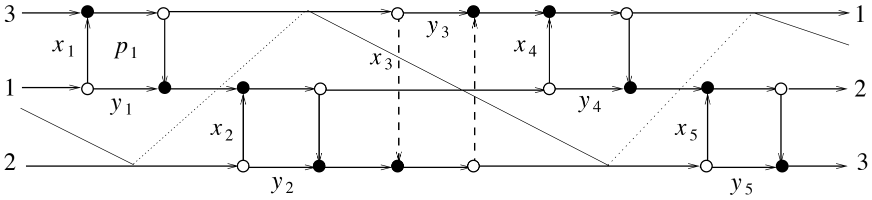
Poisson bracket (6-parameter): $\{x_i, x_j\} = c_{ij} x_i x_j, \;\; i \neq j \in \{1, 2, 3\}$



Postnikov moves (do not change the boundary measurements):

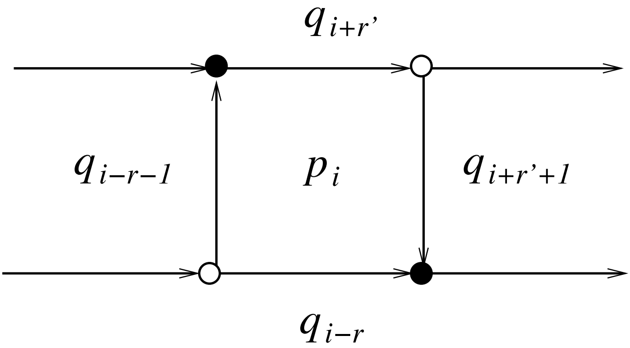


Consider a network whose dual graph is the quiver $\mathcal{Q}_{k,n}$.
It is drawn on the torus. Example, $k = 3, n = 5$:



Convention: white vertices of the graph are on the left of oriented edges of the dual graph.

The network is made of the blocks:

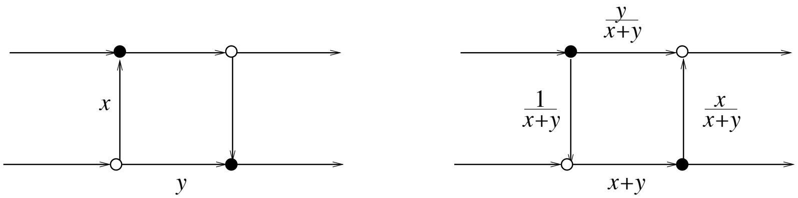


Face weights:

$$p_i = \frac{y_i}{x_i}, \quad q_i = \frac{x_{i+1+r}}{y_{i+r}}.$$

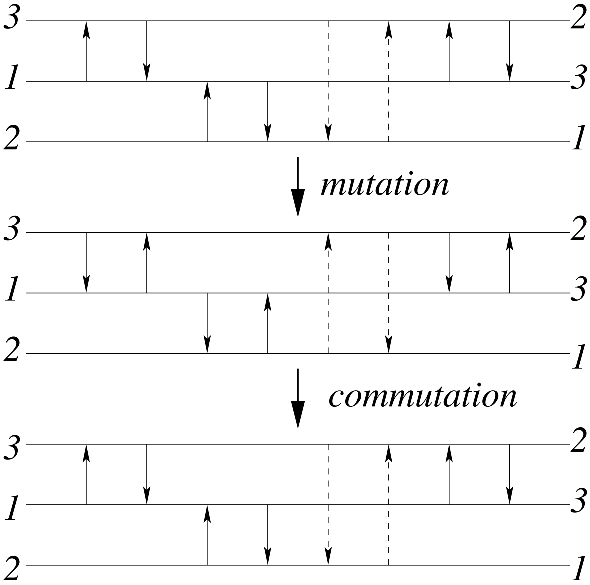
This is a projection $\pi : (x, y) \mapsto (p, q)$ with 1-dimensional fiber.

(x, y)-dynamics: mutation (Postinov type 3 move on each p-face),



followed by the Postnikov type 1 and 2 moves on the white-white and black-black edge (this interchanges p- and q-faces), including moving across the vertical cut, and finally, re-calibration to restore 1s on the appropriate edges. These moves preserve the conjugacy class of the boundary measurement matrix.

Schematically:



This results in the map T_k:

$$x_i^* = x_{i-r-1}\frac{x_{i+r}+y_{i+r}}{x_{i-r-1}+y_{i-r-1}}, \quad y_i^* = y_{i-r}\frac{x_{i+r+1}+y_{i+r+1}}{x_{i-r}+y_{i-r}}, \quad k \text{ even},$$
$$x_i^* = x_{i-r-2}\frac{x_{i+r}+y_{i+r}}{x_{i-r-2}+y_{i-r-2}}, \quad y_i^* = y_{i-r-1}\frac{x_{i+r+1}+y_{i+r+1}}{x_{i-r-1}+y_{i-r-1}}, \quad k \text{ odd}.$$

The map T_k is conjugated to the map $\overline{T}_k$: $\pi \circ T_k = \overline{T}_k \circ \pi$.
Relation with the pentagram map: the change of variables

$$x_i \mapsto Y_i, \quad y_i \mapsto -Y_i X_{i+1} Y_{i+1},$$

identifies T_3 with the pentagram map.

Complete integrability of the maps T_k

The ingredients are suggested by the combinatorics of the network.
Invariant Poisson bracket (in the "stable range" $n \geq 2k-1$):

$$\{x_i, x_{i+l}\} = -x_i x_{i+l}, 1 \leq l \leq k-2;\ \{y_i, y_{i+l}\} = -y_i y_{i+l}, 1 \leq l \leq k-1;$$
$$\{y_i, x_{i+l}\} = -y_i x_{i+l}, 1 \leq l \leq k-1;\ \{y_i, x_{i-l}\} = y_i x_{i-l}, 0 \leq l \leq k-2;$$

the indices are cyclic.
The functions $\prod x_i$ and $\prod y_i$ are Casimir. If n is even and k is odd, one has four Casimir functions:

$$\prod_{i \text{ even}} x_i, \quad \prod_{i \text{ odd}} x_i, \quad \prod_{i \text{ even}} y_i, \quad \prod_{i \text{ odd}} y_i.$$

Lax matrices, monodromy, integrals: for $k \geq 3$,

$$L_i = \begin{pmatrix} 0 & 0 & 0 & \dots & x_i & x_i + y_i \\ \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix},$$

and for $k = 2$,

$$L_i = \begin{pmatrix} \lambda x_i & x_i + y_i \\ \lambda & 1 \end{pmatrix}.$$

The boundary measurement matrix is $M(\lambda) = L_1 \cdots L_n$. The characteristic polynomial

$$\det(M(\lambda) - z) = \sum I_{ij}(x, y) z^i \lambda^j.$$

is T_k-invariant: the integrals I_{ij} are in involution.

Zero curvature (Lax) representation:

$$L_i^* = P_i L_{i+r-1} P_{i+1}^{-1}$$

where L_i are the Lax matrices and

$$P_i = \begin{pmatrix} 0 & \frac{x_i}{\lambda\sigma_i} & \frac{y_{i+1}}{\lambda\sigma_{i+1}} & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{x_{i+1}}{\sigma_{i+1}} & \frac{y_{i+2}}{\sigma_{i+2}} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{x_{i+k-4}}{\sigma_{i+k-4}} & \frac{y_{i+k-3}}{\sigma_{i+k-3}} & 0 \\ -\frac{1}{\sigma_{i+k-2}} & 0 & 0 & \dots & 0 & \frac{x_{i+k-3}}{\sigma_{i+k-3}} & 1 \\ \frac{1}{\sigma_{i+k-2}} & -\frac{1}{\lambda\sigma_{i+k-1}} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda\sigma_{i+k-1}} & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

with $\sigma_i = x_i + y_i$.

Geometric interpretations

Twisted corrugated polygons in $\mathbf{RP}^{k-1}$ and $k-1$-diagonal maps

Let $k \geq 3$. Let $\mathcal{P}_{k,n}$ be the space of projective equivalence classes of *generic* twisted n-gons in $\mathbf{RP}^{k-1}$; one has: dim $\mathcal{P}_{k,n} = n(k-1)$.
Let $\mathcal{P}^0_{k,n} \subset \mathcal{P}_{k,n}$ consist of the polygons with the following property: for every i, the vertices V_i, V_{i+1}, V_{i+k-1} and V_{i+k} span a projective plane. These are *corrugated* polygons. Projective duality preserves corrugated polygons.
The consecutive $k-1$-diagonals of a corrugated polygon intersect. The resulting polygon is again corrugated. One gets a pentagram-like $k-1$-diagonal map on $\mathcal{P}^0_{k,n}$. For $k=3$, this is the pentagram map.

Coordinates: lift the vertices V_i of a corrugated polygon to vectors $\widetilde{V}_i$ in $\mathbf{R}^k$ so that the linear recurrence holds

$$\widetilde{V}_{i+k} = y_{i-1}\widetilde{V}_i + x_i\widetilde{V}_{i+1} + \widetilde{V}_{i+k-1},$$

where x_i and y_i are n-periodic sequences. These are coordinates in $\mathcal{P}^0_{k,n}$. In these coordinates, the map is identified with T_k.
The same functions x_i, y_i can be defined on polygons in the projective plane. One obtains integrals of the "deeper" diagonal maps on twisted polygons in $\mathbf{RP}^2$.

Case $k = 2$

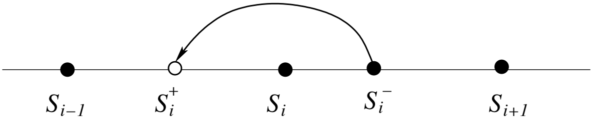
Consider the space $\mathcal{S}_n$ of pairs of twisted n-gons (S^-, S) in $\mathbf{RP}^1$ with the same monodromy. Consider the projectively invariant projection ϕ to the (x, y)-space (cross-ratios):

$$x_i = \frac{(S_{i+1} - S^-_{i+2})(S^-_i - S^-_{i+1})}{(S^-_i - S_{i+1})(S^-_{i+1} - S^-_{i+2})}$$

$$y_i = \frac{(S^-_{i+1} - S_{i+1})(S^-_{i+2} - S_{i+2})(S^-_i - S^-_{i+1})}{(S^-_{i+1} - S_{i+2})(S^-_i - S_{i+1})(S^-_{i+1} - S^-_{i+2})}.$$

Then x_i, y_i are coordinates in $\mathcal{S}_n/PGL(2, \mathbf{R})$.

Define a transformation $F_2(S^-, S) = (S, S^+)$, where S^+ is given by the following local leapfrog rule: given points $S_{i-1}, S_i^-, S_i, S_{i+1}$, the point S_i^+ is obtained by the reflection of S_i^- in S_i in the projective metric on the segment $[S_{i-1}, S_{i+1}]$:



The projection ϕ conjugates F_2 and T_2.

In formulas:

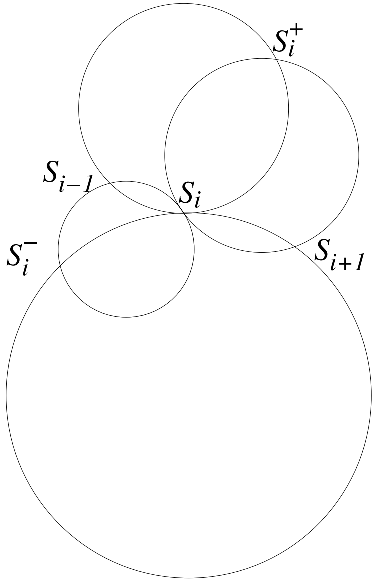
$$\frac{1}{S_i^+ - S_i} + \frac{1}{S_i^- - S_i} = \frac{1}{S_{i+1} - S_i} + \frac{1}{S_{i-1} - S_i},$$

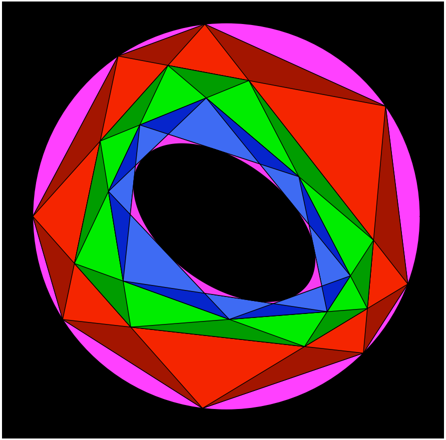
or, equivalently,

$$\frac{(S_i^+ - S_{i+1})(S_i - S_i^-)(S_i - S_{i-1})}{(S_i^+ - S_i)(S_{i+1} - S_i)(S_i^- - S_{i-1})} = -1,$$

(Toda-type equations).

In $\mathbf{CP}^1$, a circle pattern interpretation (generalized Schramm's pattern):





Thank you!