Poisson properties of cluster algebras

Paris

Day 2, April 17, 2012

Cluster algebra and compatible Poisson structure

Reference:

Cluster algebra and Poisson Geometry, M.Gekhtman, M.S., A.Vainshtein, AMS, 2010

Motivations for notion of cluster algebra

Totally Positive Matrices

An $n \times n$ matrix A is totally positive if all its minors are positive.

Note that the number of all minors grows exponentially with size. However, one can select (not uniquely) a family F of just n^2 minors of A such that A is totally positive iff every minor in the family is positive. (Fekete; B.-F.-Z.)

$n = 3$

For $n = 3$ (totally 20 minors),

$$F_1 = \{\Delta_3^3, \underline{\Delta_{23}^{23}}, \Delta_{23}^{13}, \Delta_{13}^{23}; \Delta_1^3, \Delta_3^1, \Delta_{12}^{23}, \Delta_{23}^{12}, \Delta_{123}^{123}\}$$

$$F_2 = \{\Delta_3^3, \Delta_{23}^{13}, \Delta_{13}^{23}, \underline{\Delta_{13}^{13}}; \Delta_1^3, \Delta_3^1, \Delta_{12}^{23}, \Delta_{23}^{12}, \Delta_{123}^{123}\}$$

Families F_1 and F_2

- differ in only one element
- are connected by

$$\Delta_{13}^{13}\Delta_{23}^{23} = \Delta_{13}^{23}\Delta_{23}^{13} + \Delta_3^3\Delta_{123}^{123}$$

Properties

Every other "test family" of 9 minors

- contains $\Delta_1^3, \Delta_3^1, \Delta_{12}^{23}, \Delta_{23}^{12}, \Delta_{123}^{123}$
- can be obtained from F_1 (or F_2) via a sequence of similar transformations
- defines coordinate system on $GL(3)$ bi-rationally related to natural coordinates $A = (a_{ij})_{i,j=1}^3$.
- The intersection of opposite big Bruhat cells

$$B_+ w_0 B_+ \cap B_- w_0 B_- \subset GL(3)$$

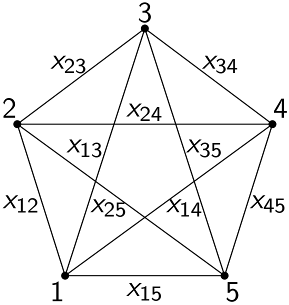
coincides with

$$\{A \in GL(3) | \Delta_1^3 \Delta_3^1 \Delta_{12}^{23} \Delta_{23}^{12} \Delta_{123}^{123} \neq 0\}$$

The number of connected components of this intersection can be computed using families F_i.

Homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_{2,n+3}]$

$\mathrm{Gr}_{2,n+3} = \{V \subset \mathbb{C}^{n+3} : \dim(V) = 2\}$. The ring $\mathcal{A} = \mathbb{C}[\mathrm{Gr}_{2,n+3}]$ is generated by the Plücker coordinates x_{ij}, for $1 \le i < j \le n+3$.
Relations: $x_{ik}x_{jl} = x_{ij}x_{kl} + x_{il}x_{jk}$, for $i < j < k < l$.



sides: scalars

diagonals: cluster variables

relations: "flips"

clusters: triangulations

Each cluster has exactly n elements, so $\mathcal{A}$ is a cluster algebra of rank n.
The monomials involving "non-crossing" variables form a linear basis in $\mathcal{A}$ (studied in [Kung-Rota]).

Double Bruhat cell in $SL(3)$

Example

$\mathcal{A} = \mathbb{C}[G^{u,v}]$, where $G^{u,v} = BuB \cap B_-vB_- =$

$$= \left\{ \begin{bmatrix} x & \alpha & 0 \\ \gamma & y & \beta \\ 0 & \delta & z \end{bmatrix} \in SL_3(\mathbb{C}) \; : \; \begin{matrix} \alpha \neq 0 & \beta \neq 0 \\ \gamma \neq 0 & \delta \neq 0 \end{matrix} \right\}$$

is a double Bruhat cell $(u, v \in \mathcal{S}_3,\ \ell(u) = \ell(v) = 2)$.
Ground ring: $\mathbb{A} = \mathbb{C}[\alpha^{\pm 1}, \beta^{\pm 1}, \gamma^{\pm 1}, \delta^{\pm 1}]$.
Five cluster variables. Exchange relations:

$$xy = \left\| \begin{matrix} x & \alpha \\ \gamma & y \end{matrix} \right\| + \alpha\gamma \qquad yz = \left\| \begin{matrix} y & \beta \\ \delta & z \end{matrix} \right\| + \beta\delta$$

$$x \left\| \begin{matrix} y & \beta \\ \delta & z \end{matrix} \right\| = \alpha\gamma z + 1 \qquad z \left\| \begin{matrix} x & \alpha \\ \gamma & y \end{matrix} \right\| = \beta\delta x + 1$$

General construction [Zelevinsky, *IMRN*, 2000]

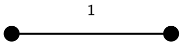
Definition

Exchange graph of a cluster algebra: $\begin{array}{ccc} \text{vertices} & \simeq & \text{clusters} \\ \text{edges} & \simeq & \text{exchanges.} \end{array}$

$\mathbb{T}_m$ *m-regular tree* with $\{1, 2, \ldots m\}$-labeled edges,

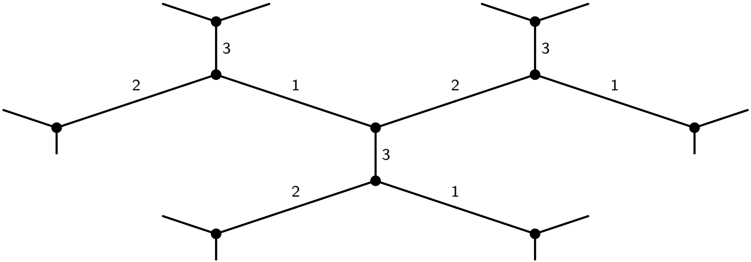
adjacent edges receive different labels

$\mathbb{T}_1$



$\mathbb{T}_2$

$\mathbb{T}_3$



Cluster algebras of geometric type

$$\mathbf{y} = (y_1, \dots, y_{n-m}) \quad \text{— frozen variables,}$$

$$\mathbf{x}(t) = (x_1(t), \dots, x_m(t)) \quad \text{— cluster variables,}$$

$$\mathbf{z}(t) = (z_1(t), \dots, z_n(t)) = (\mathbf{x}(t), \mathbf{y}) \quad \text{— extended cluster variables.}$$

Definition

Cluster algebra $\mathcal{A}$ is given by pair $(B(t), \mathbf{z}(t))$ for each cluster (vertex of exchange graph) t

- $B(t)$ is an $m \times n$ integral matrix ($m \leq n$) whose left $m \times m$ block is left-skew-symmetrizable, (we will assume it skew-symmetric for simplicity)
- $\mathbf{z}(t)$ is a vector of extended cluster variables.
- variables $z_{m+1} = y_1, \dots, z_n = y_{n-m}$ are not affected by T_i.
- both $B(t)$ and $\mathbf{z}(t)$ are subject to cluster transformations defined as follows.

Cluster transformations

Cluster change

For an edge of $\mathbb{T}_m$ $\overset{t}{\bullet}\overset{i}{\text{———}}\overset{t'}{\bullet}$ $i \in [1, \dots, m]$

$T_i : \mathbf{z}(t) \mapsto \mathbf{z}(t')$ is defined as

$$\mathbf{x}_i(t') = \frac{1}{\mathbf{x}_i(t)} \left(\prod_{b_{ik}(t)>0} z_k(t)^{b_{ik}(t)} + \prod_{b_{ik}(t)<0} z_k(t)^{-b_{ik}(t)} \right)$$

$$z_j(t') = z_j(t) \quad j \neq i,$$

Matrix mutation $B(t') = T_i(B(t))$,

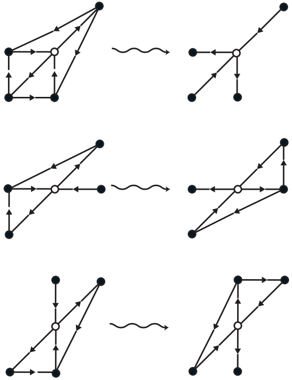
$$b_{kl}(t') = \begin{cases} -b_{kl}(t), & \text{if } (k-i)(l-i) = 0 \\ b_{kl}(t) + \dfrac{|b_{ki}(t)|b_{il}(t) + b_{ki}(t)|b_{il}(t)|}{2}, & \text{otherwise.} \end{cases}$$

Definition

Given some initial cluster t_0 put $z_i = z_i(t_0)$, $B = B(t_0)$. The cluster algebra $\mathcal{A}$ (or, $\mathcal{A}(B)$) is the subalgebra of the field of rational functions in cluster variables $z_1, \ldots, z_n$ generated by the union of all cluster variables $z_i(t)$.

Examples of $T_i(B)$

A matrix $B(t)$ can be represented by a (weighted, oriented) graph.



The Laurent phenomenon

Theorem (FZ) In a cluster algebra, any cluster variable is expressed in terms of initial cluster as a Laurent polynomial.

Poisson structure on cluster algebra

Poisson structure on a cluster algebra $\mathcal{A}$ is a skew-symmetric bracket $\{\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that Leibnitz rule:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial x_i} \{x_i, g\}$$

and Jacobi identity:

$$\{f\{g, h\}\} + \{g\{h, f\}\} + \{h\{f, g\}\} = 0$$

hold.

Remark Poisson bracket is completely determined by its values on generators $\{x_i, x_j\}$.

Compatible Poisson brackets

$\mathcal{F} =$ quotient field of $\mathcal{A}$.

Definition

Poisson bracket is log-canonical w.r.t. set of elements $f_i \in \mathcal{F}$ if $\{f_i, f_j\} = c_{ij} f_i f_j$ (equivalently, $\{\log x_i, \log x_j\} = c_{ij}$) for some constants $c_{ij} \in \mathbb{Z}$.

Remark $c_{ij} = -c_{ji}$.
Remark Given a functionally independent system of generators $f_1, \ldots, f_n \in \mathcal{F}$ there is a 1-to-1 correspondence between log-canonical Poisson structures and skew-symmetric $n \times n$ matrices.

Definition

Poisson bracket on cluster algebra is compatible with the cluster algebra structure if for every cluster it is log-canonical w.r.t. all elements of this cluster.

Sklyanin Poisson brackets

Mat_n – all $n \times n$ matrices. For any matrix X we write its decomposition into a sum of lower triangular and strictly upper triangular matrices as

$$X = X_- + X_0 + X_+$$

The R-matrix $R : Mat_n \to Mat_n$ defined by $R(X) = X_+ - X_-$
Poisson bracket on Mat_n :

$$\{f_1, f_2\}(X) = \frac{1}{2}\left(\langle R(\nabla f_1(X)X), \nabla f_2(X)X\rangle - \langle R(X\nabla f_1(X)), X\nabla f_2(X))]\rangle\right),$$

where gradient ∇ is defined w.r.t. trace form.
For matrix elements x_{ij},

$$\{x_{ij}, x_{\alpha\beta}\} = \frac{1}{2}(sign(\alpha - i) + sign(\beta - j))x_{i\beta}x_{\alpha j}$$

Observations:

- Sklyanin Poisson bracket is compatible with the "total positive" cluster algebra structure on Mat_n.
- The maximal double Bruhat cell coincides with the union of "generic" symplectic leaves.