Poisson properties of cluster algebras

Paris

April, 2012

Hamiltonian formalism and Integrable systems

Main References

V.Arnold, Mathematical Methods of Classical Mechanics, New York: Springer-Verlag, 1989

Perelomov, A. M., Integrable Systems of Classical Mechanics and Lie Algebras. Basel etc., Birkhuser Verlag 1989.

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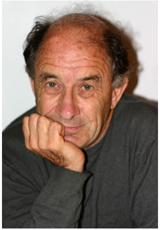

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Classical mechanics

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Phase space

We rewrite these equations as

$$\begin{cases} \dot{q} &= p \\ \dot{p} &= -\nabla U, \end{cases}$$

$p = (p_1, \ldots, p_n)$ is *the momentum*.
The space $\mathbb{R}^{2n} = \{(p, q)\}$ is *the phase space* of the system.

Poisson structure

Definition

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Example

The standard Poisson structure on $\mathbb{R}^{2n} = \{(p, q)\}$.

$$\{F, G\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

Poisson structure produces vector fields from functions

Poisson structure corresponds to the bi-vector field $\nu = \sum_i \nu_{ij} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$, such that $\{F, G\}(x) = \langle dF(x) \wedge dG(x), \nu(x) \rangle$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\Lambda^2 T_x^*(X)$ and $\Lambda^2 T_x(X)$.

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A Poisson bracket transforms a covector field into a vector field

It induces a linear map $\Psi_x : T_x^* M \to T_x M$.
For any two covectors $\xi, \eta \in T_x^* M$ let $\xi = dF(x), \eta = dG(x)$. Then Ψ_x is determined uniquely by $\eta(\Psi_x \xi) = \{F, G\}(x)$.

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any function $H \rightsquigarrow$ a vector field $\text{sgrad}\, H = \Psi_x(dH(x))$.

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For any function $f(x)$ we have
$\frac{d}{dt} f = \sum_i \frac{\partial f}{\partial x_i} \dot{x}_i = \sum_i \frac{\partial f}{\partial x_i} \{x_i, H\} = \{f, H\}$ by the Leibnitz rule.

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Consider $x = (p, q) \in \mathbb{R}^{2n}$, equipped with the standard Poisson structure $\{F, G\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$, and the hamiltonian $H(p, q) = \frac{p^2}{2} + U(q)$ (the total energy). Then the flow equations take the form

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We recognize Newton's equations of motion.

Integrals of motion

Let $I(x)$ be a function such that $\{I, H\} = 0$. Then I is preserved under the Hamiltonian flow with hamiltonian H. Indeed, $\dot{I} = \{I, H\} = 0$ and $I(t) = I(p(t), q(t)) = Const$.

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Theorem

Let a symplectic M^{2n} have n functions $F_1, \dots, F_n$ in involution (i.e., $\{F_i, F_j\} \equiv 0 \, \forall i, j$). Consider $M_f := \{x | F_i(x) = f_i, \; i = 1 \dots n\}$. Assume that all F_i are independent on M_f (i.e. $dF_i(x)$ are linearly independent for all $x \in M_f$). Then,

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- The Poisson structure in coordinates (p_i, q_i) is standard.

Equations of motion

The corresponding differential equations of motion take the form

$$\begin{cases} \dot{q}_1 & = p_1 \\ \dots & \\ \dot{q}_n & = p_n \\ \dot{p}_1 & = -2e^{2(q_1-q_2)} \\ \dot{p}_2 & = -2e^{2(q_2-q_3)} + 2e^{2(q_1-q_2)} \\ \dots & \\ \dot{p}_n & = 2e^{2(q_{n-1}-q_n)} \end{cases}$$

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$$X = \begin{pmatrix} p_1 & e^{q_1-q_2} & 0 & \dots & \\ e^{q_1-q_2} & & \dots & & \\ & & & \dots & e^{q_{n-1}-q_n} \\ & & \dots & e^{q_{n-1}-q_n} & p_n \end{pmatrix}$$

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and X_+ is a skew-symmetrization of X

$$X_+ = \begin{pmatrix} 0 & e^{q_1-q_2} & 0 & \dots & \\ -e^{q_1-q_2} & & \dots & & \\ & & & \dots & e^{q_{n-1}-q_n} \\ & & \dots & -e^{q_{n-1}-q_n} & 0 \end{pmatrix}$$

Exercise

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Remark

If Poisson structure is degenerate then any solution of Hamiltonian system lives on a symplectic leaf of the Poisson manifold.

Symplectic leafs

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- Symplectic leaves = orbits of coadjoint action of b. Tridiagonal matrices form an orbit, i.e. a symplectic leaf.
- The Lie-Poisson Poisson bracket on the space of tridiagonal matrices coincides with the Toda lattice Poisson bracket under the appropriate change of coordinates. Hamiltonian $H(x) = \frac{1}{2}Tr(X^2)$ induces the hamiltonian flow which coincides with the open Toda lattice flow.

Exercise

Check that hamiltonian flow on tridiagonal matrices equipped with Lie-Poisson bracket with hamiltonian $H(X) = \frac{1}{2} Tr(X^2)$ is given by equations of open Toda lattice.

Remark

Cauchy problem $\dot{X} = [X, X_+],\ X(0) = X_0$ has a solution of the form $X(t) = u(t)X_0u(t)^{-1}$, where $u(t)$ is an orthogonal matrix satisfying $\dot{u} = -Mu, u(0) = I, M = -\dot{u}u^{-1} = u^{-1}\dot{u}$.

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Let $det(\lambda I - X) = \prod_{k=1}^{n}(\lambda - \lambda_k) = \sum_{k=1}^{n}(-1)^k J_k(L)\lambda^{n-k}$.

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Inverse Moser Map

- Given $m(\lambda)$, define Hankel determinants

$$\Delta_i^{(l)} = \det \left(h_{\alpha+\beta+l-i-1} \right)_{\alpha,\beta=1}^{i}$$

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$$d_i = \frac{\Delta_i^{(i)} \Delta_{i-1}^{(i-2)}}{\Delta_i^{(i-1)} \Delta_{i-1}^{(i-1)}},$$

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Quadratic Poisson structures

Phase space for Toda lattice is formed by tridiagonal matrices modulo conjugation by diagonal matrices. It coincides with double Bruhat cell $C := BuB \cap B_- vB_-$ where $u = s_1 \cdot \ldots \cdot s_{n-1}$, $v = s_{n-1} \cdot \ldots \cdot s_1$ are Coxeter elements, s_i is a simple transposition $i \leftrightarrow i+1$.

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Question:

Can one consider Toda lattice as a Hamiltonian system with respect to the bracket that reflects group structure?

Poisson-Lie bracket

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Example

Sl_2. Borel subgroup $B \subset Sl_2$ is the set $\left\{ \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} \right\}$

Poisson structure on B: $\{t, x\} = tx$.

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Similarly, we define Poisson-Lie bracket for B_-.
Then, if we have embedded Poisson subgroups B and B_- they define a Poisson-Lie structure on SL_2 they generate.

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To define Poisson-Lie bracket on the whole SL_2 we use Gauss decomposition $SL_2 = B_- B_+$.

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Remark

Toda equations are Hamiltonian equations with respect to the standard quadratic Poisson-Lie bracket and Hamiltonian $tr(X)$.

R-matrix

One can construct a Poisson-Lie bracket using *$R-matrix$*.

Definition

A map $R : g \to g$ is called a *classical $R-matrix$* if it satisfies modified Yang-Baxter equation

$$[R(\xi), R(\eta)] - R\left([R(\xi), \eta] + [\xi, R(\eta)]\right) = -[\xi, \eta]$$

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R-matrix Poisson bracket

R-matrix Poisson-Lie bracket on SL_n :

$$\{f_1, f_2\}(X) = \frac{1}{2}\left(\langle R(\nabla f_1(X)X), \nabla f_2(X)X\rangle - \langle R(X\nabla f_1(X)), X\nabla f_2(X))]\rangle\right),$$

where gradient $\nabla f \in sl_n$ defined w.r.t. trace form.

Example

For any matrix X we write its decomposition into a sum of lower triangular and strictly upper triangular matrices as

$$X = X_- + X_0 + X_+$$

The standard R-matrix $R : Mat_n \to Mat_n$ defined by

$$R(X) = X_+ - X_-$$

The standard R-matrix Poisson-Lie bracket:

$$\{x_{ij}, x_{\alpha\beta}\}(X) = \frac{1}{2}(sign(\alpha - i) + sign(\beta - j))x_{i\beta}x_{\alpha j}$$

Homogeneous Poisson space

X is a homogeneous space of an algebraic group G, i.e.,

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Poisson bracket bracket compatible with the standard Poisson-Lie bracket on SL_n:

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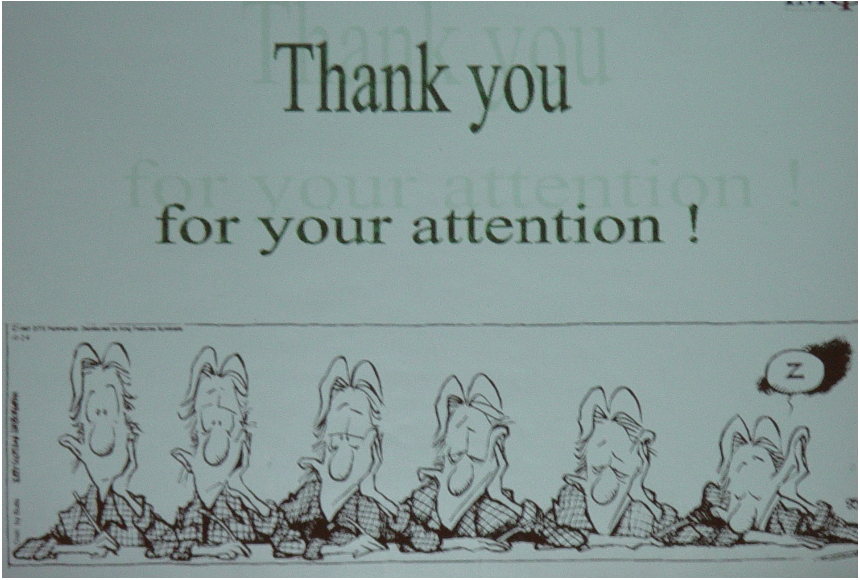
Thank you
for your attention !
Z