Poisson geometry of directed networks and integrable systems

Paris

Day 5 , Morning, April 19

Outline

Our Goal

- Introduce cluster algebra structure in the ring of regular functions on the space

$$\mathcal{R}_n = \left\{ \frac{Q(\lambda)}{P(\lambda)} \ : \deg P = n, \ \deg Q < n, \ P, Q \text{ are coprime}, \ P(0) \neq 0 \right\}$$

- Interpret Darboux-Bäcklund transformations of Coxeter-Toda lattices in terms of cluster transformations.

Definition of a Cluster Algebra (Fomin-Zelevinsky)

Ingredients

- A *seed* (of *geometric type*) - a pair $\Sigma = (\mathbf{x}, \widetilde{B})$, where $\mathbf{x} = (x_1, \ldots, x_n)$ are commuting variables and $\widetilde{B}$ is an $n \times (n+m)$ integer matrix whose $n \times n$ principal part B is skew-symmetric.
- $\mathbf{x}$ is called a *cluster*, its elements $x_1, \ldots, x_n$ are called *cluster variables*. Denote $x_{n+i} = g_i$ for $i \in [1, m]$. We say that $\widetilde{\mathbf{x}} = (x_1, \ldots, x_{n+m})$ is an *extended cluster*, and $x_{n+1}, \ldots, x_{n+m}$ are *stable variables*.

Cluster transformations

- The *adjacent cluster* in direction $k \in [1, n]$:

$$\mathbf{x}_k = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\},$$

where the new cluster variable x'_k is given by the *exchange relation*

$$x_k x'_k = \prod_{\substack{1 \le i \le n+m \\ b_{ki} > 0}} x_i^{b_{ki}} + \prod_{\substack{1 \le i \le n+m \\ b_{ki} < 0}} x_i^{-b_{ki}};$$

- $\widetilde{B}'$ is obtained from $\widetilde{B}$ by a *matrix mutation* in direction k :

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \dfrac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

- $\Sigma' = (\mathbf{x}', \widetilde{B}')$ is called *adjacent* to Σ in direction k. Two seeds are *mutation equivalent* if they can be connected by a sequence of pairwise adjacent seeds.

The *cluster algebra* (of *geometric type*) $\mathcal{A} = \mathcal{A}(\widetilde{B})$ associated with Σ is generated by all cluster variables in all seeds mutation equivalent to Σ .

Laurent phenomenon

All cluster variables are Laurent polynomials in initial cluster variables

Positivity Conjecture

All these Laurent polynomials have positive integer coefficients

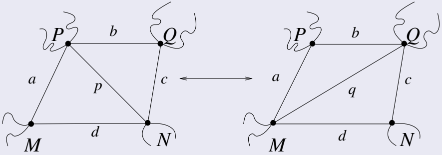
Examples of Cluster Transformations

- Short Plücker relation in $G_k(n)$

$$x_{ijJ}x_{klJ} = x_{ikJ}x_{jlJ} + x_{ilJ}x_{kjJ}$$

for $1 \leq i < k < j < l \leq m$, $|J| = k - 2$.

- Whitehead moves and Ptolemy relations in Decorated Teichmüller space:



$$f(p)f(q) = f(a)f(c) + f(b)f(d)$$

Compatible Poisson Brackets

A Poisson bracket $\{\cdot, \cdot\}$ on $\mathfrak{F}$ is *compatible* with the cluster algebra $\mathcal{A}$ if, for any extended cluster $\widetilde{\mathbf{x}} = (x_1, \dots, x_{n+m})$

$$\{x_i, x_j\} = \omega_{ij} x_i x_j \ ,$$

where $\omega_{ij} \in \mathbb{Z}$ are constants for all $i, j \in [1, n+m]$.

Theorem (Gekhtman-Sh.-Vainshtein)

Assume that $\widetilde{B}$ is of full rank. Then there is a Poisson bracket compatible with $\mathcal{A}(\widetilde{B})$.

Key Observation

Given a Poisson manifold and a coordinate system (x_i) with Poisson relations as above (log-canonical), one can try to construct a cluster algebra.

Strategy for $\mathcal{R}_n$

- Initial cluster ... inspired by a generalization of the inverse moment problem.
- Compatible Poisson bracket ... inspired by the Hamiltonian structure for Toda flows.

Standard Poisson-Lie Structure

Sklyanin bracket on GL_n

$$\{f_1, f_2\}_R(x) =$$
$$\frac{1}{2} Tr\left(R(\nabla f_1(x)\ x)\ \nabla f_2(x)\ x\right) - \frac{1}{2} Tr\left(R(x\ \nabla f_1(x))\ x\ \nabla f_2(x)\right)\ ,$$

where R solves MCYBE. The simplest classical R-matrix :

$$R_0(\xi) = (\pi_+ - \pi_-)(\xi) = \xi_+ - \xi_- = (\text{sign}(j-i)\xi_{ij})_{i,j=1}^n .$$

Toda Flows

$$\frac{d}{dt}X = \left[X,\ -\frac{1}{2}\left(\pi_+(X^k) - \pi_-(X^k)\right)\right]$$

Coxeter Double Bruhat Cells

For $u, v \in S_n$, the double Bruhat cell is defined as

$$G^{u,v} = B_+ u B_+ \cap B_- v B_-.$$

Coxeter double Bruhat cell $\iff$ u and v are products of $n-1$ distinct elementary reflections

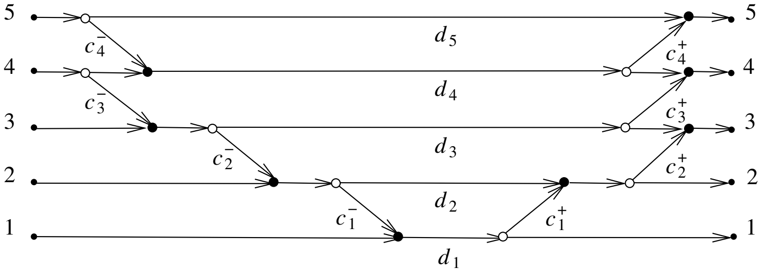
- $G^{u,v} \subset GL_n$ are Poisson submanifolds w.r.t. the Sklyanin bracket
- $G^{u,v}$ are invariant under conjugation by any diagonal matrix $h \in H$
- For Coxeter u, v, Toda flows restrict to a completely integrable system on $G^{u,v}/H$.

Examples

1. $u = v^{-1} = s_1 \cdots s_{n-1} \mapsto$ Jacobi matrices/Toda lattice
2. $u = v = s_1 \cdots s_{n-1} \mapsto$ relativistic Toda lattice
3. $u = v = (s_1 s_3 \cdots)(s_2 s_4 \cdots) \mapsto$ CMV matrices/Ablowitz-Ladik lattice

Parametrization of $G^{u,v}/H$ (by example)

$n = 5,\ v = s_4 s_3 s_1 s_2,\ u = s_3 s_2 s_1 s_4$



A generic element $X \in G^{u,v}$:

$$X = \begin{pmatrix} d_1 & x_{11}c_1^+ & x_{12}c_2^+ & 0 & 0 \\ c_1^- x_{11} & d_2 + c_1^- x_{12} & x_{22}c_2^+ & 0 & 0 \\ c_2^- x_{21} & c_2^- x_{22} & d_3 + c_2^- x_{23} & d_3 c_3^+ & 0 \\ c_3^- x_{31} & c_3^- x_{32} & c_3^- x_{33} & d_4 + c_3^- x_{34} & d_4 c_4^+ \\ 0 & 0 & 0 & c_4^- d_4 & d_5 + c_4^- x_{45} \end{pmatrix}.$$

Moser Map

$$X \mapsto m(\lambda; X) = ((\lambda \mathbf{1} - X)^{-1} e_1, e_1) = \frac{q(\lambda)}{p(\lambda)} = \sum_{j=0}^{\infty} \frac{h_j(X)}{\lambda^{j+1}} \in \mathcal{R}_n$$

linearizes any Coxeter-Toda flow.

Inverse Moser Map

- Given $m(\lambda)$, define Hankel determinants

$$\Delta_i^{(l)} = \det\left(h_{\alpha+\beta+l-i-1}\right)_{\alpha,\beta=1}^{i}$$

- Given (u, v), define an n-tuple

$$(\varkappa_1, \ldots, \varkappa_n) \ : \ \varkappa_1 = 0, \ \varkappa_{i+1} - \varkappa_i \in \{0, \pm 1\}$$

Theorem

For $X \in G^{u,v}$

$$d_i = \frac{\Delta_i^{(\varkappa_i+1)}\Delta_{i-1}^{(\varkappa_{i-1})}}{\Delta_i^{(\varkappa_i)}\Delta_{i-1}^{(\varkappa_{i-1}+1)}},$$

$$c_i := c_i^+ c_i^- = \frac{\Delta_{i-1}^{(\varkappa_{i-1})}\Delta_{i+1}^{(\varkappa_{i+1})}}{\left(\Delta_i^{(\varkappa_i+1)}\right)^2}\left(\frac{\Delta_{i+1}^{(\varkappa_{i+1}+1)}}{\Delta_{i+1}^{(\varkappa_{i+1})}}\right)^{\varepsilon_{i+1}}\left(\frac{\Delta_{i-1}^{(\varkappa_{i-1}+1)}}{\Delta_{i-1}^{(\varkappa_{i-1})}}\right)^{2-\varepsilon_i}$$

where $\varepsilon_i = \varkappa_i - \varkappa_{i+1} + 1$.

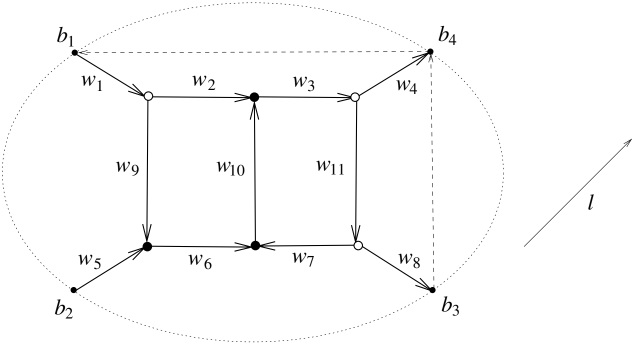
F

unctions $x_{0i} = x_{0i}(u,v) = \Delta_i^{(\varkappa_i)}$, $\quad x_{1i} = x_{1i}(u,v) = \Delta_i^{(\varkappa_i+1)}$ $(i = 1, \ldots, n-1)$ are regular functions on $\mathcal{R}_n$, that will serve as an initial cluster. Functions $x_{0n} = \Delta_n^{(n-1)}, x_{1n} = \frac{\Delta_n^{(n-2)}}{\Delta_n^{(n-1)}} = \frac{1}{\det X}$ will serve as stable variables.

Weighted network N in a disk

- $G = (V, E)$ - directed planar graph drawn inside a disk with the vertex set V and the edge set E.
- Exactly n of its vertices are located on the boundary circle of the disk. They are labelled counterclockwise $b_1, \dots, b_n$ and called boundary vertices.
- Each boundary vertex is labelled as a source or a sink.
  $I = \{i_1, \dots, i_k\} \subset [1, n]$ is a set of sources. $J = [1, n] \setminus I$ - set of sinks.
- All the internal vertices of G have degree 3 and are of two types: either they have exactly one incoming edge, or exactly one outcoming edge. The vertices of the first type are called *white*, those of the second type, *black*.
- To each $e \in E$ we assign a weight w_e.

Example



Boundary Measurements

Paths and cycles

A path P in N is an alternating sequence $(v_1, e_1, v_2, \ldots, e_r, v_{r+1})$ of vertices and edges such that $e_i = (v_i, v_{i+1})$ for any $i \in [1, r]$.
A path is called a cycle if $v_{r+1} = v_1$

Concordance number $\approx$ rotation number

For a closed oriented polygonal plane curve C , let e' and e'' be two consequent oriented segments of C, v – their common vertex. Let l be an arbitrary oriented line. Define $c_l(e', e'') \in \mathbb{Z}/2\mathbb{Z}$ in the following way:
$c_l(e', e'') = 1$ if the directing vector of l belongs to the interior of the cone spanned by e' and e'' , $c_l(e', e'') = 0$ otherwise.
Define $c_l(C)$ as the sum of $c_l(e', e'')$ over all pairs of consequent segments in C. $c_l(C)$ does not depend on l, provided l is not collinear to any of the segments in C. The common value of $c_l(C)$ for different choices of l is denoted by $c(C)$ and called the concordance number of C.

Example

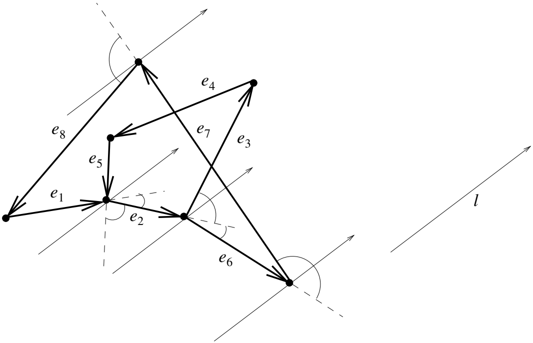


Figure: $c_l(e_1, e_2) = c_l(e_5, e_2) = 0$; $c_l(e_2, e_3) = 1$, $c_l(e_2, e_6) = 0$; $c_l(e_6, e_7) = 1$, $c_l(e_7, e_8) = 0$

Boundary Measurements (cont'd)

Weight of a path

A path P between a source b_i and a sink b_j $\rightsquigarrow$ a closed polygonal curve $C_P = P\cup$ counterclockwise path btw. b_j and b_i along the boundary.

The weight of P :

$$w_P = (-1)^{c(C_P)-1} \prod_{e \in P} w_e.$$

Boundary Measurement

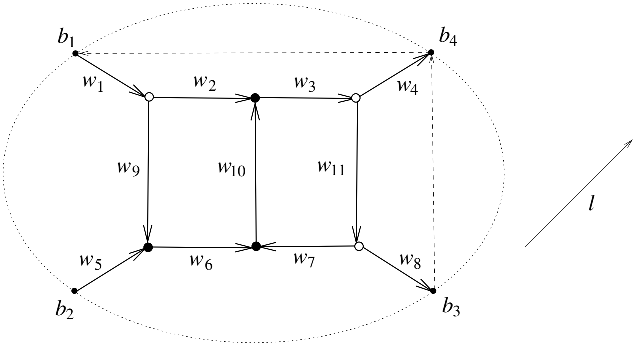
$M(i,j) = \sum$ weights of all paths starting at b_i and ending at b_j

Proposition (Postnikov)

Each boundary measurement is a rational function in the weights w_e admitting a subtraction-free rational expression.

Boundary Measurement Matrix: $M_N = (M(i_p, j_q))$

Example



$$M_N = \begin{pmatrix} \dfrac{w_3 w_4 w_5 w_6 w_{10}}{1 + w_3 w_7 w_{10} w_{11}} & \dfrac{w_3 w_5 w_6 w_8 w_{11}}{1 + w_3 w_7 w_{10} w_{11}} \\ \dfrac{w_1 w_3 w_4 (w_2 + w_6 w_9 w_{10})}{1 + w_3 w_7 w_{10} w_{11}} & \dfrac{w_1 w_3 w_8 w_{11} (w_2 + w_6 w_9 w_{10})}{1 + w_3 w_7 w_{10} w_{11}} \end{pmatrix}.$$

Network concatenation and the standard Poisson-Lie structure

The construction above is due to `Postnikov` – extension of results by `Karlin-McGregor, Lindström, Gessel-Viennot, Brenti, Berenshtein-Fomin-Zelevinsky,...` motivated by the study of total positivity.

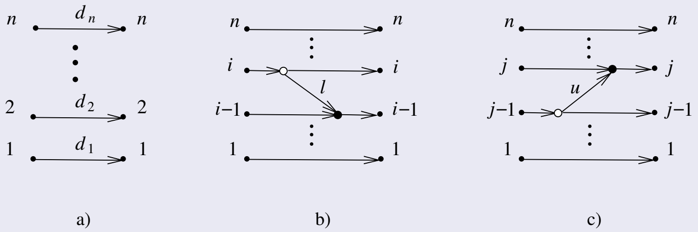
Recall:

If sources and sinks of N do not interlace:

- place N in a square rather than in a disk, with all sources located on the left side and sinks on the right side of the square
- re-label sources/sinks from bottom to top
- $M_N \rightsquigarrow A_N = M_N W_0$
  where $W_0 = (\delta_{i,m+1-j})_{i,j=1}^m$ is the matrix of the longest permutation
- concatenation of networks $\Longleftrightarrow$ matrix multiplication

Building Blocks:

Diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ and elementary bidiagonal matrices $E_i^-(\ell) := \mathbf{1} + \ell e_{i,i-1}$ and $E_j^+(u) := \mathbf{1} + u e_{j-1,j}$ correspond to:



Concatenation of several networks these types with appropriately chosen order and weights can be used to describe any element of GL_n:

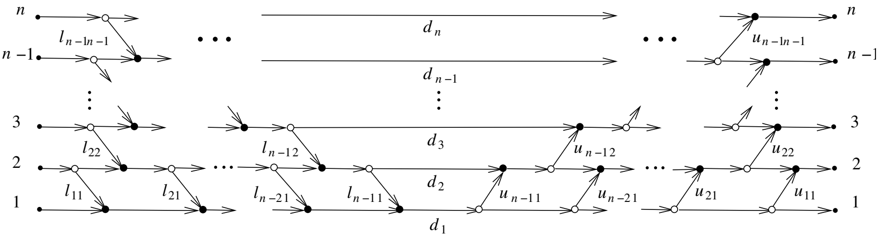


Figure: Generic planar network $\Longleftrightarrow$ Generic matrix

Standard Poisson-Lie Structure

Poisson-Lie Groups

$(\mathcal{G}, \{\cdot, \cdot\})$ is called a *Poisson-Lie group* if the multiplication map

$$\mathfrak{m} : \mathcal{G} \times \mathcal{G} \ni (x, y) \mapsto xy \in \mathcal{G}$$

is Poisson.

Building Blocks Again

Restriction of $\{\cdot,\cdot\}_{R_0}$ to subgroups

$$B_+^{(i)} = \left\{ \mathbf{1}_{i-1} \oplus \begin{pmatrix} d & c \\ 0 & d^{-1} \end{pmatrix} \oplus \mathbf{1}_{n-i-1} \right\}, B(i)_- = \left\{ \mathbf{1}_{i-1} \oplus \begin{pmatrix} d & 0 \\ c & d^{-1} \end{pmatrix} \oplus \mathbf{1}_{n-i-1} \right\}$$

is

$$\{d, c\}_{R_0} = \frac{1}{2} dc.$$

Can be described in terms of adjacent edges in corresponding networks !

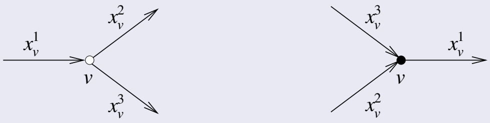
General networks in a disc ?

Concatenation

Glue a segment of the boundary of one disc to a segment of the boundary of another disc so that each source/sink in the first segment is glued to a source/sink of the second.

Half-edge weights

- Internal vertex $v \rightsquigarrow \mathbb{R}^3_v = \{x^1_v, x^2_v, x^3_v\}$:



- Equip each $\mathbb{R}^3_v$ with a Poisson bracket $\rightsquigarrow \mathcal{R} = \oplus_v \mathbb{R}^3_v$ inherits $\{\cdot,\cdot\}_\mathcal{R} = \oplus_v \{\cdot,\cdot\}_v$.

Universal Poisson Brackets

$\{\cdot,\cdot\}_{\mathcal{R}}$ is universal if

1. Each of $\{\cdot,\cdot\}_v$ depends only on the color of the vertex v.
2. The natural map $\mathcal{R} \to \mathbb{R}^{Edges}$: edge weight $=$ product of half-edge weights induces a Poisson structure on $\mathbb{R}^{Edges}$
   This is an analog of the Poisson–Lie property

Proposition

Universal Poisson brackets $\{\cdot,\cdot\}_{\mathcal{R}}$ a 6-parametric family defined by relations

$$\{x_v^i, x_v^j\}_v = \alpha_{ij} x_v^i x_v^j, \quad i,j \in [1,3], i \neq j,$$

at each white vertex v and

$$\{x_v^i, x_v^j\}_v = \beta_{ij} x_v^i x_v^j, \quad i,j \in [1,3], i \neq j,$$

at each black vertex v.

Poisson Properties of the Boundary Measurement Map

Theorem

1. For any network N in a square with n sources and n sinks and for any choice of α_{ij}, β_{ij} the map $A_N : \mathbb{R}^{Edges} \to \mathrm{Mat}_n$ is Poisson w. r. t. the Sklyanin bracket associated with the R-matrix

$$R_{\alpha,\beta} = \frac{\alpha-\beta}{2}(\pi_+ - \pi_-) + \frac{\alpha+\beta}{2} S\pi_0,$$

where $S(e_{jj}) = \sum_{i=1}^{k} \mathrm{sign}(j-i) e_{ii}, \quad j = 1, \ldots, k.$

Cluster algebra structure on boundary measurements

Transformations preserving boundary measurements

- Gauge group acts on the space of edge weights :
- Elementary transformations :

Edge weights modulo gauge group are Face Coordinates

Face weight y_f of a face f is a Laurent monomial :

$$y_f = \prod_{e \in \partial f} w_e^{\gamma_e},$$

where $\gamma_e = 1$ if the direction of e is compatible with the orientation of the boundary ∂f and $\gamma_e = -1$ otherwise.

Elementary transformations

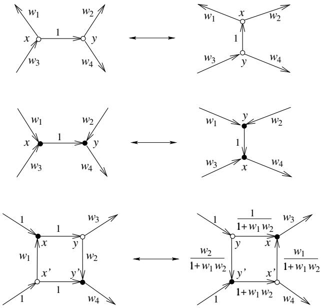


Figure: Elementary transformations

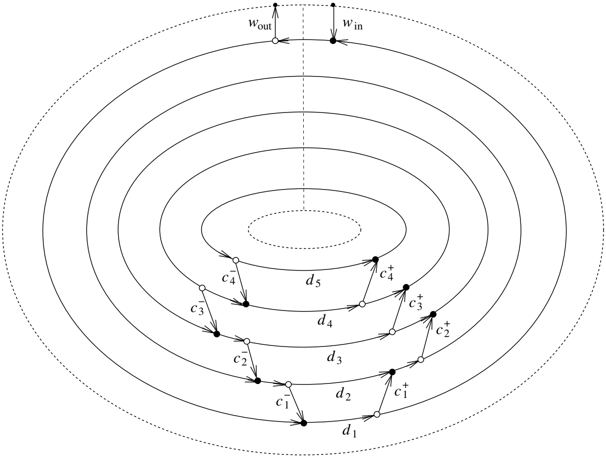
- Elementary transformation of type 3 is "Y-system type" cluster transformation for face coordinates:
- Universal Poisson structure is compatible with cluster algebra structure :

Networks on non-simply-connected higher genus surfaces?

- Simplest case: networks on a cylinder

1. Images of the boundary measurement map are rational matrix-valued functions
2. Universal Poisson brackets on edge weights lead to trigonometric R-matrix brackets in the case when sources and sinks are located at opposite ends of a cylinder
3. In the case of only one source and one sink, both located at the same component of the boundary, the corresponding Poisson bracket is relevant in the study of Toda lattices and allows to construct a cluster algebra structure in the space of rational functions.

Graphical interpretation of $m(\lambda)$ (by example)



- The weight of a path P between a source a sink:

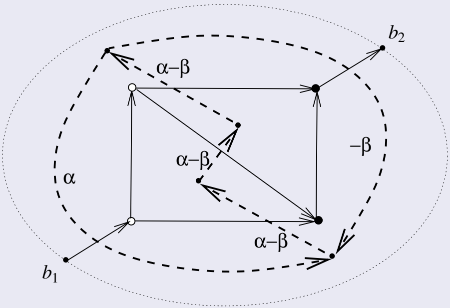
$$w_P = \pm\lambda^{ind(P\cap|)} \prod_{edge \in P} weight(edge)$$

- Boundary Measurement $:= \sum_P w_P = w_{in} w_{out} m(\lambda, X)$

Poisson structure on face coordinates

$$\{y_f, y_{f'}\} = \omega_{ff'} y_f y_{f'}.$$

where $\omega_{ff'}$ are determined by the dual graph



Theorem

Induced Poisson bracket on $\mathcal{R}_n$ is

$$\{M(\lambda), M(\mu)\} = -(\lambda\ M(\lambda) - \mu\ M(\mu))\,\frac{M(\lambda) - M(\mu)}{\lambda - \mu}.$$

It coincides with the one induced by the quadratic Poisson structure for Toda flows.

Corollary

Hankel determinants that form the initial cluster are log-canonical.

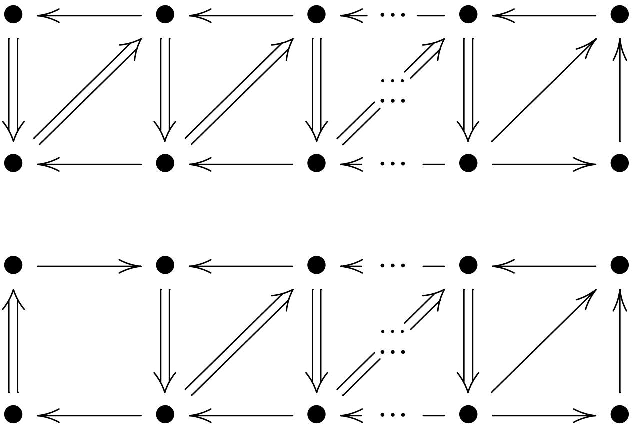
W

e have all the ingredients to build a cluster algebra for $\mathcal{R}_n$.

Cluster transformations are modeled on Jacobi's determinantal identity

$$\Delta_{i+1}^{(l)}\Delta_{i-1}^{(l)} = \Delta_i^{(l-1)}\Delta_i^{(l+1)} - \left(\Delta_i^{(l)}\right)^2,$$

Rules of transformations are defined by graphs of the form:

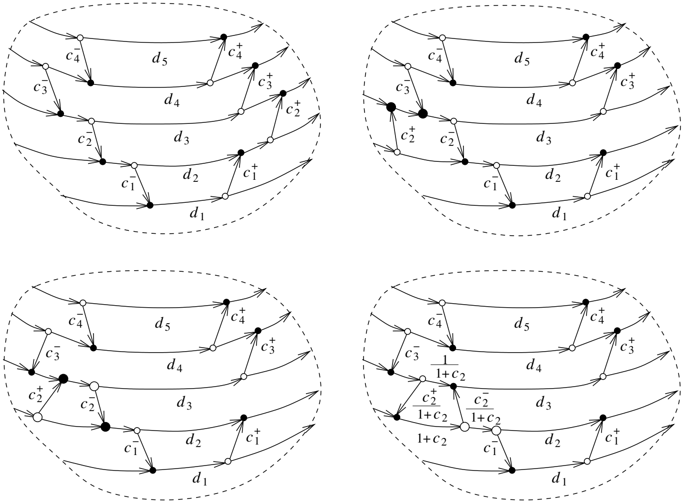


Theorem

(i) *The cluster algebra $\mathcal{A}$ we constructed does not depend on (u, v).*
(ii) *The localization of $\mathcal{A}$ with respect to the stable variables x_{2n-1}, x_{2n} coincides with the ring of regular functions on $\mathcal{R}_n$.*

Corollary

Bäcklund-Darboux transformations between Coxeter-Toda lattices $\Longleftrightarrow$ cluster transformations in $\mathcal{A}$ $\Longleftrightarrow$ network transformations: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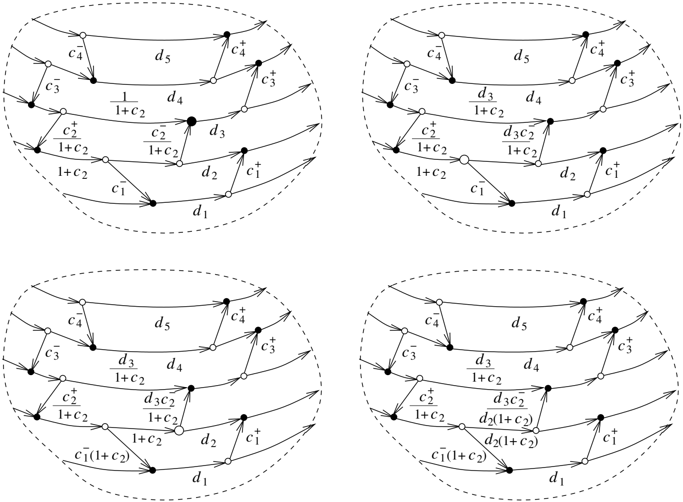
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$d_2(1+c_2)$
$c_1^-(1+c_2)$
c_1^+
d_1

Applications

- A simpler proof of the positivity conjecture for A_n Q-system - a rational recurrence appearing in the study of the XXX-model. (First proved by `DiFrancesco and Kedem`)
- Explicit formulas for Bäcklund-Darboux transformations. E. g.

Theorem

If off-diagonal entries a_i , and diagonal entries b_i of the Jacobi matrix L evolve according to the equations of the Toda lattice, then functions

$$d_i = \frac{(L^{2-i})_{[i]}(L^{2-i})_{[i-1]}}{(L^{1-i})_{[i]}(L^{3-i})_{[i-1]}}, \quad c_i = a_i \frac{(L^{-i})_{[i+1]}(L^{3-i})_{[i-1]}}{(L^{1-i})_{[i]}(L^{2-i})_{[i]}}$$

solve the relativistic Toda lattice.

References

1. Generalized Bäcklund-Darboux transformations for Coxeter-Toda flows from a cluster algebra perspective, *Acta Mathematica* , arXiv:0906.1364
2. Cluster Algebras and Poisson Geometry, *AMS Surveys and Monographs* .

Thank you!