A THEOREM OF THE SYLOW TYPE FOR INFINITE GROUPS

B. V. Kazachkov

UDC 519.4

In 1945, A. G. Kurosh, in [1] generalized as follows the well-known theorem of Ditsman, Kurosh, and Uzkov on the conjugacy and number of Sylow subgroups [2] (see, also, [3], §54).

The Kurosh Theorem. If group $\mathfrak{G}$ contains a p-subgroup A with a finite number of conjugate subgroups then, for each p-subgroup B of group $\mathfrak{G}$, one can specify at least one subgroup which is conjugate to A and which, together with B, generates a p-subgroup. If, in addition, none of the subgroups conjugate to A generates, in conjunction with A, a p-subgroup, then the number of subgroups conjugate to A is congruent to 1 module p (p a prime number).

In the present paper we shall establish conditions under which the basic assertion of the Kurosh Theorem also turns out to be true for the π-subgroups of group $\mathfrak{G}$. Here, U is an arbitrary, but fixed, set of prime numbers, either finite or infinite.

1. CONJUGACY OF π-SYLOW SUBGROUPS

THEOREM 1. Let each finite epimorphic image of group $\mathfrak{G}$ and any of its subgroups possess the property of conjugacy of π-Sylow subgroups. Then, if $\mathfrak{G}$ contains a π-subgroup A with a finite number of conjugate subgroups then, for each π-subgroup B of group $\mathfrak{G}$, one can specify at least one subgroup, conjugate to A and, together with B, generating a π-subgroup.

Proof. Let $\mathfrak{G}$ contain a finite class $\langle A\rangle$ of subgroups (not necessarily Sylow), conjugate with the π-subgroup A:

$$A,\ c_2^{-1}Ac_2, \ldots, c_\lambda^{-1}Ac_\lambda, \quad c_i \in \mathfrak{G}, \quad i = 2, 3, \ldots, \lambda.$$

To this class corresponds a finite class of conjugate normalizers of finite index $\langle N_A\rangle$. We denote by σ the intersection of all its subgroups. By the Poincaré Theorem, invariant subgroup σ also is of finite index in $\mathfrak{G}$. The intersection of any subgroup $c_i^{-1}Ac_i$ of class $\langle A\rangle$ with subgroup σ is invariant, both in $c_i^{-1}Ac_i$ and in σ. This follows, in view of the invariance of $c_i^{-1}Ac_i$ in $(c_i^{-1}Ac_i)\sigma$, from the isomorphism formula:

$$\frac{(c_i^{-1}Ac_i)\,\sigma}{\sigma} \simeq \frac{c_i^{-1}Ac_i}{(c_i^{-1}Ac_i)\cap\sigma} \quad \text{and} \quad \frac{(c_i^{-1}Ac_i)\,\sigma}{c_i^{-1}Ac_i} \simeq \frac{\sigma}{(c_i^{-1}Ac_i)\cap\sigma}.$$

Thus, to class $\langle A\rangle$ corresponds the collection of π-subgroups which are invariant in σ, i.e., the intersections of the terms of the class with subgroup σ, with the collection being, in all of group $\mathfrak{G}$, also a complete class $\langle A\cap\sigma\rangle$ of π-subgroups which are conjugate by means of these same elements c_i:

$$A\cap\sigma,\ c_2^{-1}(A\cap\sigma)c_2, \ldots, c_\lambda^{-1}(A\cap\sigma)c_\lambda.$$

It is possible to have coincident subgroups among these latter, without excluding the case that all are equal to one another. We denote their product by D. It is clear that $D \lhd \mathfrak{G}$, since it is generated by subgroups of a complete class.

Let B be some π-subgroup in $\mathfrak{G}$. We form the π-subgroup BD. If it is not yet a π-Sylow subgroup there then exists in $\mathfrak{G}$ a π-Sylow subgroup $\overline{B}$ containing it. By the isomorphism theorem, $\overline{B}\sigma/\sigma \simeq \overline{B}/\overline{D}$, where $\overline{D} = \overline{B}\cap\sigma$ is a π-subgroup having finite index in $\overline{B}$ and containing the subgroups D and $B\cap\sigma$. Invariance of $\overline{D}$ in σ and, even more to the point, in $\mathfrak{G}$, is not mandatory. We now show, however, that for any

Tomsk Pedagogical Institute. Translated from Matematicheskie Zametki, Vol. 13, No. 3, pp. 431-436, March, 1973. Original article submitted July 7, 1972.

π-subgroup $c_i^{-1}Ac_i$ of class $\langle A\rangle$ $\bar{D} \lhd (c_i^{-1} \cdot Ac_i)\bar{D}$. It suffices to remark the possibility of the following decomposition of factor group $(c_i^{-1}Ac_i)\sigma/(c_i^{-1}Ac_i) \cap \sigma$ into the direct product of its own subgroups:

$$\frac{(c_i^{-1}Ac_i)\,\sigma}{(c_i^{-1}Ac_i)\cap\sigma} = \frac{c_i^{-1}Ac_i}{(c_i^{-1}Ac_i)\cap\sigma} \times \frac{\sigma}{(c_i^{-1}Ac_i)\cap\sigma},$$

since we can then conclude that each element of the factor group $c_i^{-1}Ac_i/(c_i^{-1}Ac_i) \cap \sigma$ commutes with each element of factor group $\sigma/(c_i^{-1}Ac_i) \cap \sigma$ and, consequently, with each element of subgroup $\bar{D}/(c_i^{-1}Ac_i \cap)\sigma$ of the latter.

We denote by $\mathfrak{H}$ the subgroup, invariant in $\mathfrak{G}$, generated by all the subgroups of complete class $\langle A\rangle$. It follows from what has just been proven that $\bar{D} \lhd \mathfrak{H}\bar{B}$. Subgroups of class $\langle A\rangle$, conjugate in $\mathfrak{G}$, need not necessarily be conjugate in $\mathfrak{H}\bar{B}$. We therefore identify in class $\langle A\rangle$ the class of subgroups conjugate to A in $\mathfrak{H}\bar{B}$:

$$A,\ h_2^{-1}Ah_2, \ldots, h_k^{-1}Ah_k, \text{ when } h_i \in \mathfrak{H}\bar{B}, \quad i = 2, 3, \ldots, k,\ k \leqslant \lambda.$$

To this latter corresponds the class of subgroups in factor group $\mathfrak{H}\bar{B}/\bar{D}$:

$$\frac{A\bar{D}}{\bar{D}},\quad \frac{(h_2^{-1}Ah_2)\,\bar{D}}{\bar{D}}, \ldots, \quad \frac{(h_k^{-1}Ah_k)\,\bar{D}}{\bar{D}}.$$

We now show that each of these factor groups is finite. Let us choose an arbitrary one of them: $(h_i^{-1}Ah_i)\bar{D}/\bar{D}$. Again by the isomorphism theorem, we have the two relationships $\frac{(h_i^{-1}Ah_i)\,\bar{D}}{\bar{D}} \simeq \frac{h_i^{-1}Ah_i}{(h_i^{-1}Ah_i)\cap\bar{D}}$ and $\frac{(h_i^{-1}Ah_i)\,\sigma}{\sigma} \simeq \frac{h_i^{-1}Ah_i}{(h_i^{-1}Ah_i)\cap\sigma}$. In the second relationship, the factor group on the right is finite, so that the factor group on the right (and, this means, also on the left) of the first relationship is finite, since $(h_i^{-1}Ah_i)\cap\sigma = (h_i^{-1}Ah_i)\cap\bar{D}$, which follows from the inclusion $(h_i^{-1}Ah_i)\cap\sigma \subset D \subset \bar{D}$. We have obtained a finite class $\langle A\bar{D}/\bar{D}\rangle$ of conjugate finite subgroups in $\mathfrak{H}\bar{B}/\bar{D}$, i.e., an invariant set of elements, generating in $\mathfrak{H}\bar{B}/\bar{D}$ a finite invariant subgroup. We denote it by $T/\bar{D}$ (we have not excluded its coincidence with $\mathfrak{H}\bar{B}/\bar{D}$). Thus, the factor group $\mathfrak{H}\bar{B}/\bar{D}$ contains the subgroups $\bar{B}/\bar{D}$ and $T/\bar{D}$, with $T/\bar{D} \lhd \mathfrak{H}\bar{B}/\bar{D}$.

Let us consider further finite group $T\bar{B}/\bar{D}$. By hypothesis, all the π-Sylow subgroups in it are conjugate. Subgroup $\bar{B}$ is a π-Sylow subgroup in $\mathfrak{G}$, so that, consequently, $\bar{B}/\bar{D}$ is a π-Sylow subgroup in $T\bar{B}/\bar{D}$. Factor group $T\bar{B}/\bar{D}$ also contains $A\bar{D}/\bar{D}$, with this subgroup, if it is not yet a π-Sylow subgroup, occurring in some π-Sylow subgroup $M/\bar{D}$ of $T\bar{B}/\bar{D}$. Since, by hypothesis, $\bar{B}/\bar{D} = S^{-1}MS/\bar{D}$ or $\bar{B} = S^{-1}MS$, where $S \in T\bar{B}$, we then conclude that $\bar{B}$, a π-Sylow subgroup in $\mathfrak{G}$, contains, together with π-subgroup B, the π-subgroup $S^{-1}AS$ of class $\langle A\rangle$. But this means that the latter two π-subgroups together generate a π-subgroup. The theorem is proven.

We turn now to the corollaries of the theorem we have proven.

COROLLARY 1. If, in the conditions of Theorem 1, π-subgroup A turns out to be a π-Sylow subgroup, then the finite class $\langle A\rangle$ will exhaust all the π-Sylow subgroups of group $\mathfrak{G}$. Indeed, in this case, any π-Sylow subgroup $\mathfrak{N}$ will generate, together with any subgroup $c^{-1}Ac$ of class $\langle A\rangle$, a π-subgroup $\{\mathfrak{N}, c^{-1}Ac\}$, $c \in \mathfrak{G}$, but this means that $\mathfrak{N} = c^{-1} \cdot Ac$, since subgroup $c^{-1}Ac$ is π-Sylow, i.e., $\mathfrak{N} \subseteq \langle A\rangle$. We thus conclude that group $\mathfrak{G}$, satisfying the conditions of Theorem 1, has the property of π-conjugacy.

COROLLARY 2. Since, in any finite group, by virtue of the Sylow Theorem, all the p-Sylow subgroups are conjugate to one another, we obtain a new proof of the basic assertion of the aforementioned Kurosh Theorem if, in the conditions of Theorem 1, we take for π the set consisting of a single prime number p. We remark that thence, in its turn, follows the basic assertion of the Ditsman-Kurosh-Uzkov Theorem [2].

COROLLARY 3. Since, in any finite solvable group, in view of the theorem of P. Hall, all the π-Sylow subgroups are conjugate to one another, we then see the truth of

THEOREM 2. If locally solvable (in particular, solvable) group $\mathfrak{G}$ contains π-subgroup A possessing a finite number of conjugate subgroups then, for each π-subgroup B of group $\mathfrak{G}$, we can specify at least one subgroup which is conjugate to A and which, together with B, generates a π-subgroup.

COROLLARY 4. From Theorem 2 we readily obtain the assertion, previously proven by us (see, for example, [4]):

THEOREM 3. A locally solvable (in particular, solvable) group $\mathfrak{G}$ has the property of finite π-conjugacy.

We note that other assertions of our previously cited paper [4] also follow from Theorem 1.

2. ON THE NUMBER OF π-SYLOW SUBGROUPS

Definition. In group $\mathfrak{G}$, let all the π-Sylow subgroups form a finite class of conjugate subgroups. We call the order of this class the "Hall number" if it can be represented in the form of a product of factors each of which is congruent to 1 modulo some prime number of π and is a power of a prime number dividing one of the finite indices of the principal series of the group (the principal series between group $\mathfrak{G}$ itself and one of its subgroups $\mathfrak{H}$).

We can now formulate the fourth assertion of Theorem 9.3.1 of [5] as follows: in a finite solvable group of order mn, (m, n) = 1, the number of subgroups of order m is the "Hall number."

THEOREM 4. If each finite epimorphic image of a group and any of its subgroups possesses the property of conjugacy of π-Sylow subgroups, and their number is the "Hall number," it then follows from the existence in $\mathfrak{G}$ of a finite class of conjugate π-Sylow subgroups that the order of this class is the "Hall number."

Proof. Let group $\mathfrak{G}$, satisfying the conditions of the theorem, possess a finite class of π-Sylow subgroups conjugate with $\mathfrak{M}$:

$$\mathfrak{M},\ c_2^{-1}\mathfrak{M}c_2,\ \ldots,\ c_\lambda^{-1}\mathfrak{M}c_\lambda,\ c_i \in \mathfrak{G}, \quad i = 2, 3, \ldots, \lambda.$$

To it corresponds a finite class of normalizers which are conjugate by means of the same elements $\langle N_{\mathfrak{M}} \rangle$. We denote the intersections of the subgroups of classes $\langle \mathfrak{M} \rangle$ and $\langle N_{\mathfrak{M}} \rangle$ by D and σ respectively. By the foregoing deduction from Theorem 1, class $\langle \mathfrak{M} \rangle$ exhausts all the π-Sylow subgroups in $\mathfrak{G}$. Since invariant subgroup σ has finite index in $\mathfrak{G}$ then, by the isomorphism theorem, we obtain a finite class in $\mathfrak{G}$/D of finite π-Sylow subgroups $\mathfrak{M}/D,\ c_2^{-1}\mathfrak{M}c^2/D, \ldots,\ D_{c_\lambda^{-1}}\mathfrak{M}c_\lambda/D$. This class generates the finite subgroup $\mathfrak{H}/D \lhd \mathfrak{G}/D$.

By hypothesis, the number of π-Sylow subgroups in $\mathfrak{H}$/D is the "Hall number," i.e., $\lambda = n_1 n_2 \ldots n_k$, where each factor is congruent to 1 modulo some prime number of π and is a power of a prime which divides one of the indices of the principal series

$$\frac{\mathfrak{H}}{D} \supset \frac{\mathfrak{H}_1}{D} \supset \ldots \supset \frac{\mathfrak{H}_{l-1}}{D} \supset \frac{\mathfrak{H}_l}{D} = \frac{D}{D}$$

and, consequently, in view of the invariance of $\mathfrak{H}$ and D in $\mathfrak{G}$, one of the indices of the principal series (from $\mathfrak{G}$ to D):

$$\mathfrak{G} \subset \mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \ldots \supset \mathfrak{H} \supset \mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \ldots \supset \mathfrak{H}_l = D.$$

We conclude that the "Hall number" λ in $\mathfrak{H}$/D is also the "Hall number" in the entire group $\mathfrak{G}$.

COROLLARY. In view of the aforementioned theorem of P. Hall on finite solvable groups, we have the validity of

THEOREM 5. In a locally solvable (and, in particular, solvable) group possessing a finite class of conjugate π-Sylow subgroups, the number of such subgroups is the "Hall number."

Remark 1. In a finite group the number of p-Sylow subgroups is not necessarily the "Hall number" so that, therefore, by taking as set π a single prime number p, we do not obtain for an arbitrary group the assertion of the Kurosh Theorem on the number of p-Sylow subgroups whereas, in the case of a solvable group $\mathfrak{G}$, the stronger assertion on the number of p-Sylow subgroups is valid.

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